

Homoclinic Bifurcation at Resonant Eigenvalues

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We consider a bifurcation of homoclinic orbits, which is an analogue of period doubling in the limit of infinite period. This bifurcation can occur in generic two parameter vector fields when a homoclinic orbit is attached to a stationary point with resonant eigenvalues. The resonance condition requires the eigenvalues with positive/negative real part closest to zero to be real, simple, and equidistant to zero. Under an additional global twist condition, an exponentially flat bifurcation of double homoclinic orbits from the primary homoclinic branch is established rigorously. Moreover, associated period doublings of periodic orbits with almost infinite period are detected. If the global twist condition is violated, a resonant side switching occurs. This corresponds to an exponentially flat bifurcation of periodic saddle-node orbits from the homoclinic branch.

KEY WORDS: homoclinic orbit; period doubling; pathfollowing; global bifurcation; resonance.

AMS CLASSIFICATION: 34C15, 34C35, 58F14.

1. INTRODUCTION

By definition, a homoclinic orbit $z^*(t)$ of a flow

$$\dot{z} = F(z), \quad z \in \mathbb{R}^{m+n} \quad (1.1)$$

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tends to the same equilibrium, say $z=0$, $F(0)=0$, for both $t \rightarrow \pm\infty$. Homoclinic orbits are a key phenomenon for understanding more complicated dynamics, including chaotic motions; see, e.g., Guckenheimer and Holmes (1983), Sparrow (1982). For typical vector fields F , however, homoclinic orbits do not exist. The reason is the following. Suppose $z=0$ is a hyperbolic equilibrium, i.e., the eigenvalues of $F'(0)$ stay away from the imaginary axis. Then the m -dimensional stable manifold W^s of $z=0$ consists of all those $z_0=z(0)$ for which

$$\lim_{t \rightarrow +\infty} z^*(t) = 0$$

see, e.g., Chow and Hale (1982). Likewise, the n -dimensional unstable manifold W^u is associated to

$$\lim_{t \rightarrow -\infty} z^*(t) = 0$$

By definition, W^u and W^s intersect nontrivially (i.e., at some point $\mathbf{p}=z^*(0) \neq 0$) if $z^*(t)$ is a homoclinic orbit. Note that W^u and W^s then intersect all along the orbit $z^*(t)$. In particular, the tangent spaces $T_{\mathbf{p}}W^s$ and $T_{\mathbf{p}}W^u$ intersect nontrivially,

$$\dot{z}(t) \in T_{\mathbf{p}}W^s \cap T_{\mathbf{p}}W^u \quad \text{at } t=0 \quad (1.2)$$

In other words, W^u and W^s intersect nontransversely at \mathbf{p} , and

$$\text{codim}(T_{\mathbf{p}}W^s + T_{\mathbf{p}}W^u) = 1 \quad (1.3)$$

or, even worse, >1 . Perturbing the vector field F slightly, we can therefore push W^u slightly in the direction of that remaining codimension, keeping W^s fixed. Then W^u and W^s do not intersect anymore, locally, and the homoclinic orbit has disappeared. A precise statement of this idea is the Kupka–Smale theorem. This theorem asserts that for generic $F \in C^\infty$, stable and unstable manifolds intersect transversely, if they intersect at all. By (1.3), this excludes the possibility of homoclinic orbits. The word “generic” indicates that the theorem may fail (and it does!) only for a subset of $F \in C^\infty$, which is of the first Baire category in the weak Whitney topology. In particular, the assertion of the Kupka–Smale theorem holds for a dense set of F . See, e.g., Abraham and Robbin (1967) for more details. Note that this situation contrasts markedly with the case of discrete time flows, viz., of iterating diffeomorphisms, where transversè homoclinic points can occur—leading to Smale horseshoes and shift dynamics; see, e.g., Moser (1973).

We have seen above, heuristically, how homoclinic orbits are a

codimension 1 phenomenon in the space of vector fields. Therefore, we now consider C^∞ parametrized vector fields.

$$\dot{z} = F(\alpha, z), \quad z \in \mathbb{R}^{m+n} \tag{1.1_\alpha}$$

with parameter α . Below, we will consider parameters $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. But for the moment, let us discuss the case $\alpha \in \mathbb{R}$. By the above reasoning, homoclinic orbits will typically, i.e., generically, occur for isolated parameter values, say at $\alpha = 0$. What happens for small $\alpha \neq 0$? One possible scenario is the following. For $\alpha < 0$ we see a periodic orbit $z_\alpha(t)$ of minimal period T_α . For $\alpha \nearrow 0$, some part of the periodic orbits approaches the equilibrium $z = 0$, e.g.,

$$\lim_{\alpha \nearrow 0} z_\alpha(0) = 0$$

while the remaining points approach the closure of the homoclinic orbit of

$$\dot{z} = F(0, z)$$

Simultaneously, the minimal periods T_α tend to infinity. For $\alpha > 0$, both the periodic orbits and the homoclinic orbit just disappear. This was termed a “blue sky catastrophe” by Abraham and Marsden (1978). For $z \in \mathbb{R}^{m+n} = \mathbb{R}^2$, and under certain additional assumptions also for $m+n > 2$, Shilnikov has shown that this description holds true for generic one parameter families F . This work dates back to 1962; see, e.g., the survey (Shilnikov, 1968).

From now on we consider generic two parameter vector fields,

$$F(\alpha, \cdot), \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$$

Since homoclinic orbits are codimension one objects, we expect them to occur along one-dimensional curves $\alpha = \alpha(\tau)$ in two parameter space $\alpha \in \mathbb{R}^2$. This point of view makes homoclinic orbits amenable to a pathfollowing approach. We may now follow curves of homoclinic orbits rather than hitting them “catastrophically,” out of the blue.

The following illustrative example goes back to Arnold (1972) and Bogdanov (1976a,b), and to Takens (1974):

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \alpha_1 + \alpha_2 z_1 + z_1^2 + \sigma z_1 z_2 \end{aligned} \tag{1.4}$$

where $\sigma = \pm 1$. In Fig. 1 we show the local bifurcation diagram of system (1.4).

To the left of the fold line, two equilibria coexist. Crossing the fold to the right, they merge and disappear. Crossing the Hopf line, one of the two equilibria undergoes a Hopf bifurcation, giving rise to periodic orbits when

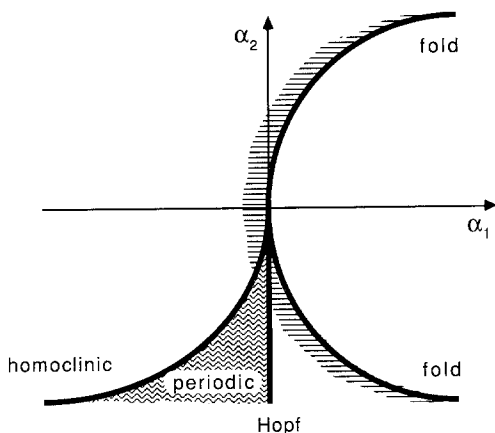


Fig. 1. A *B*-point.

we decrease α_1 . The periodic orbits terminate by a blue sky catastrophe at homoclinic orbits, which occur along a one-dimensional curve of parameters α . Following that homoclinic curve toward $\alpha=0$, the homoclinic orbits shrink down to the equilibrium $z=0$. More generally, Fig. 1 describes the generic local bifurcation diagram associated to an equilibrium (α_0, z_0) of (1.1_α) where $D_z F(\alpha_0, z_0)$ has an algebraically double eigenvalue 0. These results are due to Arnold (1972) and Bogdanov (1976a).

The term *B*-point was used by Fiedler (1986) for equilibria like (α_0, z_0) . An index for *B*-points seemed to indicate a possibility for some global pathfollowing results on homoclinic orbits; cf. Fiedler (1986, p. 74). Indeed, *B*-points give rise to branches of homoclinic orbits in generic two parameter flows just as Hopf bifurcations give rise to branches of periodic orbits in generic one parameter flows. It is then a necessary first step, toward global results on homoclinic orbits, to understand local bifurcations which a branch of homoclinic orbits can undergo on its way. We will resume this aspect at the end of the discussion in Section 7.

We now specify two such bifurcations: the *resonant side-switching* and the *resonant homoclinic doubling* (see Theorem A and Theorem B below). We fix some notations and some assumptions. First, we assume

$$\begin{aligned}
 F: \mathbb{R}^2 \times \mathbb{R}^{m+n} &\rightarrow \mathbb{R}^{m+n} \\
 (\alpha, z) &\mapsto F(\alpha, z)
 \end{aligned}
 \tag{1.5}$$

is a generic C^{M+5} vector field with two parameters $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $2 \leq M \leq \infty$

As before, genericity means that our results will hold for a residual subset of $C^{M+5}(\mathbb{R}^2 \times \mathbb{R}^{m+n}, \mathbb{R}^{m+n})$ in the weak Whitney topology of C^k -convergence, i.e., uniformly on compact sets and for bounded sets of $k \leq M + 5$.

Our second assumption concerns the existence of a homoclinic orbit Γ with real principal eigenvalues. We assume

the vector field $F(0, \cdot)$ at $\alpha = 0$ admits a homoclinic orbit

$$\Gamma = \{z^*(t): t \in \mathbb{R}\}$$

such that

$$\lim_{t \rightarrow \pm\infty} z^*(t) = 0$$

Furthermore, the linearization $D_x F(0, 0)$ at the equilibrium $z = 0$ has simple real eigenvalues $-\mu_0 < 0 < \nu_0$ such that any remaining eigenvalue μ of $D_z F(0, 0)$ satisfies

$$\text{either } \operatorname{Re} \mu < -\bar{\mu}_1 < -\mu_0 < 0$$

$$\text{or } \operatorname{Re} \mu > \bar{\nu}_1 > \nu_0 > 0$$

We call $-\mu_0, \nu_0$ the *principal eigenvalues* of $D_z F(0, 0)$, for the following reason. There exist submanifolds W^{ss} (“strong stable”) and W^{uu} (“strong unstable”) of the m -dimensional stable manifold W^s and the n -dimensional unstable manifold W^u of $z = 0$, respectively, such that

$$\begin{aligned} \lim_{t \rightarrow -\infty} e^{\bar{\nu}_1 |t|} z(t) = 0 & \quad \text{on } W^{uu}, & \quad \text{and} \\ \lim_{t \rightarrow +\infty} e^{\bar{\mu}_1 t} z(t) = 0 & \quad \text{on } W^{ss} \end{aligned} \tag{1.7}$$

It turns out that $\dim W^{ss} = m - 1$, $\dim W^{uu} = n - 1$. The tangent spaces at $z = 0$ are given by the parts of the spectrum of $D_z F(0, 0)$ with real part $< -\bar{\mu}_1$ and $> \bar{\nu}_1$, respectively. Moreover, for any $z(0) \in W^s \setminus W^{ss}$, the limit

$$\lim_{t \rightarrow +\infty} z(t)/|z(t)| \neq 0 \tag{1.8}$$

exists and is a unit eigenvector of the principal stable eigenvalue $-\mu_0$. An analogous statement holds for $z(0) \in W^u \setminus W^{uu}$ and the principle unstable eigenvalue ν_0 . For a reference see Hirsch *et al.* (1977), Shub (1987), and Brunovský and Fiedler (1986).

We now assume the resonance condition

$$\mu_0 = \nu_0 \tag{1.9}$$

for the principal eigenvalues $-\mu_0 < 0 < \nu_0$. We remember that we expect homoclinic orbits to occur along one-dimensional curves in parameter space. Therefore, resonance condition (1.9) will not contradict our genericity assumption (1.5). Of course, here we have shifted the associated equilibrium (α_0, z_0) where (1.9) holds to $(0, 0)$.

As a final prelude to our main results, we distinguish between twisted and nontwisted homoclinic orbits (cf. Figs. 2a and b, where $m = 1, n = 2$).

Let $\Gamma = \{z^*(t): t \in \mathbb{R}\}$ denote a homoclinic orbit of a vector field with simple real principal eigenvalues, as in (1.6). Choose points $\mathbf{p}, \mathbf{q} \in \Gamma$ sufficiently close to the equilibrium $z = 0$ associated to Γ , say $\mathbf{p} = z^*(0), \mathbf{q} = z^*(T)$. We assume nondegeneracy of Γ ,

$$\text{codim}(T_{\mathbf{p}}W^u + T_{\mathbf{p}}W^s) = 1 \tag{1.10a}$$

in accordance with (1.3). Motivated by the convergence result (1.8), we further assume the following general position of Γ :

$$\mathbf{p} \notin W^{ss}, \quad \mathbf{p} \notin W^{uu} \tag{1.10b}$$

Obviously, assumptions (1.10a,b) imply the same statements for \mathbf{q} . Define the two unit vectors

$$e^{\pm} := \pm \lim_{t \rightarrow \mp \infty} \dot{z}^*(t)/|\dot{z}^*(t)| \tag{1.11}$$

By (1.10b), $e^+ \in T_0W^u$ is a unit eigenvector of $\nu_0 > 0$, and $e^- \in T_0W^s$ belongs to $-\mu_0 < 0$. Now define the hyperplanes

$$T_{z^*(t)} := T_{z^*(t)}W^s + T_{z^*(t)}W^u \tag{1.12}$$

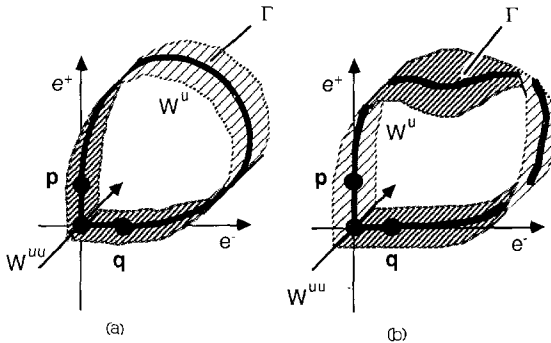


Fig. 2. Homoclinic orbits: (a) Nontwisted, (b) Twisted.

Note that $\text{codim } T_{z^*(t)} = 1$, by assumption (1.10a). For simple real principal eigenvalues, it is now a consequence of the strong λ -lemma, that generically

$$\begin{aligned} \lim_{t \rightarrow -\infty} T_{z^*(t)} &= T_0 W^{ss} \oplus T_0 W^u \\ \lim_{t \rightarrow +\infty} T_{z^*(t)} &= T_0 W^s \oplus T_0 W^{uu} \end{aligned} \tag{1.13}$$

See the strong inclination property of Deng (1989a). This is illustrated in Fig. 2, where $W^{ss} = \{0\}$ since $m = \dim W^s = 1$. For a precise statement of the strong λ -lemma see lemma 3.3 below. Choosing $\mathbf{p}, \mathbf{q} \in \Gamma$ close enough to $z = 0$, as above, (1.13) implies

$$\begin{aligned} e^- \notin T_{\mathbf{p}}, \quad \mathbb{R}^{m+n} &= T_{\mathbf{p}} \oplus \text{span}(e^-) \\ e^+ \notin T_{\mathbf{q}}, \quad \mathbb{R}^{m+n} &= T_{\mathbf{q}} \oplus \text{span}(e^+) \end{aligned} \tag{1.14}$$

Finally, note that $T_{z^*(t)}, 0 \leq t \leq T$, defines a homotopy from $T_{\mathbf{p}}$ to $T_{\mathbf{q}}$, justifying the following definition of a twist.

Definition. Let Γ be a nondegenerate homoclinic orbit in general position and with real principal eigenvalues, that is (1.6), (1.10a, b), (1.13) hold.

We call Γ *twisted*, if e^- and e^+ point to opposite sides of $T_{\mathbf{p}}$ and $T_{\mathbf{q}}$, respectively. See Fig. 2b.

If e^-, e^+ point to the same side of $T_{\mathbf{p}}, T_{\mathbf{q}}$, respectively, then we call Γ *nontwisted*. See Fig. 2a.

