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On Jaśkowski's Discussive Logics*

Abstract. We expose the main ideas, concepts and results about Jaśkowski's discussive logic, and apply that logic to the concept of pragmatic truth and to the Dalla Chiara–di Francia view of the foundations of physics.

Contents

1. Introduction

We note by T a theory based on the logic $\mathcal{L}$ whose underlying language is L. We suppose that L contains the symbol $\neg$ for negation. (L may have more than one negation, but usually one of them is taken as fundamental; when we refer to L's negation, we are thinking about that fundamental negation.) The set of all formulæ of L is noted $\mathcal{F}$. Small Greek letters stand for formulæ while capital Greek letters stand for sets of formulæ.

We say that T is *trivial* (or *over–complete*) if $T = \mathcal{F}$; otherwise T is *nontrivial* (or *not over–complete*). If there is at least a formula α such that

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both α and its negation $\neg\alpha$ belong to T, then T is *inconsistent* (or, to be more precise, $\neg$*–inconsistent*). Otherwise T is *consistent* ($\neg$*–consistent*). For most logics the concepts of "inconsistency" and "triviality" are coincident. A logic is *paraconsistent* if it can be the underlying logic of theories that are nontrivial *and* inconsistent.

Today paraconsistent logics belong to a well–developed field that has grown into an important tool in fields that range from philosophy to computer science and even in applied areas [Arruda 1980, 1982] [da Costa, Marconi 1987] [Priest et al. 1989]. *Discussive logic* (also called *discursive* or *discoursive*) is a paraconsistent logic introduced by S. Jaśkowski in 1948 [1948, 1949, 1969] who formulated the corresponding propositional calculus; da Costa and Dubikajtis [1968, 1977] extended the discussive propositional calculus to a first– and higher–order predicate calculus. Several authors beyond the initiators of the field have also contributed to discussive logic; we may quote Kotas [1971, 1974, 1975] and Furmanowski [1975].

We may now quote Jaśkowski:

> "The over–complete systems [theories] have no practical significance; no problem may be formulated in the language of an over–complete system, since every sentence is asserted in that system. Accordingly, the problem of the logic of contradictory systems [inconsistent systems] is formulated here in the following manner: the task is to find a system of the sentential calculus which: 1) when applied to contradictory systems would not always entail their over–completeness; 2) would be rich enough to enable practical inference; 3) would have an intuitive justification. Obviously, these conditions do not univocally determine the solution, since they may be satisfied in varying degrees, the satisfaction of conditon 3) being rather difficult to appraise objectively. [1969, p. 145.]"

Jaśkowski adds that:

> "As it is known, even sets of those inscriptions which have no intuitive meaning at all can be turned into a formalized deductive system. In spite of this theoretical possibility, logical researchers so far have been taking into consideration such deductive systems which are symbolic interpretations of consistent theories, so that theses in each such systems are theorems in a theory formulated in a single symbolic language free from terms whose meanings are vague. But suppose that theses which do not satisfy those conditions are included into a deductive system. It suffices, for

instance, to deduce consequences from several hypotheses that are inconsistent with one another in order to change the nature of the theses, which thus no longer reflect a uniform opinion. The same happens if the theses advanced by several participants in a discussion are combined into a single system, or if one person's opinions are so pooled into one system although that person is not sure whether the terms occurring in his various theses are not slightly differentiated in their meanings. Let such a system which cannot be said to include theses that express opinions in agreement with one another, be termed a *discursive system* [discussive or discoursive system]. To bring out the nature of the theses of such a system it would be proper to precede each thesis by the reservation: 'in accordance with the opinion of one of the participants in the discourse [discussion]' or 'for a certain admissible meaning of the terms used.' Hence, the joining of a thesis to a discussive system has a different intuitive meaning than has the assertion in an ordinary system. A *discursive assertion* includes an implicit reservation of the kind specified above, which ... has its equivalent in possibility *Pos* [$\Diamond$]. Accordingly, if a thesis α is recorded in a discussive system, its intuitive meaning ought to be interpreted so as if it were preceded by the symbol *Pos* [$\Diamond$], that is, the sense: 'it is possible that α.' This is how an impartial arbiter might understand the theses of the various participants in the discussion."

[1969, p. 149]; emphasis due to the author; clarifications added between square brackets [].)

Recently discussive logic found applications in the theory of pragmatic truth (quasi–truth), the foundations of quantum mechanics and the philosophy of science (see [da Costa, Chuaqui 1990], [da Costa, Doria 1990] and [da Costa, Dubikajtis 1968]). The present paper is part of a research program whose main guidelines stem from Hilbert's 6th Problem—the axiomatization of physics—and which has been conducted by the authors since 1988. After detailed exploration of a classical first–order language (and set theory) as a possible foundation for the axiomatics of physics ([da Costa, Doria 1992c, 19XX], [Stewart 1991]), the authors are now interested in their non–classical counterparts and the effect they will have on their foundational work. So, it is the purpose of this paper to lay those foundations; we leave their consequences to physics to a future paper. Possible applications are sketched in Sections 5 and 6, to be further developed in the future work of the authors.

2. Preliminaries

In the present Section we lay the ground for our development of discussive logic. The end of definitions, propositions and theorems is indicated with an empty triangle $\triangle$.

2.1. The first–order monadic predicate calculus

The first–order monadic predicate calculus deals only with *unary* predicate letters ([Church 1956], [Quine 1950]). Out of the primitive connectives $\vee$ and $\neg$ plus the universal quantifier $\forall$, the individual variables, monadic predicate letters (or predicate symbols), and so on, we can easily develop the monadic calculus (without equality). We will also need the following defined symbols: $\wedge$ (conjunction), $\rightarrow$ (material implication), $\leftrightarrow$ (material equivalence), and $\exists$ (there exists); individual variables are noted x, y, z etc.

DEFINITION 2.1. A **monadic structure** is a set–theoretic construct

$$\mathcal{A} = \langle A, \mathsf{P}_i \rangle_{i \in I},$$

where $\mathcal{A}$ is a nonempty set and each P_i, $i \in I \neq \emptyset$, is a subset of $\mathcal{A}$. $\triangle$

REMARK 2.2. We will abbreviate $\mathcal{A}$ as $\mathcal{A} = \langle A, \mathsf{P}_i \rangle$. $\triangle$

We suppose that the language L_M of the monadic calculus has no individual constants. (However our results can be extended to that case.) We also suppose that the variables belong to a linearly ordered denumerable sequence $x_1, x_2, x_3, \ldots$. We note by V the set of variables in L_M.

