SHARP INEQUALITIES FOR THE NORMS OF CONJUGATE FUNCTIONS AND THEIR APPLICATIONS

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1. Let C and Lp, $1\leqslant p\leqslant\infty$, be spaces of 2π-periodic functions $f:\mathbb{R}\to\mathbb{R}$ with the corresponding norms $\|\cdot\|_C$ and $\|\cdot\|_{L_p}=\|\cdot\|_p$. We denote by L_p^r, $r=1,2,\ldots;\ 1\leqslant p\leqslant\infty$, the space of functions $f\in C$, having a locally absolutely continuous derivative $f^{(r-1)}$ $(f^{(0)}=f)$ and such that $f^{(r)}\in L_p$. As usually, let $W_p^r=\{f\in L_p^r:\|f^{(r)}\|_p\leqslant 1\}$. Further, if $\omega(f, t)$ is the modulus of continuity of a function $f\in C$, and $\omega(t)$ is a given modulus of continuity, then by W^rH^ω, r = 1, 2,..., we denote the class of functions $f\in C$ such that $f^{(r)}\in C$ and $\omega(f^{(r)},t)\leqslant\omega(t)$ for every t > 0.

For a function $f\in L_1$ we denote by $\tilde f$ the function that is conjugate to f (see, for example, [1]) and we set $\tilde W_p^r=\{\tilde f:f\in W_p^r\}$, $\widetilde{W^rH^\omega}=\{\tilde f:f\in W^rH^\omega\}$.

Presently, one knows the exact solution of several extremal problems of the theory of approximations on the classes L_p^r, W_p^r, and W^rH^ω (see, for example, [2-4]). Substantially less sharp results are known for the classes of conjugate functions [5-7]. In this paper, with the aid of some known results for usual classes of functions and with the aid of the theorem of Stein and Weiss ([8], see also [9], Theorem 1.10), we have obtained a series of new results regarding the exact solution of extremal problems on classes of conjugate functions.

We introduce the following notations. If $f\in L_1$ and $f\geqslant 0$ almost everywhere, then P(f, t) is the decreasing rearrangement (see, for example, [3, pp. 92, 93]) of the restriction of f to the period. For any function $f\in L_1$ we set $\Pi(f,t)=P(f_+,t)-P(f_-,2\pi-t)$, where $f_\pm(t)=\max\{\pm f(t),0\}$. If $f,g\in L_1$, $f,g\geqslant 0$ almost everywhere and if for every $x\in[0,2\pi]$ we have

$$\int_0^x P(g,t)\,dt\leqslant\int_0^x P(f,t)\,dt,$$

then we shall write $g\prec f$.

We denote by $\varphi_{n,r}$, n, r = 1, 2,..., the r-th periodic integral with zero mean value on a period of the function $\varphi_{n,0}(t)=\operatorname{sign}\cos nt$. Instead of $\varphi_{1,r}$ we shall write φ_r. More generally, if α, β are positive numbers, then by $\varphi_{n,r;\alpha,\beta}$ we denote the r-th periodic integral with zero mean value on a period of the function $\varphi_{n,0;\alpha,\beta}(t)=\alpha\operatorname{sign}\left(\cos nt-\cos\frac{\pi\beta}{\alpha+\beta}\right)_+-\beta\operatorname{sign}\left(\cos nt-\cos\times\frac{\pi\beta}{\alpha+\beta}\right)_-$. Instead of $\varphi_{1,r;\alpha,\beta}$ we shall write $\varphi_{r;\alpha,\beta}$.

Let $\omega(t)$ be an upwardly convex modulus of continuity. We denote by $f_{n,r}(t)=f_{n,r}(\omega;t)$ the r-th period integral with zero mean value on a period of the odd $2\pi/n$-period function $f_{n,0}(t)$ such that $f_{n,0}(t)=2^{-1}\omega(2t)$ for $t\in[0,\pi/2n]$ and $f_{n,0}(t)=2^{-1}\omega(2(\pi/n-t))$ for $t\in[\pi/2n,\pi/n]$.

