Silvio Ghilardi
Marek Zawadowski

Undefinability of Propositional Quantifiers in the Modal System S4

Abstract. We show that (contrary to the parallel case of intuitionistic logic, see [7], [4]) there does not exist a translation from $S4^2$ (the propositional modal system $S4$ enriched with propositional quantifiers) into $S4$ that preserves provability and reduces to identity for Boolean connectives and $\Box$.

1980 Mathematics subject classifications. Primary 03B45, 06E25; Secondary 18C10.

Introduction

A.M. Pitts showed in [7] that there exists a translation from propositional intuitionistic logic with propositional quantifiers into ordinary propositional intuitionistic logic that reduces to identity for the quantifiers-free fragment. The result was obtained also in [4] by sheaf duality and some semantical techniques, inspired by analogous previous work of [8] on provability logic. These semantical techniques, which are nothing but applications of kinds of Ehrenfeucht games, originated from K. Fine's papers [1], [2] and they are basically used in order to deduce the existence of maximum and minimum interpolants [8] as well as the definability of propositional quantifiers [4] from a statement (called 'expansion lemma' in [8], 'combinatorial lemma' in [4]) about the games corresponding to the logic in question. In fact, this statement, which appears in this paper too (see (1) below), is *implied* by the definability of propositional quantifiers themselves. We show that it *fails* for the modal system $S4$, so to this logic Pitts' result *cannot be extended.*

Let us formulate the problem more precisely. $S4^2$ allows formulas built up from propositional letters $p_1, p_2, \ldots$ using the boolean connectives, the modal connective $\Box$ and the propositional quantifiers $\exists p_i$ (we use the notation $\alpha(\vec{p})$ to mean that the formula α contains free variables only from the list $\vec{p}$). $S4^2$ is axiomatized by taking any standard set of axioms and rules for $S4$ (see e.g. [5]) and, in addition, the axiom schemata $\alpha(\vec{p}, \beta/p_i) \to \exists p_i \alpha(\vec{p}, p_i)$ (with usual restrictions for substitution) and the following rule: from $\alpha(\vec{p}, p_i) \to \beta(\vec{p})$ infer $\exists p_i \alpha(\vec{p}, p_i) \to \beta(\vec{p})$ (where it is clear that p_i must not occur in

Presented by **Wiesław Dziobiak**; *Received October* 20, 1994; *Revised March* 10, 1995

Studia Logica **55**: 259–271, 1995.

the list $\vec{p}$). What we are looking for is a translation τ from formulas of $S4^2$ into formulas of $S4$ that satisfies the following requirements: $\tau(p) = p$ for atomic p, $\tau(\alpha \wedge \beta) = \tau(\alpha) \wedge \tau(\beta)$, $\tau(\alpha \vee \beta) = \tau(\alpha) \vee \tau(\beta)$, $\tau(\neg\alpha) = \neg\tau(\alpha)$, $\tau(\Box\alpha) = \Box\tau(\alpha)$ and , finally,

$$\vdash_{S4^2} \alpha \implies \vdash_{S4} \tau(\alpha).$$

In [3] (where the corresponding question was positively solved for the modal system K) it is shown in details (but from a slightly different point of view) that the existence of a translation is equivalent to the algebraic fact that the 'cilindrification' morphisms among finitely generated free algebras have a left adjoint satisfying an additional 'stability' condition, the so-called 'Beck-Chevalley condition'.[1] We show here how to prove what is relevant for the purposes of the present paper, namely the fact that the existence of a translation satisfying the above requirements *implies* the existence of the adjoints to the cilindrification morphisms. Simultaneously, we recall some elementary notions and fix notation.

An *interior algebra* is a Boolean algebra B endowed with a unary operator $\Box : B \longrightarrow B$ satisfying the equations: $\Box(x \wedge y) = \Box x \wedge \Box y$, $\Box\top = \top$, $\Box x \leq x$, $\Box x \leq \Box\Box x$. Such algebras appear as Lindenbaum algebras for the modal system $S4$. Typical examples of interior algebras are obtained from preordered sets, i.e. from sets endowed with a reflexive and transitive relation. If $\langle P, \leq_P \rangle$ is such a preorder, we call $\mathcal{P}^\Box(P, \leq_P)$ or simply $\mathcal{P}^\Box(P)$ the interior algebra given by the (complete) power-set Boolean algebra $\mathcal{P}(P)$ endowed with the interior operator $\Box$ defined by:[2]

$$\Box(S) = \{p \mid \forall q \in P \ (q \leq_P p \ \Rightarrow \ q \in S)\}$$

for every $S \subseteq P$. By cilindrification morphisms, we mean the morphisms among finitely generated free algebras which are the values of the free algebra functor at a set-theoretic injective map among the free generators. Otherwise said, if X, Y are finite sets, $\mathcal{F}(X)$, $\mathcal{F}(Y)$ are the related free algebras, a cilindrification morphism is any morphism $\iota : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$ for which there is an injective map $i : X \longrightarrow Y$ such that the square

[1] The existence of the adjoint only is not sufficient. To see this, note that the existence of such a translation implies interpolation. However in locally finite varieties left adjoints always exists, even if interpolation fails, since in this case 'cilindrification' is a meet-preserving map between finite lattices.

