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On the Logic of Event–causation
Jaśkowski–style Systems of Causal Logic*

> Children in their simplicity keep asking why. The person of understanding has given this up; every why, he has long found out, is merely the end of a thread that vanishes into the thick snare of infinity, which no one can truly unravel, let him tug and worry at it as much as he likes.
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> W. Busch, *The Butterfly*, translated by W. Arndt

Abstract. Causality is a concept which is sometimes claimed to be easy to illustrate, but hard to explain. It is not quite clear whether the former part of this claim is as obvious as the latter one. I will not present any specific theory of causation. Our aim is much less ambitious; to investigate the formal counterparts of causal relations between events, i.e. to propose a formal framework which enables us to construct metamathematical counterparts of causal relations between singular events. This should be a good starting point to define formal counterparts for concepts like "causal law", "causal explanation" and so on.

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This paper concerns a larger project on causal logic, Uwe Scheffler from Humboldt–University and I have been working on for several months. This collaboration explains the usage of plural forms wherever they occur in section 1 and 6. However, all results of this paper, if not stated explicitely otherwise, are due to its (single) author.

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1. Prelogical Considerations

1.1. Causal logic?

Causality is a concept which is sometimes claimed to be easy to illustrate, but hard to explain. It is not quite clear whether the former part of this claim is as obvious as the latter one. However, one thing seems to be certain: the causal nexus is not a logical relation. If this is true, then what does the title of the paper mean? In order to prepare the reader, this paper is neither about "causality in logic" (which would be entire nonsense), nor about the "logic of causation" (which doesn't seem much better). I will not present any specific theory of causation. Our aim is much less ambitious; to investigate the formal counterparts of causal relations between events, i.e. to propose a formal framework which enables us to construct metamathematical counterparts of causal relations between singular events. This should be a good starting point to define formal counterparts for concepts like "causal law", "causal explanation" and so on.

By *causal logic* we mean that part of non-classical logic which deals with causal connectives in their interaction with classical propositional connectives. (I confine myself to propositional logic only, since the formal apparatus which shall be developed subsequently will be remarkably complex. By the proposed simplification, I hope to improve the transparency of the considered constructions. However, this is not a principal limitation: in almost all cases it is easy to obtain first-order variants of the regarded calculi of causal logic.)

Most naturally the question arises: what are causal connectives? The standard answer is this: "Take a definition of the causal nexus, formalize it, and call the upshot 'causal connectives'". However, it is not very helpful in the present case. The explications of the causal nexus which occur in the literature are, in most cases, hardly precise enough for immediate logical formalization. Furthermore, although there are a lot of definitions (almost each serious philosopher created at least one of them) there is a considerable shortage in the agreement of these definitions.

It is of course possible to formalize (more or less adequately) concepts of causal nexus as stated by one or the other philosopher. It seems, however, much more interesting (and important as well) to formalize causal terminology used in empirical sciences. On the other hand, in this case it is extremely difficult to obtain the object of formalization.

Therefore, one has the following options:

- to proceed with other things in logic until formalizable (i.e. precisely and uniquely defined) concepts of causal relations become available;
- to define artificial objects and to call them "causal connectives", "causal functors" or whatever, i.e. to give a nominal definition of formal counterparts of causal relations; or,
- to describe as precisely as possible the properties of causal relations, to formalize these properties in order to obtain thereby a frame of properties which the formal counterparts of causal relations should possess and to construct subsequently in stock connectives falling into this frame.

The first option is not in keep with the importance of the problem. There seems to exist a great, and still rising need, for a formal-logical counterpart of causal nexus — think about so-called causal simulators in A.I. research. (Besides, causal calculi are themselves highly interesting objects of logical research.)

The second way was chosen e.g. by ŁUKASIEWICZ in his famous paper "Analysis and Construction of the Concept of Cause" ([7]) — one of the very first papers on causal logic. It seems to me however, that this choice merely postpones the problem instead of giving a solution. Let me briefly explain this. Causal logic in general is directed towards applications outside of logic, resembling in this respect e.g. discursive logic or deontic logic. Any appropriate formalization of (parts of) the language of empirical sciences makes the formal apparatus of these logical disciplines available in the considered sciences, rendering thereby the formal analysis of definitions, test procedures for argumentations, etc. In this sense, causal logic can be understood as "applied logic", i.e. as a service discipline for users outside of logic.

Following the standard way in logical formalization, logicians begin with "cleaning up" the area they intend to formalize. They feel free to decide what the correct usage of the considered part of the language is. Next, they elucidate the rules, how to speak "correctly". Unfortunately, logic itself has almost no ability to bring its constructions into natural language, i.e. to execute obedience to the rules concerning the usage of these artificial linguistic creations. Therefore, the upshot of such an officious indoctrination is sometimes a more or less elegant formal calculus which appears to be rather uninteresting for the intended users of the formalization. They simply do not accept the metamathematical construction as an appropriate formal counterpart of their language. In that case, the formalization would be a failure, since the success of a logical formalization depends on whether it meets the intuitive concept it intends to formalize. To that purpose however one needs a precise description of the intuitive concept — and thus the circle is closed. (The linguistic creations of logic should be strictly limited to formal languages.)

Hence, in order to formalize causal terminology of e.g., an empirical theory, it is necessary to proceed much more cautiously. One should describe as precisely as possible the real usage of language in the considered theory. Consequently, only the last way remains, and we shall follow this one.

1.2. Precausal connectives

The following two lists of formulae schemes contain formalizations of general properties of causal relations between events as occurring in "real texts" — without previously cleaning up the area. These sets of schemes are ment to be an attempt to characterize the causal frame mentioned before. Let the (connective) variable $\hookrightarrow_k$ represent any two-argument sentential connective in a formal language of a logical system.[1]

$$L^+ = \left\{ \begin{array}{l} \neg(H \hookrightarrow_k H) \\ (H \hookrightarrow_k G) \rightarrow H \\ (H \hookrightarrow_k G) \rightarrow (H \rightarrow G) \\ (H \hookrightarrow_k G) \rightarrow \neg(H \hookrightarrow_k \neg G) \\ (H \hookrightarrow_k G \wedge F) \rightarrow (H \hookrightarrow_k G) \end{array} \right\}$$

[1] By a *logical system* we understand an ordered pair consisting in a formal language and a consequence operation in this language.

The first list L^+ contains the formalizations of those few properties which are rather commonly accepted. (I shall claim this with caution. It seems that even the schemes in L^+ may be attacked by sufficiently sophisticated counterexamples.) The elements of L^+ would be "natural candidates" for axioms of causal calculi. The absence of potential axioms makes it intelligible that there are almost no axiomatic calculi in causal logic.

$$L^- = \left\{ \begin{array}{l} (H \hookrightarrow_k G) \vee (G \hookrightarrow_k H) \\ (H \hookrightarrow_k G) \rightarrow (G \hookrightarrow_k H) \\ (H \hookrightarrow_k G) \rightarrow \neg(G \hookrightarrow_k H) \\ (H \hookrightarrow_k G) \rightarrow (\neg G \hookrightarrow_k \neg H) \\ (H \hookrightarrow_k G) \rightarrow (H \wedge F \hookrightarrow_k G) \\ (H \hookrightarrow_k G) \rightarrow ((F \hookrightarrow_k H) \rightarrow (H \vee F \hookrightarrow_k G)) \\ (H \hookrightarrow_k (F \hookrightarrow_k G)) \rightarrow (F \hookrightarrow_k (H \hookrightarrow_k G) \\ (H \rightarrow G) \rightarrow (H \hookrightarrow_k G) \\ (H \wedge G) \rightarrow (H \hookrightarrow_k G) \\ \neg H \rightarrow (H \hookrightarrow_k G) \\ H \rightarrow (G \hookrightarrow_k H) \end{array} \right\}$$

These listings are *universal* inasmuch as they do not refer to any specific area of causal terminology. Taking a specific kind of causality into consideration may result in additional positive properties and possibly reduces the number of rejected properties. This procedure allows a better approximation of the aspired formalization.

Let L be any list of formula schemes containing the connective variable $\hookrightarrow_k$ and let $\rightarrow_k$ be any two-argument connective in the language of a logical system S.

DEFINITION 1 The connective $\rightarrow_k$ *meets* a list L of formulae schemes (in the system S) [symb.: $m(\rightarrow_k, L)$, respectively $(m(\rightarrow_k, L, S))$] iff the substitution of $\hookrightarrow_k$ by $\rightarrow_k$ makes every element of L true (in S).

DEFINITION 2 The connective $\rightarrow_k$ *omits* a list L of formulae schemes (in the system S) [symb.: $o(\rightarrow_k, L)$, respectively $(o(\rightarrow_k, L, S))$] iff the substitution of $\hookrightarrow_k$ by $\rightarrow_k$ makes no element of L true (in S).

