ANDRZEJ SENDLEWSKI

Axiomatic Extensions of the Constructive Logic with Strong Negation and the Disjunction Property

Abstract. We study axiomatic extensions of the propositional constructive logic with strong negation having the disjunction property in terms of corresponding to them varieties of Nelson algebras. Any such variety **V** is characterized by the property:
(PQWC) if $\mathcal{A}, \mathcal{B} \in \mathbf{V}$, then $\mathcal{A} \times \mathcal{B}$ is a homomorphic image of some well-connected algebra of **V**.
We prove:

- each variety **V** of Nelson algebras with PQWC lies in the fibre $\sigma^{-1}(\mathbf{W})$ for some variety **W** of Heyting algebras having PQWC,
- for any variety **W** of Heyting algebras with PQWC the least and the greatest varieties in $\sigma^{-1}(\mathbf{W})$ have PQWC,
- there exist varieties **W** of Heyting algebras having PQWC such that $\sigma^{-1}(\mathbf{W})$ contains infinitely many varieties (of Nelson algebras) with PQWC.

The propositional constructive logic with strong negation (CLSN) has naturally arised as a result of omitting the non-constructivity of the intuitionistic negation. It is a conservative extension of the intuitionistic logic (INT) by adding a new unary connective $\sim$ and some new axioms involving $\sim$ (see [6], [8], [9], [10] for details). By an axiomatic extension of CLSN we mean any subset of the set of all formulas built up in the usual way by means of the connectives: $\vee, \wedge, \rightarrow, \neg$ (intuitionistic connectives: *disjunction, conjunction, implication, negation*) and $\sim$ (*strong negation*) which contains CLSN and is closed under the substitution and Modus Ponens rule. Similarly we mean axiomatic extensions of INT which are called intermediate logics (whenever the set of all formulas is excluded).

An algebraic counterpart of CSLN is the variety **NA** of Nelson algebras (see [9] or [10], where Nelson algebras are called N-lattices and quasi pseudo-Boolean algebras, respectively), while of INT is the variety **HA** of Heyting algebras (Heyting algebras are often called pseudo-Boolean algebras, see [10]). With each logic L being an axiomatic extension of CLSN (INT) there is assigned a variety of Nelson algebras (Heyting algebras) which consists of

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those algebras in which all formulas from L are true. On the other hand, with each variety **V** of Nelson algebras (Heyting algebras) there is assigned a logic being an axiomatic extension of CLSN (INT) formed by all formulas which are true in every member of **V** (the content of **V**). The assignments are inverse one to another and establish a dual lattice isomorphism between the lattice of axiomatic extensions of CLSN (INT) and the lattice $Lv(\mathbf{NA})$ of all varieties of Nelson algebras (the lattice $Lv(\mathbf{HA})$ of all varieties of Heyting algebras).

The systematic investigation of the interconnections between $Lv(\mathbf{NA})$ and $Lv(\mathbf{HA})$ has been started in [12] (see also [13]). It is known that there are three lattice homomorphisms $\sigma : Lv(\mathbf{NA}) \longrightarrow Lv(\mathbf{HA})$ and $\underline{\eta}, \overline{\eta} : Lv(\mathbf{HA}) \longrightarrow Lv(\mathbf{NA})$ such that among others the following hold:

- $\sigma\underline{\eta} = \sigma\overline{\eta} = id_{Lv(\mathbf{HA})}$,
- for any $\mathbf{W} \in Lv(\mathbf{HA})$ the fibre $\sigma^{-1}(\mathbf{W})$ coincides with the interval $[\underline{\eta}(\mathbf{W}), \overline{\eta}(\mathbf{W})]$ of $Lv(\mathbf{NA})$
- if $\mathbf{V} \in \sigma^{-1}(\mathbf{W})$ then the $\sim$-free fragment of the logic corresponding to **V** coincides with the logic corresponding to **W** (the converse is also true when **V** is generated by its finite members).

These homomorphisms have been successfully applied (see [11], [12], [3]) to find and describe the axiomatic extensions of CLSN with a given property under the assumption that an algebraic equivalent of the property is known. Let us quote some characteristic results of this kind. Let **V** be a variety of Nelson algebras and L be the corresponding logic. Then the following hold true:

1. **V** is finite (L is tabular) iff $\mathbf{V} \in \sigma^{-1}(\mathbf{W})$ for some finite variety **W** of Heyting algebras ([11], [12]).
2. **V** is locally finite (L is locally tabular) iff $\mathbf{V} \in \sigma^{-1}(\mathbf{W})$ for some locally finite variety **W** ([12]).
3. **V** is minimal in the family of non-finite varieties (L is pretabular) iff $\mathbf{V} = \underline{\eta}(\mathbf{W})$ for **W** minimal in the family of non-finite varieties of Heyting algebras ([11], [12]).
4. **V** is primitive (L is structurally complete in the finitary sense with respect to Modus Ponens) iff $\mathbf{V} = \underline{\eta}(\mathbf{W})$ for some primitive **W** ([11], [12]).