[2] Notice that we use $\leq$ where standard literature uses $\geq$.

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & Y \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ \mathcal{F}(X) & \xrightarrow[\iota]{} & \mathcal{F}(Y) \end{array}$$

commutes, where $\eta_X : X \longrightarrow \mathcal{F}(X)$ and $\eta_Y : Y \longrightarrow \mathcal{F}(Y)$ are the canonical embeddings into the free algebras (i.e. they are the X and Y-component of the unit of adjunction among the forgetful and the free algebra functor). We are particularly interested in the case in which $X = \{p_1, \ldots, p_n\}$, $Y = \{p_1, \ldots, p_n, p_{n+1}\}$ and i is the obvious inclusion from $\{p_1, \ldots, p_n\}$ into $\{p_1, \ldots, p_n, p_{n+1}\}$. The existence of the left adjoint to the related cilindrification morphism $\iota : \mathcal{F}(p_1, \ldots, p_n) \longrightarrow \mathcal{F}(p_1, \ldots, p_{n+1})$[3] means that there exists a map (not an interior algebras morphism!) in the opposite sense $\iota^* : \mathcal{F}(p_1, \ldots, p_{n+1}) \longrightarrow \mathcal{F}(p_1, \ldots, p_n)$ satisfying the conditions

$$(A1) \qquad x \leq \iota(y) \implies \iota^*(x) \leq y$$

and

$$(A2) \qquad x \leq \iota\iota^*(x)$$

(notice that the latter condition can be equivalently replaced by the right-to-left side of the former).

Now, suppose that the claimed translation τ exists. We define ι^* as follows. Pick $x \in \mathcal{F}(p_1, \ldots, p_{n+1})$; free algebras can be described as Lindenbaum algebras so that x is the equivalence class of a formula $[\alpha(p_1, \ldots, p_n)]$. We take $\tau(\exists p_{n+1}\alpha)$; as (at least in principle from the requirements we asked for τ to satisfy) we cannot exclude that the quantifiers-free formula $\tau(\exists p_{n+1}\alpha)$ contains other propositional letters apart from $p_1, \ldots, p_n$, we define $\iota^*([\alpha])$ to be the equivalence class of the formula obtained from $\tau(\exists p_{n+1}\alpha)$ by replacing these extra propositional letters, say, by $\top$. To prove $(A1)$ suppose that $x = [\alpha(p_1, \ldots, p_{n+1})]$, $y = [\beta(p_1, \ldots, p_n)]$ and that $x \leq \iota(y)$. This means that $\vdash_{S4} \alpha \to \beta$ and also that $\vdash_{S4^2} \exists p_{n+1}\alpha \to \beta$. By the properties of τ, it follows that $\vdash_{S4} \tau(\exists p_{n+1}\alpha) \to \beta$ and also that $\iota^*(x) \leq y$: to see this, notice that β contains at most $p_1, \ldots, p_n$ and on the other hand we have a proof of $\tau(\exists p_{n+1}\alpha) \to \beta$ within $S4$, so we can replace extra propositional letters by $\top$ in this proof, yielding what we need.

[3]It is evident that the existence of the left adjoint to cilindrification morphisms of this kind is equivalent to the existence of the left adjoint to cilindrification morphisms of any kind (because left adjoint do compose and isomorphisms have their inverses as adjoints).

To show $(A2)$, suppose that $x = [\alpha(p_1, \ldots, p_{n+1})]$; take a propositional letter p_k $(k > n)$ not occurring in $\tau(\exists p_{n+1}\alpha)$. Now $\alpha(p_1, \ldots, p_n, p_k/p_{n+1}) \to \exists p_{n+1}\alpha$ is a theorem of $S4^2$, so $\alpha(p_1, \ldots, p_n, p_k/p_{n+1}) \to \tau(\exists p_{n+1}\alpha)$ is a theorem of $S4$. Replace in the related proof p_j (for $j > n$, $j \neq k$) by $\top$ and then p_k by p_{n+1}. This yields $x \leq \iota\iota^*(x)$.[4]

In section 2, we prove that ι^* does not exist, so that no translation τ with the above properties exists too. The question about existence of greatest and least Craig's interpolants of two given formulas is, as well, solved negatively: for, the least interpolant of $\alpha(p_1, \ldots, p_{n+1})$ and $p_1 \vee \cdots \vee p_n \vee \top$ could be clearly used in order to define $\iota^*([\alpha])$.