Clearly, twisted homoclinic orbits can occur only in space dimensions ≥ 3 . Homoclinic orbits in planar flows are always nontwisted. Note that the twist can also be expressed by watching the winding of the bundle of normal vectors to $T_{z^*(t)}$ as $z^*(t)$ moves from \mathbf{p} to \mathbf{q} .

We need a final piece of terminology. Fix a small tubular neighborhood U of our (twisted or nontwisted) homoclinic orbit $\text{clos } \Gamma = \Gamma \cup \{0\}$. An N -periodic orbit is a periodic orbit which is contained in U and has winding number N in U . Similarly, we define an N -homoclinic orbit. In particular, Γ itself is a 1-homoclinic orbit. As long as U is chosen small enough, the above definition is independent of the particular choice of U . The terminology extends canonically to small perturbations of the original vector field $F(0, \cdot)$. We can now state our main results.

Theorem A: Resonant Side-Switching. Let $F = F(\alpha, z)$ of class C^{M+5} , $2 \leq M \leq \infty$, be a generic two-parameter vector field with a nontwisted resonant homoclinic orbit at $\alpha = 0$. Then resonant side-switching occurs at $\alpha = 0$ (see Fig. 3a).

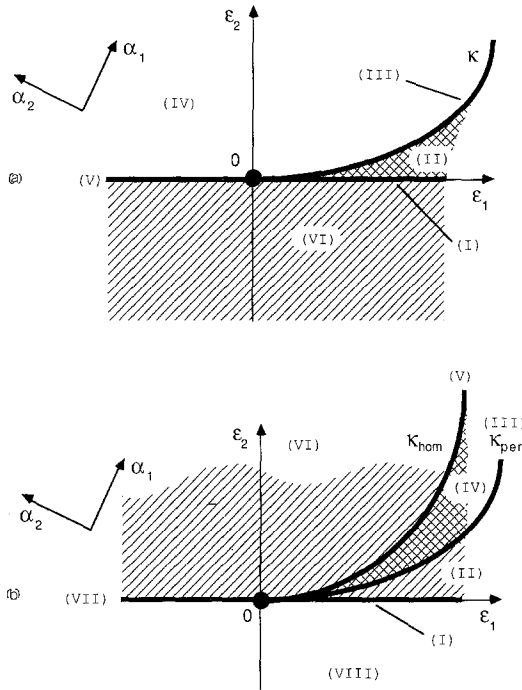


Fig. 3. (a) Resonant side-switching. (b) Resonant homoclinic doubling.

In more detail, let assumptions (1.5), (1.6), (1.9) be satisfied for a nontwisted homoclinic orbit Γ (cf. definition 1.1). Let U denote a sufficiently small tubular neighborhood of Γ .

Then there exist a C^M diffeomorphic local change of parameters

$$\varepsilon = (\varepsilon_1, \varepsilon_2) = \varepsilon(\alpha)$$

at $\alpha = 0$ and a function

$$\varepsilon_2 = \kappa(\varepsilon_1) \begin{cases} = 0 & \varepsilon_1 \leq 0 \\ & \text{for} \\ > 0 & \varepsilon_1 > 0 \end{cases} \quad (1.15)$$

of class C^M for $\varepsilon_1 > 0$, such that the numbers of 1-periodic (1-per) and of 1-homoclinic (1-hom) orbits in U for the parameter regions 0, I–VI are given by Table I. Moreover, there exists a constant $a_0 = a(0) > 1$ such that the following finite limit exists

$$\lim_{\varepsilon_1 \rightarrow 0} \kappa(\varepsilon_1) \cdot \frac{1}{\varepsilon_1} a_0^{1/\varepsilon_1} > 0 \quad (1.16)$$

Table I. Resonant Side-Switching; Number of Homoclinic and Periodic Orbits

Region	Definition	No. of 1-hom	No. of 1-per
0	$\varepsilon_1 = 0, \varepsilon_2 = 0$	1	0
I	$\varepsilon_1 > 0, \varepsilon_2 = 0$	1	1
II	$\varepsilon_1 > 0, 0 < \varepsilon_2 < \kappa(\varepsilon_1)$	0	2
III	$\varepsilon_1 > 0, \varepsilon_2 = \kappa(\varepsilon_1)$	0	1
IV	$\varepsilon_1 \in \mathbb{R}, \varepsilon_2 > \kappa(\varepsilon_1)$	0	0
V	$\varepsilon_1 < 0, \varepsilon_2 = 0$	1	0
VI	$\varepsilon_2 \in \mathbb{R}, \varepsilon_2 < 0$	0	1

The constant a_0 is given explicitly by (4.9), (4.14c) evaluated at $\alpha = \varepsilon = 0$. For the value of the limit, see (5.10).

The 1-per and 1-hom orbits, viewed as sets, depend continuously on ε in the obvious sense. Specifically: crossing line III along a one-dimensional curve from II to IV, two 1-per orbits merge and disappear at a saddle-node (or saddle-saddle) type bifurcation. Crossing I or V, one encounters a blue sky catastrophe.

Since the 1-per orbits bifurcate to different sides from the 1-hom curve $\varepsilon_2 = 0$ for $\varepsilon_1 < 0$, respectively, $\varepsilon_1 > 0$, we call this bifurcation a resonant side-switching.

Theorem B: Resonant Homoclinic Doubling. Let $F = F(\alpha, z)$ of class C^{M+5} , $2 \leq M \leq \infty$ be a generic two parameter vector field with a twisted resonant homoclinic orbit at $\alpha = 0$. Then resonant homoclinic doubling occurs at $\alpha = 0$ (see Fig. 3b).

In more detail, let Γ be as in theorem A, but twisted, and let U again denote a sufficiently small tubular neighborhood of its closure $\text{clos } \Gamma$. Then there exists a C^M diffeomorphic reparametrization $\varepsilon = \varepsilon(\alpha)$ and two functions

$$\varepsilon_2 = \kappa_i(\varepsilon_1) \begin{cases} = 0 & \varepsilon_1 \leq 0 \\ \text{for} & \\ > 0 & \varepsilon_1 > 0 \end{cases} \tag{1.17}$$

$i \in \{\text{hom, per}\}$, of class C^M for $\varepsilon_1 > 0$, such that the numbers of N -periodic (N -per) and of N -homoclinic (N -hom) orbits in U with $N = 1, 2$ for the parameter regions 0, I–VIII are given in Table II. The curves κ_i have the universal limit property

$$\lim_{\varepsilon_1 \searrow 0} \frac{\kappa_{\text{hom}}(\varepsilon_1)}{\kappa_{\text{per}}(\varepsilon_1)} = \frac{e}{2} = 1.36\dots \tag{1.18}$$

Table II. Resonant Homoclinic Doubling; Numbers of N -hom and N -per Orbits for $N = 1, 2$

Region	Definition	No. of 1-hom	No. of 2-hom	No. of 1-per	No. of 2-per
0	$\varepsilon_1 = 0, \varepsilon_2 = 0$	1	0	0	0
I	$\varepsilon_1 > 0, \varepsilon_2 = 0$	1	0	0	0
II	$\varepsilon_1 > 0, 0 < \varepsilon_2 < \kappa_{\text{per}}(\varepsilon_1)$	0	0	1	≥ 0
III	$\varepsilon_1 > 0, \varepsilon_2 = \kappa_{\text{per}}(\varepsilon_1)$	0	0	1	≥ 0
IV	$\varepsilon_1 > 0, \kappa_{\text{per}}(\varepsilon_1) < \varepsilon_2 < \kappa_{\text{hom}}(\varepsilon_1)$	0	0	1	≥ 1
V	$\varepsilon_1 > 0, \varepsilon_2 = \kappa_{\text{hom}}(\varepsilon_1)$	0	1	1	≥ 0
VI	$\varepsilon_1 \in \mathbb{R}, \varepsilon_2 > \kappa_{\text{hom}}(\varepsilon_1)$	0	0	1	≥ 0
VII	$\varepsilon_1 < 0, \varepsilon_2 = 0$	1	0	0	0
VIII	$\varepsilon_1 \in \mathbb{R}, \varepsilon_2 < 0$	0	0	0	0

Moreover, there exists a constant $a_0 = a(0) < -1$ such that the following finite limit exists

$$\lim_{\varepsilon_1 \rightarrow 0} \kappa_{\text{per}}(\varepsilon_1) |a_0|^{1/\varepsilon_1} > 0 \quad (1.19)$$

The constant a_0 is given explicitly by (4.9), (4.14c) evaluated at $\alpha = \varepsilon = 0$. For the value of the limit, see (6.8).

All homoclinic and periodic orbits depend continuously on ε as sets. Specifically, crossing lines I, V, VII yields blue sky catastrophes. Crossing line III, one encounters a period doubling bifurcation of the 1-per orbits to a sheet of 2-per orbits.

The set of 2-per orbits contains a two-dimensional continuum \mathcal{C} of orbits which extends from the period doubling 1-per orbits at parameter curve III to the blue sky catastrophe at the curve of 2-hom orbits along line V.

By a continuum, we mean a relatively compact connected set. For our notion of dimension of \mathcal{C} , we refer to Alexander and Antman (1981) and to lemma 6.5 below.

Our statements concerning 2-per solutions are somewhat weak. Actually, 2-per solutions can occur only in a wedge region which is shaped like the wedge region IV between κ_{per} and κ_{hom} but which is somewhat wider. For a precise statement see lemma 6.4 below. Due to the topological methods which we use, we cannot determine the exact number of 2-per solutions in these regions.

Both theorems, A and B, are new results as they stand. The basic underlying idea, however, that resonant principal eigenvalues $\mu_0 = \nu_0$ can lead to bifurcation, is not new. The nontwisted case was studied before by

Leontovich (1951), Nozdracheva (1982), Sanders and Cushman (1986), for planar vector fields ($z \in \mathbb{R}^2$). We refer to Section 7 for a more detailed discussion.

At this stage, it is high time to mention the pioneering results by Yanagida (1986) on homoclinic doubling in the case of a single unstable dimension ($n = 1$). In a somewhat less geometric setting Yanagida already gave algebraic conditions for existence and inexistence of 2-hom orbits. In the resonant case, these conditions amount to our twist condition. The proofs, however, are based on local C^1 -linearization in the spirit of the Grobmann-Hartman theorem. But such a linearization may be inappropriate at resonant eigenvalues. Still, we devote Section 2 to an exposition of the (nonrigorous) approach by complete linearization, for illustration purposes. The gap concerning C^1 -linearization was pointed out and closed in \mathbb{R}^3 by Kokubu (1987, 1988). These results, although based on “Shilnikov variables,” are obtained by a methodology which differs from our approach. For a more detailed discussion we refer to Section 7.

Replacing complete linearization, we build our approach on a careful analysis of Shilnikov’s parametrization of the flow near the origin; see Shilnikov (1968) and Deng (1989a). This background material is surveyed in Section 3. In Section 4, we then derive a Ljapunov–Schmidt-type reduction of the bifurcation problem down to an N -dimensional system of bifurcation equations, for N -per and N -hom solutions. These systems are highly transcendental: the bifurcation parameter enters as an exponent in the leading term. This fact causes the exponential smallness of the curves κ in theorems A and B. Theorem A is proved in Section 5, discussing the case $N = 1$. For a proof of theorem B, alias $N = 2$, see Section 6. After a detailed discussion in Section 7, we conclude with an Appendix which recalls the explicit form of all our genericity assumptions.

2. A MODEL EXAMPLE: THE PIECEWISE LINEAR CASE

In this section, we analyze the flow near a resonant homoclinic orbit Γ in \mathbb{R}^3 . Following Yanagida’s (1987) idea, we assume that our flows are piecewise linear. Due to resonance, this assumption is inadequate to the nonlinear problem. However, deviating from Yanagida, we will treat the piecewise linear case in such a way that it becomes a paradigm to our treatment of the general generic case.

Consider a homoclinic orbit Γ of a three-dimensional vector field $F = F(\alpha, z)$ at $\alpha = 0$, as in theorems A and B. Assume $F(\alpha, 0) = 0$, for all α . We introduce coordinates $z = (x, y)$, $x = x_0$, $y = (y_0, y_1)$ associated to the eigendirections of the eigenvalues $-\mu_0(\alpha) < 0 < \nu_0(\alpha) < \nu_1(\alpha)$ of the

linearization $D_z F(\alpha, 0)$. We may rescale time so that $\mu_0(\alpha) = 1$. Moreover, we assume that $v_0(\alpha) = 1 + \alpha_1$.

Our first linearity assumption is the following:

$$F(\alpha, z) = D_z F(\alpha, 0) \cdot z \quad \text{for } |z| = \max(|x|, |y|) \leq \delta_0 \quad (2.1)$$

In other words, the flow near $z = 0$ is given by

$$\begin{aligned} \dot{x}_0 &= -x_0 \\ \dot{y}_0 &= (1 + \alpha_1) y_0 \\ \dot{y}_1 &= v_1 y_1 \end{aligned} \quad (2.2)$$

Let $S_{in} := \{(x, y) : x_0 = \delta_0\}$, $S_{out} := \{(x, y) : y_0 = \delta_0\}$ denote (local) Poincaré sections, transverse to the homoclinic orbit Γ at $\mathbf{q} = (\delta_0, 0, 0)$, $\mathbf{p} = \{0, \delta_0, 0\}$ (cf. Fig. 4). Note that we assume here that \mathbf{p} lies in the linear eigenspace of v_0 for simplicity. The linear flow in the box $\{|z| \leq \delta_0\}$ defines a Poincaré map Π^{loc} from a suitable subset \bar{S}_{in} of S_{in} to S_{out} :

$$\begin{aligned} \Pi^{loc}: S_{in} &\rightarrow S_{out} \\ (x^{in}, y^{in}) &\rightarrow (x^{out}, y^{out}) \end{aligned} \quad (2.3)$$

Likewise, there is an outer Poincaré map Π^{far} from S_{out} to S_{in} , given by the flow along Γ :

$$\begin{aligned} \Pi^{far}: S_{out} &\rightarrow S_{in} \\ (x^{out,j}, y^{out,j}) &\mapsto (x^{in,j+1}, y^{in,j+1}) \end{aligned} \quad (2.4)$$

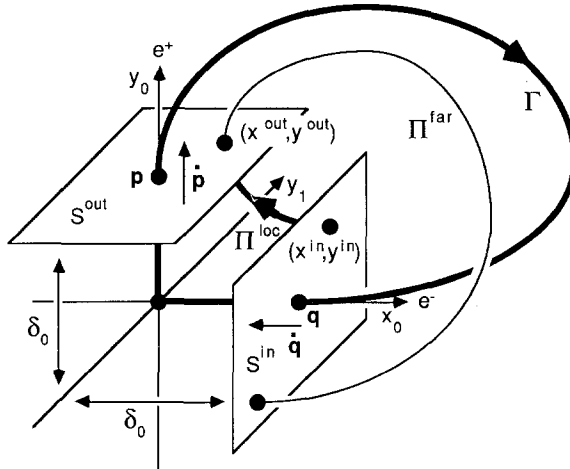


Fig. 4. Sections and return maps.

The superscripts $j, j + 1$ are introduced here because we will be interested in iterates of $\Pi^{\text{far}} \circ \Pi^{\text{loc}}$ later on.

We also assume that Π^{far} is (affine) linear,

$$\begin{pmatrix} y_0^{\text{in},j+1} \\ y_1^{\text{in},j+1} \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \Pi^{11} & \Pi^{12} \\ \Pi^{21} & \Pi^{22} \end{pmatrix} \begin{pmatrix} x_0^{\text{out},j} \\ y_1^{\text{out},j} \end{pmatrix}, \quad \text{with } \Pi^{22} \neq 0 \quad (2.5)$$

where, again for simplicity, the Π^{ik} are considered now as being independent of α . The generic assumption $\Pi^{22} \neq 0$ guarantees that W^u hits W^s in \mathbf{q} in general position so that (1.13) holds. Indeed, $T_0 W^{\text{uu}}$ is the y_1 -axis. Note the meaning of $\alpha = (\alpha_1, \alpha_2)$. For $\alpha_2 = 0$, the points

$$\mathbf{p} = (x^{\text{out}}, \delta_0, y_1^{\text{out}}) := (0, \delta_0, 0) \in W^u$$

and

$$\mathbf{q} = (\delta_0, y_0^{\text{in}}, y_1^{\text{in}}) := (\delta_0, 0, 0) \in W^s$$

lie on a 1-homoclinic branch, all along the line $\{\alpha_1 \in \mathbb{R}, \alpha_2 = 0\}$. Roughly speaking, α_2 describes the distance between W^u and W^s . The other parameter, α_1 , expresses transverse crossing of the eigenvalue $\nu_0 = 1 + \alpha_1$ through $\mu_0 = 1$ along the 1-homoclinic branch $\alpha_2 = 0$.

