the linear space L(X) is σ-bounded because the analog of the above-cited result of Arkhangel'skii [4] is valid for L(X).

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A METHOD OF CONSTRUCTION OF LYAPUNOV–KRASOVSKII FUNCTIONALS FOR LINEAR SYSTEMS WITH DEVIATING ARGUMENT

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Let us consider a linear system of differential equations with deviating argument

$$\dot{x}(t) = Ax(t) + Bx(t-\tau). \tag{1}$$

Let A be an asymptotically stable matrix. One of the methods of constructing Lyapunov-Krasovskii functionals for linear systems employs quadratic forms [1-5] of the type

$$v[x(s)] = x^{\tau}(t)Hx(t) + \int_{t-\tau}^{t} x^{\tau}(s)Gx(s)\,ds. \tag{2}$$

In view of Eq. (1), the full derivative of v[x(s)s)] has the form

$$\dot{v}[x(s)] = x^{\tau}(t)(A^{\tau}H + HA)x(t) + x^{\tau}(t)HBx(t-\tau) + x^{\tau}(t-\tau)B^{\tau}Hx(t) + x^{\tau}(t)Gx(t) - x^{\tau}(t-\tau)Gx(t-\tau),$$

or $\dot{v}[x(s)] = -z^{\tau}(t,\tau)Cz(t,\tau)$, where

$$z(t,\tau) = \begin{pmatrix} x(t) \\ x(t-\tau) \end{pmatrix}, \quad C = \begin{bmatrix} -A^{\tau}H - HA - G & -HB \\ -B^{\tau}H & G \end{bmatrix}.$$

If the symmetric matrices G, H, and C are positive definite, then v[x(s)] is positive definite, while $\dot{v}$[x(s)] is negative definite, and the zero solution x(t) ≡ 0 of the system (1) is asymptotically stable [1].

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Let us consider the set $L_{G,H}$ consisting of pairs (G, H) of symmetric matrices G and H for which the matrix

$$C(G, H) = \begin{bmatrix} -A^{T}H - HA - G & -HB \\ -B^{T}H & G \end{bmatrix}$$

is positive definite.

LEMMA 1. If the matrix C(G, H) is positive definite, then the matrices G and H will also be positive definite.

Proof. By the Hurwitz criterion, positive definiteness of G and of $-A^{T}H-HA-G$ follows from the positive definiteness of C(G, H). Since A is an asymptotically stable matrix, positive definiteness of H follows from positive definiteness of G and of $-A^{T}H-HA-G$.

LEMMA 2. If the set $L_{G,H}$ is nonempty, then it is a convex cone.

Proof. Let $(G_1, H_1) \in L_{G,H}$ and $(G_2, H_2) \in L_{G,H}$, that is, the matrices $C(G_1, H_1)$ and $C(G_2, H_2)$ are positive definite. Then for any $0 < \alpha < 1$ we have

$$C(\alpha G_1 + (1-\alpha) G_2, \alpha H_1 + (1-\alpha) H_2)$$

$$\begin{bmatrix} -A^{T}[\alpha H_1 + (1-\alpha) H_2] - [\alpha H_1 + \\ +(1-\alpha) H_2] A - \alpha G_1 - (1-\alpha) G_2 & -[\alpha H_1 + (1-\alpha) H_2] B \\ -B^{T}[\alpha H_1 + (1-\alpha) H_2] & \alpha G_1 + (1-\alpha) G_2 \end{bmatrix} = \alpha C(H_1, G_1) + (1-\alpha) C(H_2, G_2).$$

As the sum of two positive definite matrices is positive definite, $C(\alpha G_1 + (1-\alpha) G_2, \alpha H_1 + (1-\alpha) H_2)$ is positive definite. And $L_{G,H}$ is a convex set. Furthermore, for any $0 < \mu < +\infty : C(\mu G, \mu H) = \mu C(G, H)$. Therefore $L_{G,H}$ is a convex cone.

THEOREM 1. If the set $L_{G,H}$ is nonempty, the zero solution $x(t) \equiv 0$ of the system (1) is asymptotically stable.

The proof follows from Lemma 1 and from the Lyapunov–Krasovskii theorem [1].