1. Finite models

Here we present some background from [1], [2], with slight modifications. In the following we deal with triples $\langle P, \leq_P, \rho_P\rangle$, where P is a finite set, $\leq_P$ is a preorder relation on P and ρ_P is a specified root of P, meaning that for every $p \in P$, we have that $p \leq \rho_P$. When the context is clear our triples will be denoted simply as P and we drop also the subscript P in $\leq_P$ and ρ_P. Given such a triple P and a finite set L, an L-evaluation (or simply, an evaluation) on P is a function $u : P \longrightarrow L$.

We define for every natural number $n \geq 0$ and for every pair of L-evaluations $u : P \longrightarrow L$ and $v : Q \longrightarrow L$, the notion of *n-equivalence* (written $u \sim_n v$). This notion can be introduced in two equivalent ways.

- *First definition*: For $n = 0$ we put $u \sim_0 v$ iff $u(\rho_P) = v(\rho_Q)$. For $n > 0$, we introduce the n game over u, v. This game has two players, player I and player II. At each move, player I can choose either a point in P or a point in Q and player II must answer a point in the other preorder. The game terminates after n moves and the relevant rule is that, if $\langle p \in P, q \in Q\rangle$ is the last move played, then in the succesive move the two players can only choose points $\langle p', q'\rangle$ such that $p' \leq p$ and $q' \leq q$. If $\langle p_1, q_1\rangle, \ldots, \langle p_n, q_n\rangle$ are the point chosen at the end of the game, player II wins iff for every $i = 1, \ldots, n$, we have that $u(p_i) = v(q_i)$. We say that $u \sim_n v$ iff $u \sim_0 v$ *and player II has a winning strategy* for the n-game. The relations $\sim_n$ are clearly equivalence relations.

- *Second definition*: for $n = 0$, $\sim_0$ is defined as before and for $n > 0$ we

[4] The trick of replacing p_{n+1} by p_k and back is due to the fact that we cannot exclude in principle that p_{n+1} occurs in $\tau(\exists p_{n+1}\alpha)$ (in this sense, the above analysis is a little more general than that in [3]).

define $\sim_n$ by induction on n, by means of the following clause:

$$u \sim_n v \text{ iff } u \sim_0 v \;\&\; \forall p \in P\, \exists q \in Q\, (u_p \sim_{n-1} v_q) \;\& \\ \&\; \forall q \in Q\, \exists p \in P\, (u_p \sim_{n-1} v_q)$$

(here u_p is u restricted in the domain to the downward closed subset $\{p' \mid p' \leq p\}$ having p as specified root and similarly for v_q).

The notion of u being infinitely equivalent to v (written $u \sim_\infty v$) is defined as '$u \sim_n v$ holds for all $n \geq 0$'. As our preorders are finite, this is the same as jointly saying that $u \sim_0 v$ and that player II has a winning strategy for the above game with infinitely many moves.[5]

1. PROPOSITION *Given two L-evaluations $u : P \longrightarrow L, v : Q \longrightarrow L$, we have that:*

(i) $u \sim_{n+k} v \Rightarrow u \sim_n v$, *for all n, k;*

(ii) *for any fixed n and L, there are only finitely many equivalence classes of L-evaluations with rescpect to $\sim_n$.*

PROOF. (i) is immediate. As to (ii), we argue by induction on n. For $n = 0$, the claim is immediate; for $n > 0$, notice that with every u we can associate (in an injective way up to $\sim_n$ equivalence) a pair given by an element of L and a finite set of $\sim_{n-1}$-equivalence classes. ∎

Let $\mathcal{F}_L$ be the set of *closed* sets of L-evaluations, meaning the sets S of L-evaluations *having an index*, i.e. satisfying the following condition for some $n \geq 0$:

$$\forall u : P \longrightarrow L,\; \forall v : Q \longrightarrow L\; (u \in S \;\&\; u \sim_n v \;\Rightarrow\; v \in S).$$

$\mathcal{F}_L$ is an interior algebra with respects to set-theoretic finite unions, intersections and complements and whose interior operator is defined by

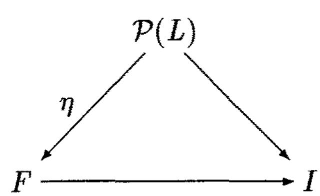
$$\Box S \;=\; \{u : P \longrightarrow L \mid \forall p \in P,\; u_p \in S\}$$

(notice that if S has index n, then $\Box S$ has index $n+1$). For every $Y \subseteq L$, $\iota_L(Y) = \{u \mid u(\rho) \in Y\}$ has index 0, so that $\iota_L : \mathcal{P}(L) \longrightarrow \mathcal{F}_L$ is a Boolean algebras morphism.