At this stage, we could proceed by piecing Π^{loc} and Π^{far} together, iterating the return map $\Pi^{\text{far}} \Pi^{\text{loc}}$ of \bar{S}_{in} . Unfortunately, this simple approach does not seem to work equally well in the fully nonlinear case, because, in iterating $\Pi^{\text{far}} \Pi^{\text{loc}}$, it is difficult to keep control of the shrinking cusp-shaped domain of definition in \bar{S}_{in} . In Shilnikov's variables, we obtain nicer domains of definition which are more suitable for applications of the implicit function theorem. Therefore, we now describe the same iterations in Shilnikov's (1968) coordinates

$$(s, x^{\text{in}}, y^{\text{out}}) = (s, \delta_0, \delta_0, y_1^{\text{out}})$$

rather than using

$$(x^{\text{in}}, y^{\text{in}}) = (\delta_0, y_0^{\text{in}}, y_1^{\text{in}}) \in S^{\text{in}}$$

The two coordinate systems are related via the linear flow (2.2) as follows:

$$\begin{aligned} y_0^{\text{in}} &= e^{-\nu_0 s} y_0^{\text{out}} = e^{-\nu_0 s} \delta_0 \\ y_1^{\text{in}} &= e^{-\nu_1 s} y_1^{\text{out}} \\ x_0^{\text{out}} &= e^{-s} x_0^{\text{in}} = e^{-s} \delta_0 \end{aligned} \quad (2.6)$$

The parameter s , for *Shilnikov time*, is the Poincaré time associated to Π^{loc} . For a more detailed exposition of this concept see Section 3.

Note that the domain of definition of $\Pi^{\text{far}} \Pi^{\text{loc}}$ takes the form of a rectangle in the Shilnikov coordinates (s, y_1^{out}) , namely, $s \geq s^*$ large enough and $|y_1^{\text{out}}| \leq \delta_0$.

Reintroducing indices $j, j+1$ to keep track of the iterations in the Shilnikov coordinates $(s_j, y_1^{\text{out},j})$, we obtain from (2.5), (2.6):

$$e^{-v_0 s_{j+1}} \delta_0 = \alpha_2 + e^{-s_j} \Pi^{11} \delta_0 + \Pi^{12} y_1^{\text{out},j} \tag{2.7a}$$

$$e^{-v_1 s_{j+1}} y_1^{\text{out},j+1} = e^{-s_j} \Pi^{21} \delta_0 + \Pi^{22} y_1^{\text{out},j} \tag{2.7b}$$

We are interested in N -periodic solutions near Γ . With $r_j := e^{-v_0 s_j} > 0$ this reads

$$\delta_0 r_{j+1} = \alpha_2 + \delta_0 \Pi^{11} r_j^{1/v_0} + \Pi^{12} y_1^{\text{out},j} \tag{2.8a}$$

$$r_{j+1}^{v_1/v_0} y_1^{\text{out},j+1} = \delta_0 \Pi^{21} r_j^{1/v_0} + \Pi^{22} y_1^{\text{out},j} \tag{2.8b}$$

with $j \pmod N$. Note that $r_j \delta_0 = y_0^{\text{in},j}$ measures the vertical distances of the periodic trajectory from the stable manifold. Recall that we are interested in solutions with small $r_j, y_1^{\text{out},j}$. In fact a solution with

$$r_0 = 0 \quad \text{and} \quad r_1, \dots, r_{N-1} > 0$$

all distinct, is an N -homoclinic orbit.

Mimicking Ljapunov–Schmidt reduction, we can solve equations (2.8b) for $(y_1^{\text{out},j})$, $j=0, \dots, N-1$, since Π^{22} is nonzero by assumption (2.5). We obtain

$$y_1^{\text{out},j} = -\delta_0 \frac{\Pi^{21}}{\Pi^{22}} r_j^{1/v_0} + O(\delta_0 |\mathbf{r}|^{(v_1+1)/v_0}) \tag{2.9}$$

where $\mathbf{r} = (r_0, \dots, r_{N-1})$ and $|\mathbf{r}| := \max r_j$. Plugging this into the remaining equations (2.8a) we find the bifurcation equations

$$r_{j+1} = \varepsilon_2 + a_0 r_j^{1+\varepsilon_1} + O(|\mathbf{r}|^{(v_1+1)/v_0}) \tag{2.10}$$

Here we use the notation

$$1/v_0 = 1/(1 + \alpha_1) =: 1 + \varepsilon_1$$

$$a_0 := (\det \Pi^{\hat{j}}) / \Pi^{22}$$

$$\varepsilon_2 := \alpha_2 / \delta_0$$

As a variant, we mention the scaling $\hat{r}_j := e^{-s_j}$. We then obtain analogously

$$\hat{r}_j = \hat{\varepsilon}_2 + \hat{a}_0 \hat{r}_{j+1}^{1+\hat{\varepsilon}_1} + O(|\hat{\mathbf{r}}|^{v_1+1}) \tag{2.10^{\wedge}}$$

where $1 + \hat{\varepsilon}_1 := 1 + \alpha_1 = v_0$, $\hat{a}_0 = a_0^{-1}$, and $\hat{\varepsilon}_2 = -\hat{a}_0 \alpha_2 / \delta_0$.

We are simply amazed at the fact that virtually the same form (2.10), (2.10[^]) of the iteration emerges from a careful Ljapunov–Schmidt reduction analysis of the fully nonlinear system; see (4.13), (4.13[^]) in corollary 4.3 and remark 4.4.

The nonzero coefficient a_0 in (2.10) has an interesting geometric interpretation. We claim

$$\begin{aligned} a_0 > 0 & \text{ if } \Gamma \text{ is a nontwisted homoclinic orbit} \\ a_0 < 0 & \text{ if } \Gamma \text{ is twisted} \end{aligned} \tag{2.11}$$

This follows directly from our definition 1.1 of a twist. See Figs. 2 and 4. Indeed, let e_1 denote the unit vector along the positive y_1 -axis; e^+ , $\dot{\mathbf{p}}$ point along the positive y_0 -axis and e^- , $-\dot{\mathbf{q}}$ along the positive x_0 -axis. Note that the orientations of the triples $(e^-, e_1, \dot{\mathbf{p}})$ and of $(e^+, e_1, \dot{\mathbf{q}})$ are equal:

$$\text{sgn}(e^- \wedge e_1 \wedge \dot{\mathbf{p}}) = \text{sgn}(e^+ \wedge e_1 \wedge \dot{\mathbf{q}}) = +1 \tag{2.12}$$

where \wedge denotes the exterior product. Consequently $\det \Pi^{ij} > 0$, by definition of Π^{ij} ; cf. (2.5). Now note that the orientation of $T_{\mathbf{p}}$, given by $e_1 \wedge \dot{\mathbf{p}}$, continues to that of $T_{\mathbf{q}}$, given by $(\Pi^{12}e^+ + \Pi^{22}e_1) \wedge \dot{\mathbf{q}}$. Therefore, (2.12) implies

$$\begin{aligned} \text{sgn}(e^+ \wedge (\Pi^{12}e^+ + \Pi^{22}e_1) \wedge \dot{\mathbf{q}}) &= \text{sgn } \Pi^{22} \cdot \text{sgn}(e^+ \wedge e_1 \wedge \dot{\mathbf{q}}) \\ &= \text{sgn } \Pi^{22} \cdot \text{sgn}(e^- \wedge e_1 \wedge \dot{\mathbf{p}}) \\ &= \text{sgn } a_0 \cdot \text{sgn}(e^- \wedge e_1 \wedge \dot{\mathbf{p}}) \end{aligned} \tag{2.13}$$

since $a_0 = (\det \Pi^{ij})/\Pi^{22}$ and $\det \Pi^{ij} > 0$

Thus e^- and e^+ , respectively, are on the same side of $T_{\mathbf{p}}$ and $T_{\mathbf{q}}$ if $a_0 > 0$. Likewise, they are on opposite sides if $a_0 < 0$. This proves our claim (2.11).

With this observation, let us now return to the bifurcation equations (2.10), (2.10[^]). Let us simple-mindedly neglect higher-order terms \mathcal{O} , altogether. We are interested only in the bifurcation diagrams in parameter space, as given in theorems A, B and Figs. 3a and b. Therefore, we can assume $|a_0| > 1$ in (2.10). Indeed, $|a_0| = 1$ is nongeneric, and $|a_0| < 1$ is equivalent to $|a_0| > 1$ when switching to (2.10[^]) and reversing the direction of iteration. It is now fairly straightforward to derive all results of theorems A, B in this simplified situation. All we have to analyse is an iteration of the monotone and convex/linear/concave map

$$r \rightarrow \varepsilon_2 + a_0 r^{1 + \varepsilon_1}$$

In fact, Tables I and II then list *all* possible N -per and N -hom orbits. We leave this simple case to the reader. For a reference see Glendinning (1987).

The analysis for O -terms included fills Sections 5 and 6. In particular, we lose control over N -per orbits with $N \geq 2$ in the nontwisted and $N \geq 3$ in the twisted case, due to these error terms. In the twisted case, the number of 2-per orbits can only be estimated. Basically, the reason is that the O -term depends not only on r_j but also on the r_k with $k \neq j$. Thus (2.10) cannot be interpreted as the iteration of a scalar function $r_j \mapsto r_{j+1}$ anymore. For a more detailed discussion and a remedy see Section 7.

3. SHILNIKOV VARIABLES AND THE STRONG λ -LEMMA

In this section, we briefly review some fundamental (but nontrivial) facts about the Shilnikov variables. These facts are basic to our Ljapunov–Schmidt reduction for resonant bifurcation of homoclinic orbits. In particular, we need careful estimates on derivatives so that we can keep track of higher-order terms during the reduction in Section 4. We also give a precise version of the strong λ -lemma, which was used in our definition of twisted versus nontwisted homoclinic orbits (see Definition 1.1). For a more detailed account, including proofs, we refer to Deng (1989a).

We fix the following normalized setting for the rest of this paper. We locally describe the original vector field

$$\dot{z} = F(\alpha, z) \tag{1.1}$$

in suitable coordinates $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Specifically (x, y) are chosen such that, near $z = 0$, F takes the form

$$\begin{aligned} \dot{x} &= A(\alpha)x + f(\alpha, x, y) \\ \dot{y} &= B(\alpha)y + g(\alpha, x, y) \end{aligned} \tag{3.1}$$

where f, g , and their first derivatives $D_{(x,y)}f, D_{(x,y)}g$ with respect to x, y vanish identically at $x = 0, y = 0$, for any (small) $|\alpha|$. Moreover, the linearizations $A(\alpha), B(\alpha)$ are assumed to have block diagonal form

$$\begin{aligned} A(\alpha) &= \left(\begin{array}{c|c} -1 & \\ \hline & A_1(\alpha) \end{array} \right) \\ B(\alpha) &= \left(\begin{array}{c|c} v_0(\alpha) & \\ \hline & B_1(\alpha) \end{array} \right) \end{aligned} \tag{3.2}$$

corresponding to the eigenspace decomposition

$$\begin{aligned} x &= (x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{m-1} \\ y &= (y_0, y_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \end{aligned} \tag{3.3}$$

of x and y . We may further assume that the local (un)stable manifolds W_{loc}^u, W_{loc}^s are given by

$$\begin{aligned} W_{loc}^u &= \{x = 0\} \\ W_{loc}^s &= \{y = 0\} \end{aligned} \tag{3.4}$$

In other words, we have

$$\begin{aligned} f(\alpha, 0, y) &= 0 \\ g(\alpha, x, 0) &= 0 \end{aligned} \tag{3.4'}$$

independently of α, x, y near zero. Note that f, g, A, B, v_0 are of class at least C^{M+4} and the spectra (spec) of A_1, B_1 satisfy

$$\begin{aligned} \operatorname{Re} \operatorname{spec} A_1 &< -\bar{\mu}_1 < -1 \\ \operatorname{Re} \operatorname{spec} B_1 &> \bar{v}_1 > v_0 > 0 \end{aligned} \tag{3.5}$$

uniformly for small $|\alpha|$ according to our spectral assumption (1.6). Also note that we have normalized the principal stable eigenvalue $-\mu_0(\alpha)$ to become -1 , by rescaling time.

With this normalization in mind, we can now reconsider Fig. 4, which, in Section 2, served us as an illustration of the flow near the origin and near the homoclinic orbit Γ at $\alpha=0$. The x_1 -components are omitted, for the sake of simplicity. Also, y_1 now denotes an $(n-1)$ -vector. We pick $\delta_0 > 0$ small enough and choose the local sections $S_{in} = \{x_0 = \delta_0\}$, $S_{out} = \{y_0 = \delta_0\}$ in the δ_0 -box $\{|x| \leq \delta_0, |y| \leq \delta_0\}$ where $|\cdot|$ denotes the max-norm. As before, (x^{in}, y^{in}) and (x^{out}, y^{out}) , respectively, denote elements of S_{in} and S_{out} . As we have indicated in Section 2, the Shilnikov variables describing the trajectories for the δ_0 -box are x^{in}, y^{out} , and $s \geq 0$; see, e.g., Shilnikov (1968). As before, $s \geq 0$ denotes the time it takes to run inside a δ_1 -box from a point $(x^{in}, \cdot) \in S_{in}$ to a point $(\cdot, y^{out}) \in S_{out}$, whenever this time is defined. With the notation

$$S_{in}^x := \{x^{in} = (\delta_0, x_1^{in}) : |x_1^{in}| \leq \delta_0\}, \quad S_{out}^y := \{y^{out} = (\delta_0, y_1^{out}) : |y_1^{out}| \leq \delta_0\}$$

we have the following.

3.1. Proposition [Deng (1989a), Theorems 2.1, 8.1]. *Let $\delta_1 > 0$ be chosen small enough, in the above setting. Then there exists a $\delta_0 > 0$ and a unique C^{M+4} -smooth map,*

$$\begin{aligned} (x, y): \{0 < t \leq s\} \times S_{in}^x \times S_{out}^y \times \mathbb{R}_{loc}^2 &\rightarrow \{(x, y) \in \mathbb{R}^{m+n} : |x| \leq \delta_1, |y| \leq \delta_1\} \\ (t; s, x^{in}, y^{out}, \alpha) &\mapsto (x, y)(t; s, x^{in}, y^{out}, \alpha) \end{aligned} \tag{3.6}$$

such that (x, y) , as a function of t , solves the differential equation (3.1) with the boundary conditions

$$\begin{aligned} x &= x^{\text{in}} & \text{at } t &= 0 \\ y &= y^{\text{out}} & \text{at } t &= s \end{aligned} \tag{3.7}$$

Smoothness of (x, y) is understood in the sense that (x, y) can be extended smoothly into a neighborhood of the specified time domain, keeping the interpretation of (x, y) being a solution of (3.1). In Deng (1989a), a C^k -version was stated, but the C^∞ -version follows analogously.