DEFINITION 3 The connective $\rightarrow_k$ *respects* a pair of lists $\langle L, L' \rangle$ of formulae schemes (in the system S) [symb.: $r(\rightarrow_k, L, L')$, respectively $(r(\rightarrow_k, L, L', S))$] iff $\rightarrow_k$ meets L and omits L' (in the system S).

DEFINITION 4 The connective $\rightarrow_k$ is *precausal* (in the system S) iff $r(\rightarrow_k, L^+, L^-)$.

By no means should the above pair (L^+, L^-) be understood as a sufficient characterization of a formal causal relation. Such a characterization, I guess, is impossible — at least if we ask for it in unrestricted generality. Maybe there are lists completely characterizing formal counterparts of some categories of e.g. juridical causation. If so, they are unknown at present. Therefore, the role of (L^+, L^-) is merely to obtain a negative selection: whatever does not respect this catalogue will surely never be the formal counterpart of any kind of causal nexus. In other words: all causal connectives must be precausal.

As already mentioned, a catalogue modified according to a considered type of causal nexus should allow a better characterization of the metamathematical counterpart of this kind of causal connection than the characterization given by the manifold based on the catalogue (L^+, L^-) and should thereby increase the chance that one of the "specialized precausal connectives" met an accepted formalization (i.e. a formalization accepted by the specialists working in the considered field). Somewhat optimistically I hope that the appropriate catalogues would be established cooperatively between logicians and scientists from the empirical theory in question. (Perhaps one has to invite some philosophers to serve as translators.)

It must be ensured however, that such an expansion produced a catalogue well-created in the following sense: there would then exist a formal system and a connective in the language of this system such that no substitution instance of any scheme from the negative list is entailed by substitution instances of formulae-schemes from the positive list (according to the consequence operation in the system).

2. Precausal Functorial Variables

It seems quite obvious that there are no precausal connectives (sa defined above) within classical propositional logic, i.e. no Boolean function respects $\langle L^+, L^- \rangle$. However, a slightly broader concept of what is classical propositional logic could be adopted: Let us call "classical" any propositional calculus which is two-valued and extensional. Then doubtlessly, the objects usually labelled "functor variables" belong to that realm. It can be shown (see [18]), that none of the original functor variables (as defined by LUKASIEWICZ in [8] and investigated e.g. by STELZNER in [14]) is a precausal connective. However, MAX introduced a class of new classical objects (see [9]), which he also calls *functorial variables*. Those objects are in fact modifications of the original construction. Surprisingly enough, two of them respect the above universal lists. That gives evidence, by the way, that the above listings L^+ and L^- are indeed well-created.

Yet in general I am sceptical whether functorial variables are promising candidates for causal connectives. First of all, they are purely technical constructions with no deeper philosophical motivation. What is more, there seems to be no easy way to change the construction of precausal functorial variables according to considered enlargements of the above listings. In any case, there are better areas in which to search for precausal connectives than classical propositional logic. (A preliminary overview can be found in [17], pp. 55–82, 119–147. The manifold of accounts, found in almost all realms of non-classical logic, can be hardly surveyed. A more detailed analysis of this issue will be a central part of our future work on causal logic.)

Are there any criteria which allow one to select the promising techniques in constructing causal connectives? Imagine you are going to hit a target which is poorly visible. Under such conditions, it would seem to be much more efficient to throw a handful of pebbles rather than a single dart. This is a close analogy to our present situation. The vision of the target still remains somewhat fuzzy. Therefore one should strictly prefer constructions which produce manifolds of objects, rather than pointwise definitions. Those constructions should be flexible, as well as manipulatable, in order to allow the production of an outcome which meets all the gradually spelled out requirements.

3. Jaśkowski-style systems

3.1. Jaśkowski's original construction

In the following three sections I will explain in more detail one example of a formal structure which seems to be highly appropriate to model causal relations. The basic idea of the construction is due to Stanisław Jaśkowski (cf. [5]). Most of his highly original results in formal logic are directed towards applications outside of logic. In each case they belong to the earliest considerations in the field (e.g. his system of natural deduction 1932, an adequate semantic for the intuitionistic logic 1933, a system of paraconsistent logic 1948). His investigations on causal logic were stimulated by one of his teachers, Stanisław Leśniewski. During his lectures held at Warsaw University, Leśniewski asked for the representability of causal connectives within an extensional framework. (In order to solve this problem Jaśkowski was constrained to assume a rather exotic concept of extensionality.)

JAŚKOWSKI defined his causal connectives at the basis of two non-classical systems: Q_f and Q^*. The former one is called "calculus of factors" whilst the latter one is called "calculus of chronological factor succession". The crucial point in each of the definitions is the use of so-called "dependent functorial variables". These objects were introduced into the literature by HEYTING (cf. [2]). JAŚKOWSKI gave the following explanation:

> Suppose that the truth of a sentence $\mathcal{P}$ depends on certain factors which cannot be determined strictly: for instance, a person is to toss a coin, and the sentence $\mathcal{P}$ means 'during the game heads will turn up more times than tails will'.
>
> For a certain sequence of random events the sentence $\mathcal{P}$ will prove true, whereas for some other sequence it will prove false. Thus the sentence $\mathcal{P}$ may be assumed to be a function that takes on the values: truth and falsehood, according to the values of the variables that stand for the random events. Since the functional relationship is not revealed by the notation, a sentence of this kind may be represented by the dependent sentential variables introduced by Heyting [...], in a way similar to that in which in mathematics the functions of the variable x are often represented by the letter y. ([4], p. 148)

In a first step JAŚKOWSKI explains, for purely technical reasons, an auxiliary calculus Q, which language contains independent sentential variables (i.e. individual variables) and dependent sentential variables (i.e. quasi-functional variables with unfixed number of arguments). The set FOR_Q of formulae of Q contains nothing but the set of all dependent variables $p, q, r \ldots$ and the following chains of signs $\neg H$, $H \wedge G$, $\forall x : H$ (where x is an independent variable) only if it contains H and G. Subsequently the class of Q–tautologies is established by means of a translation t from FOR_Q to FOR_1, the language of first-order predicate calculus PC_1:

Let $H \in FOR_Q$ and let's assume, that H contains exactly all independent variables $x_1, \ldots, x_n$.

$$t(H) =_{df} H[p/P(x_1, \ldots, x_n), \ldots, q/Q(x_1, \ldots, x_n), \ldots]$$

As usual one can define a logical calculus Q as $Q =_{df} \langle FOR_Q, Cn_Q \rangle$ with a consequence relation $Cn_Q : 2^{FOR_Q} \longrightarrow 2^{FOR_Q}$

$$H \in Cn_Q(X) \textit{ iff } \forall F \in X\,(t(F) \in PC_1 \implies t(H) \in PC_1)$$

or, as it is stated in JAŚKOWSKI's original paper [5] as a set of tautologies $Q = Cn_Q(\emptyset)$.

The next step is the definition of the causal system Q_f. The set of formulae of Q_f is generated as usual by a denumerable set of (dependent) variables AT by means of one one-argument connective $\neg$ and two two-argument connectives $\wedge$ and $\Box_f$. Formulae of the form $\Box_f(H)G$ are to be read as "G is true for all values of factors of H ". In the calculus Q_f, it is not possible to state precisely, what these "factors of H " are — it can merely be indicated: "factors of H " are those individual variables of dependent variables of H, which assignment determines the value of H. (Although those variables do not occur explicitely in Q_f.) The precise meaning of this concept emerges together with its semantical explanation.

3.2. Multidimensional f-o–frames

The construction of Q_f allows a far reaching generalization. In order to demonstrate this, we need a somewhat complex conceptual machinery. We hope however, that the constructed systems appear to be interesting enough to pay off in this effort.

Let S be a *regular* modal system in the modal language FOR_m, i.e. S contains all classical tautologies as well as the formula $\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$ and S is closed w.r.t. Substitution, Modus Ponens and $H \rightarrow G / \Box H \rightarrow \Box G$.

By a *first-order* KRIPKE-*frame* [*f-o–frame*, for short] let's understand a structure $\mathcal{F} = \langle W, Q, R, \Pi \rangle$, where $Q \subseteq W \neq \emptyset$ and $R \subseteq W^2$ and Π is closed w.r.t. set theoretical complementation, intersection, and the following operation l_R

$$l_R(V) =_{df} \{w \in W;\ \forall v \in W :\ w\, R\, v \implies v \in V\},$$

i.e. if V belongs to Π, then the set of all elements of the universe, such that whatever is accessible from one of these elements belongs to V, is itself an element of Π.

$\mathcal{F}$ is f-o–frame *for* S if all S-theorems are $\mathcal{F}$-tautologies, symb.: $\mathcal{F} \models_m S$. Let $\mathcal{K}_S$ be the class of all f-o–frames for S.