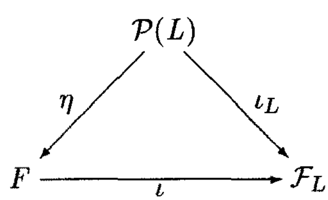
[5]Given that $u \sim_n v$ for every n, we can define an infinite strategy for player II as follows: suppose that player I picks $p \in P$ (the other case is symmetric), then for every n there exists $q_n \in Q$ such that $u_p \sim_n v_{q_n}$. As Q is finite, there is $q \in Q$ such that $u_p \sim_n v_q$ for every n; player II answers this q, etc...

2. PROPOSITION *$\mathcal{F}_L$ is the free interior algebra generated by the finite Boolean algebra $\mathcal{P}(L)$ (ι_L is the canonical embedding). In particular, if $L = \mathcal{P}(X)$, then $\mathcal{F}_L$ is the free interior algebra generated by the finite set X.*

PROOF. To understand properly the statement of the Proposition, we recall that for general reasons (left adjoint theorem, see [6]) the forgetful functor from interior algebras into Boolean algebras does have a left adjoint: the value of such left adjoint at a Boolean algebra B is called the free interior algebra generated by B. This means in particular that for our $\mathcal{P}(L)$, there exists an interior algebra F and a Boolean morphism $\eta : \mathcal{P}(L) \longrightarrow F$ such that for any other interior algebra I and any other Boolean morphism $\mathcal{P}(L) \longrightarrow I$, there exists a unique interior algebras morphism $F \longrightarrow I$ such that the triangle



commutes. In particular, as $\mathcal{F}_L$ is an interior algebra and ι_L a Boolean morphism, we have a commutative triangle:



We only have to show that ι is an isomorphism. The morphism ι is surjective because the image of ι_L generates $\mathcal{F}_L$ as an interior algebra: in fact, sets of index 0 are precisely those which are in the image of ι_L and sets S of index $n+1$ admits the following representation in terms of sets of index n:

$$S = \bigcup_{u \in S} ([u]_0 \cap \bigcap_{p \in dom(u)} \Diamond [u_p]_n \cap \Box \bigcup_{p \in dom(u)} [u_p]_n)),$$

where we used notations like $[v]_n$ for $\{v' \mid v \sim_n v'\}$ ($\Diamond$ is defined as usual as $\neg\Box\neg$).[6]

[6]Notice that all unions and intersection involved are finite, because our preordered sets are finite and because of proposition 1(ii); in particular, observe that saying that $u \sim_{n+1} u'$ is the same as saying that $[u]_0 \cap \bigcap_{p \in dom(u)} \Diamond [u_p]_n \cap \Box \bigcup_{p \in dom(u)} [u_p]_n$ is equal to $[u']_0 \cap \bigcap_{p \in dom(u')} \Diamond [u'_p]_n \cap \Box \bigcup_{p \in dom(u')} [u'_p]_n$.

We have to prove that ι is also injective. The completeness theorem with respect to finite Kripke models [5] for the modal system $S4$ ensures that free interior algebras can be embedded into a product of finite interior algebras. The same result can be extended to finitely presented interior algebras, because finitely presented interior algebras can be all described as finitely generated free interior algebras divided by a principal filter. Our F is finitely presented,[7] hence there is an embedding $\langle h_i \rangle_i : F \longrightarrow \prod_i \mathcal{P}^{\square}(P_i)$ where the P_i are finite preordered sets. The claim is proved if we show that for every i there is an interior algebras morphism $h_i' : \mathcal{F}_L \longrightarrow \mathcal{P}^{\square}(P_i)$ such that $h_i \circ \eta = h_i' \circ \iota_L$.

$$\begin{array}{ccc} \mathcal{P}(L) & \xrightarrow{\ \eta\ } & F \\ {\scriptstyle \iota_L}\downarrow & & \downarrow{\scriptstyle h_i} \\ \mathcal{F}_L & \xrightarrow[h_i']{} & \mathcal{P}^{\square}(P_i) \end{array}$$