Next, we are interested in the limit $s \rightarrow \infty$. In fact we can also solve the boundary value problem (3.7) for $s = \infty$, at least formally, by letting x, y follow the trajectory of x^{in} in the stable manifold $W_{\text{loc}}^s = \{y = 0\}$, respectively, of y^{out} (backwards) in $W_{\text{loc}}^u = \{x = 0\}$. Since we are interested in the quantities $x^{\text{out}}, y^{\text{in}}$, which complement the Shilnikov variables $s, x^{\text{in}}, y^{\text{out}}$ in our δ_0 -box, we now consider

$$\begin{aligned} x^{\text{out}}(r, x^{\text{in}}, y^{\text{out}}; \alpha) &:= x(s; s, x^{\text{in}}, y^{\text{out}}; \alpha) \\ y^{\text{in}}(r, x^{\text{in}}, y^{\text{out}}; \alpha) &:= y(0; s, x^{\text{in}}, y^{\text{out}}; \alpha) \end{aligned} \tag{3.8}$$

where $r := e^{-v_0(\alpha)s}$. Note that $r \geq 0$ is Shilnikov time, rescaled so that $r = 0$ corresponds to $s = \infty$. Similarly, let $\hat{x}^{\text{out}}, \hat{y}^{\text{in}}$ denote the same right-hand sides, but as a function of $\hat{r} := e^{-s}$ instead of r . Then $x^{\text{out}}, y^{\text{in}}, \hat{x}^{\text{out}}, \hat{y}^{\text{in}}$ extend differentiably down to $r = 0$ as follows.

3.2. Proposition [Deng (1989a), Sections 3, 4, 8]. *Let $\delta_0 > 0$ be chosen small enough, in the above setting, and choose any constant ω such that*

$$0 < \omega < \min_x \{v_0(\alpha), \mu_0(\alpha), \bar{v}_1 - v_0(\alpha), \bar{\mu}_1 - \mu_0(\alpha)\}$$

where the principal eigenvalue $-\mu_0(\alpha)$ was normalized to be -1 , above, and $\bar{\mu}_1, \bar{v}_1$ bound the remaining (nonprincipal) eigenvalues as in (1.6).

Then for any finite $0 \leq k \leq M + 1$ there exists a constant C , which is independent of $s, x_{\text{in}} \in S_{\text{in}}^x, y^{\text{out}} \in S_{\text{out}}^y$, and small α , such that (i)–(iii) below are true.

(i) *The following expansion holds for y^{in} as $r \searrow 0$:*

$$y^{\text{in}}(r, x^{\text{in}}, y^{\text{out}}; \alpha) = r(\psi(x^{\text{in}}, y^{\text{out}}; \alpha) + R_y) \tag{3.9}$$

where the remainder term $R_y = R_y(r, x^{\text{in}}, y^{\text{out}}; \alpha)$, of class C^{M+1} for $r > 0$, is estimated by the C^k -norm $\|\cdot\|_k$ in $(x^{\text{in}}, y^{\text{out}}; \alpha)$:

$$\|D_r^\beta R_y(r, \cdot, \cdot; \cdot)\|_{k-\beta} \leq Cr^{\omega-\beta} \tag{3.10}$$

for $\beta = 0, 1, \dots, k$. The leading term $\psi = (\psi_0, \psi_1)$ of class C^{M+1} has the properties:

$$\psi(x^{in}, 0; \alpha) = 0 \tag{3.11a}$$

$$\psi_1(0, y^{out}; \alpha) = 0 \tag{3.11b}$$

$$D_{y^{out}}\psi(0, 0; \alpha) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}_{n \times n} \tag{3.11c}$$

(ii) The corresponding statements for x^{out} , in the same logical context, read

$$x^{out}(r, x^{in}, y^{out}, \alpha) = r^{1/v_0(\alpha)}(\varphi(x^{in}, y^{out}; \alpha) + R_x) \tag{3.9}$$

where R_x also satisfies the estimates (3.10). The leading term

$$\varphi = (\varphi_0, \varphi_1) \in C^{M+1}$$

satisfies

$$\varphi(0, y^{out}; \alpha) = 0 \tag{3.11a}$$

$$\varphi_1(x^{in}, 0; \alpha) = 0 \tag{3.11b}$$

$$D_{x^{in}}\varphi(0, 0; \alpha) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}_{m \times m} \tag{3.11c}$$

(iii) Replacing $r = e^{-v_0(\alpha)s}$ by $\hat{r} = e^{-s}$, that is, replacing x^{out} , y^{in} by \hat{x}^{out} , \hat{y}^{in} , all the above statements remain valid for an expansion

$$\begin{aligned} \hat{x}^{out}(\hat{r}, x^{in}, y^{out}, \alpha) &= \hat{r}(\varphi + \hat{R}_x) \\ \hat{y}^{in}(\hat{r}, x^{in}, y^{out}, \alpha) &= \hat{r}^{v_0(\alpha)}(\psi + \hat{R}_y) \end{aligned} \tag{3.9}$$

Occasionally, we refer to (3.9) as Shilnikov’s expansion since expansions of this type were first introduced by Shilnikov; see, e.g., Shilnikov (1968a, 2.15).

To illustrate the geometric significance of proposition 3.2, we mention that the local strong stable manifold W_{loc}^{ss} is given by those $x^{in} \in S_{in}$ for which

$$\varphi_0(x^{in}, 0; \alpha) = 0 \tag{3.12}$$

See Deng (1989a), Corollary 4.3. Note that $\varphi = 0$ on W_{loc}^{ss} , because $\varphi_1(x^{in}, 0; \alpha) = 0$ by property (3.11b). So the rapid decay of solutions in W^{ss} , stated in (1.7) above, is reflected in the leading term φ of our exponential expansion (3.9) being zero. Actually, W_{loc}^{ss} is of class C^{M+5} and can be written as the graph of a C^{M+5} function over x_1 ; see, e.g., Shub (1987), Brunovský and Fiedler (1986). Here we refer to the regularity C^{M+5} of the vector field, which holds before any normalization of W^u, W^s . Similar statements describe W_{loc}^{uu} , given by $\psi_0(0, y^{out}; \alpha) = 0$.

What are those leading terms φ, ψ , in the completely linear case $f = 0, g = 0$? We obviously get

$$x(t; s, x^{in}; \alpha) = e^{A(\alpha)t} x^{in}$$

Hence

$$\psi(x^{in}, y^{out}; \alpha) = (x_0^{in}, 0) \in \mathbb{R}^m$$

projects onto the first (principal) x^{in} -coordinate. In other words, φ and ψ select principal components—also in the nonlinear case. Because φ, ψ are the leading terms in the r -expansion (3.9) of x^{out}, y^{in} , this is one more reason for calling the eigenvalues $-\mu_0, \nu_0$ and their eigendirections *principal*.

We now discuss the strong λ -lemma, which enters into definition 1.1 of (non-) twisted homoclinic orbits via the convergence assumption (1.13). We can rewrite (1.13) as

$$\begin{aligned} \lim_{t \rightarrow -\infty} T_{z^*(t)} W^s &= T_0 W^{ss} \oplus \text{span}(e^+) \\ \lim_{t \rightarrow +\infty} T_{z^*(t)} W^u &= \text{span}(e^-) \oplus T_0 W^{uu} \end{aligned} \tag{3.13}$$

in the notation of Section 1. We give a version of the strong λ -lemma which shows why the second limit holds, generically. The case of W^s is analogous, reversing time. Consider $\mathbf{q} = z^*(T) \in S_{in}$; see Figs. 2 and 4. The component of $W^u \cap S_{in}$ which contains \mathbf{q} is a C^{M+4} -manifold of dimension $n - 1$. Perturbing W^u slightly, if necessary, we may assume this manifold to be C^{M+4} -parametrized over y_1 . Here we have further normalized our vector field F at $\alpha = 0$ so that locally

$$\begin{aligned} W_{loc}^{uu} &= \{y_0 = 0, x = 0\}, & \text{and likewise} \\ W_{loc}^{ss} &= \{x_0 = 0, y = 0\} \end{aligned} \tag{3.14}$$

We denote the local parametrization of $W^u \cap S_{in}$ near \mathbf{q} by the map

$$\begin{aligned} (\xi, \eta): y_1 &\rightarrow (\xi(y_1), \eta(y_1)) \in S_{in} \subseteq \mathbb{R}^m \times \mathbb{R}^n \\ \eta(y_1) &= (\eta_0(y_1), y_1) \\ (\xi(0), \eta(0)) &= \mathbf{q} = (\xi(0), 0) \end{aligned}$$

Let \tilde{D}_t^{n-1} , $t \geq T$ denote the image, after time $t - T$, of $im(\xi, \eta) \subseteq S_{in}$ under the flow of F , $\alpha = 0$. Note that $z^*(t) \in \tilde{D}_t^{n-1}$. Let D_t^{n-1} denote the component of \tilde{D}_t^{n-1} in the box B_{δ_0} which contains $z^*(t)$. Finally, we recall the notation $\psi = (\psi_0, \psi_1) = \psi(x^{in}, y^{out})$ from proposition 3.2. Here, $\alpha = 0$ is suppressed.

3.3. Proposition [Strong λ -Lemma (Deng 1989a, Theorem 5.1)]. *In the above setting, assume that*

$$D_{y_0} \psi(\mathbf{q}) \notin T_{\mathbf{q}} W^u \tag{3.15}$$

Then, as $t \rightarrow +\infty$, D_t^{n-1} converges to W_{loc}^{uu} in C^k , for any fixed $0 < k \leq M + 1$.

More precisely, there exists a δ -box B_δ around $z = 0$ such that for any $\vartheta > 0$ there exists a $t^* \geq T$ with the following property. For any $t \geq t^*$ the set $B_\delta \cap D_t^{n-1}$ is given by the image of a map $y_1 \rightarrow (\xi^t, \eta^t)(y_1)$, defined for $|y_1| \leq \delta$ with $\eta_1^t(y_1) = y_1$, and the C^k -norm $\|\cdot\|_k$ satisfies

$$\|(\xi^t, \eta^t)\|_k < \vartheta$$

From this proposition, the second limit in (3.13) follows whenever the nondegeneracy condition (3.15) holds. Indeed,

$$T_{z^*(t)} W^u = \text{span}(\dot{z}^*(t)) \oplus T_{z^*(t)} D_t^{n-1}$$

The first space on the right limits onto $\text{span}(e^-)$ because $z^*(T) = \mathbf{q} \notin W^{ss}$, see (1.11). The second component, by (3.15) and the above proposition, limits onto

$$\{x = 0, y_0 = 0\} = T_0 W^{uu}$$

Incorporating time reversal, both claims of (3.13) are therefore proven.

The nondegeneracy condition (3.15) is a generic condition since we can always enforce (3.15) by a slight perturbation of the vector field F at $\alpha = 0$. Indeed, assume that δ_0 is chosen small enough and leave F unchanged inside the δ_0 -box; this fixes $D_{y_0} \psi(\mathbf{q})$ close to e^+ . But modifying F slightly along the homoclinic orbit Γ between $\mathbf{p} \in S_{out}$ and $\mathbf{q} \in S_{in}$, we can adjust $T_{\mathbf{q}} W^u$ so that it does not contain $D_{y_0} \psi(\mathbf{q})$. Therefore, (3.15) is satisfied generically.

4. LJAPUNOV-SCHMIDT REDUCTION

In this section, we derive a reduced normal form equation for N -periodic and N -homoclinic orbits of the vector field $\dot{z} = F(\alpha, z)$ near the homoclinic orbit Γ at $\alpha = (\alpha_1, \alpha_2) = 0$. We describe the flow near the origin $z = 0$ in Shilnikov's variables $x_1^{\text{in},j}, y_1^{\text{out},j}, r_j = e^{-v_0(\alpha)s_j}$, using the setting and the normalizations of Section 3. As before, j refers to iterations of the Poincaré return map

$$\Pi^{\text{tot}} = \Pi^{\text{far}} \circ \Pi^{\text{loc}}$$

where

$$\begin{aligned} \Pi^{\text{loc}}: S_{\text{in,loc}} &\rightarrow S_{\text{out}} \\ (x^{\text{in},j}, y^{\text{in},j}) &\mapsto (x^{\text{out},j}, y^{\text{out},j}) \\ \Pi^{\text{far}}: S_{\text{out,loc}} &\rightarrow S_{\text{in}} \\ (x^{\text{out},j}, y^{\text{out},j}) &\mapsto (x^{\text{in},j+1}, y^{\text{in},j+1}) \end{aligned}$$

denote the respective flow-defined maps on suitable domains. For an N -periodic orbit, we have to solve the system

$$\begin{aligned} \Psi_j = \Psi_j(\mathbf{r}, \mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}; \alpha) &:= \begin{pmatrix} x_1^{\text{in},j+1} \\ y_0^{\text{in}}(r_{j+1}, x_1^{\text{in},j+1}, y_1^{\text{out},j+1}; \alpha) \\ y_1^{\text{in}}(r_{j+1}, x_1^{\text{in},j+1}, y_1^{\text{out},j+1}; \alpha) \end{pmatrix} \\ - \Pi(x^{\text{out}}(r_j, x_1^{\text{in},j}, y_1^{\text{out},j}; \alpha), y_1^{\text{out},j}; \alpha) &= 0 \in S_{\text{in}} \quad \text{for all } j \pmod{N} \end{aligned} \quad (4.1)$$

Here $\mathbf{r}, \mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}$, denote the vectors with j th ‘‘component’’ $r_j > 0, x_1^{\text{in},j} \in \mathbb{R}^{m-1}, y_1^{\text{out},j} \in \mathbb{R}^{n-1}$. (If $r_j = 0$ for some j , then we have found a homoclinic orbit). The map Π denotes Π^{far} with the trivial x_0 -component $\Pi_{x_0}^{\text{far}} = \delta_0$ omitted. Similarly, $x_1^{\text{in}}, y_1^{\text{out}}$ can be thought of as being augmented by their trivial components

$$x_0^{\text{in}} = \delta_0, \quad y_0^{\text{out}} = \delta_0$$

whenever this is appropriate. We further normalize coordinates so that

$$\begin{aligned} x_1^{\text{in}} &= 0 \text{ resp. } y_1^{\text{out}} = 0 \text{ for} \\ &\text{our original homoclinic orbit } \Gamma \\ &\text{at } \alpha = 0 \end{aligned}$$

In other words, $(\mathbf{r}, \mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}, \alpha) = 0$ is a trivial solution of $\Psi = 0$, where Ψ has components Ψ_j , of course. We recall that, due to all the above normalizations, both our vector field F and the map Ψ are of class C^{M+4} .

In lemma 4.1 below, we solve the system

$$(\mathbf{id} - \mathbf{P}^0) \Psi(\mathbf{r}, \mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}; \alpha) = 0 \tag{4.2}$$

for $\mathbf{x}_1^{\text{in}}(\mathbf{r}, \alpha), \mathbf{y}_1^{\text{out}}(\mathbf{r}, \alpha)$, locally near the trivial solution, by the implicit function theorem, choosing a suitable projection \mathbf{P}^0 of rank N . We carefully estimate the dependence on \mathbf{r} , since differentiability with respect to r_j breaks down when $r_j = 0$. We then modify \mathbf{P}^0 slightly to become \mathbf{P}^α , a rank N projection near \mathbf{P}^0 which depends on α . In lemma 4.2, we prove a transcendental expansion for the reduced bifurcation equation

$$\Phi(\mathbf{r}, \alpha) = \mathbf{P}^\alpha \Psi(\mathbf{r}, \mathbf{x}_1^{\text{in}}(\mathbf{r}, \alpha), \mathbf{y}_1^{\text{out}}(\mathbf{r}, \alpha); \alpha) = 0 \tag{4.3}$$

see (4.8). Transforming parameters α to $\varepsilon = \varepsilon(\alpha)$ suitably, this expansion takes a normal form (4.13) which is (surprisingly) similar to the expansion (2.10) which we have obtained for the linear case. This normal form is the main goal of the present section. We finish the section by relating the sign of a coefficient in the normal form (4.13) to our geometric definition of twisted and non-twisted homoclinic orbits.

Preparing for proofs, we observe that our normalizations imply

$$\begin{aligned} x^{\text{out}}(r, x_1^{\text{in}}, y_1^{\text{out}}; \alpha) &= 0 \\ &\text{at } r = 0 \\ y^{\text{in}}(r, x_1^{\text{in}}, y_1^{\text{out}}; \alpha) &= 0 \end{aligned} \tag{4.4}$$

for all pertinent $x_1^{\text{in}}, y_1^{\text{out}}, \alpha$. Indeed, this follows directly from the definition of the Shilnikov variables since $W_{\text{loc}}^u = \{x = 0\}, W_{\text{loc}}^s = \{y = 0\}$. Less directly, we could also invoke proposition 3.2.