Thus, the study of stability of system (1) with the help of functionals of the form (2) reduces to finding out whether the set $L_{G,H}$ is nonempty.

Example. For the system $\dot{x}(t) = -ax(t) + bx(t-\tau)$ the Lyapunov–Krasovskii functional has the form $v[x(s)] = hx^2(t) + g\int_{t-\tau}^{t} x^2(s)\,ds$. The matrix C(G, H) has the form

$$C(G, H) = \begin{bmatrix} 2ah - g & -hb \\ -hb & g \end{bmatrix}.$$

The domain $L_{G,H}$ is defined by the inequalities $L_{G,H} = \{g, h : h > 0,\ g > 0,\ h^2b^2 - 2ahg + g^2 < 0\}$. If $|b| < a$, then this domain lies between the two straight lines

$$h = \frac{a + \sqrt{a^2 - b^2}}{b^2} g, \quad h = \frac{a - \sqrt{a^2 - b^2}}{b^2} g.$$

Usually the matrices H and G are found by trial and error. Let us indicate a method of choosing the matrix G that is based on a parametric representation of this matrix. We shall seek it in the form $G = \alpha(-A^{T}H - HA)$, where $0 < \alpha < 1$ is a parameter. Let us find a value α_0 for which the matric $C(\alpha_0)$ is "more stable" than for other $0 < \alpha < 1$. The matrix $C(\alpha \times (-A^{T}H - HA), H)$ takes the form

$$C(\alpha(-A^{T}H - HA), H) = \begin{bmatrix} (1-\alpha)(-A^{T}H - HA) & -HB \\ -B^{T}H & \alpha(-A^{T}H - HA) \end{bmatrix}.$$

It is not hard to see that

$$-\dot{v}[x(s)] = -\left(x(t) - \frac{1}{1-\alpha}(-A^{T}H - HA)^{-1} HBx(t-\tau)\right)^{T}(1-\alpha)$$

$$\times (-A^{T}H - HA)\left(x(t) - \frac{1}{1-\alpha}(-A^{T}H - HA)^{-1} HBx(t-\tau)\right) -$$

$$-x^{\mathrm{T}}(t-\tau)\left[\alpha(-A^{\mathrm{T}}H-HA)-(HB)^{\mathrm{T}}\frac{1}{1-\alpha}(-A^{\mathrm{T}}H-HA)^{-1}HB\right]x(t-\tau).$$

For all $0 < \alpha < 1$ the first term is a positive definite quadratic form. Let us consider conditions under which the matrix

$$(1-\alpha)C_1(\alpha)=-\alpha^2(-A^{\mathrm{T}}H-HA)+\alpha(-A^{\mathrm{T}}H-HA)-(HB)^{\mathrm{T}}(-A^{\mathrm{T}}H-HA)^{-1}HB$$

is also positive definite. The matrices $-A^{\mathrm{T}}H-HA$ and $(HB)^{\mathrm{T}}(-A^{\mathrm{T}}H-HA)^{-1}HB$ are positive definite, and the necessary condition for an extremum $\frac{\partial}{\partial\alpha}[(1-\alpha)C_1(\alpha)]=0$ gives us $\alpha_0 = 0.5$. Thus, if G is to be sought in the parametric form $G=\alpha(-A^{\mathrm{T}}H-HA)$, then we have to take $\alpha_0 = 0.5$ and the Lyapunov–Krasovskii functional will have the form

$$v[x(s)]=x^{\mathrm{T}}(t)Hx(t)-\frac{1}{2}\int_{t-\tau}^{t}x^{\mathrm{T}}(s)(A^{\mathrm{T}}H+HA)x(s)\,ds,$$

while its full derivative is

$$\dot{v}[x(s)]=-\frac{1}{4}z^{\mathrm{T}}(t,\tau)\begin{bmatrix}-A^{\mathrm{T}}H-HA & -2HB\\ -2B^{\mathrm{T}}H & -A^{\mathrm{T}}H-HA\end{bmatrix}z(t,\tau).$$

Let us consider a particular case when the matrix $H = H_E$ satisfies the matrix equation $A^{\mathrm{T}}H_E+H_EA=-E$. We shall obtain sufficient conditions of stability of the system (1) with the Lyapunov–Krasovskii functional of the form

$$v_E[x(s)]=x^{\mathrm{T}}(t)H_Ex(t)+\frac{1}{2}\int_{t-\tau}^{t}x^{\mathrm{T}}(s)x(s)\,ds.$$