In fact in this case, $\langle h_i \rangle_i \circ \eta = \langle h_i' \rangle_i \circ \iota_L = \langle h_i' \rangle_i \circ \iota \circ \eta$ which implies (by unicity) $\langle h_i \rangle_i = \langle h_i' \rangle_i \circ \iota$ and now ι is injective because it is the first component of an injection. We only have to find the h_i'. Notice that, by the duality theorem between finite Boolean algebras and finite sets, the Boolean morphism $h_i \circ \eta : \mathcal{P}(L) \longrightarrow \mathcal{P}(P_i)$ is the inverse image morphism u^{-1} for a map $u : P_i \longrightarrow L$, so we can simply put $h_i'(S) = \{p \in P_i \mid u_p \in S\}$ for every $S \in \mathcal{F}_L$. Preservation of operations and commutativity of the triangle

$$\begin{array}{ccccc} & & \mathcal{P}(L) & & \\ & {\scriptstyle \iota_L}\swarrow & & \searrow{\scriptstyle u^{-1}} & \\ \mathcal{F}_L & & \xrightarrow[h_i']{\qquad\qquad} & & \mathcal{P}(P_i) \end{array}$$

are easy. ■

According to the definition of free algebra, for every function $f : L \longrightarrow M$ between finite sets L, M, there is a unique interior algebras morphism $\mathcal{F}_f : \mathcal{F}_M \longrightarrow \mathcal{F}_L$ such that the square

[7] This is easily seen, but there is also a direct conceptual way of seeing it: an algebra F is finitely presented iff the representable functor $Hom(F, -)$ preserves filtered colimits and in our case by adjointness the functor $Hom(F, -)$ is isomorphic to the functor $Hom(\mathcal{P}(L), U(-))$, where U is the forgetful functor from interior algebras into Boolean algebras (notice that U preserves filtered colimits and that finite Boolean algebras are finitely presented).

$$\begin{array}{ccc} \mathcal{P}(M) & \xrightarrow{f^{-1}} & \mathcal{P}(L) \\ {\scriptstyle \iota_M}\downarrow & & \downarrow{\scriptstyle \iota_L} \\ \mathcal{F}_M & \xrightarrow[\mathcal{F}_f]{} & \mathcal{F}_L \end{array}$$

commutes. It is easily seen that:

3. PROPOSITION *$\mathcal{F}_f(S)$ is the inverse image along the composition with f, that is $\mathcal{F}_f(S) = \{u \mid f \circ u \in S\}$.*

PROOF. It is sufficient to check that the inverse image along composition with f is indeed an interior algebras morphism making the above square to commute. ■

We can show now that the cilindrification morphisms $\mathcal{F}(X) \longrightarrow \mathcal{F}(X,p)$ mentioned in the introduction (where X is a finite set and p is an additional free generator) are the morphisms $\mathcal{F}_f$, where $f : \mathcal{P}(X) \times 2 \longrightarrow \mathcal{P}(X)$ is the first projection and 2 is the two-element set $\{0,1\}$ (notice that $\mathcal{F}_{\mathcal{P}(X)}$ and $\mathcal{F}_{\mathcal{P}(X,p)} \simeq \mathcal{F}_{\mathcal{P}(X)\times 2}$ are the free interior algebras on X and on $X \cup \{p\}$ free generators, respectively). In fact $\mathcal{P}(\mathcal{P}(X))$ is the free Boolean algebra on X generators, $\mathcal{P}(\mathcal{P}(X,p)) \simeq \mathcal{P}(\mathcal{P}(X) \times 2)$ is the free Boolean algebra on one more generator and $f^{-1} : \mathcal{P}(\mathcal{P}(X)) \longrightarrow \mathcal{P}(\mathcal{P}(X) \times 2)$ is the unique Boolean morphism extending the inclusion from X into $X \cup \{p\}$, so according to Proposition 3, $\mathcal{F}_f$ is the unique interior algebra morphism extending such inclusion among the free generators.

2. Undefinability of propositional quantifiers

From the results of the previous section and from the remarks in the introduction, we can reduce the problem of the existence of a translation from $S4^2$ into $S4$ to the question whether morphisms of the kind $\mathcal{F}_f$ do have adjoints, in the case in which $f : L \longrightarrow M$ is a first projection function from $L = \mathcal{P}(X) \times 2$ into $M = \mathcal{P}(X)$, where X is a finite set and 2 is the two-element set $\{0,1\}$. We fix an f of the above specified kind from now on, for our arguments we need only that L has at least 2 distinct elements, consequently any X of cardinality at least 1 is good.