2. The following statement is fundamental in this paper.

<u>THEOREM 1.</u> Assume that the functions $f,g\in C$ are such that $\tilde f,\tilde g\in C$; $f\neq 0$ and $\tilde f\neq 0$ almost everywhere, $\int_0^{2\pi}\operatorname{sign}\tilde f(t)\,dt=0$, and

$$\|\tilde f\|_1=\int_0^{2\pi}P(|f|,t)\,P(|(\operatorname{sign}\tilde f)^\sim|,t)\,dt. \tag{1}$$

In this case if $|g|\prec|f|$, then $\|\tilde g\|_1\leqslant\|\tilde f\|_1$.

<u>Remark 1.</u> The conditions imposed in Theorem 1 on the function f are satisfied for n, r = 1, 2,... by the above defined functions $\varphi_{n,r}$ and $f_{n,r}(\omega)$. This circumstance and the presence in many cases of inequalities of the type $|g|\prec|f|$ stipulates a large collection of applications of Theorem 1.

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Proof. Assume first that $\tilde{g} \neq 0$ almost everywhere. We have

$$\|\tilde{g}\|_1 = \int_0^{2\pi} \tilde{g}(t)\,\text{sign}\,\tilde{g}(t)\,dt = \left|\int_0^{2\pi} g(t)(\text{sign}\,\tilde{g}(t))^{\sim} dt\right| \leqslant \int_0^{2\pi} P(|g|, t)\,P(|(\text{sign}\,\tilde{g})^{\sim}|, t)\,dt.$$

Taking into account condition $|g| \prec |f|$ of the theorem and Proposition 5.4.7 from [3], we obtain

$$\|\tilde{g}\|_1 \leqslant \int_0^{2\pi} P(|f|, t)\,P(|(\text{sign}\,\tilde{g})^{\sim}|, t)\,dt.$$

From the theorem of Stein and Weiss ([8]; see also [9], Theorem 1.10), taking into account the conditions $\int_0^{2\pi} \text{sign}\,\tilde{f}(t)\,dt = 0$ and $\tilde{f} \neq 0$ almost everywhere, there follows easily that P × $(|(\text{sign}\,\tilde{g})^{\sim}|, t) \leqslant P(|(\text{sign}\,\tilde{f})^{\sim}|, t)$ for any $t \in [0, 2\pi]$. Therefore, taking into account also condition (1), we find

$$\|\tilde{g}\|_1 \leqslant \int_0^{2\pi} P(|f|, t)\,P(|(\text{sign}\,\tilde{f})^{\sim}|, t)\,dt = \|\tilde{f}\|_1,$$

and, in the given case, the theorem is proved.

Assume now that $\tilde{g} \not\equiv 0$ but $\tilde{g} = 0$ on a set of positive measure. We note that the operator of taking the conjugate function acts continuously from C into L_1 so that there exists $K_1 > 0$ such that if $g, h \in C$ and $\|g - h\|_C < \varepsilon$, then $\|\tilde{g} - \tilde{h}\|_1 \leqslant K_1\varepsilon$ for any $\varepsilon > 0$. Further, since $\int_0^x$ P × $(|f|, t)\,dt$ is a nondecreasing, upwardly convex, positive function, there exists $K_2 > 0$ such that $x \leqslant K_2 \int_0^x P(|f|, t)\,dt$ for any $x \in [0, 2\pi]$.

We consider an arbitrary $\varepsilon > 0$ and we find a nonconstant trigonometric polynomial T such that $\|g - T\|_C < \varepsilon$. Then $\tilde{T} \neq 0$ almost everywhere, $\|\tilde{g} - \tilde{T}\|_1 \leqslant K_1\varepsilon$ and for all $x \in [0, 2\pi]$ we have

$$\int_0^x P(|T|, t)\,dt \leqslant \int_0^x P(|g|, t)\,dt + \int_0^x P(|g - T|, t)\,dt \leqslant \int_0^x P(|f|, t\ dt + \varepsilon x$$
$$\leqslant \int_0^x P(|f|, t)\,dt + K_2\varepsilon \int_0^x P(|f|, t)\,dt \leqslant \int_0^x P((1 + K_2\varepsilon)|f|, t)\,dt.$$