For later use, we repeat that the map Ψ is C^{M+4} with respect to $\mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}, \alpha$ in its domain of definition. As long as $\mathbf{r} > 0$ holds componentwise, Ψ is also C^{M+3} jointly in \mathbf{r} . When some \mathbf{r} -components tend to zero, we remember from proposition 3.2 that the derivatives of Ψ with respect to $\mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}, \alpha$ behave continuously, up to any finite order $k \leq M + 1$. The first derivative of Ψ with respect to $\mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}$ is given by

$$\begin{aligned} D_{(\mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}})} \Psi_j \cdot (\hat{\mathbf{x}}_1^{\text{in}}, \hat{\mathbf{y}}_1^{\text{out}}) &= -D_x \Pi D_{x_1} x^{\text{out}} \cdot \hat{x}_1^{\text{in}, j} \\ &\quad - (D_x \Pi D_{y_1} x^{\text{out}} \cdot \hat{y}_1^{\text{out}, j} + D_{y_1} \Pi \cdot \hat{y}_1^{\text{out}, j}) \\ &\quad + \left(\begin{array}{c} \hat{x}_1^{\text{in}, j+1} \\ D_{x_1} y^{\text{in}} \cdot \hat{x}_1^{\text{in}, j+1} + D_{y_1} y^{\text{in}} \cdot \hat{y}_1^{\text{out}, j+1} \end{array} \right) \\ &= -D_{y_1} \Pi \hat{y}_1^{\text{out}, j} + \begin{pmatrix} \hat{x}_1^{\text{in}, j+1} \\ 0 \end{pmatrix} \text{ (at } \mathbf{r} = 0) \end{aligned} \tag{4.5}$$

Indeed, the derivatives of the functions x^{out} and y^{in} vanish identically at $\mathbf{r} = 0$, by (4.4) above.

We can now define the projection \mathbf{P}^0 for the Ljapunov–Schmidt reduction; cf. (4.2) above. To define the j th component P_j^0 of \mathbf{P}^0 , we first claim that

$$D_{(x_1^{\text{in}}, y_1^{\text{out}})} \Psi_j$$

has maximal rank (corank = 1 in S_{in}) at

$$(\mathbf{r}, \mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}, \alpha) = 0$$

that is, at the homoclinic orbit Γ . Provided this claim holds true, we can then define

$$\begin{aligned} P_j^0: & \text{ the orthogonal projection onto the one-dimensional} \\ & \text{ orthogonal complement in } S_{\text{in}} \text{ of the image} \\ & \text{ of } D_{(x_1^{\text{in}}, y_1^{\text{out}})} \Psi_j \text{ at } (\mathbf{r}, \mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}, \alpha) = 0 \end{aligned} \tag{4.6}$$

Note that P_j^0 denotes the same projection for all j , by definition and by our expression (4.5) for $D_{(x_1^{\text{in}}, y_1^{\text{out}})} \Psi_j$. Moreover, $\text{rank } P_j^0 = 1$ and $\text{rank } \mathbf{P}^0 = N$. We now show that our claim actually holds, that is, $D_{(x_1^{\text{in}}, y_1^{\text{out}})} \Psi_j$ does possess maximal rank. By (4.5), it is sufficient to show that

$$D_{y_1} \Pi_{y_1} \text{ is an isomorphism}$$

when Π_{y_1} denotes the y_1 -component of Π . To identify $D_{y_1} \Pi_{y_1}$ as an isomorphism, we first observe that

$$\text{im } D_{y_1} \Pi = T_{\mathbf{q}} W^u \cap S_{\text{in}}$$

This follows because we have normalized $W_{\text{loc}}^u \cap S^{\text{out}}$ to be given by $x = 0$, $y_0 = \delta_0$. Second, motivated by the strong λ -lemma, we have assumed in (3.13) that

$$\lim_{t \rightarrow \infty} T_{z^*(t)} W^u = \text{span}(e^-) \oplus T_0 W^{uu}$$

Since we have normalized W_{loc}^{uu} in (3.14) to be given by

$$W_{\text{loc}}^{uu} = \{x = 0, y_0 = 0\}$$

and since $\mathbf{q} = z^*(T)$ for some large enough T , this implies that we have an isomorphism

$$\text{im } D_{y_1} \Pi = T_{\mathbf{q}} W^u \cap S_{\text{in}} \rightarrow T_0 W^{uu} = \{x = 0, y_0 = 0\}$$

defined by putting $x = 0$, $y_0 = 0$ (that is, orthogonal projection). Therefore, $\text{im } D_{y_1} \Pi_{y_1}$ is indeed an isomorphism, and our definition of P_j^0 is justified.

4.1. Lemma. *In the above setting, the system*

$$(\mathbf{id} - \mathbf{P}^0) \Psi(\mathbf{r}, \mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}; \alpha) = 0 \tag{4.2}$$

has unique solutions $\mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}$, for any given $\mathbf{r} \geq 0$, α , near $\mathbf{r} = 0$, $\alpha = 0$. The functions

$$\begin{aligned} \mathbf{x}_1^{\text{in}} &= \mathbf{x}_1^{\text{in}}(\mathbf{r}, \alpha) \\ \mathbf{y}_1^{\text{out}} &= \mathbf{y}_1^{\text{out}}(\mathbf{r}, \alpha) \end{aligned}$$

are C^{M+4} in (\mathbf{r}, α) for $\mathbf{r} > 0$. They are continuous in \mathbf{r} and C^k in α for $\mathbf{r} \geq 0$ and any finite $k \leq M + 1$.

Putting $\mathbf{r} = 0$, denote

$$\begin{aligned} \mathbf{x}_1^{\text{in}}(\alpha) &:= \mathbf{x}_1^{\text{in}}(0, \alpha) \\ \mathbf{y}_1^{\text{out}}(\alpha) &:= \mathbf{y}_1^{\text{out}}(0, \alpha) \end{aligned}$$

Then the components

$$x_1^{\text{in}}(\alpha) := x_1^{\text{in},j}(\alpha)$$

and

$$y_1^{\text{out}}(\alpha) := y_1^{\text{out},j}(\alpha)$$

are the same for all j . Moreover, the following estimates hold for any small $\gamma > 0$ and $|\mathbf{r}| \leq r^0(\gamma)$, $|\alpha| \leq \alpha^0(\gamma)$:

$$\begin{aligned} |\mathbf{x}_1^{\text{in}}(\mathbf{r}, \alpha) - \mathbf{x}_1^{\text{in}}(\alpha)| &\leq C |\mathbf{r}|^{1-\gamma} \\ |\mathbf{y}_1^{\text{out}}(\mathbf{r}, \alpha) - \mathbf{y}_1^{\text{out}}(\alpha)| &\leq C |\mathbf{r}|^{1-\gamma} \end{aligned} \tag{4.7}$$

Here $C = C(\gamma)$ denotes a large positive constant and $|\mathbf{r}|$ is the max-norm. If we replace the differences of $\mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}$ in (4.7) by their derivatives $D_z^i D_r^\beta$ of total order up to k , then the same estimate holds, except that $|\mathbf{r}|^{1-\gamma}$ gets replaced by $|\mathbf{r}|^{\omega-\gamma-|\beta|}$ where $|\beta| = \beta_0 + \dots + \beta_{N-1}$ for the multi-index β .

Proof. Existence, uniqueness, and regularity of

$$\mathbf{x}_1^{\text{in}}(\mathbf{r}, \alpha), \mathbf{y}_1^{\text{out}}(\mathbf{r}, \alpha)$$

follow from the implicit function theorem (Berger, 1977, p. 115) with the remarks preceding the lemma. Here we think of Ψ as being extended to all \mathbf{r} in a neighborhood of 0, allowing for components r_j to become negative. In fact, we can just extend the maps

$$(x^{\text{out}}, y^{\text{in}}) = (x^{\text{out}}, y^{\text{in}})(r, x^{\text{in}}, y^{\text{out}}, \alpha)$$

to negative values of $r \in \mathbb{R}$ by their leading terms:

$$(x^{\text{out}}, y^{\text{in}}) := (-|r|^{1/\nu_0(\alpha)} \varphi(x^{\text{in}}, y^{\text{out}}; \alpha), r\psi(x^{\text{in}}, y^{\text{out}}; \alpha)), \quad \text{for } r < 0$$

This way, $(x^{\text{out}}, y^{\text{in}})$ remain C^{M+1} in $(x^{\text{in}}, y^{\text{out}}; \alpha)$ and all derivatives depend continuously on r . Thus Berger (1977, p. 115) applies. For $\nu_0(\alpha) < 1$, the derivatives up to order M in $(x^{\text{in}}, y^{\text{out}}; \alpha)$ depend C^1 on r .

For $\mathbf{r} = 0$ we have $y^{\text{in}} = 0$ and $x^{\text{out}} = 0$. Hence (4.2) reads

$$(\text{id} - P_j^0) \left(\begin{pmatrix} x_1^{\text{in}, j+1} \\ 0 \end{pmatrix} - H(0, y_1^{\text{out}, j}; \alpha) \right) = 0 \quad \text{for } j \pmod N \quad (4.2j)$$

Since (4.2j) defines the same equation, for each j , the components

$$x_1^{\text{in}, j}(\alpha), y_1^{\text{out}, j}(\alpha)$$

are indeed independent of j .

It remains to prove the estimates (4.7). We first estimate, for small $\mathbf{r} > 0$, the partial derivative

$$\begin{aligned} |D_{\mathbf{r}} \Psi_j| &\leq |D_{r_{j+1}} y^{\text{in}}| + C |D_{r_j} x^{\text{out}}| \\ &\leq C(1 + |r_j|^{-\gamma}) \leq C |\mathbf{r}|^{-\gamma} \end{aligned}$$

according to proposition 3.2. Here and below, C denotes possibly different constants. From the implicit function theorem and the above estimate for $D_{\mathbf{r}} \Psi$ we can now conclude

$$\begin{aligned} |\mathbf{x}_1^{\text{in}}(\mathbf{r}, \alpha) - \mathbf{x}_1^{\text{in}}(\alpha)| &\leq \int_0^1 |D_{\tau} \mathbf{x}_1^{\text{in}}(\tau \mathbf{r}, \alpha) \mathbf{r}| d\tau \\ &\leq C \int \tau^{-\gamma} d\tau \cdot |\mathbf{r}|^{1-\gamma} \leq C |\mathbf{r}|^{1-\gamma} \end{aligned}$$

The estimate for $\mathbf{y}_1^{\text{out}}$ is analogous. The estimates (4.7) for (higher) derivatives of $\mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}}$ follow from the higher derivative estimates (3.10) on the functions x^{out} and y^{in} . This completes the proof. ■

We can now define the projection \mathbf{P}^α which enters into our reduced bifurcation equation

$$\Phi(\mathbf{r}, \alpha) = \mathbf{P}^\alpha \Psi(\mathbf{r}, \mathbf{x}_1^{\text{in}}(\mathbf{r}, \alpha), \mathbf{y}_1^{\text{out}}(\mathbf{r}, \alpha); \alpha) = 0 \quad (4.3)$$

We define the j th component P_j^α analogously to the projection P_j^0 which was defined in (4.6) above. We let P_j^α denote the orthogonal projection onto the one-dimensional orthogonal complement in S_{in} of the image

$$\text{im } D_{(\mathbf{x}_1^{\text{in}}, \mathbf{y}_1^{\text{out}})} \Psi_j \quad \text{at } \mathbf{r} = 0, \mathbf{x}_1^{\text{in}} = \mathbf{x}_1^{\text{in}}(\alpha), \mathbf{y}_1^{\text{out}} = \mathbf{y}_1^{\text{out}}(\alpha)$$

Let $p(\alpha) \in S_{\text{in}}$ be a unit vector which spans this complement. Note that $p(\alpha)$ can be chosen to be C^M in α . Like P_j^0 , the projection P_j^α does not depend on j . Describing P_j^α by the unit vector $p(\alpha) \in S_{\text{in}}$, our reduced bifurcation equation becomes the N -dimensional system

$$\Phi_j(\mathbf{r}, \alpha) = p(\alpha)^T \Psi_j(\mathbf{r}, \mathbf{x}_1^{\text{in}}(\mathbf{r}, \alpha), \mathbf{y}_1^{\text{out}}(\mathbf{r}, \alpha); \alpha) = 0, \quad j(\text{mod } N) \quad (4.3j)$$

Note that all x_1 -components $p_{x_1}(\alpha)$ of $p(\alpha)$ are zero, by construction and by the form (4.5) of $D_{(x_1^{\text{in}}, y_1^{\text{out}})} \Psi_j$ at $\mathbf{r} = 0$. Moreover,

$$p(\alpha)^T D_{y_1} \Pi(0, y_1^{\text{out}}(\alpha); \alpha) = 0$$

by (4.5) and the definition of $p(\alpha)$. Picking δ_0 small enough, we can therefore assume the components $p_y(\alpha)$ to be close to e^+ , that is, the unit vector in the y_0 -direction. Indeed, this is true for $\alpha = 0$ by our discussion of $\text{im } D_{y_1} \Pi$ preceding lemma 4.1. Hence, it is true for all small $|\alpha|$.

We can now state the long-desired transcendental expansion for the reduced bifurcation function Φ_j with respect to r .

4.2 Lemma. *The reduced bifurcation function Φ_j given in (4.3j) above has the expansion*

$$\Phi_j(\mathbf{r}, \alpha) = -c_0(\alpha) + c_1(\alpha)r_{j+1} - c_2(\alpha)r_j^{1/v_0(\alpha)} + O(|\mathbf{r}|^{1+\omega}), \quad j(\text{mod } N) \quad (4.8)$$

with

$$\begin{aligned} c_0(\alpha) &:= p(\alpha)^T \Pi(0, y_1^{\text{out}}(\alpha); \alpha) \\ c_1(\alpha) &:= p_y(\alpha)^T \psi(x^{\text{in}}(\alpha), y^{\text{out}}(\alpha); \alpha) \\ c_2(\alpha) &:= p(\alpha)^T D_{x^{\text{out}}} \Pi(0, y_1^{\text{out}}(\alpha); \alpha) \cdot \varphi(x^{\text{in}}(\alpha), y^{\text{out}}(\alpha); \alpha) \end{aligned} \quad (4.9)$$

all of class C^M . Here

$$x^{\text{in}}(\alpha) = (\delta_0, x_1^{\text{in}}(\alpha)), \quad y^{\text{out}}(\alpha) = (\delta_0, y_1^{\text{out}}(\alpha))$$

as usual. The functions φ and ψ were introduced in proposition 3.2. The expansion (4.8) is understood in the following sense. There exists a small $\omega > 0$ such that the remainder term O is estimated, up to a constant factor, by $|\mathbf{r}|^{1+\omega}$. This estimate holds uniformly for $|\alpha| < \alpha_0(\omega)$ including derivatives in α of order up to any finite $k \leq M$.

For $\mathbf{r} > 0$, all terms in the expansion (4.8) are C^M . The higher derivatives $D_\alpha^i D_{\mathbf{r}}^\beta \Phi_j$, up to total order k , satisfy the analogously differentiated expansion with the error term replaced by

$$O(|\mathbf{r}|^{1+\omega-|\beta|})$$

where $|\beta| = \beta_0 + \dots + \beta_{N-1}$.

