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ON THE EXPONENTIAL DICHOTOMY OF PULSE EVOLUTION SYSTEMS

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The equivalence of regularity and exponential dichotomy is established for linear pulse differential equations with unbounded operators in a Banach space. The separatrix manifolds of a linear pulse system exponentially dichotomous on a semiaxis are studied in a finite-dimensional space. The conditions of weak regularity of this system are given.

1. Equation in a Banach Space

Consider an equation with pulse influence

$$dx/dt = A(t)x + f(t),$$

$$\Delta x|_{t=t_i} = x(t_i) - x(t_i - 0) = B_i x(t_i - 0) + g_i, \tag{1}$$

where $x \in \mathfrak{B}$, $\mathfrak{B}$ is a Banach space with a norm $\|\ \|$, $A(t)$ and B_i, $i \in \mathbb{Z}$, $t \in \mathbb{R}$, are linear (generally speaking, unbounded) operators in the space $\mathfrak{B}$, $\{t_i\}$ is a strictly increasing sequence of real numbers satisfying the condition

$$\limsup_{T \to \infty} i(t, t+T)/T = p < \infty, \tag{2}$$

where $i(t, t+T)$ is the number of pulses on the interval $(t, t+T)$, $C'(\mathfrak{B})$ is the space of functions $\mathbb{R} \to \mathfrak{B}$ strongly continuous on $\mathbb{R} \setminus \{t_i\}$, discontinuous but right-continuous at the points t_i, and $l(\mathfrak{B})$ is the space of functions $g(t_i) = g_i$ defined on $\{t_i\}$ with values in $\mathfrak{B}$ and the norm $\|g\| = \sup_{i \in \mathbb{Z}} \|g_i\|$.

We assume the following:

(i) The right-solvability condition for Eq. (1) is satisfied, i.e., for any initial value $u(\tau_0) \in \mathfrak{B}$, the homogeneous equation corresponding to Eq. (1)

$$du/dt = A(t)u,$$

$$\Delta u|_{t=t_i} = u(t_i) - u(t_i - 0) = B_i u(t_i - 0) \tag{3}$$

has a unique solution $u(t)$, $t \geq \tau_0$, strongly continuous for $t \neq t_i$ and $u(t_i) = (I + B_i)u(t_i - 0)$;

(ii) The solution operators $U(t, \tau_0)$ are strongly continuous in t for $t \neq t_i$ and satisfy the estimate

$$\|U(t, \tau_0)\| \leq l_0 \exp(c_0(t - \tau_0)), \quad t \geq \tau_0. \tag{4}$$

Sufficient conditions of this are presented in [1, 2].

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The inhomogeneous equation (1) has the solution

$$x(t) = U(t,\tau_0)x_0 + \int_{\tau_0}^{t} U(t,s)f(s)\,ds + \sum_{\tau_0 \le t_i \le t} U(t,t_i)g_i.$$

The operator L is defined as an unbounded operator in the space $C'(\mathfrak{B})$ with values in $C'(\mathfrak{B}) \oplus l(\mathfrak{B})$,

$$Lu = (T_1u, T_2u) = (du/dt - A(t)u,\ u(t_i) - (I+B_i)u(t_i-0)).$$

As in [3], we give the following definition.

Definition 1. *The operator L is called exponentially dichotomous on an axis if there exists a variable decomposition $\mathfrak{B} = \mathfrak{N}_1(\tau) \oplus \mathfrak{N}_2(\tau)$ of the space $\mathfrak{B}$ ($P_1(\tau)$ and $P_2(\tau)$ are the corresponding projectors) such that the solution $u(t)$ of the homogeneous equation (3) satisfies the following conditions:*

(i) $U(t,\tau_0)\mathfrak{N}_i(\tau_0) \subseteq \mathfrak{N}_i(t), \quad t \ge \tau_0, \quad i = 1,2;$

(ii) $\sup \| P_i(\tau_0) \| < \infty, \quad i = 1,2;$

(iii) The inequality

$$\| u(t) \| \le l_1 \| u(\tau_0) \| \exp(-c_1(t-\tau_0)), \quad t \ge \tau_0 \tag{5}$$

holds for $u(\tau_0) \in \mathfrak{N}_1(\tau_0)$;

(iv) For every initial value $u(\tau_0) \in \mathfrak{N}_2(\tau_0)$, *there exists a unique extension to the whole axis such that*

$$\| u(t) \| \le l_1 \| u(\tau_0) \| \exp(c_1(t-\tau_0)), \quad t \le \tau_0. \tag{6}$$