It is well known that for any modal system S there is a class of f-o–frames $\tilde{\mathcal{K}}$, such that the class of all S-theorems equals the class of $\tilde{\mathcal{K}}$-tautologies, i.e. the intersection of all classes of $\mathcal{F}$-tautologies, for $\mathcal{F} \in \tilde{\mathcal{K}}$. Obviously, $\tilde{\mathcal{K}} \subseteq \mathcal{K}_S$. Since

$$S \subseteq \{H \in FOR_m;\ \mathcal{K}_S \models_m H\} \subseteq \{H \in FOR_m;\ \tilde{\mathcal{K}} \models_m H\} = S$$

we obtain that $\mathcal{K}_S$ is adequate for S.

DEFINITION 5 Let $\mathcal{F}_i = \langle W_i, Q_i, R_i, \Pi_i \rangle$; $i \leq n$ be $\mathcal{K}_S$–frames.
The structure $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n = \langle \mathcal{W}, \mathcal{Q}_1, ..., \mathcal{Q}_n, \mathcal{R}_1, ..., \mathcal{R}_n, \mathcal{P} \rangle$ is called *n–dimensional product* of $\mathcal{F}_i$; i $\leq n$ *iff*

1° $\mathcal{W} = W_1 \times \ldots \times W_n$;

2° $\forall\, i \leq n : \mathcal{Q}_i = W_1 \times \ldots \times W_{i-1} \times Q_i \times W_{i+1} \times \ldots \times W_n$;

3° $\forall\, i \leq n : \mathcal{R}_i = id_1 \times \ldots \times id_{i-1} \times R_i \times id_{i+1} \times \ldots \times id_n$;

4° $\mathcal{P} = \Pi_1 \times \ldots \times \Pi_n$.

It is not hard to prove that $\mathcal{F}^n = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ is a f-o–frame, too. In particular, $\mathcal{F}^n$ fulfills all the required closure conditions.

Let $\mathcal{K}_S^n$ be the class of all n–dimensional products in $\mathcal{K}_S$. In case $n = 0$ we appoint $\mathcal{K}_S^0 = \{\langle \{\emptyset\}, \emptyset, \emptyset, \{\emptyset, \{\emptyset\}\} \rangle\}$ with a one-element universe and without relations. In the class $\mathcal{K}_S^n$ we interpret the formal language L_f of the system Q_f.

DEFINITION 6 The ordered pair $\langle \mathcal{F}^n, v \rangle$ is denoted by $\mathcal{M}^n$ and called *n–dimensional model based on* $\mathcal{F}^n = \langle \mathcal{W}, \mathcal{Q}_1, ..., \mathcal{Q}_n, \mathcal{R}_1, ..., \mathcal{R}_n, \mathcal{P} \rangle$ *iff*
$v : AT \longrightarrow \mathcal{P}$.

DEFINITION 7 Let $p \in AT$; H, $F \in FOR$; $\tilde{x} \in \mathcal{W}$. We define *the truth of* H *in a point* $\tilde{x}$ *of the model* $\mathcal{M}$ $[\mathcal{M}^n \models H[x]]$ recursively.

1° $\mathcal{M}^n \models p[\tilde{x}]$ *iff* $\tilde{x} \in v(p)$;
2° $\mathcal{M}^n \models \neg H[\tilde{x}]$ *iff* $\mathcal{M}^n \not\models H[\tilde{x}]$;
3° $\mathcal{M}^n \models H \wedge F[\tilde{x}]$ *iff* $\mathcal{M}^n \models H[\tilde{x}]$ *and* $\mathcal{M}^n \models F[\tilde{x}]$.

In order to interpret the remaining case of the non-classical operator $\Box_f$ some abbreviations are helpful. For all $i \leq n$ we use as technical signs diamonds $\Diamond_i$ and boxes $\Box_i$:

$\mathcal{M}^n \models \Diamond_i H[\tilde{x}]$ *iff* $\exists \tilde{y} \in \mathcal{W}$: $\tilde{x}\, \mathcal{R}_i\, \tilde{y}$ *and* $\mathcal{M}^n \models H[\tilde{y}]$

and

$\Box_i H =_{df} \neg \Diamond_i \neg H$.

DEFINITION 8 Let $k \leq n$.

$$f^n(k, H) =_{df} \Diamond_1 \ldots \Diamond_n (\Diamond_k H \wedge \Diamond_k \neg H).$$

$f^n(k, H)$ is to be read as "k has influence on the truth-value of H" or as "k is a factor of H".

DEFINITION 9 Let $\kappa = \{k_1, \ldots, k_m\} \subseteq \{1, \ldots, n\}$.

$$f^n(\kappa, H) =_{df} f^n(k_1, H) \wedge \ldots \wedge f^n(k_m, H)$$
$$\Box_\kappa H =_{df} \Box_{k_1} H \wedge \ldots \wedge \Box_{k_m} H .$$

The definitions 8 and 9 completely explain the concept of the "set of factors of a formula in a point of a model": $\kappa \subseteq \{1, \ldots, n\}$ is the *set of factors of H in the point $\tilde{x}$ of the model $\mathcal{M}^n$*, if and only if

$$\mathcal{M}^n \models f^n(\kappa, H)[\tilde{x}].$$

DEFINITION 10 Let H, G, n and κ be as before.

$$\Box^n_{f(H)} G =_{df} G \wedge \bigwedge_{\kappa \subseteq \{1 \ldots n\}} f^n(\kappa, H) \rightarrow \Box_\kappa G.$$

The formula $\Box^n_{f(H)} G$ is to be read as "G is true and it is necessary for any set of factors of H".

Now we are ready to define the case for the operator $\Box_f$, left open in definition 7.

DEFINITION 11 (continuation of definition 7)
Let H, $F \in FOR_f$ and let $\tilde{x} \in \mathcal{W}$. We define *the truth of* $\Box_{f(F)} H$ *in a point* $\tilde{x}$ *of the model* $\mathcal{M}$:
4^o $\mathcal{M}^n \models \Box_{f(H)} G[\tilde{x}]$ *iff* $\mathcal{M}^n \models \Box^n_{f(H)} G[\tilde{x}]$

The definitions 7 and 11 made clear what is meant when stating that a formula from FOR_f is true in a point of a model. As usual one can now define the truth of a formula in a model, in a frame, and, finally, in a class of frames.

The acceptance relation explained in definitions 7 and 11 determines a consequence operation $\mathcal{J}_S : P(FOR_f) \longrightarrow P(FOR_f)$ for each regular modal system S.

DEFINITION 12 Let $X \subseteq FOR_f$, ω denotes the set of natural numbers.

$$\mathcal{J}_S(X) =_{df} \bigcap_{n \in \omega} \mathcal{J}^n_S(X),$$

where

$$H \in \mathcal{J}^n_S(X) \text{ iff } \forall \mathcal{F}^n = \langle \mathcal{W}, \ldots, \mathcal{P} \rangle \in \mathcal{K}^n_S \; \forall v \in \mathrm{HOM}(AT, \mathcal{P}) \; \forall \tilde{x} \in \mathcal{W}:$$
$$\langle \mathcal{F}^n, v \rangle \models X[\tilde{x}] \implies \langle \mathcal{F}^n, v \rangle \models H[\tilde{x}]$$

Let $S5$ be the well-known modal system of LEWIS. $S5$ is obviously regular and therefore it determines a consequence operation $\mathcal{J}_{S5}$.

THEOREM 1 ([16]) $\mathcal{J}_{S5}(\emptyset) = Q_f$

The construction of the inference relation $\mathcal{J}_S$ is based on a generalization of JAŚKOWSKI's ideas. This fact, together with the result established in theorem 1, motivates the following definition:

DEFINITION 13 Let S be a regular modal system. The set of formulae $\mathcal{J}_S(\emptyset)$ is called the JAŚKOWSKI-*style system designated by* S.

We shall write $\mathcal{J}_S$ instead of $\mathcal{J}_S(\emptyset)$ wherever this doesn't lead to misunderstandings.

3.3. Basic systems

For further investigations on the JAŚKOWSKI-style systems we need a method which allows us to find the tautologies of these systems. It seems hopeless to test the validity of formulae *ex definitione*. Fortunately, there exists an appropriate method for a large class of modal systems.

DEFINITION 14 The modal calculus S is called *basic system* iff
(1) $S \subseteq TR$ [2];
(2) there is a class $\mathcal{K}$ of reflexive, full [3] KRIPKE-frames such that
  (i) $S = \{H \in FOR_m;\ \mathcal{K} \models H\}$;
  (ii) $\mathcal{K}$ is closed w.r.t. direct products [4].

THEOREM 2 (REPRESENTATION THEOREM ([17]))
Let S be a basic system.
$\forall H \in FOR_f\ \exists r(H) \in \omega\ :\ H \in \mathcal{J}_S \Longleftrightarrow \mathcal{K}_S^{r(H)} \models H.$

The function $r\ :\ FOR_f \longrightarrow \omega$ is effectively computable. For any formula H the number $\log_2 r(H)$ is not greater than the number of different non-classical subformulae in H. (The precise definition of $r(H)$ can be found in [17].)