Proof. To prove the expansion (4.8), we write (omitting arguments)

$$\Phi_j = p^T \Psi_j = p_y^T y^{\text{in}} - p^T \Pi$$

and expand the y^{in} -term and the Π -term, separately, with respect to \mathbf{r} . We consider y^{in} first, using Shilnikov's expansion (proposition 3.2) and the estimate (4.7) from lemma 4.1.

$$\begin{aligned} & y^{\text{in}}(r_{j+1}, x_1^{\text{in},j+1}(\mathbf{r}, \alpha), y_1^{\text{out},j+1}(\mathbf{r}, \alpha); \alpha) \\ &= r_{j+1}(\psi(x^{\text{in},j+1}(\mathbf{r}, \alpha), y^{\text{out},j+1}(\mathbf{r}, \alpha); \alpha) + R_y) \\ &= \psi(x^{\text{in}}(\alpha), y^{\text{out}}(\alpha); \alpha) r_{j+1} + O(|\mathbf{r}|^{1+\omega}) \end{aligned} \tag{4.10}$$

This yields the term $c_1(\alpha)r_{j+1}$ in our expansion.

Next we expand $p^T \Pi$ at $\mathbf{r} = 0$:

$$\begin{aligned} p^T \Pi &= p(\alpha)^T \Pi(x^{\text{out}}(r_j, x_1^{\text{in},j}(\mathbf{r}, \alpha), y_1^{\text{out},j}(\mathbf{r}, \alpha); \alpha), y_1^{\text{out},j}(\mathbf{r}, \alpha); \alpha) \\ &= p(\alpha)^T \Pi(x^{\text{out}}(0, \dots), y_1^{\text{out}}(\alpha); \alpha) \\ &\quad + p(\alpha)^T D_{(x^{\text{out}}, y_1^{\text{out}})} \Pi(x^{\text{out}}(0, \dots), y_1^{\text{out}}(\alpha); \alpha) \\ &\quad \cdot \begin{pmatrix} x^{\text{out}}(r_j, x_1^{\text{in},j}(\mathbf{r}, \alpha), y_1^{\text{out},j}(\mathbf{r}, \alpha); \alpha) \\ y_1^{\text{out},j}(\mathbf{r}, \alpha) - y_1^{\text{out}}(\alpha) \end{pmatrix} \\ &\quad + O(|\mathbf{r}|^{1+\omega}) \end{aligned} \tag{4.11}$$

Indeed, $x^{\text{out}}(0, \dots) = 0$ and the terms following $D\Pi$ will now be estimated to be small of order $O(|\mathbf{r}|^{1-\gamma})$, where $\gamma > 0$ was chosen small in lemma 4.1. Then their squared norms contribute less than $O(|\mathbf{r}|^{1+\omega})$, proving (4.11). Concerning $y_1^{\text{out},j}(\mathbf{r}, \alpha) - y_1^{\text{out}}(\alpha)$ we have an $O(|\mathbf{r}|^{1-\gamma})$ -estimate from lemma 4.1. The term $x^{\text{out}}(r_j, \dots)$ can be estimated similarly as $y^{\text{in}}(r_{j+1}, \dots)$ has been estimated in (4.10) above, that is,

$$\begin{aligned} & x^{\text{out}}(r_j, x_1^{\text{in},j}(\mathbf{r}, \alpha), y_1^{\text{out},j}(\mathbf{r}, \alpha); \alpha) \\ &= \varphi(x^{\text{in}}(\alpha), y^{\text{out}}(\alpha); \alpha) r_j^{1/\nu_0(\alpha)} + O(|\mathbf{r}|^{1+\omega}) \end{aligned} \tag{4.12}$$

since $\nu_0(0) = 1$. With (4.12) at hand, we can now resume our estimate (4.11), using that $p(\alpha)^T D_{y_1} \Pi(0, y_1^{\text{out}}(\alpha); \alpha) = 0$ by the properties of the projection $p(\alpha)^T$. We get

$$\begin{aligned} p^T \Pi &= c_0(\alpha) + p(\alpha)^T D_{x^{\text{out}}} \Pi(0, y_1^{\text{out}}(\alpha); \alpha) \\ &\quad \cdot \varphi(x^{\text{in}}(\alpha), y^{\text{out}}(\alpha); \alpha) r_j^{1/\nu_0(\alpha)} + O(|\mathbf{r}|^{1+\omega}) \\ &= c_0(\alpha) + c_2(\alpha) r_j^{1/\nu_0(\alpha)} + O(|\mathbf{r}|^{1+\omega}) \end{aligned}$$

This proves the transcendental expansion (4.8).

The claims concerning differentiability of the expansion follow from the corresponding differentiability statements in lemma 4.1 and proposition 3.2. The proof is therefore complete. ■

4.3. Corollary. *For generic (normalized) vector fields $F = F(\alpha, z)$ of class C^{M+4} , there exists a diffeomorphic local change of parameters $\varepsilon = \varepsilon(\alpha) \in \mathbb{R}^2$ of class C^M , such that the bifurcation equation (4.8) takes the equivalent normal form*

$$r_{j+1} = \varepsilon_2 + a(\varepsilon)r_j^{1+\varepsilon_1} + O(|\mathbf{r}|^{1+\omega}), \quad j \pmod N \tag{4.13}$$

with $|a(0)| \neq 0, 1$. The error term has the same meaning as in lemma 4.2.

Specifically, the genericity assumptions and the parameter transformation are given as follows. We define

$$\begin{aligned} (a) \quad \varepsilon_1 &:= \frac{1}{v_0(\alpha)} - 1 \\ (b) \quad \varepsilon_2 &:= \frac{c_0(\alpha)}{c_1(\alpha)} \\ (c) \quad a(\varepsilon) &:= \frac{c_2(\alpha)}{c_1(\alpha)} \end{aligned} \tag{4.14}$$

under the generic assumptions

$$\begin{aligned} (a) \quad &D_x c_0(0) \text{ and } D_x v_0(0) \text{ are linearly independent} \\ (b) \quad &c_1(0) > 0 \\ (c) \quad &\left| \frac{c_2(\alpha)}{c_1(\alpha)} \right| \neq 0, 1 \end{aligned} \tag{4.15}$$

We note explicitly that $\varepsilon = 0$ at $\alpha = 0$, since $v_0(0) = 1$ and $c_0(0) = 0$.

Proof. We first prove that $v_0 = 1, c_0 = 0$ at $\alpha = 0$. Then we indicate why the nondegeneracy assumptions (4.15a–c) are generic. The proof of our claims about the coordinate transformation $\varepsilon = \varepsilon(\alpha)$ is then an obvious consequence of the standard inverse function theorem, applied to equations (4.14a, b) in ε and α .

Actually, $v_0 = 1$ at $\alpha = 0$ since $v_0 = \mu_0$, by our resonance assumption (1.9), and since we have normalized μ_0 to be one. Moreover, Γ is a 1-homoclinic orbit at $\alpha = 0$. Therefore, $\mathbf{r} = 0$ is a solution of the reduced bifurcation equation (4.8) at $\alpha = 0$. (We still remember that infinite

Shilnikov time s_j corresponds both to a homoclinic orbit and to $r_j = 0$.) Hence $c_0 = 0$ at $\alpha = 0$.

We now address genericity of our assumptions (4.15a–c): we claim that (4.15a–c) hold, possibly after a slight perturbation of the original vector field F which does not affect any of the remaining assumptions. Let us compute $D_\alpha c_0(0)$ first, for $c_0(\alpha)$ as defined in (4.9). We obtain

$$D_\alpha c_0(0) = p(0)^T D_\alpha \Pi(0, 0; \alpha)|_{\alpha=0} \tag{4.16}$$

since $y_1(0) = 0$, $\Pi(0, 0; 0) = 0$ and since, by definition of $p(0)$, also

$$p(0)^T D_{y_1} \Pi = 0 \quad \text{at} \quad (x^{\text{out}}, y_1^{\text{out}}; \alpha) = (0, 0; 0)$$

On the other hand, $v_0(\alpha)$ is the principal unstable eigenvalue of the linearization $D_z F(z, \alpha)$ at the equilibrium $z = 0$. Adjusting both $D_\alpha D_z F$ and $D_\alpha \Pi$ slightly, at these points, linear independence of $D_\alpha c_0(0)$ and $D_\alpha v_0(0)$ can easily be achieved.

Next we prove $c_1 = p_y^T \psi > 0$. Indeed, we have already picked δ_0 small enough so that $p(\alpha)$ is close to e^+ , the unit vector in the positive y_0 -direction. See the properties of $p(\alpha)$ listed just before lemma 4.2. Therefore, $c_1(0)$ is close to $\psi_0(0, (\delta_0, 0); 0)$, which is close to $\delta_0 > 0$, by property (3.11c) of ψ . This proves $c_1 > 0$.

Similarly, we can assume $c_2 = p^T D_{x^{\text{out}}} \Pi \cdot \varphi \neq 0$ since $\varphi(x^{\text{in}}(\alpha), y^{\text{out}}(\alpha); \alpha)$ points roughly in the x_0 -direction and since $D_{x^{\text{out}}} \Pi(0, y_1^{\text{out}}(\alpha); \alpha)$ can be adjusted accordingly. Adjusting $D_{x^{\text{out}}} \Pi$ once more, we can finally guarantee

$$|c_2(0)/c_1(0)| \neq 1$$

completing the proof. ■

4.4. Remark. We note that we may assume $|a(0)| > 1$, without loss of generality, for our discussion of the normal form (4.13) in Section 5. Indeed, use the alternative scaling $\hat{r}_j := e^{-s_j}$ instead of $r_j := e^{-v_0(\alpha)s_j}$. Then expansion (4.8) reads

$$\Phi_j(\hat{\mathbf{r}}, \alpha) = -c_0(\alpha) + c_1(\alpha) \hat{r}_{j+1}^{v_0(\alpha)} - c_2(\alpha) \hat{r}_j + O(|\hat{\mathbf{r}}|^{1+\omega}), \quad j \pmod N \tag{4.8^{\wedge}}$$

where the $c_i(\alpha)$ are defined by (4.9), as before. This yields the normal form

$$\hat{r}_j = \hat{\varepsilon}_2 + \hat{a}(\hat{\varepsilon}) \hat{r}_{j+1}^{1+\hat{\varepsilon}_1} + O(|\hat{\mathbf{r}}|^{1+\omega}), \quad j \pmod N \tag{4.13^{\wedge}}$$

where $\hat{\varepsilon}$, \hat{a} now relate to α , $v_0(\alpha)$ by

$$\begin{aligned}
 \text{(a)} \quad \hat{\varepsilon}_1 &:= v_0(\alpha) - 1 \\
 \text{(b)} \quad \hat{\varepsilon}_2 &:= -\frac{c_0(\alpha)}{c_2(\alpha)} \\
 \text{(c)} \quad \hat{a}(\hat{\varepsilon}) &:= \frac{c_1(\alpha)}{c_2(\alpha)} = \frac{1}{a(\varepsilon)}
 \end{aligned}
 \tag{4.14'}$$

This proves our remark about $a(0)$.

As a direct consequence of the normal form (4.13) for resonant homoclinic bifurcation we obtain corollary 4.5.

4.5. Corollary. *Along the line $\varepsilon_2 = 0$, there exists a 1-homoclinic orbit ($N = 1$) given by $r_0 = 0$ (primary homoclinic branch). As ε_1 passes through zero along this branch, the principal eigenvalues $-\mu_0 = -1$ and $v_0 = (1 + \varepsilon_1)^{-1}$ cross their resonance transversely. In the transformed parameter ε , this reads*

$$\begin{aligned}
 -\mu_0(\varepsilon) + v_0(\varepsilon) &= 0 \\
 D_{\varepsilon_1}(-\mu_0(\varepsilon) + v_0(\varepsilon)) &\neq 0, \quad \text{at } \varepsilon = 0
 \end{aligned}
 \tag{4.17}$$

4.6. Lemma. *The twist of the homoclinic orbit Γ determines the sign of $a_0 = a(0)$, namely,*

$$\begin{aligned}
 a_0 > 0 & \quad \text{if } \Gamma \text{ is nontwisted} \\
 a_0 < 0 & \quad \text{if } \Gamma \text{ is twisted}
 \end{aligned}$$

Proof. Since $a = c_2/c_1$ and since $c_1 > 0$ by corollary 4.3 (4.15b), we only have to prove that c_2 is negative if Γ is twisted and positive if Γ is nontwisted.

To relate $c_2 = p^T D_{x_{\text{out}}} \Pi \cdot \varphi$ to the twist we observe the following, near $\alpha = 0$. First, $p(\alpha)$ is close to e^+ and spans the orthogonal complement to the space $T_q = T_q W^s + T_q W^u$. This follows from the properties of $p(\alpha)$, which are listed just above lemma 4.2, since T_q is spanned by the x -space and $\text{im } D_{y_1} \Pi$. Second, φ points in the positive x_0 -direction, again by proposition 3.2. In other words, $(\varphi, 0)$ is close to $\delta_0 e^-$. Third, e^- complements the space $T_p = T_p W^s + T_p W^u$, by the strong λ -lemma. Consequently, $D_{x_{\text{out}}} \Pi \cdot \varphi$ complements T_q , too.

The proof can now be completed as follows. The complement e^- induces an orientation r_p on T_p , say, such that $e_p \wedge e^-$ is positive. Here e_p denotes an alternating linear $(m + n - 1)$ -form: the volume form on T_p .

Following the linearized flow along Γ , e_p continues to an orientation e_q on T_q such that $e_q \wedge D\Pi e^-$ is also positive. Since $(\varphi, 0)$ is close to e_- , this implies positivity of $e_q \wedge D_{x^{out}}\Pi\varphi$. In other words, e^- and $D_{x^{out}}\Pi\varphi$ point to the same side of T_p and T_q , respectively. Since $p(\alpha)$ is orthogonal to T_q , we know, on the other hand, that

$$c_2(\alpha) = p(\alpha)^T \cdot D_{x^{out}}\Pi\varphi$$

is positive resp. negative if $p(\alpha)$ and $D_{x^{out}}\Pi\varphi$ point to the same resp. to opposite sides of T_q . Since $p(\alpha)$ is close to e^+ we can combine all this as follows:

$$\begin{aligned} \text{sign}(e_q \wedge e^+) &= \text{sign}(e_q \wedge p(\alpha)) \\ &= \text{sign } c_2(\alpha) \cdot \text{sign}(e_q \wedge D_{x^{out}}\Pi\varphi) \\ &= \text{sign } c_2(\alpha) \cdot \text{sign}(e_p \wedge e^-) \end{aligned}$$

By definition 1.1, Γ is nontwisted if $\text{sign}(e_q \wedge e^+) = \text{sign}(e_p \wedge e^-)$ and twisted otherwise. Since $\text{sign } c_2 = \text{sign } a$, this completes the proof. ■

5. THE SIDE-SWITCHING BIFURCATION

In this section, we complete the proof of theorem A on resonant side-switching. We recall from corollary 4.3 that all 1-homoclinic (1-hom) and 1-periodic (1-per) solutions correspond, locally near $\alpha=0$ and near the homoclinic orbit Γ , to solutions r, ε of the reduced bifurcation equation

$$r = \varepsilon_2 + a(\varepsilon)r^{1+\varepsilon_1} + O(r^{1+\omega}) \tag{5.1}$$

Here $\varepsilon = \varepsilon(\alpha) = (\varepsilon_1, \varepsilon_2)$ are the new normalized parameters, and $r = e^{-v(\alpha)s} \geq 0$ describes the exponentially rescaled Shilnikov time associated to the 1-hom resp. the 1-per orbits. Note that we have written r instead of r_j , since we are only interested in 1-per and 1-hom orbits, in this section.

To prove theorem A, we first observe that r, ε_1 parametrize the (local) solution set of (5.1). Indeed, a straightforward application of the implicit function theorem enables us to solve (5.1) for ε_2 by a C^0 -function

$$\varepsilon_2 = \varepsilon_2(r, \varepsilon_1), \quad r \geq 0 \tag{5.2}$$

near $\varepsilon = 0, r = 0$. For $r > 0$, the function ε_2 is C^M .