Definition 2. *The operator L is called regular on $\mathbb{R}$ if, for an arbitrary function $f(t) \in C'(\mathfrak{B})$ and any sequence $\{g_i\} \in l(\mathfrak{B})$, the linear inhomogeneous equation (1) has a unique solution $x(t)$ bounded on $\mathbb{R}$ and the following estimate holds:*

$$\| x(t) \|_{C'} \le K(\| L_1 x \|_{C'} + \| lx \|_l). \tag{7}$$

The relationship between regularity and exponential dichotomy for bounded operators $A(t)$ and B_i, $i \in \mathbb{Z}$, was considered earlier in [4–7]. In [4–6], there was an additional requirement of existence of uniformly bounded operators $(I+B_i)^{-1}$. In the present paper, this is not required.

Theorem 1. *The regularity of the operator L is equivalent to its exponential dichotomy.*

Proof. Let the operator L be exponentially dichotomous on an axis. Let us construct its Green's function. It follows from condition (iv) that there exists a bounded operator $\Omega(t,\tau_0) = U(t,\tau_0)P_2(\tau_0)$ for any $\tau_0 \in \mathbb{R}$. Green's function has the form

$$G(t, s) = \begin{cases} U(t, s)P_1(s), & t \geq s, \\ -\Omega(t, s), & t \leq s. \end{cases}$$

Conditions (i)–(iv) yield the estimate

$$\| G(t, s) \| \leq l \exp(c_1 |t - s|). \tag{8}$$

The function

$$x(t) = \int_{-\infty}^{\infty} G(t, s) f(s)\, ds + \sum_{i=-\infty}^{\infty} G(t, t_i) g_i$$

is a bounded solution of Eq. (1), provided that condition (2) is satisfied. This can be checked by analogy with [8, Sec. 26].

Let the operator L be regular. Consider the manifolds

$$\mathfrak{N}_1(\tau_0) = \{x(\tau_0) \in \mathfrak{B},\ Lx = 0,\ \sup_{t \geq \tau_0} \|x(t)\| < \infty\}, \tag{9}$$

$$\mathfrak{N}_2(\tau_0) = \{x(\tau_0) \in \mathfrak{B},\ Lx = 0,\ \sup_{t \leq \tau_0} \|x(t)\| < \infty\}. \tag{10}$$

Note that relation (7) implies that $x(t)$ in (10) can be uniquely extended to $(-\infty, t_0)$.

As in [3], one can prove the following inequalities:

$$\|x(t)\| \leq l \|x(\tau_0)\|, \quad t \geq \tau_0, \quad x(\tau_0) \in \mathfrak{N}_1(\tau_0), \tag{11}$$

$$\|x(t)\| \leq l \|x(\tau_0)\|, \quad t \leq \tau_0, \quad x(\tau_0) \in \mathfrak{N}_2(\tau_0). \tag{12}$$

Inequalities (11) and (12) imply estimate (5) for $x(\tau_0) \in \mathfrak{N}_i(\tau_0)$ and (6) for $x(\tau_0) \in \mathfrak{N}_2(\tau_0)$. The manifolds $\mathfrak{N}_1(\tau_0)$ and $\mathfrak{N}_2(\tau_0)$ complement each other.

2. Equation (1) in Finite-Dimensional Space $\mathfrak{B} = \mathbb{R}^m$

Assume that $\|A(t)\| \leq K$, $\forall t \in \mathbb{R}$, $f(t) \in C(\mathbb{R})$, $\|g_i\| \leq K$, $\forall i \in \mathbb{Z}$, $\|B_i\| \leq K$, $\forall i \in \mathbb{Z}$, $\|\cdot\|$ is the norm of a matrix or of a vector. The pulses may degenerate $(\det(I + B_i) = 0)$ for some (or all) $i \in \mathbb{Z}$; therefore, solutions of Eq. (1) are not extendable to the negative semiaxis or are ambiguously extendable.

Definition 3. *Equation (3) for $\mathfrak{B} = \mathbb{R}^m$ is called exponentially dichotomous for $t \geq \tau_0$ if the space $\mathbb{R}^m$ can be represented as the direct sum $\mathbb{R}^m = U_\tau \oplus S_\tau$ for any $\tau \geq \tau_0$ so that the following conditions are satisfied:*

(i) A solution $u(t)$ of Eq. (3), $u(\tau) \in S_\tau$, satisfies the estimate

$$\| u(t) \| \leq K \| u(\tau) \| \exp(-\nu(t - \tau)), \quad t \geq \tau; \tag{13}$$

(ii) The estimate

$$\| u(t)\| \geq K\| u(\tau)\| \exp(\nu(t-\tau)), \quad t \geq \tau \tag{14}$$

holds for $u(\tau) \in U_\tau$.