DEFINITION 2.3. An **assignment over** $\mathcal{A}$ is a function $s : V \rightarrow A$. $\triangle$

Let P_i, $i \in I$, be the family of unary predicate symbols of L_M; to each P_i in $\mathcal{A}$ there corresponds a P_i in L_M. We are going to use $\Rightarrow$ and $\Leftrightarrow$ as abbreviations for the metalogical notions of (resp.) implication and equivalence. Then:

DEFINITION 2.4. If α is a formula of L_M and s is an assignment over $\mathcal{A}$, then we define by recursion on α the relation "$\mathcal{A}, s \models \alpha$" ($\mathcal{A}, s$ satisfy α):

1. $\mathcal{A}, s \models P_i(x_j) \Leftrightarrow s(x_i) \in \mathsf{P}_i$;
2. $\mathcal{A}, s \models \beta \vee \gamma \Leftrightarrow \mathcal{A}, s \models \beta$ or $\mathcal{A}, s \models \gamma$;
3. $\mathcal{A}, s \models \neg\beta \Leftrightarrow \mathcal{A}, s \not\models \beta$ ($\mathcal{A}, s$ do not satisfy β);

4. $\mathcal{A}, s \models \forall x_j \beta \Leftrightarrow$ For every assignment s' which differs from s at most in $x_{s'}$ we have $\mathcal{A}, s' \models \beta$. $\triangle$

DEFINITION 2.5. If for every assignment s over $\mathcal{A}$, we have that $\mathcal{A}, s \models \alpha$, then we say that $\mathcal{A}$ **satisfies** α, and write $\mathcal{A} \models \alpha$. $\triangle$

DEFINITION 2.6. If for any structure $\mathcal{A} = \langle A.\mathsf{P}_i \rangle$ we have that $\mathcal{A} \models \alpha$, we say that α **is valid**. $\triangle$

DEFINITION 2.7. If for any structure $\mathcal{A}$ and any assignment s over $\mathcal{A}$, if $\mathcal{A}, s \models \gamma$, for all γ in Γ, then $\mathcal{A}, s \models \alpha$, we write $\Gamma \models \alpha$. $\triangle$

Notice that the usual (correct and complete) axiom systems for the polyadic calculus are (correct and complete) axiomatizations for the monadic calculus when we restrict their polyadic letters to the monadic ones. If the symbol $\vdash$ of syntactic consequence is defined as in Henkin and Montague [1956], we have:

PROPOSITION 2.8. *For the monadic calculus,* $\Gamma \vdash \alpha \Leftrightarrow \Gamma \models \alpha$. $\triangle$

Behman (and Löwenheim) showed that the monadic calculus is decidable (cf. Church [1956, p. 284]).

DEFINITION 2.9. The first-order **uniform predicate calculus** is a subcalculus of the monadic calculus in which there is only one individual variable, say x.

Its language will be denoted by L_U. $\triangle$

The previous definitions can be easily adapted to the uniform calculus. For example, Definition 2.4 is changed to:

DEFINITION 2.10. If α is a formula of L_U, $\mathcal{A} = \langle A, P_i \rangle$ is a structure for L_U, and $a \in A$ is an assignment over $\mathcal{A}$, then we define by recursion on α the relation "$\mathcal{A}, a \models \alpha$" ($\mathcal{A}, a$ satisfy α):

1. $\mathcal{A}, a \models P_i(x) \Leftrightarrow a \in \mathsf{P}_i$;
2. $\mathcal{A}, a \models \beta \vee \gamma \Leftrightarrow \mathcal{A}, a \models \beta$ or $\mathcal{A}, a \models \gamma$;
3. $\mathcal{A}, a \models \neg\beta \Leftrightarrow \mathcal{A}, a \not\models \beta$ ($\mathcal{A}, a$ do not satisfy β);
4. $\mathcal{A}, a \models \forall x \beta \Leftrightarrow$ For every $b \in A$ we have $\mathcal{A}, b \models \beta$. $\triangle$

A *sentence* is a formula without free variables. Then,

PROPOSITION 2.11. *If α is a sentence of the monadic calculus, then there is a sentence α' of the uniform calculus such that $\models \alpha \leftrightarrow \alpha'$.*

Moreover α' can be recursively obtained from α. $\triangle$

PROPOSITION 2.12. *If α is a formula of the monadic calculus, then we can recursively obtain a formula α' of the uniform calculus such that $\models \alpha$ in the monadic calculus if and only if $\models \alpha'$ in the uniform calculus.* $\triangle$

Propositions 2.11 and 2.12 follow from Behman's decision method. A correct and complete axiomatization for the uniform calculus is presented below; see Quine [1950] on this calculus.

2.2. The propositional calculus $S5$

$S5$ is a modal propositional calculus. Its language L_5 has the following primitive symbols:

1. Connectives: $\vee$, $\neg$ and $\Box$ (necessity); $\rightarrow$, $\wedge$ and $\Diamond$ (possibility) are introduced by definition, as usual.

2. Propositional variables: $p_1, p_2, p_3, \ldots$ Their set is noted A.

3. Parentheses.

The notion of formula is obtained in the usual way.

DEFINITION 2.13. A **Kripke structure for** $S5$ or an $S5$–structure is a set–theoretic structure

$$\mathcal{K} = \langle W, v \rangle,$$

where W is a nonempty set whose elements are called "worlds," and v is a valuation function

$$v : W \times A \rightarrow \{0, 1\}.$$

If $v(w, p_j) = 1 (= 0)$, we say that p_j is **true** (**false**) in the world w. If $w \in W$, we put:

$$\tilde{w} = \{p_j : v(w, p_j) = 1\}. \triangle$$

DEFINITION 2.14. If α is a formula of L_5 and if $w \in W$, we define the relation $K, w| \vdash \alpha$ (K, w force α) by recursion on α as follows:

1. $K, w| \vdash p_j \Leftrightarrow v(w, p_j) = 1$.

2. $K, w| \vdash \beta \vee \gamma \Leftrightarrow K, w| \vdash \beta$ or $K, w| \vdash \gamma$.

3. $K, w| \vdash \neg\beta \Leftrightarrow K, w| \not\vdash \beta$.

4. $K, w| \vdash \Box\beta \Leftrightarrow$ for every $t \in W$, $K, t| \vdash \beta$. $\triangle$

DEFINITION 2.15.

1. If $K, w| \vdash \alpha$ for every $w \in W$, then we say that K **forces** α and put $K| \vdash \alpha$.

2. If for any $S5$–structure K, $K| \vdash \alpha$, then we say that the formula α is $S5$–valid.

3. Given the set of formulæ $\Gamma \cup \{\alpha\}$, we say that α is a **semantic consequence in $S5$ of Γ** (and write $\Gamma \models_{S5} \alpha$), if for any $S5$–structure K and any world w of K, $K, w| \vdash \alpha$ whenever $K, w| \vdash \gamma$, for any $\gamma \in \Gamma$. $\triangle$

We can give $S5$ the following axiomatization [Hughes, Cresswell 1968]:

AXIOM SYSTEM 2.16

1. If α is an instance of a classical tautology, then α is an axiom.

2. $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$.

3. $\Box\alpha \to \alpha$.

4. $\Diamond\alpha \to \Box\Diamond\alpha$.