We now consider the surface (5.2) in some more detail. We recall from lemma 4.6 that $a(\varepsilon)$ is positive, since Γ is assumed to be nontwisted. By

remark 4.4 above, we can further assume $a > 1$, by a rescaling argument. This implies that

$$\varepsilon_2 = \varepsilon_2(r, \varepsilon_1) \leq 0 \quad \text{for } r \geq 0, \quad \varepsilon_1 \leq 0$$

as long as $r \leq \rho_0$ is small enough. Indeed,

$$\varepsilon_2 = r(1 - a(\varepsilon)r^{\varepsilon_1} + O(r^\omega)) \leq 0 \tag{5.3}$$

for $\varepsilon_1 \leq 0 \leq r \leq \rho_0$ and suitably small ρ_0 , since $a(0) > 1$. This proves our claims in the left part ($\varepsilon_1 \leq 0$) of region IV; see Table I and Fig. 3a of a resonant side switching for these claims.

Our strategy for completing the proof of theorem A, now, is the following. In lemma 5.2. below we prove that:

$$\varepsilon_2 = \varepsilon_2(r, \varepsilon_1) = 0 \tag{5.4}$$

has a unique nontrivial small solution $r = r(\varepsilon_1) > 0$, for small $\varepsilon_1 > 0$. Of course, there is also the trivial solution $r = 0$ which describes the 1-hom branch; cf. corollary 4.5. In lemma 5.3 we show that, for fixed ε_1 , the C^M -function

$$r \mapsto \varepsilon_2 = \varepsilon_2(r, \varepsilon_1), \quad r > 0 \tag{5.5}$$

has zero derivative only along a C^M -curve

$$\varepsilon_2 = \kappa(\varepsilon_1), \quad \varepsilon_1 > 0$$

where the derivative changes sign. We also show that κ is exponentially flat as required in (1.16). By strict monotonicity of $\varepsilon_2(\cdot, \varepsilon_1)$ off the curve κ , lemma 5.2 and 5.3 then imply all claims concerning multiplicities in the various regions of Table I and Fig. 3a. Indeed, Fig. 3a is just the projection of the surface $\varepsilon_2 = \varepsilon_2(r, \varepsilon_1)$ into $(\varepsilon_1, \varepsilon_2)$ -space with fold line κ .

We need the following basic lemma about solutions of a transcendental equation.

5.1. Lemma. *Let $b = b(\varepsilon)$ be a C^1 -function such that $b(0) > 1$. Consider the*

$$1 = b(\varepsilon)r^{\varepsilon_1} + O(r^\omega), \quad \varepsilon_1 > 0 \tag{5.6}$$

with $\omega > 0$. As before, the error term $O(r^\omega)$ is understood to allow differentiation. Then, for $\varepsilon_1 \leq 0$, equation (5.6) cannot have solutions with small $r \geq 0$. For $\varepsilon_1 > 0$, equation (5.6) can be solved for $\rho := r^{\varepsilon_1}$ as a C^1 -func-

tion of ε , locally near $\varepsilon = (\varepsilon_1, \varepsilon_2) = 0$, $\rho(0) = 1/b(0)$, by the implicit function theorem. If moreover $\varepsilon_2 = o(\varepsilon_1)$ for $\varepsilon_1 \searrow 0$, then the following limit exists

$$\lim_{\varepsilon_1 \searrow 0} r(\varepsilon) \cdot b(0)^{1/\varepsilon_1} = \exp(-D_{\varepsilon_1} b(0)/b(0)) \tag{5.7}$$

Proof. Solving (5.6) for $\rho := r^{\varepsilon_1}$ is standard if we extend (5.6) to $\varepsilon_1 \leq 0$ formally by defining

$$O(r^\omega) = O(\rho^{\omega/\varepsilon_1}) := 0 \quad \text{for } \varepsilon_1 \leq 0$$

We obtain

$$\rho(\varepsilon) = \frac{1}{b(0)} \left(1 - \frac{D_\varepsilon b(0)}{b(0)} \varepsilon + o(\varepsilon) \right)$$

for small $|\varepsilon|$. If we also assume $\varepsilon_2 = o(\varepsilon_1)$, then we have

$$r(\varepsilon) = \rho(\varepsilon)^{1/\varepsilon_1} = b(0)^{-1/\varepsilon_1} \left(1 - \frac{D_{\varepsilon_1} b(0)}{b(0)} \varepsilon_1 + o(\varepsilon_1) \right)^{1/\varepsilon_1}$$

which proves our convergence claim. ■

We now return to our reduced bifurcation equation.

$$r = \varepsilon_2 + a(\varepsilon)r^{1+\varepsilon_1} + O(r^{1+\omega}) \tag{5.1}$$

5.2. Lemma. *Along the line $\varepsilon_2 = 0$, equation (5.1) has the following solution set (locally near $\varepsilon = 0, r = 0$):*

- (i) *the trivial solution $r = 0$, for all ε_1*
- (ii) *a nontrivial positive solution $r = r_0(\varepsilon_1)$, for $\varepsilon_1 > 0$, with limiting behavior*

$$\lim_{\varepsilon_1 \searrow 0} r_0(\varepsilon_1) \cdot a(0)^{1/\varepsilon_1} = \exp(-D_{\varepsilon_1} a(0)/a(0)) \tag{5.8}$$

Proof. For $\varepsilon_2 = 0$, we divide (5.1) by r ;

$$1 = a(\varepsilon)r^{\varepsilon_1} + O(r^\omega)$$

Then all claims follow from lemma 5.1 with $b := a$. ■

It remains to consider fold points of the surface of 1-per solutions

$$r \mapsto \varepsilon_2 = \varepsilon_2(r, \varepsilon_1) \tag{5.5}$$

projected onto ε -space. In other words, we look for (r, ε_1) such that

$$D_r \varepsilon_2(r, \varepsilon_1) = 0$$

5.3. Lemma. *Locally near $\varepsilon = 0$, $r = 0$, the fold points (r, ε_1) of the 1-per surface are given in the form $r = r_*(\varepsilon_1)$, $\varepsilon_1 > 0$. The function r_* is of class C^M and has the limiting behavior*

$$\lim_{\varepsilon_1 \searrow 0} r_*(\varepsilon_1) a(0)^{1/\varepsilon_1} = \exp(-1 - D_{\varepsilon_1} a(0)/a(0)) \tag{5.9}$$

At $r = r_*(\varepsilon_1)$, the derivative $r \mapsto D_r \varepsilon_2(r, \varepsilon_1)$ changes sign.

The projected fold curve

$$\varepsilon_1 \mapsto \varepsilon_2 = \kappa(\varepsilon_1) := \varepsilon_2(r_*(\varepsilon_1), \varepsilon_1)$$

is also C^M , for $\varepsilon_1 > 0$, and has the limiting behavior

$$\lim_{\varepsilon_1 \searrow 0} \kappa(\varepsilon_1) \cdot \frac{1}{\varepsilon_1} a(0)^{1/\varepsilon_1} = \exp(-1 - D_{\varepsilon_1} a(0)/a(0)) \tag{5.10}$$

Proof. Solving (5.1) for ε_2 , by the implicit function theorem, we have obtained

$$r = \varepsilon_2(r, \varepsilon_1) + a(\varepsilon_1, \varepsilon_2(r, \varepsilon_1)) r^{1+\varepsilon_1} + O(r^{1+\omega}) \tag{5.11a}$$

Recall that $\varepsilon_2 = \varepsilon_2(r, \varepsilon_1)$ is C^M as long as $r > 0$. Hence the fold condition, $D_r \varepsilon_2 = 0$, holds if and only if, in addition,

$$1 = (1 + \varepsilon_1) a(\varepsilon_1, \varepsilon_2) r^{\varepsilon_1} + O(r^\omega), \quad r > 0 \tag{5.11b}$$

with $\varepsilon_2 = \varepsilon_2(r, \varepsilon_1)$. Indeed,

$$1 = D_r \varepsilon_2 + (1 + \varepsilon_1) a r^{\varepsilon_1} + D_{\varepsilon_2} a \cdot D_r \varepsilon_2 \cdot r^{1+\varepsilon_1} + O(r^\omega) \tag{5.11b'}$$

and the term $D_{\varepsilon_2} a \cdot D_r \varepsilon_2 \cdot r^{1+\varepsilon_1}$ can be absorbed in the error term $O(r^\omega)$, for fixed $0 < \omega < 1$. Note that the right-hand side of (5.11b) is strictly increasing with r . Hence (5.11b) can have, at most, one solution $r = r_*(\varepsilon_1)$, and $r \mapsto D_r \varepsilon_2(r, \varepsilon_1)$ changes sign at $r = r_*(\varepsilon_1)$.

More specifically, we can apply lemma 5.1 and solve (5.11b) for $\rho = r^{\varepsilon_1}$, $\varepsilon_1 > 0$, as a function of ε_1 and ε_2 near $\varepsilon = 0$. Note that

$$r_*(\varepsilon_1)^{\varepsilon_1} = \rho(\varepsilon_1, \varepsilon_2(r_*(\varepsilon_1), \varepsilon_1)) \tag{5.12}$$

and, by standard differentiation of (5.11b) with respect to ε at $\varepsilon = 0$,

$$\begin{aligned} (0, 0) &= (a\rho, 0) + \rho D_\varepsilon a + a D_\varepsilon \rho \\ &= (1, 0) + \rho D_\varepsilon a + a D_\varepsilon \rho \end{aligned} \tag{5.13}$$

since $a\rho = 1$ at $\varepsilon = 0$. In particular, $r_*(\varepsilon_1)$ depends differentiably on $\varepsilon_1 > 0$, by (5.12), since $D_r \varepsilon_2(r_*(\varepsilon_1), \varepsilon_1) = 0$. In fact, $r_*(\varepsilon_1)$ is still C^M for $\varepsilon_1 > 0$.

It remains to prove the exponential asymptotics (5.9), (5.10) for $r_*(\varepsilon_1)$ and $\kappa(\varepsilon_1) = \varepsilon_2(r_*(\varepsilon_1), \varepsilon_1)$. Of course, we would like to utilize lemma 5.1 again. As a first step, we claim

$$\kappa(\varepsilon_1) = \varepsilon_2(r_*(\varepsilon_1), \varepsilon_1) = o(\varepsilon_1), \quad \text{for } \varepsilon_1 \searrow 0 \tag{5.14}$$

Indeed, κ extends continuously down to $\kappa(0) := 0$ just as $r_*(\varepsilon_1) \in [0, r_0(\varepsilon_1)]$ does. Moreover, κ is of class C^M for $\varepsilon_1 > 0$ with derivative

$$\kappa'(\varepsilon_1) = D_{\varepsilon_1} \varepsilon_2 = -r_*^{1+\varepsilon_1}(D_{\varepsilon_1} a + D_{\varepsilon_2} a \cdot D_{\varepsilon_1} \varepsilon_2 + a \log r_*) + O(r_*^{1+\omega})$$

Here we have used $D_r \varepsilon_2(r^*(\varepsilon_1), \varepsilon_1) = 0$, and we have differentiated (5.11a) to compute $D_{\varepsilon_1} \varepsilon_2$. Obviously,

$$\lim_{\varepsilon_1 \searrow 0} \kappa'(\varepsilon_1) = 0$$

since $r_*(\varepsilon_1) \rightarrow 0$ as $\varepsilon_1 \searrow 0$. Therefore, κ is differentiable down to $\varepsilon_1 = 0$, $\kappa'(0) = 0$. This proves claim (5.14).

Now, lemma 5.1 with $b := (1 + \varepsilon_1)a$ applies to the solution $r = r_*$ of (5.11b). This implies

$$\lim_{\varepsilon_1 \searrow 0} r_*(\varepsilon_1) a(0)^{1/\varepsilon_1} = \exp(-1 - D_{\varepsilon_1} a(0)/a(0))$$

as was claimed in (5.9), since $\varepsilon_2 = o(\varepsilon_1)$ by (5.14). To prove (5.10) we compute from (5.11a)

$$\begin{aligned} \lim_{\varepsilon_1 \searrow 0} \kappa(\varepsilon_1) &= \lim_{\varepsilon_1 \searrow 0} \frac{1}{\varepsilon_1} a(0)^{1/\varepsilon_1} \\ &= \lim_{\varepsilon_1 \searrow 0} \varepsilon_2(r_*(\varepsilon_1), \varepsilon_1) a(0)^{1/\varepsilon_1} / \varepsilon_1 \\ &= \lim_{\varepsilon_1 \searrow 0} r_*(\varepsilon_1) a(0)^{1/\varepsilon_1} (1 - a(\varepsilon) r_*(\varepsilon_1)^{\varepsilon_1} + O(r_*^\omega)) / \varepsilon_1 \\ &= \lim_{\varepsilon_1 \searrow 0} r_*(\varepsilon_1) a(0)^{1/\varepsilon_1} (1 - a(\varepsilon) \rho(\varepsilon) + O(r_*^\omega)) / \varepsilon_1 \\ &= \lim_{\varepsilon_1 \searrow 0} r_*(\varepsilon_1) a(0)^{1/\varepsilon_1} ((-\rho(0) D_\varepsilon a(0) - a(0) D_\varepsilon \rho(0)) \cdot \varepsilon + o(\varepsilon)) / \varepsilon_1 \\ &= \lim_{\varepsilon_1 \searrow 0} r_*(\varepsilon_1) a(0)^{1/\varepsilon_1} (\varepsilon_1 + o(\varepsilon)) / \varepsilon_1 \\ &= \lim_{\varepsilon_1 \searrow 0} r_*(\varepsilon_1) a(0)^{1/\varepsilon_1} = \exp(-1 - D_{\varepsilon_1} a(0)/a(0)) \end{aligned}$$

Here we have used (5.12), (5.13) to introduce ρ and to compute $D_\varepsilon \rho(0)$. The second last equality holds because $\varepsilon_2 = o(\varepsilon_1)$, by claim (5.14) above. Thus $\kappa(\varepsilon_1)$ is exponentially flat at $\varepsilon_1 = 0$, and the proofs of lemma 5.3 and of theorem A are complete. ■

6. THE HOMOCLINIC DOUBLING BIFURCATION

In this section, we complete the proof of theorem B on homoclinic doubling. We recall from corollary 4.3 that all N -hom (homoclinic) and N -per (periodic) solutions with $N = 1, 2$ correspond, locally near $\alpha = 0$ and near the 1-hom Γ , to solutions (r_0, r_1, ε) of the reduced bifurcation equation

$$\begin{aligned} 0 &= -r_1 + \varepsilon_2 + a(\varepsilon)r_0^{1+\varepsilon_1} + h_0(\mathbf{r}, \varepsilon) \\ 0 &= -r_0 + \varepsilon_2 + a(\varepsilon)r_1^{1+\varepsilon_1} + h_1(\mathbf{r}, \varepsilon) \end{aligned} \tag{6.1}$$

Here $\mathbf{r} = (r_0, r_1)$ are the exponentially rescaled Shilnikov times associated to the 1-per ($r_0 = r_1 > 0$), the 1-hom ($r_0 = r_1 = 0$), the 2-per ($0 < r_0 \neq r_1 > 0$), or the 2-hom ($r_0 = 0 < r_1$ or $r_1 = 0 < r_0$) orbits. As usual, $\varepsilon = \varepsilon(\alpha) = (\varepsilon_1, \varepsilon_2)$ are the new normalized parameters. Since Γ is assumed to be twisted, we can assume $a(0) < -1$; see remark 4.4 and lemma 4.6. The error terms

$$h_j(\mathbf{r}, \varepsilon) = O(|\mathbf{r}|^{1+\omega})$$

have the following symmetry at solutions $(\mathbf{r}, \varepsilon)$ of (6.1):

$$h_0(r_0, r_1, \varepsilon) = h_1(r_1, r_0, \varepsilon) \tag{6.2}$$

Indeed, $\mathbf{r} = (r_0, r_1)$ is a solution whenever $\bar{\mathbf{r}} = (r_1, r_0)$ is a solution, because (6.1) was derived from a Poincaré-type section of a flow.