5. **V** has the Amalgamation Property (L has the Craig Interpolation Property) iff **V**$=\underline{\eta}(\mathbf{W})$ or **V**$=\overline{\eta}(\mathbf{W})$ for some **W** havig the Amalgamation Property (proved independently in [3] and [12]).

In this paper we investigate varieties of Nelson algebras that correspond to logics with disjunction property, that is, to logics L such that for all formulas α, β : $\alpha \vee \beta \in$ L implies $\alpha \in$ L or $\beta \in$ L. This property is equivalent to the constructivity of the strong negation in L, i.e. $\sim(\alpha \wedge \beta) \in$L iff $\sim\alpha \in$L or $\sim\beta \in$L.

1. Basic notions and facts

Nelson algebrs are considered in the language with binary operations: $\vee$ (*join*), $\wedge$ (*meet*), $\rightarrow$ (*weak relative pseudocomplementation*), unary operations: $\neg$ (*weak pseudocomplementation*), $\sim$ (*De Morgan negation*), and two constants 0, 1; while Heyting algebras with operations: $\vee$, $\wedge$, $\Rightarrow$ (*relative pseudocomplementation*), $-$ (*pseudocomplementation*) and constants 0, 1. Between Nelson and Heyting algebras there exists a close connection (see [12], [13], comp. [1], [2], [3], [11], [14]). Let us recall fundamental facts.

For any Heyting algebra $\mathcal{B}=(B, \vee, \wedge, \Rightarrow, -, 0, 1)$ and Boolean congruence Θ of $\mathcal{B}$ (Θ is Boolean if the quotient algebra $\mathcal{B}/\Theta$ is a Boolean algebra), the algebra

$$N_\Theta(\mathcal{B}) = (N_\Theta(B), \vee, \wedge, \rightarrow, \neg, \sim, (0,1), (1,0)),$$

where

$$N_\Theta(B) := \{(a,b) \in B \times B; a \wedge b = 0 \textit{ and } a \vee b \equiv 1(\Theta)\}$$

and for all $(a,b),(c,d) \in N(B)$:

$$\begin{aligned}
(a,b) \vee (c,d) &:= (a \vee c, b \wedge d),\\
(a,b) \wedge (c,d) &:= (a \wedge c, b \vee d),\\
(a,b) \rightarrow (c,d) &:= (a \Rightarrow c, a \wedge d),\\
\neg(a,b) &:= (-a, a),\\
\sim(a,b) &:= (b, a),
\end{aligned}$$

is a Nelson algebra. Each Nelson algebra $\mathcal{A}$ is up to isomorphism of the form $N_\Theta(\mathcal{B})$, for some $\mathcal{B}$ and Θ. Necessary $\mathcal{B}$ can be taken to be the Heyting algebra of the form

$$\mathcal{A}^\star = (A^\star, \vee^\star, \wedge^\star, \Rightarrow^\star, 0, 1),$$

where

$$A^\star := \{a^\star; a \in A\} \; (a^\star abbreviaties \sim \neg a),$$

and, for any $a, b \in A^\star$:

$$\begin{aligned} a \vee^\star b &:= (a \vee b)^\star (= a \vee b), \\ a \wedge^\star b &:= (a \wedge b)^\star, \\ a \Rightarrow^\star b &:= (a \to b)^\star, \\ -^\star a &:= (\neg a)^\star. \end{aligned}$$

While Θ is the restriction to the set $A^\star$ of the congruence on $\mathcal{A}$ generated by $\{(\neg a, \sim a); a \in A\}$. In this case the quoted isomorphism establishes the map $A \ni a \longmapsto (a^\star, (\sim a)^\star) \in N_\Theta(\mathcal{A}^\star)$. On the other hand, each Heyting algebra $\mathcal{B}$ is isomorphic to $N_\Theta(\mathcal{B})^\star$, for any Boolean congruence Θ of $\mathcal{B}$, via the map $B \ni b \longmapsto (b, -b) \in N_\Theta(B)$.

Among Boolean congruences on an arbitrary Heyting algebra there are two distinguished congruences, namely, the greatest one which is a full congruence and the least one being an intersection of all Boolean congruences. When Θ is the greatest congruence and when Θ is the least congruence, instead of $N_\Theta(\mathcal{B})$ we write $\overline{N}(\mathcal{B})$ and $\underline{N}(\mathcal{B})$, respectively. Note that $\overline{N}(\mathcal{B}) = \{(a,b) \in B \times B; a \wedge b = 0\}$ and $\underline{N}(\mathcal{B}) = \{(a,b) \in B \times B; (a \Rightarrow b) \wedge (b \Rightarrow a) = 0\}$. Nelson algebras isomorphic to algebras of the form $\underline{N}(\mathcal{B})$ will be called regular (in [3], they are called normal). The class of all regular Nelson algebras forms a variety (equational class), denoted by **RNA**.