The manifold U_τ in condition (ii) is selected ambiguously, it is one of the dual spaces to S_τ. In condition (i), the solution $u(t)$ can degenerate into zero. The dimensionalities of the manifolds S_τ and U_τ can vary together with τ. These variations are determined by the following lemma:

Lemma 1. *For the dimensionalities of the stable and unstable manifolds, we have*

$$\dim S_t \geq \dim S_\tau \quad \text{for } t \leq \tau,$$
$$\dim U_t \geq \dim U_\tau \quad \text{for } t \leq \tau. \tag{15}$$

Proof. Obviously, if there are no pulse points between t and τ in the condition of the lemma, then the inequalities turn into equalities. The strict inequalities may appear for $t = t_i - 0$ and $\tau = t_i$. Let us consider this case. The image of the space S_{t_i-0} under the mapping $(I+B_i)$ is contained in S_{t_i}. Its dimensionality is $\dim(I+B_i)S_{t_i-0} = \dim S_{t_i-0} - \dim\ker(I+B_i)$. In S_{t_i}, we construct a complement to $(I+B_i)S_{t_i-0}$. This subspace has the dimensionality $\dim S_{t_i} - \dim S_{t_i-0} + \dim\ker(I+B_i)$ and belongs to the set $\mathbb{R}^m \setminus (I+B_i)\mathbb{R}^m$. Therefore, the inequality

$$\dim S_{t_i} - \dim S_{t_i-0} + \dim\ker(I+B_i) \leq \dim\ker(I+B_i),$$

or $\dim S_{t_i} \leq \dim S_{t_i-0}$, is valid. The second inequality in (15) can be proved by using the fact that S_τ and U_τ complement each other in $\mathbb{R}^m$. The lemma is proved.

Lemma 2. *Let system (3) be defined for* $t \geq 0$ *and exponentially dichotomous on the semiaxis* $t \geq \tau_1 > 0$. *Then it is exponentially dichotomous on the semiaxis* $t \geq s$ *for all* $s \in [0, \tau_1]$.

Proof. Let $\tau_1 \in [t_j, t_{j+1})$. Then solutions can be uniquely extended to the left of the point $t = \tau_1$ to the segment $[t_j, \tau_1]$ as solutions of a linear differential system with a bounded matrix. With regard to estimate (4), one can easily conclude that Eq. (3) is dichotomous on $[t_j, \infty)$ because this is so on $[\tau_1, \infty)$. In this procedure, the constants in inequalities (15) and (16) may change. Since the matrix $(I+B_i)$ is degenerate, not all solutions on $[t_j, \infty)$ have preimages at the point $t_j - 0$. Let S_{t_j} and U_{t_j} be stable and unstable manifolds of the system at the points t_j, respectively. The manifold S_{t_j-0} is the sum of preimages of vectors from S_{t_j} and elements of $\ker(I+B_j)$. Taking into account that the matrix on the right-hand side of system (3) is bounded, one can easily see that inequality (15) holds for S_{t_j-0}.

The elements of the manifold U_{t_j} that have preimages under the action of the matrix $(I+B_j)$ form a vector space. Let $v_1, \ldots, v_l$ be its basis. Consider the algebraic system

$$(I+B_j)y = v_k, \quad k = \overline{1, l}. \tag{16}$$

For each v_k, among solutions of system (16), we choose a solution y_k with the least norm. By reducing $(I + B_j)$ to the Jordan form, we can show that the inequality

$$\|v_k\| \geq M\|y_k\|, \quad k = \overline{1, l}, \tag{17}$$

is valid with the constant independent of v_k. A span of the vectors y_k forms a manifold U_{t_j-0}. By using (17), we obtain inequality (14) for vectors from U_{t_j-0}; furthermore, $S_{t_j-0} \oplus U_{t_j-0} = \mathbb{R}^m$ by construction. Thus, we proved that system (3) is exponentially dichotomous for $t_j - 0$. The points $t < t_j - 0$ can be considered similarly. The lemma is proved.