Rules:

1. *Modus ponens.* α, $\alpha \to \beta$ / β.

2. *Gödel's rule.* α / $\Box\alpha$. $\triangle$

The concept of syntactic consequence is introduced without difficulty in $S5$, but it is convenient to treat Gödel's rule according to Henkin and Montague [1956], as if $\Box$ were the universal quantifier. When α is a syntactic consequence of Γ in $S5$, we write $\Gamma \vdash_{S5} \alpha$.

PROPOSITION 2.17. $\Gamma \vdash_{S5} \alpha \Leftrightarrow \Gamma \models_{S5} \alpha$.

PROOF. By an adaptation of the standard proofs of the weak completeness of the above axiomatization for $S5$. ■

There is an obvious bijection between the set of formulæ of L_U and that of L_5: given a formula α of L_U we obtain the corresponding formula α^* of L_5 by replacing any subformula $P_i(x)$ of α by p_i and any universal quantification $\forall x$ by $\Box$.

We have:

PROPOSITION 2.18. *If α is a formula of L_U and if α^* is the corresponding formula in L_5, then $\models_U \alpha$ (in the uniform calculus) iff $\models_{S5} \alpha^*$ (in $S5$).*

PROOF. (I) Given the structure $\mathcal{A} = \langle A, \mathsf{P}_i \rangle$, we obtain a Kripke–structure $K_{\mathcal{A}} = \langle W, v \rangle$ as follows: if $a \in \mathcal{A}$, we put $w_a = \{p_i : a \in \mathsf{P}_i\}$, and W is the set of all w_a for a in A. Therefore, $w_a = \tilde{w}_a$ and $v(w_a, p_i) = 1$ iff $p_i \in \tilde{w}_a$.

Hence, $K_{\mathcal{A}}, w_a| \vdash p_i \Leftrightarrow p_i \Leftrightarrow a \in \tilde{w}_a \in \mathsf{P}_i$. On the other hand, $\mathcal{A}, a| \vdash P_i(x) \Leftrightarrow a \in \mathsf{P}_i$. Hence, $K_{\mathcal{A}}, w_a| \vdash p_i \Leftrightarrow \mathcal{A}, a \models P_i(x)$. Then, if we take into account Definitions 2.10 and 2.14, we conclude that if $\models_{S5} \alpha^*$, then $\models_U \alpha$.

(II) Let $K = \langle W, v \rangle$ be a Kripke–structure. We denote by $\tilde{W}$ the set of all $\tilde{w}$, for w in W. Then there is a structure $\mathcal{A}_K = \langle A, \mathsf{P}_i \rangle$, defined as follows: $A = \tilde{W}$ and $\mathsf{P}_i = \{\tilde{w} : p_i \in \tilde{w}\}$. One has:

$$\mathcal{A}_K, \tilde{w}| \vdash p_i \Leftrightarrow p_i \in \tilde{w} \Leftrightarrow \tilde{w} \in \mathsf{P}_i \Leftrightarrow p_i \in \tilde{w} \Leftrightarrow K, w| \vdash p_i.$$

Again from Definitions 2.10 and 2.14, we conclude that if $\models_U \alpha$, then $\models_{S5} \alpha^*$.

(I) and (II) imply that $\models_U \alpha \Leftrightarrow \models_{S5} \alpha^*$. ■

DEFINITION 2.19. If Γ is a set of formulæ of the uniform calculus, then

$$\Gamma^* = \{\alpha^* : \alpha \in \Gamma\}. \triangle$$

COROLLARY 2.20. $\Gamma \models_U \alpha \Leftrightarrow \Gamma^* \models_{S5} \alpha^*$. △

COROLLARY 2.21. $\Gamma \vdash_U \alpha \Leftrightarrow \Gamma^* \vdash_{S5} \alpha^*$. △

COROLLARY 2.22. *$S5$ is decidable.* △

COROLLARY 2.23. *The standard deduction rules of the monadic predicate calculus are true in $S5$.* △

(For example: the deduction theorem.) The preceding results show that $S5$ and the uniform calculus are essentially the same logical system. In particular we easily deduce an axiomatization for the uniform calculus out of another for $S5$. Also, out of the standard model theory for the uniform calculus we may get a corresponding model theory for $S5$. Results such as the Löwenheim–Skolem, Łós–Tarski, Łós–Suszko–Tarski and Craig theorems can be immediately adapted for $S5$ [Shoenfield 1967]. As am example, we formulate Craig's interpolation lemma for $S5$: if α and β are formulæ of $S5$, the set of their common propositional variables is noted $\text{Var}(\alpha, \beta)$.

PROPOSITION 2.24. (CRAIG'S INTERPOLATION LEMMA FOR $S5$.) *If α and β are formulæ of $S5$ such that $\text{Var}(\alpha,\beta) \neq \emptyset$, then if $\models_{S5} \alpha \to \beta$, there is a propositional formula γ whose propositional variables belong to $\text{Var}(\alpha,\beta)$ such that $\models_{S5} \alpha \to \gamma$ and $\models_{S5} \gamma \to \beta$.*

If $\text{Var}(\alpha,\beta) = \emptyset$ and $\models_{S5} \alpha \to \beta$, then either $\models_{S5} \neg\alpha$ or $\models_{S5} \beta$.

PROOF.. Suffices to use Kleene's version for the Craig lemma [Klee 1967, pp. 257–258]. ■

DEFINITION 2.25. A **Henle algebra** is a structure

$$\mathcal{H} = \langle A, -, \wedge, * \rangle,$$

where $\langle A, -, \wedge \rangle$ is a Boolean algebra and $*$ is a unary operator over A such that:

$$*1 = 1,$$

$$*x = 0,$$

for all $x \neq 0$, where 0 and 1 are respectively the first and last elements of $\mathcal{H}$. △

Then:

PROPOSITION 2.26. *If α is a formula of $S5$, then the following are equivalent:*

1. *α is a theorem of $S5$.*

2. *α is valid in every Henle algebra.*

3. *α is valid in every finite Henle algebra.*

4. α is valid in the denumerable Henle algebra

$$\mathcal{H} = \langle A, -, \wedge, * \rangle$$

where $\langle A, -, \wedge \rangle$ is the Boolean algebra of all finite and cofinite sets of natural numbers.

PROOF. We use Behmann's decision method for the uniform calculus, as adapted to $S5$. ■

DEFINITION 2.27. If C is a propositional logic, then C has the **finite model property** if for every formula α of C, if α is not a thesis of C, then there is a finite model **M** such that **M** validates all theses of C but does not validate α. △

PROPOSITION 2.28. *$S5$ has the finite model property.* △

REMARK 2.29. Since $S5$ is axiomatizable, the last proposition provides us with a second method for proving that $S5$ is decidable. △

PROPOSITION 2.30. (DUGUNDJI.) *$S5$ does not have a finite characteristic matrix.* △

For details about the last two theorems see [da Costa 1975]. If we take into account condition 2 in Proposition 2.26, we can develop an algebraic semantics for $S5$ through Henle algebras. With the help of those algebras we define the relation of semantic consequence and so on. Condition 4 in the same proposition shows that $S5$ is a many–valued logic with an infinite set of truth–values.