The operators $(\ )^\star$, $\underline{N}$ and $\overline{N}$ can be naturally extended to functors: $(\ )^\star$:**NA**$\longrightarrow$ **HA**, $\underline{N}$, $\overline{N}$:**HA**$\longrightarrow$**NA**. In [13] it has been proved the following

THEOREM 1.1. *The functor $(\ )^\star$ is topological with $\underline{N}$ and $\overline{N}$ as left and right adjoint functors, respectively. Moreover, each of the functors $(\ )^\star$, $\underline{N}$ and $\overline{N}$, preserves injective and surjective homomorphisms and direct products of algebras.*

For any class $\mathbf{K}$ of algebras, $\mathsf{I}(\mathbf{K})$, $\mathsf{H}(\mathbf{K})$, $\mathsf{S}(\mathbf{K})$ and $\mathsf{P}(\mathbf{K})$ denote, respectively, the classes of isomorphic images, homomorphic images, subalgebras and direct products of members of $\mathbf{K}$. If $\mathbf{K}$ is a class of Nelson algebras and $\mathbf{L}$ is a class of Heyting algebras, then we define:

$$\begin{aligned} \mathbf{K}^\star &:= \{\mathcal{A}^\star; \mathcal{A} \in \mathbf{K}\}, \\ \underline{N}(\mathbf{L}) &:= \{\underline{N}(\mathcal{B}); \mathcal{B} \in \mathbf{L}\}, \\ \overline{N}(\mathbf{L}) &:= \{\overline{N}(\mathcal{B}); \mathcal{B} \in \mathbf{L}\}. \end{aligned}$$

The following hold true (see [12], [13])

THEOREM 1.2. *(i) For any class* $\mathbf{K}$ *of Nelson algebras,* $\mathsf{HSP}(\mathbf{K}^\star) = \mathsf{HSP}(\mathbf{K})^\star$*. Hence, if* $\mathbf{K}$ *is a variety of algebras then* $\mathbf{K}^\star$ *is a variety of Heyting algebras.*

(ii) For any class $\mathbf{L}$ *of Heyting algebras,* $\mathsf{HSP}(\underline{N}(\mathbf{L})) = \mathsf{I}(\underline{N}(\mathsf{HSP}(\mathbf{L})))$ *and* $\mathsf{HSP}(\overline{N}(\mathbf{L})) = \mathsf{IS}(\overline{N}(\mathsf{HSP}(\mathbf{L})))$*.*
Hence, if $\mathbf{L}$ *is a variety, then the varieties of Nelson algebras generated by* $\underline{N}(\mathbf{L})$ *and* $\overline{N}(\mathbf{L})$ *are equal to* $\mathsf{I}(\underline{N}(\mathbf{L}))$ *and* $\mathsf{IS}(\overline{N}(\mathbf{L}))$*, respectively.*

Let $\sigma : Lv(\mathbf{NA}) \longrightarrow Lv(\mathbf{HA})$ and $\underline{\eta}, \overline{\eta} : Lv(\mathbf{HA}) \longrightarrow Lv(\mathbf{NA})$ be maps defined by the formulas:

$$\begin{aligned} \sigma(\mathbf{V}) &:= \mathbf{V}^\star, \\ \underline{\eta}(\mathbf{W}) &:= \mathsf{I}(\underline{N}(\mathbf{W})), \\ \overline{\eta}(\mathbf{W}) &:= \mathsf{IS}(\overline{N}(\mathbf{W})). \end{aligned}$$

Basic properties of σ, $\underline{\eta}$ and $\overline{\eta}$ summarizes the following theorem proved in [12] (see also [13]).

THEOREM 1.3. *The maps* $\sigma, \underline{\eta}$ *and* $\overline{\eta}$ *are complete lattice homomorphisms (they preserve joins and meets of arbitrary non-empty families of algebras) such that* $\sigma\underline{\eta} = \sigma\overline{\eta} = id_{Lv(\mathbf{HA})}$ *.*
Moreover, for any varieties $\mathbf{W}$*,* $\mathbf{U}$ *of Heyting algebras there hold:*

(i) The fibre $\sigma^{-1}(\mathbf{W})$ *coincides with the interval* $[\underline{\eta}(\mathbf{W}), \overline{\eta}(\mathbf{W})]$ *of* $Lv(\mathbf{NA})$*, in particular* $\sigma^{-1}(\mathbf{HA}) = [\mathbf{RNA},\ \mathbf{NA}]$*.*