Definition 4. *The operator L is called weakly regular on the semiaxis $t \geq \tau_0$ if, for arbitrary $f(t) \in C'[\tau_0, \infty)$ and $\{g_i\} \in l[\tau_0, \infty)$, the linear inhomogeneous equation (1) has at least one solution $u(t)$ bounded for $t \geq \tau_0$.*

Theorem 2. *In order that system (1) be weakly regular, it is necessary and sufficient that the system be exponentially dichotomous on a semiaxis and the dimensionalities of the stable subspace S_τ and the unstable subspace U_τ be independent of $\tau \geq \tau_0$.*

Proof. *Sufficiency.* Let $\dim S_\tau = r$ and $\dim U_\tau = m - r$ for any $\tau \geq \tau_0$. The manifold S_τ is uniquely determined (unlike its complement U_τ) and invariant. We introduce a basis $u_1(t), \ldots, u_r(t)$ in S_τ and complement it to a basis in $\mathbb{R}^m$ by vector functions $u_{r+1}, \ldots, u_m(t)$. We construct a matrix $Q(t)$ so that its ith column is $u_i(t)| \det Q(t)| \geq a > 0$. Let us change the variables $x = Q(t)y$ in system (1). In terms of the variables (y_1, y_2), the manifold S_τ has the form $(y_1(t), 0)$. In the new variables (y_1, y_2), we obtain the following triangular system:

$$dy_1 / dt = A_1(t)y_1 + A_{12}(t)y_2 + f_1(t),$$

$$\Delta y_1|_{t=t_i} = B_i^1 y_1 + B_i^{12} y_2 + g_i^1; \tag{18}$$

$$dy_2 / dt = A_2(t)y_2 + f_2(t),$$

$$\Delta y_2|_{t=t_i} = B_i^2 y_2 + g_i^2. \tag{19}$$

The homogeneous system (19) (for $f_2(t) = 0$ and $g_i^2 = 0$, $i \in \mathbb{Z}$) has no nontrivial bounded solutions; all its solutions increase exponentially. In this case, $\det(I + B_i^2) \neq 0$ for any $i \in \mathbb{Z}$ and the solutions can be uniquely extended to the semiaxis $t \geq \tau_0$. A unique bounded solution of the inhomogeneous system (19) is given by the relation

$$y_2(t) = \int_t^\infty Y_2(t, s) f_2(s) ds + \sum_{t \leq t_i < \infty} Y_2(t, t_i) g_i^2,$$

where $Y_2(t, s)$ is the evolution matrix of the homogeneous system (19).

The homogeneous system (18) is exponentially stable for $f_1 = 0$, $g_i^2 = 0$, and $y_2(t) = 0$. Thus, all solutions of the inhomogeneous system are bounded and can be expressed as

$$y_1(t) = \int_{t_0}^{t} Y_1(t,s)(A_{12}(s)y_{20}(s) + f_1(s))\,ds + \sum_{t_0 \le t_i \le t} Y_1(t,t_i)(B_i^{12} y_{20}(t_i) + g_i^1),$$

where $Y_1(t,s)$ is the evolution matrix of the homogeneous system (18) (for $y_2 = 0$), $y_{20}(t)$ is a bounded solution of system (19).

Necessity. As in [5], we prove that solutions of the homogeneous system (3) bounded for $t \ge \tau_0$ satisfy estimate (13) while solutions from some complement to the space of bounded solutions satisfy estimate (14). Note that the fact that the matrix $(I + B_i)$ must be nondegenerate is, actually, not used in the proof in [5].

Let us show it is necessary that the dimensionalities of S_τ and U_τ be constant. Suppose that, for $t = t_i$, this condition is violated, i.e., in view of Lemma 1, $\dim S_{t_i} < \dim S_{t_i - 0}$. Let $\dim\ker(I + B_i) = k$, $\dim S_{t_i} = s$, and $\dim S_{t_i - 0} = s + s_1$. Under the action of $(I + B_i)$, the subspace $S_{t_i - 0}$ is decomposed into a sum of a k-dimensional subspace, which turns to zero, and an $(s + s_1 - k)$-dimensional subspace, which has a nonzero image in S_{t_i}. The space S_{t_i} is decomposed into a direct sum of an $(s + s_1 - k)$-dimensional subspace $S_{t_i}^1$, which has a preimage in $S_{t_i - 0}$, and a $(k - s_1)$-dimensional subspace $S_{t_i}^2$, which has no preimage.

A solution of system (1) has the form

$$x(t) = U(t,t_i)x_i + \int_{t_i}^{t} U(t,s)f(s)ds + \sum_{t_i < t_j < t} U(t,t_j)g_j$$

$$= U(t, t_i - 0)x_{i-0} + \int_{t_i - 0}^{t} U(t,s)f(s)ds + \sum_{t_i \le t_j < t} U(t,t_j)g$$

$$= U(t,t_i)(I + B_i)x_{i-0} + \int_{t_i - 0}^{t} U(t,s)f(s)ds + \sum_{t_i \le t_j < t} U(t,t_j)g_j.$$