We begin with a lemma on L-evaluations that will ensure us that certain sets of evaluations have index:

1. LEMMA *For all evaluation $u : P \longrightarrow L$ there exists N such that for all evaluation $v : Q \longrightarrow L$ $(u \sim_N v \Rightarrow u \sim_\infty v)$.*

PROOF. We prove the lemma by taking $N = 2n$, where n is the height of the antisymmetric quotient of P. We argue by induction on n.

Suppose that $n = 1$: this means that $\leq_P$ is total in P; we suppose that $u \sim_2 v$. We shall produce a strategy that always gives positions of the kind $\langle p, q\rangle$, where $u_p \sim_0 v_q$. Suppose such a position is reached (or that $p = \rho_P$ and that $q = \rho_Q$, if we are at the first move). If player I plays in Q, then we have an answer producing a position of the same kind because $\leq_P$ is total and from $u \sim_1 v$ it follows that the images of u and v are equal. If player I plays $p' \in P$, then there must be $q' \leq q$ such that $u(p') = v(q')$. Otherwise there wouldn't be a 2-strategy: player I would start with q and would play p' in the second move (he is allowed to do that independently of the answer of player II, because $\leq_P$ is total).

Suppose now that $n > 1$ and let us assume the statement for evaluations on preordered sets having antisymmetric quotient of lower height. For $p_1, p_2 \in P$, let us write $p_1 \sim p_2$ iff $p_1 \leq p_2$ and $p_2 \leq p_1$. Clearly, if $p_1 \sim p_2$ and $u_{p_1} \sim_0 u_{p_2}$, then we also have $u_{p_1} \sim_\infty u_{p_2}$. We define a strategy that produces positions $\langle p, q\rangle$, where either $u_p \sim_\infty v_q$ or ($p \sim \rho_P$ and $u_p \sim_{2n-1} v_q$). Suppose $\langle p, q\rangle$ is such a position: the interesting case is when $p \sim \rho_P$ and $u_p \sim_{2n-1} v_q$ (otherwise we already know how to go on).

- *Case 1:* player I plays $q' \leq q$ in Q; then there is $p' \in P$ such that $u_{p'} \sim_{2n-1} v_{q'}$, as we assumed that $u \sim_{2n} v$ and as $p \sim \rho_P$. Player II is allowed to answer such p' and $\langle p', q'\rangle$ is a position of the claimed kind (by induction hypothesis, in case $p' \not\sim \rho_P$).

- *Case 2:* player I plays $p' \leq p$ in P. Then, there is $q' \leq q$ such that $u_{p'} \sim_{2(n-1)} v_{q'}$ as $u_p \sim_{2n-1} v_q$. In case $p' \not\sim \rho_P$, induction hypothesis shows that $u_{p'} \sim_\infty v_{q'}$, so player II can answer p'. If $p' \sim \rho_P$, we argue as follows: from the assumption $u \sim_{2n} v$, it follows that there is $p'' \leq \rho_P \sim p'$ such that $u_{p''} \sim_{2n-1} v_{q'}$.

- *Subcase 2.1:* Suppose that $p'' \not\sim \rho_P$. Then, by induction, $u_{p''} \sim_\infty v_{q'}$. Moreover, from $u_{p''} \sim_\infty v_{q'} \sim_{2(n-1)} u_{p'}$ and the transitivity of $\sim_{2(n-1)}$, we get $u_{p''} \sim_{2(n-1)} u_{p'}$. Applying induction hypothesis, this yields $u_{p''} \sim_\infty u_{p'}$. By transitivity of $\sim_\infty$, from $v_{q'} \sim_\infty u_{p''} \sim_\infty u_{p'}$, we get $v_{q'} \sim_\infty u_{p'}$, so q' is a good answer for player II.

- *Subcase 2.2:* Suppose that $p'' \sim \rho_P$. In this case $u(p'') = v(q') = u(p')$ and $p'' \sim \rho_P \sim p'$, so $u_{p''} \sim_\infty u_{p'}$. This, combined with $u_{p''} \sim_{2n-1} v_{q'}$, shows that the position $\langle p', q'\rangle$ is again of the desired kind, so that player II answers q'. ■

2. Corollary. *For every L-evaluation u, the set $[u]_\infty = \{v \mid v \sim_\infty u\}$ (as well as its complement) has an index.* ■

We transform the problem of the existence of the left adjoint to $\mathcal{F}_f$ into a combinatorial statement about our games and afterwards we show that such statement is false.

3. Proposition *$\mathcal{F}_f$ has a left adjoint iff the following statement is true (let's write $f(u)$ for $f \circ u$):*

$$(1) \qquad \forall n\, \exists N\, \forall u\, \forall v\, (f(u) \sim_N v \;\Rightarrow\; \exists u'\, (u' \sim_n u \;\&\; f(u') \sim_\infty v))$$

(here u, u' are supposed to be L-evaluations and v an M-evaluation).