Furthermore, we have the following useful lemma.

LEMMA 1 ([17]) *Let H, G and κ be as before.*
$\mathcal{M}^n \models f(\kappa, H) \Longrightarrow \mathcal{M}^n \models \Box_{f(H)}G \equiv \Box_\kappa G.$

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[2] TR is the *trivial* modal logic, obtained by adding $\Box H \equiv H$ to the modal system K.

[3] $\mathcal{F} = \langle W, Q, R, \Pi \rangle$ is called *full* KRIPKE-frame iff $\Pi = 2^W$.

[4] i.e. for $\mathcal{F}_i = \langle W_i, Q_i, R_i \rangle \in \mathcal{K}$; $i \leq n$, their *direct product*, i. e. the following structure: $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n = \langle \mathcal{W}, \bigcup_{i \leq n} \mathcal{Q}_i, \mathcal{S} \rangle$, where $\mathcal{W}$ and $\mathcal{Q}_i$ are explained in definition 5 and $\mathcal{S}$ is defined as $\langle x_1, \ldots, x_n \rangle\ \mathcal{S}\ \langle y_1, \ldots, y_n \rangle$ *iff* $\forall i \leq n : x_i\ R_i\ y_i$, belongs to $\mathcal{K}$, too.

Theorem 2, together with lemma 1, allow us to generalize the well-known diagram method for testing modal formulae for the multi-modal case. In general, however, such a method does not establish the decidability of the corresponding causal system. The topic is quite complicated for dimensions higher than 3 and will not be discussed here in further detail. Let me just notice that the product construction in definition 5 is a special case of the generalized product defined in [1]. According to the fundamental theorem of that paper, the decision problem of the first order theory of the generalized product can be reduced to the decision problems of the first-order theories of its factors. It is not hard to show that for all $n \in \omega$ the system T^n is decidable and so is $\mathcal{J}_T$. However, $S5^3$ is not decidable — it follows from theorem 1 together with a result due to PIECZKOWSKI that this system is a reduction class of the f-o predicate calculus ([11]) — and hence $\mathcal{J}_{S5}$ is undecidable as well.

The non-classical box-operator $\Box_f$ can be used to define further connectives.

DEFINITION 15 $H \rightarrow_f G =_{df} \Box_{f(H)}(H \rightarrow G)$.

On the other hand, $\rightarrow_f$ can be used to define $\Box_f$ as well.

LEMMA 2 *Let S be any basic system.*
$\Box_{f(H)}G \equiv \Box_{f(H)}(H \rightarrow G) \wedge \Box_{f(H)}(\neg H \rightarrow G) \in \mathcal{J}_S(\emptyset)$.

The connective $\rightarrow_f$ already has some interesting properties. Some of its other properties are, however, quite counterintuitive with respect to an intended causal interpretation. In any case, this connective is still a good starting-point for searching precausal connectives. To give just an example,
$\Box_{f(H)}G \wedge H \rightarrow_f G \wedge \neg\Diamond_{f(H)}(\Diamond_{f(H)}H \rightarrow G)$
seems to be a good candidate.

4. Modifications

There are lots of possible topics with which to proceed. One of them is the axiomatization of JAŚKOWSKI-style systems. ([17] contains some partial results.) Furthermore, as we mentioned above, the *Entscheidungsproblem* of these systems seems to be highly interesting.

Nevertheless, our primary interest is devoted to the formalization of causal relations. According to NUTE's classification (cf.[10]) the JAŚKOWSKI-systems, although numerous, are situated rather close to each other. Therefore it makes sense to ask for further modifications of the construction.

4.1. First order Jaśkowski-style systems

In the introduction, we claimed that the limitation of our investigations to propositional languages only is not a principal one, but is merely to increase the transparency of the presented material. In the following, we demonstrate how to accomplish a first-order version of a system which is based on relational frames.

Let VAR be a denumerable set of individual variables $x, y, z, \ldots$ and let PRE be a denumerable set of predicate symbols $P, Q, R, \ldots$ of fixed arity $ar(P)$ etc. The set AT of atomic formulae consists of all sequences $P(x_{i1}, \ldots, x_{i\,ar(P)})$, where $P \in PRE$ and $x_{i1}, \ldots, x_{i\,ar(P)}$ are arbitrary individual variables.

The set FOR_1 of all formulae is defined to be the least set including AT and containing $\neg H$ as well as $H \wedge G$, $\Box_{f(H)}G$ and $\forall x : H$, whenever it contains H and G. Further connectives may be defined as usual.

Next we interpret FOR_1 in appropriately enlarged first-order frames. The construction goes along the way well known from modal logic (cf. e.g. [6], pp. 164–168).

Let S be a *normal* basis system. The class $\mathcal{K}^n_{1S}$ consists in all structures thus defined:

DEFINITION 16 The structure $\mathcal{F}^n_1 = \langle \mathcal{W}, \mathcal{R}_1, ..., \mathcal{R}_n, \mathcal{D}, \mathcal{U} \rangle$ is called *enlarged n–dimensional product* iff

(1) $\langle \mathcal{W}, \mathcal{R}_1, ..., \mathcal{R}_n, \rangle \in \mathcal{K}^n_S$;
(2) $\mathcal{D}$ is a non-empty domain of individuals;
(3) $\mathcal{U}$ is an assignment such that
  (i) $\forall x \in VAR : \mathcal{U}(x) \in \mathcal{D}$;
  (ii) $\forall P \in PRE : \mathcal{U}(P) \subseteq \mathcal{D}^{ar(P)} \times \mathcal{W}$.

On $\mathcal{F}^n_1$ we base the construction of an *enlarged n–dimensional model* $\mathcal{M}^n_1$, by adding a valuation $v_1 : AT \longrightarrow 2^{\mathcal{W}}$ explained as

$$v_1(P(x_{i1}, \ldots, x_{i\,ar(P)})) = \{\tilde{x} \in \mathcal{W};\ \langle \mathcal{U}(x_{i1}), \ldots, \mathcal{U}(x_{i\,ar(P)}), \tilde{x} \rangle \in \mathcal{U}(P)\}.$$

DEFINITION 17 We explain the *acceptance of a formula* $H \in FOR_1$ *in a point* $\tilde{x}$ *of a model* $\mathcal{M}^n_1$ [$\mathcal{M}^n_1 \models_1 H[\tilde{x}]$, for short]. All Boolean cases are defined as usual.

$\mathcal{M}^n_1 \models_1 \Box_{f(H)}G[\tilde{x}]$ *iff* $\mathcal{M}^n_1 \models_1 \Box^n_{f(H)}G[\tilde{x}]$,
where $\Box^n_{f(H)}G$ is explained as in definition 11. Furthermore,

$\mathcal{M}^n_1 \models_1 \forall x : H[\tilde{x}]$ *iff* $\mathcal{N}^n_1 \models_1 H[\tilde{x}]$,
where $\mathcal{N}^n_1$ differs from $\mathcal{M}^n_1$ at most in its value for $\mathcal{U}(x)$.

Now it should be obvious, how to go through the notion of acceptance in a model, in a frame, in a class of frames up to a consequence operation $\mathcal{F}_{1S}$ in FOR_1 and, finally, to arrive at a *first-order* JAŚKOWSKI-*style system* $\mathcal{F}_{1S}(\emptyset)$.

4.2. Intuitionistic Jaśkowski-style systems

Among all the systems of non-classical propositional logic, the intuitionistic calculi gained a distinguished position. As the only non-classical system intuitionism is accepted as a logical basis of metamathematics. In this sense, intuitionistic logic is the only alternative to classical propositional calculus. Moreover, constructivistic argumentation, based on intuitionistic logic, can be met outside mathematics as well. For this reason, we should take a closer look at causal-logical systems connected in a way with the basic ideas underlying intuitionistic logic.

The construction described in subsection 3.2. can be modified accordingly. We take as a starting point the adequate KRIPKE-semantic for the intuitionistic logic *INT* (cf. e.g. [13]).

Let $\mathcal{K}_i$ be the class of all reflexive, transitive, and antisymmetric KRIPKE-frames. Taking $\mathcal{F}_1, \ldots, \mathcal{F}_n$ from $\mathcal{K}_i$, we form an n-dimensional i–frame $\mathcal{F}^n = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ as described in definition 5. It's easy to see that $\mathcal{F}^n$ is reflexive, transitive, and antisymmetric as well (i.e. all of its relations are of that kind).