Consequently, according to what has been proved, $\|\tilde{T}\|_1 \leqslant (1 + K_2\varepsilon)\|\tilde{f}\|_1$. But then $\|\tilde{g}\|_1 \leqslant \|\tilde{T}\|_1 + \|\tilde{g} - \tilde{T}\|_1 \leqslant K_1\varepsilon + (1 + K_2\varepsilon)\|\tilde{f}\|_1$ and, by virtue of the arbitrariness of $\varepsilon > 0$, we have $\|\tilde{g}\|_1 \leqslant \|\tilde{f}\|_1$. The theorem is proved.

We give an other statement of the type of Theorem 1.

THEOREM 2. Assume that the functions $f, g \in C$ are such that $\tilde{f}, \tilde{g} \in C$; $f \neq 0$ and $\tilde{f} \neq 0$ almost everywhere, $\int_0^{2\pi} \text{sign}\,\tilde{f}(t)\,dt = 0$, and

$$\|\tilde{f}\|_1 = \int_0^{2\pi} \Pi(f, t)\,\Pi((\text{sign}\,\tilde{f})^{\sim}, t)\,dt. \tag{2}$$

In this case, if $(g - \lambda)_\pm \prec (f - \lambda)_\pm$ for any $\lambda \in \mathbb{R}$, then $\|\tilde{g}\|_1 \leqslant \|\tilde{f}\|_1$.

Remark 2. The conditions imposed in Theorem 2 on the function f are satisfied for all n = 1, 2,... and r = 2, 4,... by the function $\varphi_{n,r;\alpha,\beta}$, $\alpha, \beta > 0$.

Proof. As in the proof of Theorem 1, we can assume that $\tilde{g} \neq 0$ almost everywhere. We have

$$\|\tilde{g}\|_1 = \int_0^{2\pi} \tilde{g}(t)\,\text{sign}\,\tilde{g}(t)\,dt = -\int_0^{2\pi} g(t)(\text{sign}\,\tilde{g}(t))^{\sim} dt. \tag{3}$$

With the aid of the known properties of rearrangements, it is easy to prove that for any $F \in L_1$ and any function $f \in L_1$, with zero mean value on the period, we have $\int_0^{2\pi} f(t)F(t)\,dt \leqslant \int_0^{2\pi} \Pi(f, t)\,\Pi(F, t)\,dt$

Taking also into account that from the already mentioned theorem of Stein and Weiss there follows the equality $\Pi(-(\operatorname{sign} g)^{\sim}, t) = \Pi((\operatorname{sign} g)^{\sim}, t)$, from (3) we obtain

$$\|\tilde{g}\|_1 \leqslant \int_0^{2\pi} \Pi(g, t)\, \Pi((\operatorname{sign} \tilde{g})^{\sim}, t)\, dt. \tag{4}$$

Again from the theorem of Stein and Weiss there follows, taking into account condition $\int_0^{2\pi} \operatorname{sign} \times \tilde{f}(t)dt = 0$, that for any $\lambda \in \mathbb{R}$ we have $((\operatorname{sign} \tilde{g})^{\sim} - \lambda)_{\pm} \prec ((\operatorname{sign} \tilde{f})^{\sim} - \lambda)_{\pm}$. Taking into condition $(g-\lambda)_{\pm} \prec (f-\lambda)_{\pm}$ of the theorem and also Proposition 5.4.7 from [3], we can prove the inequality

$$\int_0^{2\pi} \Pi(g, t)\, \Pi((\operatorname{sign} \tilde{g})^{\sim}, t)\, dt \leqslant \int_0^{2\pi} \Pi(f, t)\, \Pi((\operatorname{sign} \tilde{f})^{\sim}, t)\, dt. \tag{5}$$

Combining (4), (5), and (2), we conclude the proof.