By (1), the full derivative of $v_E[x(s)]$ has the form

$$\dot{v}_E[x(s)]=-\frac{1}{4}z^{\mathrm{T}}(t,\tau)\begin{bmatrix}E & -2H_EB\\ -2B^{\mathrm{T}}H_E & E\end{bmatrix}z(t,\tau).$$

Let us use the notation

$$-2H_EB=D=\begin{bmatrix}d_{11} & d_{12} & \dots & d_{1n}\\ d_{21} & d_{22} & \dots & d_{2n}\\ \cdot & \cdot & \dots & \cdot\\ d_{n1} & d_{n2} & \dots & d_{nn}\end{bmatrix},$$

and let us denote by

$$D_{j_1j_2\dots j_k}^{i_1i_2\dots i_s}=\begin{bmatrix}d_{i_1j_1} & d_{i_1j_2} & \dots & d_{i_1j_k}\\ d_{i_2j_1} & d_{i_2j_2} & \dots & d_{i_2j_k}\\ \cdot & \cdot & \dots & \cdot\\ d_{i_sj_1} & d_{i_sj_2} & \dots & d_{i_sj_k}\end{bmatrix}$$

the rectangular $(s \times k)$-matrix obtained from the matrix D by taking the rows $i_1, i_2, \dots, i_s$ and the columns $j_1, j_2, \dots, j_k$. Let O_s be the square $(s \times s)$-matrix with zero elements.

<u>LEMMA 3.</u> The determinant of the matrix

$$\Delta_{sk}=\begin{bmatrix}O_s & D_{j_1j_2\dots j_k}^{i_1i_2\dots i_s}\\ (D_{j_1j_2\dots j_k}^{i_1i_2\dots i_s})^{\mathrm{T}} & O_k\end{bmatrix}$$

is equal to zero if $k \neq s$ and $\det\Delta_{sk}=(-1)^k(\det D_{j_1j_2\dots j_k}^{i_1i_2\dots i_k})$ if $k = s$.

<u>Proof.</u> Taking into account properties of the determinant, we use Gaussian elimination to reduce the block $D_{j_1j_2\dots j_k}^{i_1i_2\dots i_s}$ of the matrix Δ_{sk} to triangular form. Hence, if, for example, $k < s$, we obtain that the last $s - k$ rows of the reduced block $D_{j_1j_2\dots j_k}^{i_1i_2\dots i_s}$ consist of zero elements, with the possible exception of the last one, that is, they are linearly dependent and

$\det \Delta_{sk} = 0$. If $k = s$, then performing Gaussian elimination on each of the blocks, we find that $\det \Delta_{sk} = (-1)^k (\det D^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k})^2$.

THEOREM 2. If the roots z_i, $i = \overline{1, n}$ of the algebraic equation

$$z^n + p_1 z^{n-1} + \ldots + p_n = 0 \tag{3}$$

satisfy the conditions $0 < z_i < 1$, $i = \overline{1, n}$, the system (1) of differential equations is asymptotically stable. Here

$$p_m = (-1)^m \sum_{\substack{i_1 \neq i_2 \neq \ldots \neq i_m \\ j_1 \neq j_2 \neq \ldots \neq j_m}} (\det D^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_m})^2, \tag{4}$$

$D^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_m} = \begin{bmatrix} d_{i_1 j_1} & d_{i_1 j_2} & \ldots & d_{i_1 j_m} \\ d_{i_2 j_1} & d_{i_2 j_2} & \ldots & d_{i_2 j_m} \\ \cdot & \cdot & \ldots & \cdot \\ d_{i_m j_1} & d_{i_m j_2} & \ldots & d_{i_m j_m} \end{bmatrix}$ is the submatrix of the matrix $D = -2H_E B$, obtained by taking the rows $i_1, i_2, \ldots, i_m$ and the columns $j_1, j_2, \ldots, j_m$.