On $\mathcal{F}^n$ we explain an i–model $\mathcal{M}^n$ as an ordered pair consisting in $\mathcal{F}^n$ and a monotonic valuation $v_i : AT \longrightarrow 2^{\mathcal{W}}$. That means v_i has to fulfill

$$\forall p \in AT\, \forall \tilde{x}, \tilde{y} \in \mathcal{W}\, \forall j \leq n : \tilde{x}\, \mathcal{R}_j\, \tilde{y} \textit{ and } \tilde{x} \in v_i(p) \Longrightarrow \tilde{y} \in v_i(p).$$

Next, we define the acceptance of formulae from the language $L_i =_{df} \langle AT, \sim, +, \cdot, \succ, \Box_f, \Diamond_f \rangle$ in a point of an n-dimensional i–model. The symbols $\sim, +, \cdot$ and $\succ$ represent the intuitionistic connectives while $\Box_f$ and $\Diamond_f$ denote two-argument non-classical connectives.

DEFINITION 18 Let $H \in FOR_i$ be a formula generated in the usual way from AT by $\sim, +, \cdot, \succ, \Box_f$ and $\Diamond_f$. The *acceptance of H in the point $\tilde{x}$ of the model $\mathcal{M}^n$* [$\mathcal{M}^n \models_i H[\tilde{x}]$, for short] is explained recursively:

1^0	$\mathcal{M}^n \models_i p[\tilde{x}]$	*iff*	$\tilde{x} \in v_i(p)$	for $p \in AT$;
2^0	$\mathcal{M}^n \models_i \sim H[\tilde{x}]$	*iff*	$\mathcal{M}^n \not\models_i H[\tilde{y}]$	for all $\tilde{y}$, such that there is an $j \leq n$ with $\tilde{x}\mathcal{R}_j\tilde{y}$;
3^0	$\mathcal{M}^n \models_i H + G[\tilde{x}]$	*iff*	$\mathcal{M}^n \models_i H[\tilde{x}]$ *or* $\mathcal{M}^n \models_i G[\tilde{x}]$;	
4^0	$\mathcal{M}^n \models_i H \cdot G[\tilde{x}]$	*iff*	$\mathcal{M}^n \models_i H[\tilde{x}]$ *and* $\mathcal{M}^n \models_i G[\tilde{x}]$;	

5^0 $\mathcal{M}^n \models_i H \succ G[\tilde{x}]$ *iff* $\mathcal{M}^n \models_i H[\tilde{y}] \Longrightarrow \mathcal{M}^n \models_i G[\tilde{y}]$; for all $\tilde{y}$, such that there is an $j \leq n$ with $\tilde{x}\mathcal{R}_j\,\tilde{y}$.

The remaining cases of "modalized" formulae $\Box_{f(H)}G$ and $\Diamond_{f(H)}G$ shall be defined as correspondingly as possible to the classical case (cf. definition 11). Again, some abbreviations will be helpful. Quite similarly as in the case of the classical acceptance relation, $\models$, we introduce n pairs of technical signs $\Box_k$, $\Diamond_k$; $k \leq n$.

$\mathcal{M}^n \models_i \Box_k(H)[\tilde{x}]$ *iff* $\forall \tilde{y} \in \mathcal{W}$: $\tilde{x}\,\mathcal{R}_k\,\tilde{y} \Longrightarrow \mathcal{M}^n \models_i H[\tilde{y}]$

and

$\mathcal{M}^n \models_i \Diamond_k(H)[\tilde{x}]$ *iff* $\exists \tilde{y} \in \mathcal{W}$: $\tilde{x}\,\mathcal{R}_k\,\tilde{y}$ *and* $\mathcal{M}^n \models_i H[\tilde{y}]$.

As one could have expected, $\Box_k$ and $\Diamond_k$ are no longer interdefinable in i–models. (There exist two-dimensional countermodels.) Therefore, besides the abbreviations stated in definition 8 and 9 we arrange: $\Diamond_\kappa$ is short for $\Diamond_{k_1}, \ldots, \Diamond_{k_l}$, in case of $\kappa = \{k_1, \ldots, k_l\}$.

Now the missing cases can be easily supplied.

DEFINITION 19 (continuation of definition 18)
Let $f^n(\kappa, H)$ be as explained in definition 9.

6^o $\mathcal{M}^n \models_i \Box_{f(H)}G$ *iff* $\mathcal{M}^n \models_i G \wedge \bigwedge_{\kappa \subseteq \{1\ldots n\}} f^n(\kappa, H) \rightarrow \Box_\kappa G[\tilde{x}]$;

7^o $\mathcal{M}^n \models_i \Diamond_{f(H)}G$ *iff* $\mathcal{M}^n \models_i G \vee \bigvee_{\kappa \subseteq \{1\ldots n\}} f^n(\kappa, H) \wedge \Diamond_\kappa G[\tilde{x}]$.

We proceed as usual and arrive finally at the following definition.

DEFINITION 20 Let $X \subseteq FOR_i$.

$$\mathcal{J}_i(X) =_{df} \bigcap_{n\in\omega} \mathcal{J}_i^n(X),$$

where

$$H \in \mathcal{J}_i^n(X) \text{ iff } \forall \mathcal{F}^n = \langle \mathcal{W}, \mathcal{R}_1 \ldots, \mathcal{R}_n \rangle \in \mathcal{K}_i^n\ \forall v_i\ \forall \tilde{x} \in \mathcal{W}: \langle \mathcal{F}^n, v_i \rangle \models_i X[\tilde{x}] \Longrightarrow \langle \mathcal{F}^n, v_i \rangle \models_i H[\tilde{x}]$$

$\mathcal{J}_i(\emptyset)$ is called the *intuitionistic* JAŚKOWSKI-*style system* and represented symbolically by $\mathcal{J}_{INT}$. It follows immediately from the above definitions that all L_i-substitutions of intuitionistically invalid formulae fail to be theorems of $\mathcal{J}_{INT}$. Thus, to take an example,

$$(\Box_{f(p)}q) + \sim (\Box_{f(p)}q) \notin \mathcal{J}_{INT}.$$

On the other hand, all intuitionistic tautologies belong to $\mathcal{J}_{INT}$.

$$INT \subseteq \mathcal{J}_{INT}.$$

For further investigations of the system $\mathcal{J}_{INT}$, the most important question is whether there exists a method for testing formulae. Fortunately, one can modify the proof of theorem 2 to obtain the following corollary:

THEOREM 3 ([18])

$$\forall H \in FOR_i \, \exists r(H) \in \omega \,:\, H \in \mathcal{J}_{INT} \Longleftrightarrow \mathcal{K}_i^{r(H)} \models_i H.$$

We skip all the technical details here and mention only that the core of the inductive proof is the evidence for the monotonicity of the induced valuation function in the considered m–dimensional model. This property can be shown as follows:

For $m \leq n$ let g be a function from $\{1, \ldots, n\}$ onto $\{1, \ldots, m\}$. Let $\mathcal{M}^n = \langle \mathcal{W}, \mathcal{R}_1, ..., \mathcal{R}_n, v \rangle$ be an n–dimensional model. We define an m–dimensional model $\mathcal{M}_g^m$ as follows:

DEFINITION 21 $\mathcal{M}_g^m =_{df} \langle \mathcal{W}^m, \mathcal{S}_1, \ldots, \mathcal{S}_n, v_g \rangle$, where for $i, j \leq m$

(1) $\langle \tilde{x}^1, \ldots, \tilde{x}^m \rangle \; \mathcal{S}_i \; \langle \tilde{y}^1, \ldots, \tilde{y}^m \rangle$ *iff* $\forall j \neq i \; (\; \tilde{x}^j = \tilde{y}^j \; \wedge$
$\forall k \leq n \,:\, \tilde{x}_k^i \; R_k \; \tilde{y}_k^i \;)$;

(2) $v_g(p) = \{ \langle \tilde{x}^1, \ldots, \tilde{x}^m \rangle \in \mathcal{W}^m ; \; \langle \tilde{x}_1^{g(1)}, \ldots, \tilde{x}_n^{g(n)} \rangle \in v(p) \}$.

LEMMA 3 *v_g is monotonic, provided that v is also monotonic.*

PROOF.
Assume that $\langle \tilde{x}^1, \ldots, \tilde{x}^m \rangle \in v_g(p)$ and that $\langle \tilde{x}^1, \ldots, \tilde{x}^m \rangle \; \mathcal{S}_i \; \langle \tilde{y}^1, \ldots, \tilde{y}^m \rangle$ for a given $i \leq m$.

From the above definition we obtain

$$\langle \tilde{x}_1^{g(1)}, \ldots, \tilde{x}_n^{g(n)} \rangle \in v(p) \qquad [A]$$

and

$$\forall j \neq i \;:\; \tilde{x}^j = \tilde{y}^j \qquad [B_1]$$

as well as

$$\forall k \leq n : \; \tilde{x}_k^i \; R_k \; \tilde{y}_k^i \qquad [B_2]$$

Let $\{j_1, \ldots, j_l\}$ be the set $g^{-1}(i)$ in its natural ordering. Then it follows from $[B_2]$

$\tilde{x}_{j_1}^i = \tilde{x}_{j_1}^{g(j_1)}$ *and* $\tilde{x}_{j_1}^i R_{j_1} \tilde{y}_{j_1}^i$ *and* $\tilde{y}_{j_1}^{g(j_1)} = \tilde{y}_{j_1}^i$

.