Therefore,

$$(I + B_i)x_{i-0} = x_i + g_i. \tag{20}$$

Let $g_i = 0$ and $f(t) = 0$ for $t > t_i$. For $t > t_i$, system (1) has a set of bounded solutions that start from S_{t_i}. Since system (1) is weakly regular, the algebraic system (20) has a solution for arbitrary g_i and $x_i \in S_{t_i}$. Let us select $g_i \notin \operatorname{Im}(I + B_i)$. Then $(g_i + S_{t_i}^1)$ has no preimage under the action of $(I + B_i)$. The dimensionality of the hyperplane $(g_i + S_{t_i}^2)$ is $m - s_1$. The dimensionality of $\operatorname{Im}(I + B_i)$ is $m - k$ and, thus, it is always possible to choose g_i so that $(g_i + S_{t_i}^2)$ and $\operatorname{Im}(I + B_i)$ do not intersect, i.e., it is possible to find inhomogeneities $f(t)$ and g_i such that system (1) has no bounded solutions. This contradicts the assumption of weak regularity. The theorem is proved.

3. Exponential Dichotomy of a Perturbed System

Parallel with system (3), we consider the perturbed system

$$dy/dt = (A(t) + \tilde{A}(t))\,y, \qquad t \neq t_i,$$
$$\Delta y|_{t=t_i} = (B_i + \tilde{B}_i)\,y. \tag{21}$$

Theorem 2. *Let system (3) be exponentially dichotomous for $u \in \mathbb{R}^m$ and $t \geq 0$. Then system (21) is also exponentially dichotomous for $\|\tilde{A}(t)\|_C \leq \delta$ and $\|\tilde{B}_i\|_l \leq \delta$ with sufficiently small $\delta > 0$.*

Proof. As follows from Lemma 1, the dimensionality of the stable manifold S_τ of the exponentially dichotomous system (3) does not increase with τ. Since the dimensionality of the space is finite, it is stabilized beginning from certain τ_0 and the dimensionalities of S_τ and U_τ are constant for $\tau \geq \tau_0$ (it is possible that $S_\tau = 0$ or $S_\tau = \mathbb{R}^m$). By Theorem 2, system (3) is weakly regular on the semiaxis $\tau \geq \tau_0$ and has Green's function $G(t, s)$ satisfying estimate (8). If $t \geq \tau_0$ and δ are small, a direct decomposition $\mathbb{R}^m = \tilde{U}_\tau \oplus \tilde{S}_\tau$ exists for system (3), which determines the exponential dichotomy. The subspace $\tilde{S}_\tau$ consists of initial values $y(\tau)$ of solutions of Eq. (21) tending to zero as $t \to +\infty$. These solutions can be obtained from the integral equation

$$y(t) = U(t, \tau)u(\tau) + \int_\tau^\infty G(t, s)\tilde{A}(s)y(s)\,ds + \sum_{\tau < t_i < \infty} G(t, t_i)\tilde{B}_i y(t_i), \tag{22}$$

where $u(\tau) \in S_\tau$. It can be shown that the right-hand side of (22) is a contraction operator for sufficiently small δ, provided that conditions (2) and (8) are satisfied. Therefore, for each $u(\tau) \in S_\tau$, Eq. (22) has a unique solution $y(t)$ that satisfies estimate (13). Thus, $\tilde{S}_\tau$ is constructed.

All solutions from a complement to $\tilde{S}_\tau$ grow exponentially. These solutions are determined by the equation

$$y(t) = U(t, \tau)y(\tau) + \int_\tau^t U(t, s)\tilde{A}(s)y(s)\,ds + \sum_{\tau < t_i < t} U(t, t_i)\tilde{B}_i y(t_i).$$

It can be shown by analogy with [9, p. 259] that estimate (14) holds for these solutions. The theorem is proved.

Example. Consider the equation

$$dy/dt = 0, \quad t \neq n \in \mathbb{Z}, \quad \Delta y|_{t=n} = b_n y + g_n. \tag{23}$$

Assume that $b_n = 2$ for $n \geq 1$ and $b_n = 0$ for $n \leq 0$, $g_n = 0$, $n \neq 1$, and $g_1 = 2$. Equation (23) is exponentially dichotomous on the semiaxis $t \geq \tau_0$ for all τ_0. Clearly, $U_t = 0$ for $t < 1$ and $\dim U_t = 1$ for $t \geq 1$. On the semiaxis $t \geq 1$, the equation has a unique bounded solution $y(t) = -1$, $1 \leq t < 2$, which is zero for $t \geq 2$. There are no bounded solutions on the semiaxis $t \geq 0$. Indeed, for $t = 1$, all solutions pass through the point $y(1) = 2$, but a solution with this initial condition is unbounded.

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