(ii) If $\mathbf{W} \subset \mathbf{U}$ *then the map* $e_{\mathbf{WU}} : \sigma^{-1}(\mathbf{W}) \longrightarrow \sigma^{-1}(\mathbf{U})$ *defined by* $e_{\mathbf{WU}}(\mathbf{V}) = \mathbf{V} \vee \underline{\eta}(\mathbf{U})$ *is a complete lattice embedding.*

(iii) The map $e_{\mathbf{W}} : Lv(\mathbf{W}) \longrightarrow \sigma^{-1}(\mathbf{W})$ *defined by* $e_{\mathbf{W}}(\mathbf{V}) = \underline{\eta}(\mathbf{W}) \vee \overline{\eta}(\mathbf{V})$ *is a complete lattice embedding.*

(iv) The map $i_{\mathbf{W}} : Lv(\overline{\eta}(\mathbf{W})) \longrightarrow Lv(\mathbf{W}) \times \sigma^{-1}(\mathbf{W})$ *defined by* $i_{\mathbf{W}}(\mathbf{V}) = (\mathbf{V}^\star,\ \mathbf{V} \vee \underline{\eta}(\mathbf{W}))$ *is a complete lattice embedding.*

An algebra is subdirectly irreducible if its congruence lattice has exactly one atom. A Heyting algebra $\mathcal{B}$ is subdirectly irreducible iff in the set $B \setminus \{1_{\mathcal{B}}\}$ there exists the greatest element. For Nelson algebras we have the following (see [1], [11], [12]).

PROPOSITION 1.4. *Let* $\mathcal{A}$ *be a Nelson algebra. The following are equivalent:*
(i) $\mathcal{A}$ *is subdirectly irreducible,*
(ii) $\mathcal{A}^\star$ *is subdirectly irreducible,*
(iii) In the set $A \setminus \{1_{\mathcal{A}}\}$ *there exists the greatest element.*

Using the properties of the characteristic formula of a finite subdirectly irreducible Heyting (see [16]) or Nelson algebra (in view of the above proposition it can be defined analogously as in the case of Heyting algebras) we have

COROLLARY 1.5. *For any class* $\mathbf{K}$ *of Heyting or Nelson algebras the finite subdirectly irreducible members of* $\mathsf{HSP}(\mathbf{K})$ *belong to* $\mathsf{HS}(\mathbf{K})$.

2. Remarks on varieties with PQWC

A Heyting or Nelson algebra is said to be well-connected if for any of its elements a and b, $a = 1$ or $b = 1$ whenever $a \vee b = 1$. Note that any subdirectly irreducible Heyting or Nelson algebra is well-connected and the converse also holds in the case of finite algebras.

THEOREM 2.1. *Let* $\mathbf{V}$ *be a variety of Heyting or Nelson algebras. The following conditions are equivalent:*

(*i*) *The logic corresponding to* $\mathbf{V}$ *has DP.*

(*ii*) *There exists a class* $\mathbf{K}$ *such that* $\mathbf{V} = \mathsf{HSP}(\mathbf{K})$ *and the direct product* $\mathcal{A}_1 \times \mathcal{A}_2$ *of any members* $\mathcal{A}_1$, $\mathcal{A}_2$ *of* $\mathbf{K}$ *is a homomorphic image of some well-connected algebra from* $\mathbf{V}$.

(*iii*) *For any class* $\mathbf{K}$ *with* $\mathbf{V} = \mathsf{HSP}(\mathbf{K})$, *the direct product* $\mathcal{A}_1 \times \mathcal{A}_2$ *of any members* $\mathcal{A}_1$, $\mathcal{A}_2$ *of* $\mathbf{K}$ *is a homomorphic image of some well-connected algebra from* $\mathbf{V}$.

(*iv*) *The direct product* $\mathcal{A}_1 \times \mathcal{A}_2$ *of any members* $\mathcal{A}_1$, $\mathcal{A}_2$ *of* $\mathbf{V}$ *is a homomorphic image of some well-connected algebra from* $\mathbf{V}$.

(*v*) *Each member of* $\mathbf{V}$ *is a homomorphic image of some well-connected algebra from* $\mathbf{V}$.

Moreover, in the case when $\mathbf{V}$ *is a variety of Nelson algebras each of the above conditions is equivalent to*

(*vi*) *The strong negation* $\sim$ *is constructive in the logic corresponding to* $\mathbf{V}$.