REMARK 2.31. As we have reduced the Kripke semantics for $S5$ to the extant semantics for the uniform calculus, it is also possible to reduce the Kripke semantics of other modal propositional calculi to an appropriate version of the first–order predicate calculus with monadic predicate letters and a single binary predicate letter. △

2.3. $S5Q^=$: $S5$ with quantification and necessary equality

We now describe $S5Q^=$, that is, $S5$ with quantification and necessary equality. (For more details on $S5Q^=$ see [Hughes, Cresswell 1968].) The language of $S5Q^=$ has the same primitive symbols as those of the usual first–order predicate calculus with equality and individual constants, plus the symbol that denotes 'necessity." (To simplify we exclude function symbols.) Introduction of defined symbols is made as usual; conventions for writing formulæ are also standard, as well as the way we state the postulates for the calculus (axiom schemes and primitive rules of inference). The postulates we adopt for $S5Q^=$ are:

AXIOM SYSTEM 2.32

1. If α is an instance of a classical tautology, then α is an axiom.
2. $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$.
3. $\Box\alpha \to \alpha$.
4. $\Diamond\alpha \to \Box\Diamond\alpha$.
5. $\forall x\alpha(x) \to \alpha(t)$.
6. $\alpha,\ \alpha \to \beta/\beta$.
7. $\alpha/\Box\alpha$.
8. $\alpha \to \beta(x)/\alpha \to \forall x\beta(x)$.
9. $x = x$.
10. $x = y \to (\alpha(x) \leftrightarrow \alpha(y))$. $\triangle$

Notation here is obvious; for instance, in axiom scheme 5, t is either a variable free for x in $\alpha(x)$ or an individual constant.

The concept of syntactic consequence $\vdash$ is treated as in [Henkin, Montague 1956]; therefore $\Box$ is subject to restrictions similar to those of $S5$. The usual metatheorems of the predicate calculus (e.g., the deduction theorem) remain valid. Therefore:

PROPOSITION 2.33. *The following rules are valid for $S5Q^=$:*

1. $\Gamma \cup \{\alpha\} \vdash \beta \Rightarrow \Gamma \vdash \beta$.
2. $\{\alpha, \alpha \to \beta\} \vdash \beta$.
3. $\{\alpha, \beta\} \vdash \alpha \wedge \beta$.
4. $\{\alpha \wedge \beta\} \vdash \alpha$.
5. $\{\alpha \wedge \beta\} \vdash \beta$.

6. $\Gamma \cup \{\alpha\} \vdash \gamma$ *and* $\Gamma \cup \{\beta\} \vdash \gamma \Rightarrow \Gamma \cup \{\alpha \vee \beta\} \vdash \gamma$.

7. $\{\alpha\} \vdash \alpha \vee \beta$.

8. $\{\beta\} \vdash \alpha \vee \beta$.

9. $\Gamma \cup \{\alpha\} \vdash \beta$ *and* $\Gamma \cup \{\alpha\} \vdash \neg\beta \Rightarrow \Gamma \vdash \neg\alpha$.

10. $\{\alpha, \neg\alpha\} \vdash \beta$.

11. $\{\neg\neg\alpha\} \vdash \alpha$.

12. $\{\alpha\} \vdash \neg\neg\alpha$.

13. $\{x = y\} \vdash \Box(x = y)$. $\triangle$

Kripke semantics for $S5$ is easily extended to $S5Q^{=}$ (see [Hughes, Cresswell 1968]); the notion of semantic consequence $\models$ is also imediately defined.

PROPOSITION 2.34. *If* $\Gamma \cup \{\alpha\}$ *is a set of formulæ of* $S5Q^{=}$, *then we have in* $S5Q^{=}$*that* $\Gamma \vdash \alpha \Leftrightarrow \Gamma \models \alpha$.

PROOF. See [da Costa, Chuaqui 1990]. ■

Notice that with the help of the preceding result we can obtain a proof of the strong correctness and completeness for a two–sorted first–order logic in which there is a single variable of the first sort.

Equality is necessary in $S5Q^{=}$, but there is no difficulty in making it into a system where equality is contingent.

3. The system J

The language of the discussive propositional calculus J is the language of $S5$.

DEFINITION 3.1. $\Diamond\Gamma = \{\Diamond\alpha : \alpha \in \Gamma\}$. $\triangle$

J can be semantically defined as:

DEFINITION 3.2. $\Gamma \models_{\mathsf{J}} \alpha \Leftrightarrow \Diamond\Gamma \models_{S5} \Diamond\alpha$. $\triangle$

PROPOSITION 3.3. $\models_{\mathsf{J}} \alpha \Leftrightarrow \models_{S5} \Diamond\alpha$. $\triangle$

PROPOSITION 3.4. $\Gamma \models_{\mathsf{J}} \alpha$ *if and only if there are* $\gamma_1, \gamma_2, \ldots, \gamma_n$ *in* Γ *such that* $\models_{S5} \Diamond\gamma_1 \wedge \Diamond\gamma_2 \wedge \ldots \wedge \Diamond\gamma_n \rightarrow \Diamond\alpha$.

PROOF. Consequence of the definition of J. ∎

COROLLARY 3.5. *$\Gamma \models_J \alpha$ if and only if there is a finite set $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subset \Gamma$ such that $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \models_J \alpha$.* △

COROLLARY 3.6. $\models_{S5} \alpha \Rightarrow \models_J \alpha$

PROOF. In $S5$ we have that $\models_{S5} \alpha \Rightarrow \models_J \Diamond\alpha$. ∎

Due to the preceding definitions and results we see that J satisfies Jaśkowski's main motivations in the construction of his logic. Furthermore J can be looked upon as a kind of logic of vague concepts, given the essential equivalence between the uniform calculus and $S5$; also, obviously, as a logic of discussion in Jaśkowski's sense.

J has several axiomatizations (cf. [da Costa 1975] and [da Costa, Dubikajtis 1977]). The one we present here is however a new one, axiomatization $\mathcal{A}$. The postulates for $\mathcal{A}$ are:

AXIOM SYSTEM 3.7.

1. If α is an axiom of $S5$, then $\Box\alpha$.
2. $\Box\alpha, \Box(\alpha \rightarrow \beta)/\Box\beta$.
3. $\Box\alpha/\alpha$.
4. $\Diamond\alpha/\alpha$.
5. $\Box\alpha/\Box\Box\alpha$. △

LEMMA 3.8. *If $\vdash_{\mathcal{A}} \alpha$ (one reads: "α is provable in $\mathcal{A}$"), then $\models_{\mathcal{A}} \Diamond\alpha$.*

PROOF. By induction on the length of the derivation of α in $\mathcal{A}$. ∎

LEMMA 3.9. *If $\models_{S5} \alpha$, then $\vdash_{\mathcal{A}} \Box\alpha$.*

PROOF. By induction on the length of the derivation of α in $S5$ ($\models_{S5} \alpha \Leftrightarrow \vdash_{S5} \alpha$). ∎

PROPOSITION 3.10. $\vdash_{\mathcal{A}} \alpha \Leftrightarrow \models_J \alpha$.