.

.

$\tilde{x}_{j_l}^i = \tilde{x}_{j_l}^{g(j_l)}$ *and* $\tilde{x}_{j_l}^i R_{j_l} \tilde{y}_{j_l}^i$ *and* $\tilde{y}_{j_l}^{g(j_l)} = \tilde{y}_{j_l}^i$.

As stated above, j_1 is the smallest number s.t. $g(j_1) = i$. Therefore [B$_1$] entails:

$$\langle \tilde{x}_1^{g(1)}, \ldots, \tilde{x}_n^{g(n)} \rangle \,\mathcal{R}_{j_1} \langle \tilde{y}_1^{g(1)}, \ldots, \tilde{y}_{j_1}^{g(j_1)}, \tilde{x}_{j_1+1}^{g(j_1+1)}, \ldots, \tilde{x}_n^{g(n)} \rangle.$$

We proceed with j_2 etc. until we finally arrive at j_l, which is the greatest number with this property:

$$\langle \tilde{y}_1^{g(1)}, \ldots, \tilde{y}_{j_1}^{g(j_{l-1})}, \tilde{x}_{j_{l-1}+1}^{g(j_{l-1}+1)}, \ldots, \tilde{x}_n^{g(n)} \rangle \,\mathcal{R}_{j_l} \langle \tilde{y}_1^{g(1)}, \ldots, \tilde{y}_n^{g(n)} \rangle.$$

Because of [A], and of the first of the above lines, the monotonicity of v yields that

$$\langle \tilde{x}_1^{g(1)}, \ldots, \tilde{x}_{j_1}^{g(j_{l-1})}, \tilde{y}_{j_1}^{g(j_1)}, \tilde{x}_{j_1+1}^{g(j_1+1)}, \ldots, \tilde{x}_n^{g(n)} \rangle \in v(p).$$

Procceding step by step on this way the last of the above lines leads us to

$$\langle \tilde{y}_1^{g(1)}, \ldots, \tilde{y}_n^{g(n)} \rangle \in v(p)$$

which means

$$\langle \tilde{y}^1, \ldots, \tilde{y}^m \rangle \in v_g(p).$$ ■

4.3. Paraconsistent Jaśkowski-style systems

Paraconsistent systems (as long as paraconsistency is interpreted in a rational, that means non-dialethical, way) are of great methodological interest. They seem to be suitable to formalize e.g. discourse situations as occurring in rational discussions. Thus, paraconsistent systems may be useful as metamathematical background for considerations on the methodology of empirical sciences, i.e. a field of application which they share with causal logic. Therefore it may be interesting to look for a possible amalgamation of both of these accounts.

Beside JAŚKOWSKI's construction of causal systems there is another highly original construction by this author which received slightly more attention in history (however, still not as much as it deserves) — his discussive logic $\mathcal{D}_2$. The calculus $\mathcal{D}_2$ was one of the earliest contributions to what is called today paraconsistent logic and it was, in fact, the first *formalized* paraconsistent system.

Here is a short description reformulated in modern symbolic language. Let FOR_d be the set of all formulas freely generated from a denumerable set of propositional variables by means of some Boolean complete set of propositional functors and two additional two-argument "discussive" connectives: the discussive conjunction $\wedge_d$ and the discussive implication $\rightarrow_d$. Next we explain a transformation $\tau : FOR_d \longrightarrow FOR_m$ from FOR_d to propositional modal language FOR_m. τ leaves unchanged propositional variables as well as the Boolean connectives. Let $\Diamond$ be the *S5* possibility operator. Then we establish:

(1)	$\tau(H)$	$=_{df}$	H,	for $H \in AT$;
(2)	$\tau(\neg H)$	$=_{df}$	$\neg\tau(H)$;	
(3)	$\tau(H \wedge G)$	$=_{df}$	$\tau(H) \wedge \tau(G)$;	
(4)	$\tau(H \wedge_d G)$	$=_{df}$	$\tau(H) \wedge \Diamond\tau(G)$;	
(5)	$\tau(H \rightarrow_d G)$	$=_{df}$	$\Diamond\tau(H) \rightarrow \tau(G)$.	

The above transformation explains JAŚKOWSKI's original somewhat "oblique" connectives — they can be easily exchanged with more familiar ones without losing anything essential.

DEFINITION 22 $D_2 =_{df} \{H \in FOR_d; \Diamond\tau(H) \in S5\}$.

JAŚKOWSKI's idea can be generalized in several directions: It is possible to use a large class of modal systems, among them even non-normal ones, to obtain interesting discussive calculi. Furthermore, keeping in mind that JAŚKOWSKI's construction is a Polish one, it seems more natural to take in discussive logic in the traditional Polish style as a logical system (i.e. a consequence operation in a formal language) rather than merely as a set of formulae. For each modal calculus S containing $S3$, we are able to define an inference operation, $\mathcal{D}_S$, in the discussive language FOR_d and to give a direct semantical characterization for the system $\langle FOR_d, \mathcal{D}_S\rangle$ in terms of KRIPKE-frames[5].

THEOREM 4 $\mathcal{D}_{S5}(\emptyset) = D_2$.

For all calculi S containing $S3$, the following deduction theorem holds true:

THEOREM 5 $\forall S\, \forall X \subseteq FOR_d\, \forall H, G \in FOR_d$
$(H \rightarrow_d G) \in \mathcal{D}_S(X) \Longleftrightarrow G \in \mathcal{D}_S(X \cup \{H\})$.

Usually we interpret the deduction theorem as a characterization of the implication within the considered system. But now it goes the other way around: the discussive implication possesses all the properties expected from a discussive inference operation. By theorem 4 they are transmitted to the consequence relation. Moreover, the original JAŚKOWSKI system, D_2, is one of the above defined systems. For these reasons, we call the calculi

[5][19] contains some more details.

$\langle FOR_d, \mathcal{D}_S \rangle$ *discussive* JAŚKOWSKI-*systems*. Each of them generates a class of higher-degree discussive systems. It turns out, however, that JAŚKOWSKI's original construction was remarcably stable: for normal basic systems S the whole manifold of higher-degree discussive calculi collapses into D_2. Some further properties of these calculi are dicussed in [19].

First we observe that FOR_d can be redefined as a part of FOR_f. Indeed, $\neg\Box_{f(F)}\neg F \rightarrow G$ serves as a definition for $F \rightarrow_d G$, and nothing else is needed, because $F \wedge_d G$ can be defined in terms of the remaining FOR_d–connectives as $\neg(F \rightarrow_d \neg G)$.

Let us note one further possible generalization of definition 22. Let $FOR_{m(n)}$ be the n–modal language containing n pairs of modal operators $\{\Box_i, \Diamond_i\}$ $i \leq n$ instead of merely $\Box$ and $\Diamond$. Then, the above transformation τ induces straightforwardly the following $\tau_n : FOR_d \longrightarrow FOR_{m(n)}$: instead of the $\Diamond$ in (4) and (5) of the definition of τ, we place everywhere $\Diamond_1, \ldots, \Diamond_n$. Thereby we obtain a multi-modal inference relation $\mathcal{D}^n_S$ instead of mono-modal $\mathcal{D}_S$.

At present such a generalization is quite spurious: all modal operators $\Box$ and $\Diamond$ are substituted by complete strings of modal operators $\Box_1, \ldots, \Box_n$ and $\Diamond_1, \ldots, \Diamond_n$, respectively. Of course, nothing essential has changed: $\mathcal{D}^n_S = \mathcal{D}_S$. However, the knack shall be useful after a while.

The most important properties of the $\mathcal{D}_S$ for our present purposes is their definability in terms of classes of KRIPKE-frames. This will enable us to accomplish the amalgamation we hoped for.

DEFINITION 23 Let $X \subseteq FOR_f$, ω denotes the natural numbers.

$$\mathcal{JD}_S(X) =_{df} \bigcap_{n \in \omega} \mathcal{JD}^n_S(X),$$

where

$$H \in \mathcal{JD}^n_S(X) \text{ iff } \forall \mathcal{F}^n = \langle \mathcal{W}, \ldots, \mathcal{P} \rangle \in \mathcal{K}^n_S \, \forall v \in HOM(AT, \mathcal{P}) \, \forall \tilde{x} \in \mathcal{W}: \langle \mathcal{F}^n, v \rangle \models X[\tilde{x}] \implies \langle \mathcal{F}^n, v \rangle \models \Diamond_1 \ldots \Diamond_n H[\tilde{x}]$$

The systems constructed that way turn out to be both paraconsistent and causal.