PROOF. In case when $\mathbf{V}$ is a variety of Heyting algebras the equivalence of (i), (ii), (iii) and (iv) follows by the Maksimova's theorem (Theorem 1 in [7]), and the fact that the contents of a class $\mathbf{K} \subset \mathbf{V}$ and the variety $\mathbf{V}$ coincide iff $\mathbf{V} = \mathsf{HSP}(\mathbf{K})$. The equivalence of (iv) and (v) is obvious. In the case when $\mathbf{V}$ is a variety of Nelson algebras the same arguments also work, and the equivalence of (i) and (vi) is an immediate consequence of De Morgan laws and the double negation law for strong negation which hold in any Nelson algebra. ∎

If a class $\mathbf{K}$ of Heyting or Nelson algebras satisfies one of the equivalent conditions of the above theorem, then we say that products of members of $\mathbf{K}$ (the variety $\mathsf{HSP}(\mathbf{K})$) are quotients of well-connected algebras (PQWC for short).

LEMMA 2.2. *(i) A Nelson algebra $\mathcal{A}$ is well-connected iff the Heyting algebra $\mathcal{A}^\star$ is well-connected.*

(ii) For each Heyting algebra $\mathcal{B}$ and each Boolean congruence Θ on $\mathcal{B}$ the algebra $N_\Theta(\mathcal{B})$ is well-connected iff $\mathcal{B}$ is well-connected.

PROOF. (i) is a consequence of the fact that for any element a of a Nelson algebra, $a = 1$ iff $a^\star = 1$. And (ii) follows from (i) because $N_\Theta(\mathcal{B})^\star$ is isomorphic to $\mathcal{B}$. ■

LEMMA 2.3. *Let $\mathbf{K}$ be a class of Nelson algebras. If the product $\mathcal{A}_1 \times \mathcal{A}_2$ of any algebras $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{K}$ is a homomorphic image of some well-connected algebra from $\mathsf{HSP}(\mathbf{K})$, then the product $\mathcal{B}_1 \times \mathcal{B}_2$ of any algebras $\mathcal{B}_1, \mathcal{B}_2 \in \mathbf{K}^\star$ is a homomorphic image of some well-connected algebra from $\mathsf{HSP}(\mathbf{K}^\star)$.*

PROOF. Let $\mathcal{B}_i = \mathcal{A}_i^\star$ for some $\mathcal{A}_i \in \mathbf{K}$, $i = 1, 2$. By the assumption there exists a well-connected algebra $\mathcal{A}$ in $\mathsf{HSP}(\mathbf{K})$ and a surjective homomorphism $h : \mathcal{A} \longrightarrow \mathcal{A}_1 \times \mathcal{A}_2$. Since the functor $(\ )^\star$ preserves surjective homomorphisms and direct products, the algebra $(\mathcal{A}_1 \times \mathcal{A}_2)^\star$ is a homomorphic image of the algebra $\mathcal{A}^\star$ with respect to the homomorphism $h^\star$ and is isomorphic to $\mathcal{B}_1 \times \mathcal{B}_2$. Therefore $\mathcal{B}_1 \times \mathcal{B}_2$ is a homomorphic image of the algebra $\mathcal{A}^\star \in \mathsf{HSP}(\mathbf{K})^\star$. But, by Theorem 1.2, $\mathsf{HSP}(\mathbf{K})^\star = \mathsf{HSP}(\mathbf{K}^\star)$. Moreover, by Lemma 2.2, $\mathcal{A}^\star$ is well-connected which completes the proof. ■

LEMMA 2.4. . *Let $\mathbf{L}$ be a class of Heyting algebras and let N denote $\underline{N}$ or $\overline{N}$. If the product $\mathcal{B}_1 \times \mathcal{B}_2$ of any algebras $\mathcal{B}_1, \mathcal{B}_2 \in \mathbf{L}$ is a homomorphic image of some well-connected algebra from $\mathsf{HSP}(\mathbf{L})$, then the product $\mathcal{A}_1 \times \mathcal{A}_2$ of any algebras $\mathcal{A}_1, \mathcal{A}_2 \in N(\mathbf{L})$ is a homomorphic image of some well-connected algebra from $\mathsf{HSP}(N(\mathbf{L}))$.*

PROOF. Apply the arguments from the proof of the previous lemma and the following facts: both $\underline{N}$ and $\overline{N}$ preserves surjective homomorphisms and direct products of algebras (Theorem 1.1); the algebras $\underline{N}(\mathcal{B})$ and $\overline{N}(\mathcal{B})$ are well-connected whenever $\mathcal{B}$ is well-connected (Lemma 2.2); $\underline{N}(\mathsf{HSP}(\mathbf{L})) \subset \mathsf{HSP}(\underline{N}(\mathbf{L}))$ and $\overline{N}(\mathsf{HSP}(\mathbf{L})) \subset \mathsf{HSP}(\overline{N}(\mathbf{L}))$ (Theorem 1.2(ii)). ■