Proof. The existence of a left adjoint means that for every $X \in \mathcal{F}_L$ we can define $\exists_f(X) \in \mathcal{F}_M$ so that the relation

$$(2) \qquad \exists_f(X) \subseteq S \iff X \subseteq \mathcal{F}_f(S)$$

holds for every $S \in \mathcal{F}_M$. We show that *we must* have

$$(3) \qquad \exists_f(X) \;=\; \{v \mid \exists u \in X\, (f(u) \sim_\infty v)\}$$

(notice that, conversely, if we are allowed to define $\exists_f(X)$ in this way, i.e. if $\exists_f(X)$ as defined in (3) has an index, then (2) holds too). In fact, if we fix v and take S to be $\neg[v]_\infty$ (we can take into consideration such S because of the previous Corollary), we have, as a special case of (2), that

$$\exists_f(X) \subseteq \neg[v]_\infty \iff X \subseteq \mathcal{F}_f(\neg[v]_\infty)$$

But $\exists_f(X) \subseteq \neg[v]_\infty$ is equivalent to $v \notin \exists_f(X)$ and $X \subseteq \mathcal{F}_f(\neg[v]_\infty)$ is equivalent to $\forall u\, (f(u) \sim_\infty v \;\Rightarrow\; u \notin X)$. So we get

$$v \notin \exists_f(X) \iff \forall u\, (f(u) \sim_\infty v \;\Rightarrow\; u \notin X),$$

that is the relation (3) as claimed.

So the left adjoint exists iff for every set X of L-evaluations the set $\exists_f(X)$ as defined in (3) has an index, provided X has. As there are only finitely many distinct sets X having as index a fixed n (see proposition 1.1(ii)), we can suppose that the index of $\exists_f(X)$ is given uniformly in dependence on n. Finally, it is easily seen that for every fixed n, the statement

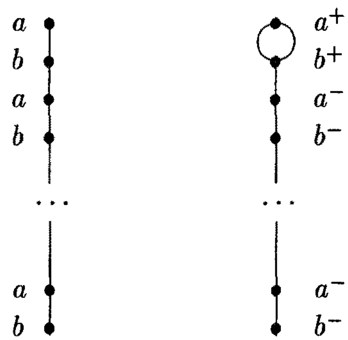
"for every X having index n, $\exists_f(X)$ has index N"

is equivalent to the statement

"$\forall u \forall v\, (f(u) \sim_N v \;\Rightarrow\; \exists u'\, (u' \sim_n u \;\&\; f(u') \sim_\infty v))$"

(the up direction is immediate, for the other one take $[u]_n$ as X). This completes the proof of the lemma. ∎

Now we can build a counterexample to (1). For every natural number n, $\underline{n}$ is the linear poset $1 \leq 2 \leq \cdots \leq n$. We introduce for any $k \geq 0$, an M-evaluation $v^k : Q^k \longrightarrow M$ and an L-evaluation $u^k : Q^k \longrightarrow L$. Here L, M, f must be as explained at the beginning of the section. We recall that $L = M \times 2$ and f is the first projection. We indicate the elements of M as $a, b, \ldots$ and we abbreviate $\langle a, 1\rangle$ as a^+ and $\langle a, 0\rangle$ as a^-. Thus, f simply 'removes the sign'. Q^k is $\underline{2k+2}$ and $v^k(i) = b$ if i is odd and $v^k(i) = a$ if k is even. P_k is $\underline{2k} + \{p, q\}$; we have for $x, y \in P^k$, $x \leq_{P^k} y$ iff ($y = p$ or $y = q$ or ($x, y \in \underline{2k}$ and $x \leq y$)). If $x \in \underline{2k}$, $u^k(x)$ is equal to a^- if $v^k(x) = a$ and is equal to b^- if $v^k(x) = b$; finally we put $u^k(p) = a^+$, $u^k(q) = b^+$. p is the specified root of P^k (of course, $2k+2$ is the specified root of Q^k).



4. LEMMA $f(u^k) \sim_{2k+1} v^k$.

PROOF. We collect some simple facts.

(i) *If* $i \leq 2k$, *then* $v_i^k \sim_\infty f(u_i^k)$: this is trivial.

(ii) *If* $i \leq j \leq 2k$ *and* i, j *are both odd or even, then* $v_i^k \sim_{i-1} f(u_j^k)$: this is shown by induction on i. For $i = 1$, the claim is trivial; for $i > 1$, the only relevant case is when player I chooses $s \in P^k$ such that $s > i$ (otherwise the obvious answer is suggested by (i)). In this case, player II answers i or $i - 1$, depending on the fact that s is even or odd, and wins by induction hypothesis.