THEOREM 6 *For all S containing S3:*

$$\mathcal{J}_S \subseteq \mathcal{JD}_S.$$

PROOF. [straightforward] ∎

We need the following lemma 5 in order to establish the connection between $\mathcal{D}_S$ and $\mathcal{JD}_S$. In the proof of lemma 5, we make use of a fact proved in [17]:

LEMMA 4 *For all* $H \in FOR_f$ *we have*

(a) $\mathcal{K}^n_S \models \Box_{f(H)}H \equiv \Box_1 \ldots \Box_n H$

(b) $\mathcal{K}^n_S \models \Box_{f(\neg H)}H \equiv \Box_{f(H)}H$

LEMMA 5 *Let* $\models_m$ *be the usual multi-modal logic turnstile. Then for all* $H \in FOR_d$ *we obtain:*

$\mathcal{K}^n_S \models_m \tau_n(H) \iff \mathcal{K}^n_S \models H.$

PROOF. [by induction on formula-length]
The only interesting case is $H := F \rightarrow_d G$.

$\mathcal{K}^n_S \models_m \tau_n(F \rightarrow_d G) \iff$

$\mathcal{K}^n_S \models_m \Diamond_1 \ldots \Diamond_n \tau_n(F) \rightarrow \tau_n(G) \iff$

$\forall \mathcal{M}^n \in \mathcal{K}^n_S \, \forall \tilde{x} \, (\mathcal{M}^n \models_m \Diamond_1 \ldots \Diamond_n \tau_n(F)[\tilde{x}] \Rightarrow \mathcal{M}^n \models_m \tau_n(G)[\tilde{x}]) \iff$

$\forall \mathcal{M}^n \in \mathcal{K}^n_S \, \forall \tilde{x} \, (\mathcal{M}^n \models_m \tau_n(F)[\tilde{y}] \Rightarrow \mathcal{M}^n \models_m \tau_n(G)[\tilde{x}]) \iff$

[for some $\tilde{y} \in \mathcal{R}_1 \ldots \mathcal{R}_n([\tilde{x}])$]

$\forall \mathcal{M}^n \in \mathcal{K}^n_S \, \forall \tilde{x} \, (\mathcal{M}^n \models F[\tilde{y}] \Rightarrow \mathcal{M}^n \models G[\tilde{x}]) \iff$

[by induction hypothesis]

$\forall \mathcal{M}^n \in \mathcal{K}^n_S \, \forall \tilde{x} \, (\mathcal{M}^n \models \Diamond_1, \ldots, \Diamond_n F[\tilde{x}] \Rightarrow \mathcal{M}^n \models G[\tilde{x}]) \iff$

$\forall \mathcal{M}^n \in \mathcal{K}^n_S \, \forall \tilde{x} \, (\mathcal{M}^n \models \Diamond_{f(F)} F[\tilde{x}] \Rightarrow \mathcal{M}^n \models G[\tilde{x}]) \iff$

[by lemma 4]

$\forall \mathcal{M}^n \in \mathcal{K}^n_S \, \forall \tilde{x} : \mathcal{M}^n \models F \rightarrow_d G[\tilde{x}] \iff$

$\mathcal{K}^n_S \models F \rightarrow_d G.$ ■

Now we can easily prove the following

THEOREM 7 $\mathcal{JD}_S(\emptyset) \cap FOR_d = \mathcal{D}_S(\emptyset)$.

What is more, for paraconsistent causal systems designated by basic systems, we again obtain a representation theorem similar to theorem 2 and thereupon a formulae-testing method for these systems.

5. Generalized Jaśkowski-style systems

5.1. Basic definitions

There was one more causal system created by JAŚKOWSKI, namely Q^*. We shall first reformulate JAŚKOWSKI's original definition (see [5]) in multi-modal terminology and subsequently try to generalize his construction. The language of Q^* arises from L_f by adding two further two-argument non-classical connectives $\Box_e$ and $\Box_d$. In order to interpret FOR^* in points of n-dimensional models we have to explain the interpretations of those additional connectives.

DEFINITION 24 Let k = 1,..., n.
$c^n(k, H) =_{df} \Box_k \ldots \Box_n H \vee \Box_k \ldots \Box_n \neg H.$

$c^n(k, H)$ is to be read as "the truth of H does not depend on $k, \ldots, n$". The signs $c^n(0, H)$ and $c^n(n+1, H)$ are treated as empty.

DEFINITION 25 Let k = 0,..., n.
$e^n(k, H) =_{df} \neg c^n(k, H) \wedge c^n(k+1, H).$

$e^n(k, H)$ can be read as "the truth of H doesn't depend on $k+1, \ldots, n$, it depends however on $k, \ldots, n$", or "k is the *efficient factor* of H". The set $k, k+1, \ldots, n$ is called the *definitive set* of H.

The following definitions will establish the meaning of the concepts "*efficient factor* of H *in the point* $\tilde{x}$ *of the model* $\mathcal{M}$" [$e(H, \mathcal{M}^n, \tilde{x})$, for short] and "*definitive set* of H *in the point* $\tilde{x}$ *of the model* $\mathcal{M}$" [$d(H, \mathcal{M}^n, \tilde{x})$]:
Let $k \leq n, \kappa = \{k, \ldots, n\}$:
$e(H, \mathcal{M}^n, \tilde{x}) = k$ *iff* $\mathcal{M}^n \models e^n(k, H)[\tilde{x}]$
as well as
$d(H, \mathcal{M}^n, \tilde{x}) = \kappa$ *iff* $\mathcal{M}^n \models e^n(k, H)[\tilde{x}]$.

As in the case of JAŚKOWSKI-style systems we confine ourself to reflexive structures only.

DEFINITION 26 Let n be any natural number.
$\Box^n_{e(H)} G =_{df} \bigwedge_{0 \leq k \leq n} e^n(k, H) \rightarrow \Box_k G$
and
$\mathcal{M}^n \models \Box_{e(H)} G[\tilde{x}]$ *iff* $\mathcal{M}^n \models \Box^n_{e(H)} G[\tilde{x}]$.

The above definition is in good accordance with JAŚKOWSKI's intuitions: $\Box_{e(H)} G$ expresses the truth of G for all values of H's efficient factor — if $e^n(k, H)$ holds in a situation $\tilde{x}$, then G is true in all situations differing from $\tilde{x}$ only by the value of their k^{th} component, being the efficient factor of H under those circumstances. Definition 26 meets JAŚKOWSKI's original definition, i.e.

$$\Box^n_{e(H)} G =_{df} \bigvee_{1 \leq k \leq n} \Box_k G \wedge e^n(k, H):$$

This can be shown by proving the following lemma.

LEMMA 6 *Let $\mathcal{F}^n$ be a n–dimensional reflexive f-o-frame, i.e. $\forall i \leq n : \tilde{x} \in \mathcal{W}\backslash \mathcal{Q}_i \Longrightarrow \tilde{x}\mathcal{R}_i\tilde{x}$.*
Then

$$\bigvee_{1\leq k\leq n} \Box_k G \wedge e^n(k, H) \equiv \bigwedge_{0\leq k\leq n} e^n(k, H) \rightarrow \Box_k G.$$

Thereby, the efficient factor of H may be understood as the factor finally determining the value of H under the given circumstances. JAŚKOWSKI pointed out that the efficient factor of a real event is its cause in the sense of ROMAN INGARDEN (cf. [5]): INGARDEN claimed that the cause is

> [...] not the entire sufficient condition of its effect, but merely an event being the final factor completing the already existing circumstances into the entire, active, complete condition of its effect. ([3], p. 76)

INGARDEN's concept of a cause is not subject to the construction of causal connectives in JAŚKOWSKI-style systems. It is not even expressible in the language L^* — it does not fit into any syntactical category. Nevertheless, JAŚKOWSKI's remark is interesting; at least with respect to the subsequent construction of generalized JAŚKOWSKI-style systems. We will show that multi-dimensional KRIPKE-structures provide an adequate semantic framework for Q^*. On the other hand, those structures do not contain any temporal elements. Hence it comes out that INGARDEN's concept of a cause can be formalized by means of a semantic framework without any temporal relations. Therefore the questions arises, whether the assumption of temporal orderings in the course of events is essential in INGARDEN's definition, or whether the temporal ordering could be left out. Keeping in mind INGARDEN's enormous difficulties with the problems of temporal succession in singular cause-effect-relations (cf.[3], pp. 44 ff.) such a modification would be highly desirable.[6]

Before proceeding with an outline of the completeness of Q^* w.r.t. appropriate classes of multidimensional KRIPKE-frames, we have to complete the remaining cases of interpretation. $\Box_{d(H)}G$ means that G is true for all values of H's definitive set. Our notation transforms JAŚKOWSKI's original formalization into

$$\Box^n_{d(H)}G =_{df} \Box_n G \wedge \bigwedge_{1\leq k\leq n-1} c^n(k+1, H) \rightarrow \Box_k \ldots \Box_n G.$$

[6] JAŚKOWSKI's formalization, trying to be adequate, achieves thereby a position from which can be asked relevant and possibly important questions concerning the original philosophical issue — a nice example of how logic and philosophy can interact.