As an immediate consequence of Theorems 1.3, 2.1 and Lemmas 2.3 and 2.4 we have

THEOREM 2.5. (*i*) *Each variety* $\mathbf{V}$ *of Nelson algebras with PQWC belongs to the fibre* $\sigma^{-1}(\mathbf{W})$ *for some variety* $\mathbf{W}$ *of Heyting algebras having PQWC.*

(*ii*) *For any variety* $\mathbf{W}$ *of Heyting algebras having PQWC the varieties of Nelson algebras being the bounds of the fibre* $\sigma^{-1}(\mathbf{W})$ *have PQWC.*

Taking into account the well-known results concerning intermediate logics (see [7], pp. 70, 71) in virtue of Theorems 1.3 and 2.5 we also have

COROLLARY 2.6. (*i*) *Each non-trivial variety of Nelson algebras with PQWC contains the variety* $\underline{\eta}(\mathcal{D})$*, where* $\mathcal{D}$ *is the variety of Heyting algebras corresponding to the Dummett's logic.*

(*ii*) *There exist continnum non-trivial varieties of Nelson algebras with as well as without PQWC.*

(*iii*) *The least non-trivial variety of Nelson algebras with PQWC does not exist.*

(*iv*) *The variety* $\mathbf{V}$ *of Nelson algebras is minimal in the family of all non-trivial varieties with PQWC iff* $\mathbf{V} = \underline{\eta}(W)$ *for some variety* $\mathbf{W}$*, that is minimal in the family of non-trivial varieties of Heyting algebras having PQWC.*

3. The fibre $\sigma^{-1}(\mathbf{M})$

Let $\mathbf{M}$ be the variety of Heyting algebras corresponding to the intermediate logic which was defined in [7] and used to prove that the Gabby-de Jong logic of finite binary trees is not maximal among logics with DP. In this section we prove that in the fibre $\sigma^{-1}(\mathbf{M})$ there are infinitely many varieties of Nelson algebras with PQWC. In the proof we shall use the poset technique (Kripke semantics), so to be precise, let us recall some principal notions. A subset Q of a poset $\mathcal{P} = (P, \leqslant)$ is increasing if $x \in Q$ and $x \leqslant y$ implies $y \in Q$. With each poset $\mathcal{P}$ there is associated the Heyting algebra $HA(\mathcal{P})$ of all increasing subsets of $\mathcal{P}$. If $\mathcal{P} \sqcup \mathcal{Q}$ is a disjoint union of posets then algebras $HA(\mathcal{P} \sqcup \mathcal{Q})$ and $HA(\mathcal{P}) \times HA(\mathcal{Q})$ are isomorphic. For any subset Q of $\mathcal{P}$, the relation $\Theta(Q)$ defined by

$$a \equiv b(\Theta(Q)) \ \ iff \ \ a \cap Q = b \cap Q,$$

for all a, b increasing in $\mathcal{P}$, is a congruence on $HA(\mathcal{P})$ if and only if Q is increasing in $\mathcal{P}$. In this case, $HA(\mathcal{P})/\Theta(Q)$ is isomorphic to $HA(\mathcal{Q})$, where $\mathcal{Q}$ is a subposet of $\mathcal{P}$ with underlying set Q (the order is inherited from $\mathcal{P}$). The congruence $\Theta(Q)$ is Boolean iff Q is a subset of the set $Max(\mathcal{P})$ of all maximal elements in $\mathcal{P}$. For finite poset $\mathcal{P}$, $HA(\mathcal{P})$ is well-connected (equivalently, subdirectly irreducible) iff $\mathcal{P}$ has the least element. For any poset $\mathcal{P}$ and any subset S of $Max(\mathcal{P})$ we use the symbol $N_S(\mathcal{P})$ to denote the Nelson algebra $N_{\Theta(S)}(HA(\mathcal{P}))$. Note that $N_{\emptyset}(\mathcal{P}) = \overline{N}(HA(\mathcal{P}))$ and $N_{Max(\mathcal{P})}(HA(\mathcal{P})) = \underline{N}(HA(\mathcal{P})$.

LEMMA 3.1. (see [11], [13]) (*i*) *Let Q be increasing in $\mathcal{P}$, and let S be a subset of $Max(\mathcal{P})$. Then the Nelson algebra $N_{S\cap Q}(\mathcal{Q})$ is a homomorphic image of the algebra $N_S(\mathcal{P})$.*

(*ii*) *Let $\mathcal{P}$ and $\mathcal{Q}$ be disjoint posets and let $S \subset Max(\mathcal{P})$, $T \subset Max(\mathcal{Q})$. Then the Nelson algebras $N_{(S\cup T)}(\mathcal{P}\sqcup\mathcal{Q})$ and $N_S(\mathcal{P})\times N_T(\mathcal{Q})$ are isomorphic.*

Let $\boldsymbol{n}$ denote the poset $\{0, 1, 2, ..., n-1\}$ with the natural ordering of numbers; and let $\mathcal{P}_n$ be a subposet of the product $\boldsymbol{n}\times\boldsymbol{n}$ with the underlying set $P_n = \{(i,j); i+j < n\}$. For any posets $\mathcal{P}_m$ and $\mathcal{P}_n$ define the poset $\mathcal{P}_m \oplus \mathcal{P}_n$ whose underlying set is the disjoint union of underlying sets of $\mathcal{P}_m$, $\mathcal{P}_n$ and $\boldsymbol{m}\times\boldsymbol{n}$, and whose order is indicated by the diagram on the figure 1.