PROOF. If $\vdash_{\mathcal{A}} \alpha$, then by Lemma 3.8, $\models_{S5} \Diamond\alpha$. So, by definition, $\models_J \alpha$.

Conversely, if $\models_J \alpha$, then by definition $\models_{S5} \Diamond\alpha$. So, by Lemma 3.9, $\vdash_{\mathcal{A}} \Box\Diamond\alpha$. By postulate 3 of $\mathcal{A}$, $\vdash_{\mathcal{A}} \Diamond\alpha$, and by rule 4, $\vdash_{\mathcal{A}} \alpha$. ■

DEFINITION 3.11. $\vdash_J \alpha$ means $\vdash_{\mathcal{A}} \alpha$. We write $\Gamma \vdash_J \alpha$ iff there are $\gamma_1, \ldots, \gamma_n$ in Γ such that $\vdash_J \Diamond\gamma_1 \wedge \Diamond\gamma_2 \wedge \ldots \wedge \Diamond\gamma_n \to \Diamond\alpha$. △

PROPOSITION 3.12. $\Gamma \vdash_J \alpha \Leftrightarrow \Gamma \models_J \alpha$.

PROOF. Consequence of Proposition 3.10 and Definition gef. ■

PROPOSITION 3.13. Modus ponens $\alpha, \alpha \to \beta/\beta$ *isn't valid in* J.

PROOF. We may have in the uniform calculus U: $\models_U \exists x\alpha$ and $\models_U \exists x(\alpha \to \beta)$, but not $\exists x\beta$. ■

PROPOSITION 3.14. *The rules* $\alpha, \beta/\alpha \wedge \beta$ *and* $\alpha, \neg\alpha/\beta$ *are not valid in* J. △

PROPOSITION 3.15. *The deduction theorem*

$$\Gamma \cup \{\alpha\} \vdash_J \beta \Rightarrow \Gamma \vdash_J \alpha \to \beta$$

isn't true in J. △

J is essentially a tool to reason with inconsistent sets of premises while avoiding triviality. It is a calculus which can be made as the underlying logic of inconsistent nontrivial theories.

We now enrich J with a new set of more convenient connectives whose meaning is intuitively clear:

DEFINITION 3.16

1. **Discussive implication:** $\alpha \to_d \beta \equiv \Diamond\alpha \to \beta$.
2. **Discussive conjunction:** $\alpha \wedge_d \beta \equiv \Diamond\alpha \wedge \beta$.
3. **Discussive equivalence:** $\alpha \leftrightarrow \beta \equiv (\alpha \to_d \beta) \wedge_d (\beta \to_d \alpha)$.
4. **Possibility:** $\nabla\alpha \equiv \neg\Diamond\alpha$. △

PROPOSITION 3.17. $\to_d$. $\wedge_d$, $\vee$, $\leftrightarrow_d$ *and* ∇ *possess in* J *all the classic properties of* $\to$, $\wedge$, $\vee$, $\leftrightarrow$ *and* $\neg$, *respectively.*

PROOF. Direct computation.

So, the classical propositional calculus can be seen as embedded in a natural way into J. ■

However there is a *caveat*:

PROPOSITION 3.18. *The following formulæ and rules are not valid in* J:

1. $\alpha \to_d (\beta \to_d \alpha \wedge \beta)$.
2. $\alpha \to_d (\neg\alpha \to_d \beta)$.
3. $(\alpha \wedge \beta \to_d \gamma) \to_d (\alpha \to_d (\beta \to_d \gamma))$.
4. $\Gamma \cup \{\alpha\} \models_{\mathsf{J}} \beta$ *and* $\Gamma \cup \{\alpha\} \models_{\mathsf{J}} \neg\beta \Rightarrow \Gamma \models_{\mathsf{J}} \neg\alpha$.
5. $(\alpha \leftrightarrow_d \neg\alpha) \to_d \beta$.
6. $\alpha \to_d \neg\alpha) \to_d \neg\beta$.

PROOF. By the semantics of J. ■

PROPOSITION 3.19.

1. J *is decidable.*
2. J *has no finite characteristic matrix, but has the finite model property.*

PROOF. Through Propositions 2.18, 2.28, 2.30. ■

It is now clear that J has two semantics, one based on Kripke structures and another one based on the notion of Henle's algebra. However in the present paper we will primarily be interested in the first one. Also, by the same steps we took to build a model theory for $S5$ we can construct a similar one for J. We give as an example Craig's interpolation lemma for J:

PROPOSITION 3.20. *If* α *and* β *be formulæ of* J *such that* $\mathrm{Var}(\alpha,\beta) \neq \emptyset$ *then, if* $\models_{\mathsf{J}} \alpha \to_d \beta$, *there is a formula* γ *whose propositional variables are in* $\mathrm{Var}(\alpha,\beta)$ *such that* $\models_{\mathsf{J}} \alpha \to_d \gamma$ *and* $\models_{\mathsf{J}} \gamma \to_d \Diamond\beta$.

If $\mathrm{Var}(\alpha,\beta) = \emptyset$ *and* $\models_{\mathsf{J}} \alpha \to_d \beta$, *then either* $\models_{\mathsf{J}} \neg\alpha$ *or* $\models_{\mathsf{J}} \beta$.

PROOF. Apply Proposition 2.24. ■

DEFINITION 3.21. $\overline{\Gamma} = \{\alpha : \Gamma \vdash_{\mathsf{J}} \alpha\}$. △

DEFINITION 3.22. If $\overline{\Gamma}$ is the set of all formulæ, then Γ is **trivial**. If not, Γ is **nontrivial**. $\triangle$

DEFINITION 3.23. If there is a formula α such that both $\Gamma \vdash_J \alpha$ and $\Gamma \vdash_J \neg\alpha$, then Γ is **inconsistent**. If not, Γ is **consistent**. $\triangle$

DEFINITION 3.24. If there is a formula α such that $\Gamma \vdash_J \alpha$ and $\Gamma \vdash_J \nabla\alpha$, then Γ is **strongly inconsistent**. $\triangle$

COROLLARY 3.25. *Γ is trivial if and only if Γ is strongly inconsistent.*

PROOF. Immediate. ■

PROPOSITION 3.26. *There are inconsistent but nontrivial sets of formulæ.*

PROOF. If p is a propositional variable then $\{p, \neg p\}$ is inconsistent and nontrivial. ■

DEFINITION 3.27. If K is a Kripke structure, then K is a model for Γ if, for every $\gamma \in K$ there is a world w of K such that $K, \mathsf{w}| \vdash \gamma$. $\triangle$

PROPOSITION 3.28

1. *Γ has a model iff Γ is nontrivial.*

2. *Therefore there are inconsistent sets of formulæ which have models.*

PROOF. Apply the Kripke semantics for J. ■

The results just proved show that J is a paraconsistent logic.