3. We proceed to applications. Inequalities for the norms of the derivatives of the type of the Kolmogorov inequality (see [3, Sec. 6.2]) are well known and play an important role in the theory of approximations. We prove sharp inequalities for the norm of the derivatives of conjugate function.

THEOREM 3. Let r = 2, 3,... and let $f \in L_\infty^r$. Then for k = 1,...,r − 1 we have

$$\|\tilde{f}^{(k)}\|_1 \leqslant \frac{\|\tilde{\varphi}_{r-k}\|_1}{\|\varphi_r\|_\infty^{1-k/r}} \|f\|_\infty^{1-k/r} \|f^{(r)}\|_\infty^{k/r}. \tag{6}$$

Inequality (6) is sharp on L_∞^r. It turns into an equality for the fucntions $f = a\varphi_{n,r}$, $a \in \mathbb{R}$, n = 1, 2,... .

Proof. Let $f \in L_\infty^r$, $f \not\equiv \text{const}$, and let $\|f^{(r)}\|_\infty \leqslant 1$. We select b > 0 so that $\|f\|_\infty = b^r \|\varphi_r\|_\infty$, i.e., $b = (\|f\|_\infty / \|\varphi_r\|_\infty)^{1/r}$. Then, by virtue of theorem of Korneichuk and Ligun (see [3], Theorem 5.5.1), for any k = 1,...,r − 1 we have $|f^{(k)}| \prec b^{r-k} |\varphi_{r-k}|$. From here, taking into account Remark 1 and Theorem 1, we derive $\|\tilde{f}^{(k)}\|_1 \leqslant b^{r-k} \|\tilde{\varphi}_{r-k}\|_1$, or

$$\|\tilde{f}^{(k)}\|_1 \leqslant \frac{\|\tilde{\varphi}_{r-k}\|_1}{\|\varphi_r\|_\infty^{1-k/r}} \|f\|_\infty^{1-k/r}. \tag{7}$$

Applying for every $f \in L_\infty^r$ the inequality (7) to the function $f/\|f^{(r)}\|_\infty$, we obtain (6). The theorem is proved.

COROLLARY 1. If $f \in L_\infty^r$, r = 2, 3,..., and k = 1,...,r − 1, then

$$\bigvee_0^{2\pi} (\tilde{f}^{(k-1)}) \leqslant \frac{\bigvee_0^{2\pi} (\tilde{\varphi}_{r-k+1})}{\|\varphi_r\|_\infty^{1-k/r}} \|f\|_\infty^{1-k/r} \|f^{(r)}\|_\infty^{k/r}.$$

The inequality is sharp on L_∞^r.

Now we prove the asymmetric analogue of Theorem 3 (regarding the known "asymmetric" inequalities for the norms of derivatives, see [3, Sec. 6.2], [10, 11]). Let $E(f) = \inf_{\lambda \in \mathbb{R}} \|f - \lambda\|_\infty$ and $\|f\|_{\infty;\alpha,\beta} = \|\alpha f_+ + \beta f_-\|_\infty$, where α, β are positive numbers.

THEOREM 4. Let $f \in L_\infty^r$, r = 1, 2,...; k = 1, 2,...,r − 1; k ≡ r (mod 2); let α, β be positive numbers. Then

$$\|\tilde{f}^{(k)}\|_1 \leqslant \frac{\|\tilde{\varphi}_{r-k;\alpha,\beta}\|_1}{E(\varphi_{r;\alpha,\beta})^{1-k/r}} E(f)^{1-k/r} \|f^{(r)}\|_{\infty;\alpha^{-1},\beta^{-1}}^{k/r}. \tag{8}$$

Inequality (8) is sharp on L_∞^r.