According to PIECZKOWSKI, the meaning of the symbol defined above is entirely clear: G is true for all values of the components belonging to the definitive set of H ([12], p. 172). One can hardly agree with his claim. The connection between JAŚKOWSKI's intuitions on the one hand, and the above definition on the other, seems somewhat fuzzy. Even if one (assumed a printing mistake and therefore) substituted $c^n(k+1,H)$ by $e^n(k+1,H)$ — or even by $c^n(k,H)$ — one would not obtain any better accordance.[7] It seems however that the following definition fits in with JAŚKOWSKI's intuitions:

DEFINITION 27 Let H, G and n be as before.

$$\Box^n_{d(H)} G =_{df} \bigwedge_{1\leq k\leq n} e^n(k,H) \to \Box_k \ldots \Box_n G.$$

In reflexive n–dimensional structures $\Box^n_{d(H)}G$ is weaker than $\boxdot^n_{d(H)}G$. The remaining acceptance condition can now be stated as follows:

DEFINITION 28 $\mathcal{M}^n \models \Box_{d(H)} G[\tilde{x}]$ *iff* $\mathcal{M}^n \models \Box^n_{d(H)} G[\tilde{x}]$.

Definitions 7, 10, 26 and 28 altogether establish the acceptance of a formula of L^* in a point of a n–dimensional model.

In any class of f-o–frames $\Box H \to H$ is a tautology and hence belongs to the modal system designated by this class. On the other hand, any class of f-o–frames adequate for a regular modal system containing $\Box H \to H$ is reflexive. Therefore we will label each regular system containing $\Box H \to H$ "Regular system" (with capital R).

The acceptance relation $\models$ defined in $\mathcal{K}^n_S$, for $n \in \omega$ establishes a consequence operation in FOR^*.

DEFINITION 29 Let S be a Regular modal system.

$$\Im^n_S(X) =_{df} \{H \in FOR^* : \mathcal{K}^n_S \models X \Longrightarrow \mathcal{K}^n_S \models H\}$$

and

$$\Im_S(X) =_{df} \bigcap_{n\in\omega} \Im^n_S(X).$$

A transformation of FOR^* into $FOR_{m(n)}$ which treats JAŚKOWSKI's connective $\Box_d$ as the n–modal connective $\boxdot^n_d$ leads to the identity between JAŚKOWSKI's original calculus Q^* and the system JAŚKOWSKI's $\Im_{S5}(\emptyset)$ de-

[7] JAŚKOWSKI had elaborated this topic in the late thirties. Almost all of his manuscripts burnt in the Warsaw Uprising in 1944. After the war he reproduced only a part of them. Although fragments of his scientific inheritance are preserved in the Mathematical Institute of Toruń University, I was not able to find any further comments concerning the definition. So perhaps this matter will remain in the dark.

fined above. The proof of this fact runs analogously to the proof of Theorem 1.

Yet interpreting $\Box_d$ according to definition 27 we lose this identity. One can show that

$$\Box_{d(H\vee\neg H)}G \rightarrow \Box_{f(G)}G \in \Im_{S5}(\emptyset) \setminus Q^*.$$

Hence $\Im_S$ is not a straightforward generalization of JAŚKOWSKI's calculus Q^*. Despite of this fact JAŚKOWSKI's intuitions concerning $\Box_{d(H)}G$ justify

DEFINITION 30 We call $\Im_S(\emptyset)$ the *general* JAŚKOWSKI-*style system designated by* S.

5.2. Some more connectives

Definition 30 seems justified because the underlying construction obviously generalizes JAŚKOWSKI's original attempt and because, in spite of the fact that $\Im_{S5}(\emptyset) \neq Q^*$, the system $\Im_{S5}(\emptyset)$ fits JAŚKOWSKI's intuitions better than his own calculus Q^*.

JAŚKOWSKI's aim in creating Q^* was to achieve more adequate counterparts of causal relations than were obtainable in the Q_f-system. All the constructions of what he called "causal connectives" can be restated in any general JAŚKOWSKI-style system.

The sets of $\Im_{S5}$–theorems vary for different designating Regular modal systems. Therefore we obtain slightly varying properties of a given connective in different general JAŚKOWSKI-style systems, or — if one likes to say so — whole classes of connectives are obtained for each single connective defined in Q^*. Having once defined a causal connective as closely as possible to our intuitions, allows finetuning of its properties by choosing the appropriate designating system. Hence, by generalizing JAŚKOWSKI's construction, we accomplish a large number of potential formal counterparts of causal relations.

All definitions of formal connectives stated in JAŚKOWSKI-style systems can be reformulated in general JAŚKOWSKI-style systems. Given any basic system all properties of the L_f–connectives remain unchanged when passing from $\mathcal{J}_S$ to $\Im_S$. However, the enlarged language L^* offers new possibilities for defining precausal connectives.

First of all, in perfect analogy to definition 15, one may explain two further non-classical implications connected with $\Box_e$ and $\Box_d$ respectively (cf. lemma 2).

LEMMA 7 *Let S be any basic system.*
$\Box_{i(H)}G \equiv (H \rightarrow_i G) \wedge (\neg H \rightarrow_i G) \in \Im_S$, *for* $i \in \{f, e, d, \}$.

Hence it would be possible to work within an equivalent language, which would be generated by adding $\rightarrow_f$, $\rightarrow_e$, and $\rightarrow_d$ to classical propositional language.

Here are some of JAŚKOWSKI's examples of causal-logical connectives (cf. [5], pp. 89–90):

$H \leftarrow_i G$	$=_{df} \neg H \rightarrow_i \neg G$,	for $i \in \{f, e, d\}$;
$H \equiv_i F$	$=_{df} (H \rightarrow_i G) \wedge (H \leftarrow_i G)$,	for $i \in \{f, e, d\}$;
$H \wedge_i G$	$=_{df} H \wedge G \wedge \Diamond_{i(H)}(\neg H \wedge \neg F)$,	for $i \in \{f, e, d\}$;
$H \rightarrow_d \wedge_d F$	$=_{df} (H \rightarrow_d G) \wedge (H \wedge_d G)$;	
$H \leftarrow_e \wedge_e G$	$=_{df} (H \leftarrow_e G) \wedge (H \wedge_e G)$.	

According to JAŚKOWSKI, the last connective fits as the formal counterpart to the contemporary notion of causality in jurisprudence.

5.3. A serious technical problem

The box-operator $\Box$, defined as $\Box H =_{df} \Box_{f(H)}H$, has in JAŚKOWSKI-style systems the intuitive meaning of the usual necessity-operator: $\Box H$ can be read as "H is necessary" (cf. [17]).

The connectives $\Box_e$ and $\Box_d$ do not have the same "definitorial power":

$$\Box_{e(H)}H \not\equiv \Box_{f(H)}H \not\equiv \Box_{d(H)}H, \text{ in all } \Im_S.$$

This fact has far reaching consequences. It can be shown that in general JAŚKOWSKI-style systems the representation theorem (theorem 2) has no analogue. In fact, just the addition of even one of the connectives $\Box_e$, $\Box_d$, or $\Box_d$ to L_f results in this effect.

That results in enormous problems in practical theorem-testing within general JAŚKOWSKI-style systems. Under these circumstances we have the following possibilities:

1) to find a modified concept of basis-system, such that there is an analogue to the representation theorem provable for the resulting class of JAŚKOWSKI-style systems.
2) to search for a recursively computable function r^*, such that

$$\forall H \in FOR^* : r^*(H) > r(H) \text{ and } H \in \Im_S(\emptyset) \Longleftrightarrow \mathcal{K}_S^{r^*(H)} \models H.$$

However, both possibilities seem to be somewhat problematic. It comes out that each appropriate modification of the concept of basis-system must exclude at least all modal logics containing *S4*. The class of remaining systems would diminish regrettably. If there is, on the other hand, any natural number r^* with the required properties, then the computational effort in testing the validity of formulae would increase considerably.

Be that as it may, as long as there is no method for theorem-proving available, generalized JAŚKOWSKI-style systems are useless for causal-logical purposes (whereas they may well be — and in fact are — interesting for purely logical investigation).

6. No Further Outlook

At the present state of our work one should avoid further outlooks: the account contains obviously many more open problems then solved ones. But even when the logical work is done, almost all interesting problems remain open. Logic can make propositions — the choice of a specific causal connective as the formalization of the considered kind of causal nexus is not the business of pure logic alone.

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