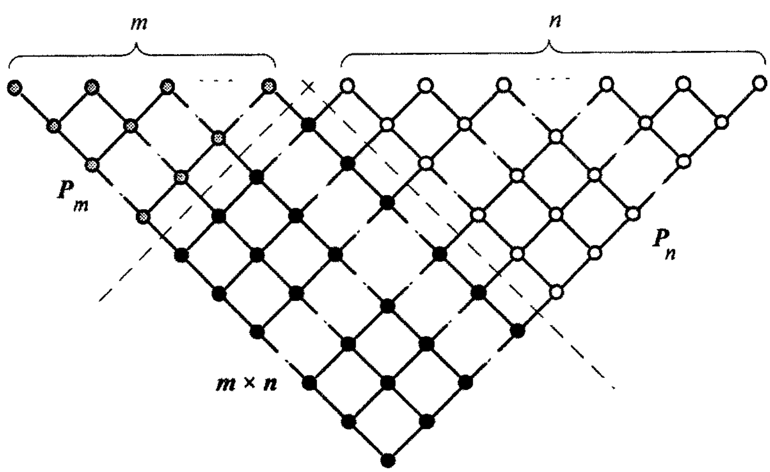

Figure 1.

Clearly, for any $m, n < \omega$, the subsets P_m and P_n are disjoint and increasing in $\mathcal{P}_m \oplus \mathcal{P}_n$. Hence their union is also increasing. Moreover, the poset $\mathcal{P}_m \oplus \mathcal{P}_n$ is isomorphic to $\mathcal{P}_{m+n}$. This, by Theorem 2.1, guarantees that the variety

$$\mathsf{M} := \mathsf{HSP}(\{HA(\mathcal{P}_n); n < \omega\})$$

has PQWC (see [7], Theorem 3).

For each poset $\mathcal{P}_n$, let $M_n = Max(\mathcal{P}_n)$ and let $' : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ be a map defined by the formula $(i,j)' = (j,i)$. This map is an automorphism of the poset $\mathcal{P}_n$. Moreover, the following simple fact holds true

LEMMA 3.2. *For any $S \subset M_n$, $n < \omega$, the Nelson algebras $N_S(\mathcal{P}_n)$ and $N_{S'}(\mathcal{P}_n)$ (where S' is the image of S under the automorphism $'$) are isomorphic.*

To define a required sequence of varieties of Nelson algebras in the fibre $\sigma^{-1}(\mathbf{M})$ we need some definitions. An element $x \in M_n$ is said to be a corner element of $\mathcal{P}_n$ if there exists only one element in $\mathcal{P}_n$ covered by x (z is covered by x, equivalently, x covers z iff $z \leqslant x$ and $z \leqslant u \leqslant x$ implies $u = z$ or $u = x$). For any $S \subset M_n$, elements $x, y \in M_n \setminus S$ are said to be conjugate if $x = y$ or there is $z \in P_n$ covered by x and y. Let ρ be a transitive closure of the conjugate relation on the set $M_n \setminus S$. Obviously, ρ is an equivalence relation on the set $M_n \setminus S$. We say that the set S has a partition index k if $k = max\{|[x]\rho|; x \in M_n \setminus S\}$. Now, for any n and m, let $\mathbf{K}_m^{(n)}$ be a class of all Nelson algebras of the form $N_S(\mathcal{P}_n)$ with S containing exactly one corner element of $\mathcal{P}_n$ and having a partition index $\leqslant m$. Define

$$\begin{aligned} \mathbf{K}_m &:= \bigcup\{\mathbf{K}_m^{(n)}; n < \omega\} \\ \mathbf{V}_m &:= \mathsf{HSP}(\mathbf{K}_m). \end{aligned}$$

THEOREM 3.3. *For all $m < \omega$, the varieties $\mathbf{V}_m$ have PQWC and form ω-chain in the fibre $\sigma^{-1}(\mathbf{M})$ whose join is $\overline{\eta}(\mathbf{M})$.*