The Kripke semantics for J is a totally classical set–theoretic construct. Nevertheless, if we proceed as in Rescher and Brandon [1980], we can obtain out of that classical semantics a paraconsistent one. It is enough to consider structures obtained from Kripke structures by collapsing (in each structure) their worlds into a single one. We thus get 'inconsistent,' that is, paraconsistent worlds, and it is possible to reword the semantics of J in terms of those 'hyper–complete' worlds.

The actual choice between both semantical systems or their merits and defficiencies depend on philosophical, not technical, criteria, which we do not discuss in this paper.

4. The system J*

The language of J*, the first–order discussive predicate calculus with necessary equality, is the language of $S5Q^=$.

DEFINITION 4.1

1. $\Gamma \models_{J*} \alpha \Leftrightarrow \Diamond\Gamma \models_{S5Q^=} \Diamond\alpha$.
2. In particular, $\models_{J*} \alpha \Leftrightarrow \models_{S5Q^=} \Diamond\alpha$. $\triangle$

All results in the previous Section can be applied to J*. For example:

PROPOSITION 4.2. *$\Gamma \models_{J*} \alpha$ iff there are $\gamma_1, \gamma_2, \ldots, \gamma_n$ in Γ such that:*

$$\models_{S5Q^=} \Diamond\gamma_1 \wedge \Diamond\gamma_2 \wedge \ldots \wedge \Diamond\gamma_n \rightarrow \Diamond\alpha. \; \triangle$$

We now present a complete and correct axiomatization for J*:

AXIOM SYSTEM 4.3

1. *Axiom scheme.* If α is an axiom of $S5Q^=$, then $\Box\alpha$.

Rules:

1. $\Box\alpha$. $\Box(\alpha \rightarrow \beta/\Box\beta$.
2. $\Box\alpha/\alpha$.
3. $\Diamond\alpha/\alpha$.
4. $\Box\alpha/\Box\Box\alpha$.
5. $\Box(\alpha \rightarrow \beta(x)/\Box(\alpha \rightarrow \forall x\beta(x))$, where x isn't free in α. $\triangle$

PROPOSITION 4.4. *Within J*. $\rightarrow_d$, $\wedge_d$, $\vee$, $\leftrightarrow_d$, ∇. $\exists$ and $\forall$ have all the classical properties of $\rightarrow$, $\vee$, $\wedge$, $\leftrightarrow$, $\neg$, $\exists$ and $\forall$, respectively.* $\triangle$

Notice that $S5Q^=$ is contained in J*.

PROPOSITION 4.5. *J* and $S5Q^=$ are not decidable.* $\triangle$

PROPOSITION 4.6. *Both* J* *and* $S5Q^=$ *have algebraic semantics relative to which they are sound and complete.* $\triangle$

PROPOSITION 4.7. *There are inconsistent but nontrivial sets of formulæ in* J*. $\triangle$.

PROPOSITION 4.8. *For* J*, Γ *is nontrivial iff there is a Kripke structure for* J* *which is a model of* Γ.

Therefore there are inconsistent sets of formulæ which have models. $\triangle$

The extension of J* to a higher–order logic presents no difficulties (see [da Costa, Dubikajtis 1977]). We plan to formalize a discussive version of set theory in a forthcoming paper; for a different kind of a first–order discussive logic, see [da Costa, Chuaqui 1990].

5. Pragmatic truth

One of the most important applications of discussive logic is in the field of pragmatic truth (or quasi–truth). For details on the formal theory of pragmatic truth see [da Costa 1989a], [da Costa, Chuaqui 1990] and [Mikenberg et al. 1986]. In what follows we restrict ourselves to an overview of some aspects of those references in order to illustrate the power of discussive logic.

Following [da Costa, Chuaqui 1990], suppose that we are studying a previously specified domain of knowledge Δ within an empirical science (say, particle physics). Therefore we are concerned with certain *real objects* (vapor trajectories in a Wilson chamber, spectral lines, etc.); we note their set by A_1. There are some relations among the elements of A_1 which are relevant to our work; we model them as partial relations R_i, $i \in I$. (We suppose that every relation has a fixed arity r_i.) The R_i are partial,that is, they are not necessarily defined for all r_i–tuples in A_1.

(Partial relations express what we know or what we accept as true about the actual relations of Δ for the members of A_1.) Thus the partial structure $\langle A_1, R_i\rangle$, $i \in I$, encompasses what we know or accept as true about the actual structure of Δ.

In order to better organize our knowledge about Δ we must add some *ideal* elements to our structure $\langle A_1, R_i\rangle_{i\in I}$ (e.g., quarks and wavefunctions in quantum physics and in particle physics). The set of those new (ideal) objects is noted A_2. Of course $A_1 \cap A_2 = \emptyset$, and we put $A = A_1 \cup A_2$. We must also add new partial relations R_j, $j \in J$, to our picture. (Some of them extend the R_i, $i \in I$.)

Moreover there are some sentences (closed formulæ) in the non-modal language L in which we talk about the structure $\langle A, R_k\rangle$, $k \in I \cup J$. $(I \cap J = \emptyset.)$ We accept those as true in the sense of the correspondence theory of truth. For example, in the case of true *decidable* propositions (propositions whose truth or falsehood can be decided are said to be decidable). We also include here some 'general' sentences that express laws or theories already accepted as true; those are the *primary* sentences P.

When we take into account those remarks we are led to the following more technical formulation:

DEFINITION 5.1. A **simple pragmatic structure** (sps) is a set–theoretic construct of the form:

$$\mathcal{A} = \langle A_1, A_2, R_i, R_j, P\rangle,$$

where the nonempty sets A_1 and A_2, the partial relations R_i and R_j, $i \in I$ and $j \in J$, and the set of sentences P satisfy the conditions above. △

However from the strictly mathematical point of view it is more convenient to define a sps as follows:

DEFINITION 5.2. A sps is a structure:

$$\mathcal{A} = \langle A, R_k, P\rangle_{k\in K},$$

where A is a nonempty set, R_k is a partial relation defined on A for every k, and P is a set of sentences of a language L of the same similarity type as that of $\mathcal{A}$; L is interpreted in $\mathcal{A}$.

A is the **universe** of $\mathcal{A}$. △

(For some k, R_k may be empty; P may also be empty while L is a first–order non–modal language with identity in the general situation.)

From here onwards a sps will always be an $\mathcal{A}$ as in Definition 5.1. To simplify, partially defined functions are not included in a sps.