For α = β = 1 we obtain inequality (6). Let $B_r(x) = 2\sum_{k=1}^{\infty} k^{-r} \cos(kx - \pi r/2)$ be the Bernoulli functions. Taking into account that $E(\varphi_{r;1,\beta}) \to E(B_r)$ and $\|\tilde{\varphi}_{r-k;1,\beta}\|_1 \to \|\tilde{B}_{r-k}\|_1$ for $\beta \to \infty$; $E(\varphi_{r;\alpha,1}) \to E(B_r)$ and $\|\tilde{\varphi}_{r-k;\alpha,1}\|_1 \to \|\tilde{B}_{r-k}\|_1$ for $\alpha \to \infty$, we obtain the following corollary.

COROLLARY 2. Under the assumptions of Theorem 4 we have the sharp inequality

$$\|\tilde{f}^{(k)}\|_1 \leqslant \frac{\|\tilde{B}_{r-k}\|_1}{E(B_r)^{1-k/r}} E(f)^{1-k/r} \|(f^{(r)})_\pm\|_\infty^{k/r}.$$

Proof of Theorem 4. Let $f \in L_\infty^r$ and $\|f^{(r)}\|_{\infty;\alpha^{-1},\beta^{-1}} \leqslant 1$. We select b > 0 from the condition $E(f) = b^r E(\varphi_{r;\alpha,\beta})$. Then, in view of Theorem 1 from [7], for every $\lambda \in \mathbb{R}$ we have $(f^{(k)} - \lambda)_\pm \prec b^{r-k} \times (\varphi_{r-k;\alpha,\beta} - \lambda)_\pm$. Taking into account Remark 2 and Theorem 2, we obtain for k ≡ r (mod 2)

$$\|\tilde{f}^{(k)}\|_1 \leqslant b^{r-k} \|\tilde{\varphi}_{r-k;\alpha,\beta}\|_1 = \frac{\|\tilde{\varphi}_{r-k;\alpha,\beta}\|_1}{E(\varphi_{r;\alpha,\beta})^{1-k/r}} E(f)^{1-k/r}. \tag{9}$$

If $f \in L_\infty^r$ is arbitrary, then applying (9) to the function $f/\|f^{(r)}\|_{\infty;\alpha^{-1},\beta^{-1}}$, we obtain inequality (8). Equality in (8) is attained for the functions $f = a\varphi_{n,r;\alpha,\beta}$, $n = 1, 2, \ldots;\ a > 0$.

The applications of the inequalities of Kolmogorov type include (see [3], Chap. 6) results regarding the best approximation of one class of functions by another, inequalities for the upper bounds of seminorms, etc. Similar applications have the inequalities from Theorems 3 and 4. We give only an estimate of the approximation in the space L_1 of the class $\tilde{W}_\infty^k$ by the class NW_1^r, N > 0, whose proof can be carried out similarly to the proof of Theorem 6.1 from [12].

THEOREM 5. Let r = 2, 3,...; k = 1,...,r − 1; N > 0. Then

$$\sup_{f \in \tilde{W}_\infty^k} \inf_{g \in NW_1^r} \|f - g\|_1 \leqslant \frac{r-k}{r} \|\tilde{\varphi}_k\|_1^{r/(r-k)} \left(\frac{k}{rN\|\varphi_r\|_\infty}\right)^{k/(r-k)}.$$

4. We denote by $H_{2n,r}^S$, n = 1, 2,...; r = 0, 1,..., the set of 2π-periodic polynomial splines of order r and defect 1 with nodes at the points k = 0, ±1, ±2,... .

It is known ([4], Theorem 2.6.7) that for the splines $s \in H_{2n,r}^S$ we have the inequality

$$|s^{(k)}| \prec \frac{\|s\|_\infty}{\|\varphi_{n,r}\|_\infty} |\varphi_{n,r}^{(k)}|, \quad k = 1, \ldots, r.$$

Taking into account Remark 1, we can see that Theorem 1 can be applied to the functions $s^{(k)} \times (x)$ and $\frac{\|s\|_\infty}{\|\varphi_{n,r}\|_\infty} |\varphi_{n,r}^{(k)}(x)|$. Therefore, the following theorem holds.