PROOF. By the definitions $\mathbf{K}_m \subset \mathbf{K}_{m+1}$, hence $\mathbf{V}_m \subset \mathbf{V}_{m+1}$. By Lemma 3.1 (i), the algebra $N_\emptyset(\mathcal{P}_{m+1}) \in \mathsf{H}(N_S(\mathcal{P}_{m+2}))$, where S is a singleton of a corner element of $\mathcal{P}_{m+2}$. Obviously, $N_S(\mathcal{P}_{m+2}) \in \mathbf{K}_{m+1}$. Hence $N_\emptyset(\mathcal{P}_{m+1}) \in \mathbf{V}_{m+1}$. On the other hand, $N_\emptyset(\mathcal{P}_{m+1}) \notin \mathbf{V}_m$. Indeed, by Proposition 1.4 $N_\emptyset(\mathcal{P}_{m+1})$ is subdirectly irreducible. Therefore, if $N_\emptyset(\mathcal{P}_{m+1}) \in \mathbf{V}_m$ then, by Lemma 1.5, $N_\emptyset(\mathcal{P}_{m+1}) \in \mathsf{HS}(\mathbf{K}_m)$ which is impossible. This proves $\mathbf{V}_m \neq \mathbf{V}_{m+1}$.

Clearly, for every $S \subset M_m$, $N_S(\mathcal{P}_m)$ is a subalgebra of $N_\emptyset(\mathcal{P}_m)$. Therefore $\mathbf{V}_m \subset \overline{\eta}(\mathbf{M})$ for every m. Moreover, since $\overline{\eta}(\mathbf{M})$ is generated by algebras $N_\emptyset(\mathcal{P}_m)$, where $m < \omega$, and $N_\emptyset(\mathcal{P}_m) \in \mathbf{V}_m$, the lattice join of all varieties $\mathbf{V}_m$ is equal to $\overline{\eta}(\mathbf{M})$.

To prove $\mathbf{V}_m$ has PQWC, observe that for any two algebras from $\mathbf{K}_m$, say $N_S(\mathcal{P}_n)$ and $N_T(\mathcal{P}_k)$ (assume $P_n \cap P_k = \emptyset$), at least one the of algebras $N_{S\cup T}(\mathcal{P}_n \oplus \mathcal{P}_k)$ or $N_{S'\cup T}(\mathcal{P}_n \oplus \mathcal{P}_k)$ belongs to $\mathbf{K}_m$. If $N_{S\cup T}(\mathcal{P}_n \oplus \mathcal{P}_k) \in \mathbf{K}_m$, then, since $P_n \cup P_k$ is increasing in $\mathcal{P}_n \oplus \mathcal{P}_k$, the algebra $N_{S\cup T}(\mathcal{P}_n \sqcup \mathcal{P}_k)$ is a homomorphic image of $N_{S\cup T}(\mathcal{P}_n \oplus \mathcal{P}_k)$ (Lemma 3.1 (i)). But by Lemma 3.1 (ii), $N_{S\cup T}(\mathcal{P}_n \sqcup \mathcal{P}_k)$ is isomorphic to $N_S(\mathcal{P}_n) \times N_T(\mathcal{P}_k)$. Thus, by Theorem

2.1, $\mathbf{V}_m$ has PQWC. The case $N_{S'\cup T}(\mathcal{P}_n \oplus \mathcal{P}_k) \in \mathbf{K}_m$ implies PQWC for $\mathbf{V}_m$, goes in virtue of Lemma 3.2, by the same argument. ■

Remark. The method used to prove $\sigma^{-1}(\mathbf{M})$ has infinitely many varieties with PQWC also works in many other fibres over varieties of Heyting algebras with PQWC which are generated by finite algebras. For example, the most interesting fibre $\sigma^{-1}(\mathbf{HA}) = [\mathbf{RNA}, \mathbf{NA}]$ (by Theorem 1.3 (iii) it has cardinality $2^{\aleph_0}$) also has this property. To prove this fact it sufficies to take a family of posets determining the well-known Jaśkowski's algebras generating the variety **HA**. This family contains posets $\mathcal{J}_n$, $n < \omega$, where $\mathcal{J}_1$ is a one element poset, while $\mathcal{J}_{n+1}$ is a disjoint union of n copies of $\mathcal{J}_n$ with a new smallest element adjoined. Then choosing subsets S in $Max(\mathcal{J}_n)$ $(n < \omega)$ majorized by an appropriate parameter related with natural number m, it is routine to define classes $\mathbf{R}_m$, and whence varieties $\mathsf{HSP}(\mathbf{R}_m)$, which form a ω-chain of varieties with PQWC in the considered fibre. The join of this chain is the variety **NA**.

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Faculty of Mathematics and Informatics
Nicholas Copernicus University
87-100 Toruń, Chopina 12/18, Poland
email: sendlew@mat.uni.torun.pl