Let L be a non–modal first–order language with equality but without function symbols. An interpretation of L in an sps $\mathcal{A} = \langle A, R_k, P\rangle$ associates an individual of the universe A of $\mathcal{A}$ to every constant in L and a relation R_k in $\mathcal{A}$ to each predicate symbol in L. (We suppose that the correspondence between predicates and relations is 1–1 and onto.)

Now let L and $\mathcal{A} = \langle A, R_k, P\rangle$ be respectively a language and a sps. Then:

DEFINITION 5.3. $\mathcal{B}$ is a **total structure** if it is a structure where all relations R_k are total, that is, where relations of arity n are defined for all n–tuples of elements of $\mathcal{B}$. △

Suppose moreover that L is also interpreted in $\mathcal{B}$. Then:

DEFINITION 5.4. $\mathcal{B}$ is $\mathcal{A}$–**normal** if the following conditions are simultaneously satisfied:

1. The universe of $\mathcal{B}$ is A.
2. The (total) relations of $\mathcal{B}$ extend the corresponding partial relations of $\mathcal{A}$.
3. If c is an individual constant of L then in both $\mathcal{A}$ and $\mathcal{B}$, c is interpreted by the same element.
4. If $\alpha \in P$, then $\mathcal{B} \models \alpha$. △

REMARK 5.5. Necessary and sufficient conditions for the existence of $\mathcal{A}$–normal structures are seen in [Mikenberg et al. 1986]; it is not always the case that a sps $\mathcal{A}$ has an $\mathcal{A}$–normal structure. However from here on we always suppose that our sps satisfy the Mikenberg–da Costa–Chuaqui [1986] conditions. △

Let L and $\mathcal{A}$ be a language and a sps where L is interpreted.

DEFINITION 5.6. A sentence α of L is **pragmatically true** in the sps $\mathcal{A}$ according to $\mathcal{B}$ if:

1. $\mathcal{B}$ is $\mathcal{A}$–normal;
2. $\mathcal{B} \models \alpha$.

α is **pragmatically true in** $\mathcal{A}$ if there is an $\mathcal{A}$–normal structure $\mathcal{B}$ where α is true (again in the sense of Tarski).

If α is not pragmatically true in α according to $\mathcal{B}$ (is not pragmatically true in the sps $\mathcal{A}$) we say that α is pragmatically false in $\mathcal{A}$ according to $\mathcal{B}$ (is pragmatically false in $\mathcal{A}$). △

Since $\mathcal{A}$ usually models an empirical domain Δ (or whenever $\mathcal{A}$ does model such a domain), we easily verify that α is pragmatically true in $\mathcal{A}$ (or Δ) if α saves the appearances in Δ. In other words, if everything occurs in Δ as if (*als ob*) α were true in the sense of the correspondence theory of truth. Therefore that formalization we just sketched for the concept of pragmatic truth captures some of the views on the question of truth that have been developed by philosophers as H. Vaihinger and C. S. Peirce, as a more detailed analysis would confirm.

To close the circle: we can quite naturally identify the $\mathcal{A}$-normal models of a sps with the worlds of a Kripke structure for $S5$ with quantification (or with that same structure for $S5Q^=$). It then follows that pragmatic truth in a sps and pragmatic truth in any sps (pragmatic validity) are essentially connected with satisfiability in a Kripke structure and satisfiability in all Kripke structures, respectively. So, discussive logic is the logic of pragmatic truth, and to develop the theory of pragmatic truth we must use a modal language, that of J*, which then extends L above.

For more details see [da Costa, Chuaqui 1988] and [Mikenberg et al. 1986].

6. The foundations of physics

M. L. Dalla Chiara and G. Toraldo di Francia define in their book *Le Teorie Fisiche* [1981] a physical structure as a quadruple $\mathcal{A} =< M, S, Q, \rho >$, where S is a set that represents the actual physics to be portrayed by $\mathcal{A}$, M is a mathematical species of structures in the sense of Suppes-Bourbaki (see [da Costa, Chuaqui 1988], [da Costa, Doria, de Barros 1990], [da Costa, Doria 1991, 1992a]), Q is an ordered set of "physical quantities," defined on S, and ρ is the rule that relates the mathematical portion M of the structure to its physically meaningful portion.

However, as one can see from their work, M must follow the rules of classical mathematics—and therefore of classical logic, as it deals with things like vectorspaces, bundles, connections and the like, that are traditionally formalized within a classical axiom system like that of Zermelo–Fraenkel; in connection with Q we must cope with contradictory situations—that is, the underlying logic of the *theory* of $\mathcal{A}$ should be a paraconsistent logic.

Well, which paraconsistent system? One that easily fits the picture is Jaśkowski's. We show in the present Section that the Dalla Chiara and di Francia underlying logic can be taken to be Jaśkowski's discussive logic.

Another way to define a first-order discussive calculus with necessary equality, which we denote by J** is the following: the language of J** is that of $S5Q^=$, as introduced above. If α is a formula, then $\sqcup\alpha$ denotes any formula composed by α preceded by any sequence of universal quantifiers, such that all variables of $\sqcup\alpha$ are bound.

Given a formula α of J^{**}, α is a thesis of J^{**} if and only if $\Diamond \sqcup \alpha$ is a thesis of $S5Q^{=}$. In symbols,

$$\vdash_{\mathsf{J}^{**}} \Leftrightarrow \vdash_{S5Q^{=}} \Diamond \sqcup \alpha.$$

We say that the formula α is a syntactic consequence in J^{**} of the set of formulæ Γ and write $\Gamma \vdash_{\mathsf{J}^{**}}$, if there are $\gamma_1, \ldots, \gamma_n$ in Γ such that

$$\Diamond \sqcup ((\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow \alpha)$$

is valid in $S5Q^{=}$.

J^{**} can be axiomatized as follows:

Axiom System 6.1.

1. If α is an instance of a (propositional) tautology, then $\Box \sqcup \alpha$ is an axiom.

2. $\Box \sqcup \alpha$, $\Box \sqcup (\alpha \rightarrow \beta)/\Box \sqcup \beta$.

3. $\Box \sqcup (\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta))$.

4. $\Box \sqcup (\Box\alpha \rightarrow \alpha)$.

5. $\Box \sqcup (\Diamond\alpha \rightarrow \Box\Diamond\alpha)$.

6. $\Box \sqcup (\forall x \alpha(x) \rightarrow \alpha(t))$.

7. $\Box \sqcup \alpha/\alpha$.

8. $\Box \sqcup \alpha/\Box \sqcup \Box\alpha$.

9. $\Diamond \sqcup \alpha/\alpha$.

10. $\Box \sqcup (\alpha \rightarrow \beta(x))/\Box \sqcup (\alpha \rightarrow \forall x \beta(x))$.

11. Vacuous quantification may be introduced or suppressed in any formula.

12. $\Box \sqcup (x = x)$.