Proof. Let us expand the characteristic equation

$$\begin{vmatrix} (1-\lambda) E & -2H_E B \\ -2B^{\mathrm{T}} H_E & (1-\lambda) E \end{vmatrix} = (1-\lambda)^{2n} + q_1 (1-\lambda)^{2n-1} + \ldots + q_{2n}.$$

Transforming, we obtain for an arbitrary coefficient q_i, $i = \overline{1, 2n}$

$$q_i = \sum_{\substack{i_1 \neq i_2 \neq \ldots \neq i_s \\ j_1 \neq j_2 \neq \ldots \neq j_k}} \det \begin{bmatrix} O_s & D^{i_1 i_2 \ldots i_s}_{j_1 j_2 \ldots j_k} \\ (D^{i_1 i_2 \ldots i_s}_{j_1 j_2 \ldots j_k})^{\mathrm{T}} & O_k \end{bmatrix}, \qquad s + k = i.$$

For odd coefficients q_{2m+1} there are no s and k such that simultaneously $s = k$ and $s + k = 2m + 1$. Therefore by Lemma 3 they are equal to zero, while

$$q_{2m} = \sum_{\substack{i_1 \neq i_2 \neq \ldots \neq i_m \\ j_1 \neq j_2 \neq \ldots \neq j_m}} \det \begin{bmatrix} O_m & D^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_m} \\ (D^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_m})^{\mathrm{T}} & O_m \end{bmatrix} = (-1)^m \sum_{\substack{i_1 \neq i_2 \neq \ldots \neq i_m \\ j_1 \neq j_2 \neq \ldots \neq j_m}} (\det D^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_m})^2.$$

Thus, we obtain the algebraic equation $z^n + p_1 z^{n-1} + \ldots + p_n = 0$, where $p_m = q_{2m}$, $z = (1-\lambda)^2$. Let $z_1, z_2, \ldots, z_n$ be the roots of this equation. Since the matrix

$$C_E = \begin{bmatrix} E & D \\ D^{\mathrm{T}} & E \end{bmatrix}$$

is symmetric, its eigenvalues are real, and therefore $z_i > 0$, $i = \overline{1, n}$. Therefore $\lambda_i = 1 - \sqrt{z_i}$, $\lambda_{i+n} = 1 + \sqrt{z_i}$, $i = \overline{1, n}$, and for positive definiteness of the matrix C_E it is necessary and sufficient that $0 < z_i < 1$.

Example. Let us consider the system

$$\begin{cases} \dot{x}(t) = -x(t) + y(t) + ay(t-\tau), \\ \dot{y}(t) = -2y(t) + bx(t-\tau). \end{cases}$$

The Lyapunov–Krasovskii functional has the form

$$v[x(s), y(s)] = \frac{1}{2} x^2(t) + \frac{1}{3} x(t) y(t) + \frac{1}{3} y^2(t) + \int_{t-\tau}^{t} [x^2(s) + y^2(s)]\, ds.$$

The required polynomial is $z^2 + p_1 z + p_2 = 0$, where $p_1 = -\frac{5}{9}(b^2 + 2a^2)$, $p_2 = \frac{25}{81} a^2 b^2$. The asymptotic stability conditions have the form $b^2 + 2a^2 + \sqrt{b^4 + 4a^4} < 18/5$.

The condition $0 < z_i < 1$, $i = \overline{1, n}$ imposed on the roots of Eq. (3) is hard to verify in systems of high dimension. Let us give a different condition, which is easier to check.

THEOREM 3. In order that the linear system (1) be asymptotically stable, it is sufficient that the polynomial $u^n + b_1 u^{n-1} + \ldots + b_n$, where $b_i = \sum_{j=0}^{i} C_{n-j}^{i-j} p_j$, p_j defined by (4) satisfy the Hurwitz criterion.

Proof. The condition $z_i > 0$, $i = \overline{1, n}$ follows from the facts that the matrix C_E is symmetric and that $z = (1 - \lambda)^2$. The condition $z_i < 1$ is transformed into the condition $u_i < 0$ by the change of variables $z = u + 1$. The condition for the roots of the transformed polynomial $u^n + b_1 u^{n-1} + \ldots + b_n$, $b_i = \sum_{j=0}^{i} C_{n-j}^{i-j} p_j$, $i = \overline{1, n}$, to be negative is given by the Hurwitz criterion.

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