THEOREM 6. Let $s \in H_{2n,r}^S$, r = 1, 2,... . Then

$$\|\tilde{s}^{(k)}\|_1 \leqslant \frac{\|\tilde{\varphi}_{n,r}^{(k)}\|_1}{\|\varphi_{n,r}\|_\infty} \|s\|_\infty, \quad k = 1, \ldots, r-1.$$

The inequality is sharp.

Theorem 1 has other applications of a similar type. Regarding the known inequalities for the derivatives of conjugate trigonometric polynomials, see [13, 14].

5. Let $\sigma_{2n,r}(f, t)$ be a spline from $H_{2n,r}^S$, interpolating $f \in C$ at the zeros of the function $\varphi_{n,r+1}(t + \pi/2n)$. It is well known (see, for example, [4], Theorem 5.1.2) that if $f \in W_\infty^{r+1}$, then $|f(t) - \sigma_{2n,r}(f, t)| \leqslant |\varphi_{n,r+1}(t + \pi/2n)|$ at each point t. From here there follows

$$|f - \sigma_{2n,r}(f)| \prec |\varphi_{n,r+1}|, \quad r = 1, 2, \ldots. \tag{10}$$

Further, from Lemma 5.1.18 [4] there follows easily that

$$|f' - \sigma'_{2n,r}(f)| \prec |\varphi_{n,r}|, \quad r = 1, 2, \ldots. \tag{11}$$

Taking into account the relations (10), (11) and Remark 1, we can see that to the pairs of functions $f - \sigma_{2n,r}(f)$, $\varphi_{n,r+1}$ and $f' - \sigma'_{2n,r}(f)$, $\varphi_{n,r}$ we can apply Theorem 1. Thus, the following theorem holds, similar to Theorem 5.1.7 from [4].

THEOREM 7. We have the equalities

$$\sup_{f \in W_\infty^{r+1}} \|\tilde{f} - \tilde{\sigma}_{2n,r}(f)\|_1 = \|\tilde{\varphi}_{n,r+1}\|_1, \quad r = 1, 2, \ldots;$$

$$\sup_{f\in W_\infty^{r+1}} \|\tilde{f}' - \tilde{\sigma}'_{2n,r}(f)\|_1 = \|\tilde{\varphi}_{n,r}\|_1, \quad r = 2, 3, \ldots.$$

The comparison of Theorem 7 and Theorem 5 from [7] shows that the set $\tilde{H}^S_{2n,r} = \{\tilde{s}: s \in H^S_{2n,r}\}$ has for the class $\tilde{W}^{r+1}_\infty$ the same approximation properties as the set of trigonometric polynomials of order $\leqslant n-1$ and the set $H^S_{2n,\mu}$, $\mu \geqslant r+1$. Moreover, in Theorem 7 we have a linear approximation method.

Assume now that $\sigma_{2n,1}(f) \in H^S_{2n,1}$ is an interpolational polygonal line for $f \in W^1H^\omega$.

THEOREM 8. If ω(f) is an upwardly convex modulus of continuity, then

$$\sup_{f\in W^1H^\omega} \|\tilde{f} - \tilde{\sigma}_{2n,1}(f)\|_1 = \|\tilde{f}_{n,1}(\omega)\|_1, \quad n = 1, 2, \ldots.$$

Indeed, from Lemma 5.2.8 of [4] there follows that $|f - \sigma_{2n,1}(f)| < |f_{n,1}(\omega)|$. Therefore (taking into account Remark 1), Theorem 1 can be applied to the pair of functions $f - \sigma_{2n,1}(f)$ and $f_{n,1}(\omega)$.

In conclusion we give one more inequality of the type of the Jackson inequality, unimprovable on the space C^1 of continuously differentiable 2π-periodic functions. This inequality can be also obtained with the aid of Lemma 5.2.8 from [4] and Theorem 1.

THEOREM 9. Let $f \in C^1$, n = 1, 2,.... Then

$$\|\tilde{f} - \tilde{\sigma}_{2n,1}(f)\|_1 \leqslant \frac{\|\tilde{\varphi}_1\|_1}{2n}\, \omega(f', \pi/n).$$

The inequality is sharp on C^1.

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