(iii) *For every* $i \in Q^k$ *and for* $x = p, q$, $v_i^k \sim_0 f(u_x^k)$ *implies* $v_i^k \sim_{i-1} f(u_x^k)$: this is clear for $i = 1, 2$. Suppose that $i > 2$. If player I plays in P^k, by (i), we can examine only the case in which he plays $s \in P^k$ for $s > i$ or $s = p, q$. In the latter case, the answer is i or $i - 1$: player II wins by induction hypothesis. In the former case, the answer is again i or $i-1$, depending if s is even or odd. Player II wins by (ii). Finally, if player

I plays $i' \leq i$ in Q^k, then if $i' \leq 2k$, the answer is suggested by (i) and if $i' = 2k+1, 2k+2$, the answers are q, p, respectively (in the former case, we can apply induction, the latter case is irrelevant because it is an identical move).

The statement of the lemma is the particular case $i = 2k+2$ of (iii). ■

5. THEOREM *$\mathcal{F}_f$ does not have a left adjoint.*

PROOF. We show that (1) fails for $n = 3$. For reductio, suppose N exists so that (1) is satisfied. Take k such that $2k+1 \geq N$, $u = u^k$, $v = v^k$. Now $f(u^k) \sim_N v^k$ holds by the previous lemma. We show that there cannot be any L-evaluation $u' : P' \longrightarrow L$ such that $u' \sim_3 u$ and $f(u') \sim_\infty v$. We claim that otherwise there must be $p_1, p_2 \in P'$ such that $p_1 \leq p_2$, $p_2 \leq p_1$, $u'(p_1) = a^+$, $u(p_2) = b^+$: this is in contradiction to $f(u') \sim_\infty v^k$, because in Q^k player II cannot play points labelled $a, b, a, b, \ldots$ for ever. To prove our claim, we use the fact that $u' \sim_3 u$. In particular, there must be some point p_1 in P' whose u'-value is a^+ (certainly, for instance, the specified root). As our preordered sets are finite, we can pick a minimal such p_1, minimal with respect to the partial order relation in the antisymmetric quotient of P'. In other words, p_1 must be such that $u'(p_1) = a^+$ and such that for every $p' \leq p_1$ (if $u'(p') = a^+$, then $p_1 \leq p'$). Player II has a 3-strategy, so he must be able to answer successfully to the following moves: first, player I plays $p_1 \in P'$, then (as the forced answer is the root p of P^k), he plays $q \in P^k$ and finally $p \in P^k$. If p_2, p' are the answers of player II, we have $p' \leq p_2 \leq p_1$, $u'(p_2) = b^+$, $u'(p') = a^+$ and also $p_1 \leq p'$ by the minimality of p_1. In conclusion, we have $p_1 \leq p_2$ and $p_2 \leq p_1$ with $u'(p_1) = a^+$ and $u'(p_2) = b^+$, as claimed. ■

References

[1] FINE K., 'Logics containing K4, Part I', *Journal of Symbolic Logic* **34**, 31-42 (1974);

[2] FINE K., 'Logics containing K4, Part II', *Journal of Symbolic Logic* **50**, 619-651 (1985);

[3] GHILARDI S., 'An algebraic theory of normal forms', *Annals of Pure and Applied Logic* **71 (3)**, 189-245 (1995);

[4] GHILARDI S., ZAWADOWSKI M., 'A sheaf representation and duality for finitely presented Heyting algebras', to appear in *Journal of Symbolic Logic;*

[5] HUGHES G.E., CRESSWELL M.J., *A companion to modal logic*, Methuen, London (1984);

[6] MCLANE S., *Categories for the working mathematician*, Springer, Berlin (1971);

[7] PITTS A.M., 'On an interpretation of second order quantification in first order intuitionistic propositional logic', *Journal of Symbolic Logic* **57**, **1**, 33-52 (1992);

[8] SHAVRUKOV V.YU., *Subalgebras of diagonalizable algebras of theories containing arithmetic*, Dissertationes Mathematicae, CCCXXIII, Polska Akademia Nauk, Instytut Matematyczny, Warsaw (1993).

SILVIO GHILARDI
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DEGLI STUDI DI MILANO
VIA C. SALDINI 50
20133 MILANO, ITALY
ghilardi@vmimat.mat.unimi.it

MAREK ZAWADOWSKI
INSTYTUT MATEMATYKI
UNIWERSYTET WARSZAWSKI
UL. S.BANACHA 2
00-913 WARSZAWA, POLAND
ZAWADO@MIMUW.EDU.PL