13. $\Box \sqcup (x = y \rightarrow (\alpha(x) \leftrightarrow \alpha(y))$. $\triangle$

PROPOSITION 6.2. *In* J^{**}, $\rightarrow_d$, $\wedge_d$, $\vee$, ∇, $\leftrightarrow_d$, $\exists$ *and* $\forall$ *have all properties of classical (material) implication, conjunction, disjunction, negation, (material) equivalence, and of the existential and universal quantifiers, respectively.* $\triangle$

Classical logic is thus contained in J^{**}. Moreover, when we restrict our attention to *stable* formulas, i.e., formulas α such that $\Box(\alpha \leftrightarrow_d \Diamond\alpha)$ is true, J^{**} reduces to classical first-order logic.

Now, if we take into account the meaning of the symbols of J^{**}, we are naturally led to define in it a *pragmatic theory* as a set of sentences closed under syntactic consequence in J^{**}. For details on that, see [da Costa 1975] and [da Costa, Chuaqui 1990]. Notice in particular that any pragmatic theory is closed under discussive modus ponens, that is, from α and $\alpha \rightarrow_d \beta$, we infer β.

REMARK 6.3. It is easy to verify that J^* and J^{**} are essentially the same calculus. If we limit ourselves to sentences, J^* and J^{**} coincide. For the applications, J^{**} is more convenient that J^*, since there is no problem on the 'discussive' interpretation of the free variables. $\triangle$

6.1. Physical structures and theories

We summarize here the key ideas from [Dalla Chiara, Toraldo di Francia 1981]. A physical structure $\mathcal{A}$ is a set-theoretic structure of the form

$$\mathcal{A} = < M, S, < Q_0, Q_1, ..., Q_n >, \rho >,$$

where:

1. M is an instance of a mathematical species of structures, in the sense of Suppes–Bourbaki (see [da Costa, Chuaqui 1990]);

2. S is a set of physical situations (a physical situation is a set of physical states assumed by a physical system in a certain time interval);

3. Each Q_i, $0 \leqslant i \leqslant n$, denotes an operationally defined quantity whose domain of definition is some subset of S;

4. ρ is a function that associates to each term one needs in order to characterize S, and in particular to each Q_i, a set-theoretic construct in M.

Loosely speaking, the situation here goes as follows: we have a language, L, which we use to talk about $\mathcal{A}$, with the help of a sublanguage $L_0 \subset L$ which we use to talk about M. Also in L there is an infinite denumerable set of terms (and, in particular, of variables) that correspond to each Q_i, that is to say, to the physics portrayed in $\mathcal{A}$. We also suppose that L includes the Zermelo–Fraenkel set theory, so that we can discuss all instances of M, and that Q_0 represents time.

Now if we try to measure Q_i at a time t_i, $1 \leqslant i \leqslant n$, we get an interval $I(i, t_i)$. The length of $I(i, t_i)$ depends on the specific measurement technique and on the nature of the quantity involved (cf. [Dalla Chiara, Toraldo di Francia 1981]). If we measure Q_0 (that is to say, time), we get as a result a time interval.

The Q_i are denoted by terms of L; we agree that t and t_i will represent time instants, and that $q_i(t_j)$ expresses in L the value of Q_i at t_j.

Now let $\alpha(t, q_i(t_i))$ be a formula of L with no free variables besides t and t_i. Dalla Chiara and di Francia consider the case of partial formulas, that is, of formulas that aren't defined for all values of its variables and parameters, but for simplicity we restrict out attention to total formulas, that means, those that are not partial formulas. We then say that $\alpha(t, q_i(t_j)$ is *true* in the situation $s \in S$ if there are values t^0 of t in I_t, and q_i^0 of Q_i in $I(i, t_i)$, $1 \leqslant i \leqslant n$, such that $\alpha(t^0, q_i^0)$ is true in M (in the sense of Tarski). We also say that $\alpha(t, q_i(t_j))$ is true in $\mathcal{A}$ if it is true for every $s \in S$.

Paraconsistency enters the Dalla Chiara–di Francia approach whenever we get $\tilde{t}$ in I_t and $\tilde{q}_i$ in $I(t, t_i)$, so that $\neg\alpha(\tilde{t}, \tilde{q}_i(\tilde{t}_i))$ is also true in $\mathcal{A}$. Thus, both α and $\neg\alpha$ are true in $\mathcal{A}$.

Finally, given the preceding definition for truth in the language of a physical theory according to Dalla Chiara and di Francia, we then define notions like that of models for a physical theory, validity for a physical theory and the like.

6.2. The underlying logic of the Dalla Chiara and di Francia characterization for physical theories

A physical theory $\mathcal{T}$ (in the above sense) can be characterized as follows: we start from a language L; the set of axioms A in L splits as $A = A_L \cup A_M \cup A_\varphi$, where A_L, A_M and A_φ are, respectively, the set of logical axioms, of mathematical axioms, and of physical axioms.

We have also supposed given a sublanguage $L_0 \subset L$; the logic of L_0, in which we deal with the mathematical species of structures of $\mathcal{T}$, must be classical logic. Thus, we include in A_M all classically valid formulas.

At this point, and as a result of our preceding analyses, we also require that all theses of J^{**} be included in A_L, so that we may be able to cope with the paraconsistent behavior we found in the Dalla Chiara–di Francia approach.

Now, since a physical theory has as its physical models structures that are analogous to $\mathcal{A}$ above, A_M must contain all axioms for the species of structures of M. Finally, the elements of A_φ are physically motivated propositions.

The theorems of $\mathcal{T}$ that belong to L_0 have to be closed under classical syntactic consequence. Moreover, in general, the theorems of $\mathcal{T}$ must be closed under J^{**}'s syntactic consequence. Such a closure is a pragmatic theory.

L_0 cannot contain terms that refer to the operationally–defined quantities Q_i (since such quantities induce our language's paraconsistent behavior). For any formula $\alpha \in L_0$, we require that $\Box(\alpha \leftrightarrow_d \Diamond\alpha)$ be true (that is quite reasonable; it suffices to check the intuitive meaning for that discussive equivalence). In other words, α has to be stable, and, as we noticed above, L_0 is classical.

On the other side, if in a given formula β we have terms that denote some of the Q_i, we will in general find that both β and $\neg\beta$ are true in a model of $\mathcal{T}$, and should belong to A_φ.

To sum it up: the underlying logic of a physical theory in the Dalla Chiara and di Francia approach,is most adequately represented by Jaśkowski's discussive logic.

REMARK 6.4. After the present ideas was developed we received a paper by M. L. Dalla Chiara and R. Giuntini [1989] where they argue that the logic of quantum mechanics is a paraconsistent logic; in that paper they deal with the *logic of observables*, while in the present note we investigate the *underlying logic of quantum theory*, which encompasses the previous kind of logic. △

REMARK 6.5. If we want to incorporate the case in which there are partial formulæ (see above) then we have to employ a non–alethic logic, i.e., a logic that is simultaneously paraconsistent and paracomplete [da Costa 1989b]. △

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