Our strategy of proof for theorem B is the following. In lemma 6.1 we show that the 1-per solutions $r_0 = r_1 > 0$ form a (local) sheet $\varepsilon_2 = \varepsilon_2(r, \varepsilon_1) > 0$, parametrized over $r \geq 0$ and ε_1 . By monotonicity arguments, this sheet projects one-to-one onto the (local) half-space $\{(\varepsilon_1, \varepsilon_2) \mid \varepsilon_2 > 0\}$. The 1-per sheet extends down to the 1-hom branch $r = 0, \varepsilon_2 = 0$, which was found in corollary 4.5. In lemma 6.2, we find a unique C^M -curve $\varepsilon_2 = \kappa_{\text{per}}(\varepsilon_1) > 0, \varepsilon_1 > 0$, where an ordinary period doubling bifurcation occurs the 1-per sheet. The curve κ_{per} is shown to have the correct exponential asymptotic behavior for $\varepsilon_1 \searrow 0$. Lemma 6.3, on the other hand, detects a unique C^M -curve $\varepsilon_2 = \kappa_{\text{hom}}(\varepsilon_1) > 0, \varepsilon_1 > 0$, where 2-hom orbits occur. We also show the universal scaling property

$$\lim_{\varepsilon_1 \searrow 0} \frac{\kappa_{\text{hom}}(\varepsilon_1)}{\kappa_{\text{per}}(\varepsilon_1)} = \frac{e}{2} \tag{6.3}$$

In lemma 6.4, we show that there exist C^M -curves $0 < \underline{\kappa}(\varepsilon_1) < \bar{\kappa}(\varepsilon_1), \varepsilon_1 > 0$, such that 2-per solutions do not occur except possibly for pairs $(\varepsilon_1, \varepsilon_2)$ with $\underline{\kappa}(\varepsilon_1) < \varepsilon_2 < \bar{\kappa}(\varepsilon_1)$. Our estimating curves $\underline{\kappa}, \bar{\kappa}$ have the same asymptotics as the curves $\frac{1}{2}\kappa_{\text{per}}, 2\kappa_{\text{hom}}$, but we cannot force them to

coincide with these latter curves. Finally, we show in lemma 6.5 that there exists a two-dimensional continuum \mathcal{C} of 2-per solutions (r, ε) which connects the 2-hom solutions on the κ_{hom} -curve to the period doubling solutions on the κ_{per} -curve. This 2-per continuum \mathcal{C} is confined to the region of ε between the two curves $\underline{\kappa}, \bar{\kappa}$. To find \mathcal{C} , we use topological methods which are due to Alexander and Antman (1981). Therefore, we cannot estimate the precise number of 2-per solutions.

The following three lemmas are much in the spirit of Section 5; we assume that the reader is now familiar with the arguments given there. Our first lemma deals with 1-per/1-hom solutions, that is, solutions of

$$r = \varepsilon_2 + a(\varepsilon)r^{1+\varepsilon_1} + O(r^{1+\omega}) \tag{6.4}$$

Recall that $a(0) < -1$, in this section.

6.1. Lemma. *Locally near $\varepsilon = 0, r = 0$ the solutions of (6.4) can be parametrized over (r, ε_1) such that*

$$\varepsilon_2 = \varepsilon_2(r, \varepsilon_1) \tag{6.5}$$

Here ε_2 is a C^0 -function of $r \geq 0$ and of ε_1 ; it is C^1 in $r \geq 0, \varepsilon_1 > 0$, and C^M for $r > 0$.

Moreover, $r \mapsto \varepsilon_2(r, \varepsilon_1)$ is strictly increasing for any fixed ε_1 . Therefore, the set of solutions (r, ε) projects one-to-one down to those $\varepsilon = (\varepsilon_1, \varepsilon_2)$ with $\varepsilon_2 \geq 0$.

Proof. The parametrization (6.5) together with the stated differentiability properties follows directly from the implicit function theorem, applied to equation (6.4). Concerning differentiability, we of course remember the extension of our equations to $r < 0$ which was discussed in the proof of lemma 5.1. Strict monotonicity of $\varepsilon_2(\cdot, \varepsilon_1)$ follows by implicit differentiation of (6.4) with respect to r for $r > 0$. Indeed,

$$1 = (1 + D_{\varepsilon_2} a \cdot r^{1+\varepsilon_1} + O(r^{1+\omega})) D_r \varepsilon_2 + (1 + \varepsilon_1) a(\varepsilon) r^{\varepsilon_1} + O(r^\omega)$$

implies that $D_r \varepsilon_2$ is positive, for all sufficiently small r, ε_1 , since $a(0) < 0$. This completes the proof. ■

Next, we are interested in period doubling bifurcations on the 1-per sheet $\varepsilon_2 = \varepsilon_2(r, \varepsilon_1)$. These bifurcations can be detected in system (6.1). The 1-per sheet are those solutions of (6.1) for which $r_0 = r_1 = r$. For $r > 0$, let

$$\Delta = \Delta(r, \varepsilon_1) \tag{6.6}$$

denote the determinant of the 2×2 -Jacobian of the right-hand side of (6.1) with respect to (r_0, r_1) at $r_0 = r_1 = r, \varepsilon_2 = \varepsilon_2(r, \varepsilon_1)$.

6.2. Lemma. *The determinant Δ on the 1-per sheet $\varepsilon_2 = \varepsilon_2(r, \varepsilon_1)$, $r > 0$, vanishes only along a unique C^M -curve*

$$r = \tilde{r}(\varepsilon_1) > 0, \quad \varepsilon_1 > 0$$

and Δ changes sign there.

Viewed in the projection onto ε -space, this curve is given by a C^M -curve

$$\varepsilon_2 = \kappa_{\text{per}}(\varepsilon_1) := \varepsilon_2(\tilde{r}(\varepsilon_1), \varepsilon_1), \quad \varepsilon_1 > 0$$

The asymptotics of \tilde{r} and κ_{per} for $\varepsilon_1 \searrow 0$ are

$$\lim_{\varepsilon_1 \searrow 0} \tilde{r}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} = \exp(-1 - D_{\varepsilon_1} a(0)/a(0)) \tag{6.7}$$

$$\lim_{\varepsilon_1 \searrow 0} \kappa_{\text{per}}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} = 2 \exp(-1 - D_{\varepsilon_1} a(0)/a(0)) \tag{6.8}$$

Proof. Differentiating (6.1) with respect to (r_0, r_1) at $r_0 = r_1 = r > 0$ we see that

$$\Delta = \det \begin{pmatrix} (1 + \varepsilon_1) ar^{\varepsilon_1} & -1 \\ -1 & (1 + \varepsilon_1) ar^{\varepsilon_1} \end{pmatrix} + O(r^\omega)$$

Thus Δ vanishes if and only if

$$(a) \quad 1 = (1 + \varepsilon_1) a(\varepsilon) r^{\varepsilon_1} + O(r^\omega) \tag{6.9}$$

or

$$(b) \quad -1 = (1 + \varepsilon_1) a(\varepsilon) r^{\varepsilon_1} + O(r^\omega)$$

Since $a < 0$, the first equation (6.9a) cannot have solutions with small r . Solving (6.9b) together with the original fixed point equation

$$r = \varepsilon_2 + a(\varepsilon) r^{1 + \varepsilon_1} + O(r^{1 + \omega}) \tag{6.4}$$

this time with $a(0) < -1$, is quite similar to the situation which came up for the fold curve κ in the proof of lemma 5.3; see (5.11a, b). We indicate only the necessary ramifications.

First, we note that (6.9b) can have a solution for small r and $|\varepsilon_1|$ only when $\varepsilon_1 > 0$, since $a(0) < -1$. As usual, we can then solve (6.9b) uniquely for $r = \tilde{r}(\varepsilon_1)$ at $\varepsilon_2 = \varepsilon_2(r, \varepsilon_1)$, by monotonicity. Likewise, we obtain a differentiable $\rho = \rho(\varepsilon_1, \varepsilon_2)$ such that, analogously to (5.12),

$$\tilde{r}(\varepsilon_1)^{\varepsilon_1} = \rho(\varepsilon_1, \varepsilon_2(\tilde{r}(\varepsilon_1), \varepsilon_1)) \tag{6.10}$$

The function $\tilde{r}(\varepsilon_1)$ is of class C^M for $\varepsilon_1 > 0$. At $\varepsilon = 0$, we have $\tilde{r} = 0$ and

$$(1, 0) = \rho D_{\varepsilon} a + a D_{\varepsilon} \rho \quad (6.11)$$

since $a\rho = -1$, this time, in (5.13).

As in (5.14), we claim

$$\kappa_{\text{per}}(\varepsilon_1) = \varepsilon_2(\tilde{r}(\varepsilon_1), \varepsilon_1) = o(\varepsilon_1), \quad \text{for } \varepsilon_1 \searrow 0 \quad (6.12)$$

Unfortunately, we cannot use $D_r \varepsilon_2(\tilde{r}, \varepsilon_1) = 0$, this time, since we are not at a fold point. However, to show (6.12), we can first differentiate (6.10) with respect to ε_1 to obtain

$$D_{\varepsilon_1} \tilde{r}(\varepsilon_1) = O\left(\frac{1}{\varepsilon_1^2} |\rho|^{1/\varepsilon_1}\right), \quad |\rho| < 1 \quad (6.13)$$

Here we have used

$$\lim_{\varepsilon_1 \searrow 0} D_r \varepsilon_2(\tilde{r}(\varepsilon_1), \varepsilon_1) = 2 \quad (6.14)$$

which follows by differentiating (6.4). Also, we have used that

$$\lim_{\varepsilon_1 \searrow 0} D_{\varepsilon_1} \varepsilon_2(\tilde{r}(\varepsilon_1), \varepsilon_1) = 0$$

holds, by the same reasoning as in the proof of (5.14) above. Now we can conclude

$$\lim_{\varepsilon_1 \searrow 0} \kappa'_{\text{per}}(\varepsilon_1) = \lim_{\varepsilon_1 \searrow 0} (D_r \varepsilon_2 \cdot D_{\varepsilon_1} \tilde{r} + D_{\varepsilon_1} \varepsilon_2) = 0$$

since the first term was estimated in (6.13)–(6.14) above and the second term yields zero, as before. This proves (6.12).

With these preparations, lemma 5.1 applies to (6.9b) with

$$b(\varepsilon) := (1 + \varepsilon_1) \cdot (-a(\varepsilon))$$

and yields

$$\lim_{\varepsilon_1 \searrow 0} \tilde{r}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} = \exp(-1 - D_{\varepsilon_1} a(0)/a(0))$$

This proves expansion (6.7) for $\tilde{r}(\varepsilon_1)$.

To prove expansion (6.8) for $\kappa_{\text{per}}(\varepsilon_1)$, we compute similarly to the proof of lemma 5.3

$$\begin{aligned} \lim_{\varepsilon_1 \searrow 0} \kappa_{\text{per}}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} &= \lim_{\varepsilon_1 \searrow 0} \varepsilon_2(\tilde{r}(\varepsilon_1), \varepsilon_1) |a(0)|^{1/\varepsilon_1} \\ &= \lim_{\varepsilon_1 \searrow 0} \tilde{r}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} (1 - a(\varepsilon) \tilde{r}(\varepsilon_1)^{\varepsilon_1} + O(\tilde{r}^\omega)) \\ &= \lim_{\varepsilon_1 \searrow 0} \tilde{r}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} (1 - a(\varepsilon) \rho(\varepsilon) + O(\tilde{r}^\omega)) \\ &= 2 \exp(-1 - D_{\varepsilon_1} a(0)/a(0)) \end{aligned}$$

since $\lim_{\varepsilon_1 \searrow 0} a(\varepsilon) \rho(\varepsilon) = -1$ for $\varepsilon = (\varepsilon_1, \varepsilon_2(\tilde{r}(\varepsilon_1), \varepsilon_1))$, $\rho = \tilde{r}^{\varepsilon_1}$, by (6.9b). Therefore (6.8) holds, and the lemma is proved. ■

Note that lemma 6.2 implies a (local) period doubling bifurcation from the 1-per sheet. Indeed, fix $\varepsilon_1 > 0$ and increase the remaining parameter ε_2 through the bifurcation point $\varepsilon_2 = \kappa_{\text{per}}(\varepsilon_1)$. Then the determinant Δ of the linearization of system (6.1) at the corresponding 1-per solutions $r_0 = r_1 = r$ changes sign. This change of topological degree implies local bifurcation of 2-per solutions: a period doubling bifurcation. We will return to these arguments in much more detail in lemma 6.5.

We can also look at the other limiting case of 2-per solutions, namely, at 2-hom orbits. Recall that $r_0 = 0 < r_1$, for a 2-hom. (Alternatively, we could of course consider the equivalent case $r_0 > 0 = r_1$.)

6.3. Lemma. *2-hom solutions $r_0 = 0 < r_1$ occur only along a unique C^M -curve*

$$\begin{aligned} \varepsilon_2 &= \kappa_{\text{hom}}(\varepsilon_1) > 0 \\ r_1 &= \hat{r}(\varepsilon_1) > 0, \quad \varepsilon_1 > 0 \end{aligned}$$

The asymptotics of \hat{r} and κ_{hom} for $\varepsilon_1 \searrow 0$ are

$$\lim_{\varepsilon_1 \searrow 0} \hat{r}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} = \exp(-D_{\varepsilon_1} a(0)/a(0)) \tag{6.15}$$

$$\lim_{\varepsilon_1 \searrow 0} \kappa_{\text{hom}}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} = \exp(-D_{\varepsilon_1} a(0)/a(0)) \tag{6.16}$$

In particular, we have the universal limit

$$\lim_{\varepsilon_1 \searrow 0} \frac{\kappa_{\text{hom}}(\varepsilon_1)}{\kappa_{\text{per}}(\varepsilon_1)} = \frac{e}{2} = 1.36\dots \tag{6.17}$$

Proof. Equations (6.1) with $r_0 = 0$ yield

$$\begin{aligned} \text{(a)} \quad 0 &= -r_1 + \varepsilon_2 + O(r_1^{1+\omega}) \\ \text{(b)} \quad 0 &= \varepsilon_2 + a(\varepsilon)r_1^{1+\varepsilon_1} + O(r_1^{1+\omega}) \end{aligned} \tag{6.18}$$

since $|\mathbf{r}| = r_1$. From (6.18a) we find

$$\varepsilon_2 = \varepsilon_2(r_1, \varepsilon_1) = r_1 + O(r_1^{1+\omega}) \tag{6.19}$$

where $O(r_1^{1+\omega})$ depends only on r_1 and ε_1 . Plugging this into (6.18b) and using differentiability of $a(\varepsilon)$, we obtain

$$0 = r_1 + a(\varepsilon_1, 0)r_1^{1+\varepsilon_1} + O(r_1^{1+\omega})$$

or, equivalently,

$$1 = -a(\varepsilon_1, 0)r_1^{\varepsilon_1} + O(r_1^\omega) \tag{6.20}$$

Solving (6.20) for $r_1 = \hat{r}_1(\varepsilon_1)$, by lemma 5.1 again, we find the asymptotics (6.15) for $\hat{r}_1(\varepsilon_1)$. In particular,

$$\lim_{\varepsilon_1 \searrow 0} \hat{r}_1(\varepsilon_1) = 0$$

Now (6.19) yields the same asymptotics for

$$\kappa_{\text{hom}}(\varepsilon_1) = \varepsilon_2(\hat{r}_1(\varepsilon_1), \varepsilon_1)$$

since

$$\lim_{\varepsilon_1 \searrow 0} \varepsilon_2(\hat{r}_1(\varepsilon_1), \varepsilon_1) |a(0)|^{1/\varepsilon_1} = \lim_{\varepsilon_1 \searrow 0} \hat{r}_1(\varepsilon_1) |a(0)|^{1/\varepsilon_1}$$

This proves the κ_{hom} -asymptotics (6.16). Finally, the limit

$$\lim_{\varepsilon_1 \searrow 0} \frac{\kappa_{\text{hom}}(\varepsilon_1)}{\kappa_{\text{per}}(\varepsilon_1)} = \lim_{\varepsilon_1 \searrow 0} \frac{\kappa_{\text{hom}}(\varepsilon_1) \cdot |a(0)|^{1/\varepsilon_1}}{\kappa_{\text{per}}(\varepsilon_1) \cdot |a(0)|^{1/\varepsilon_1}} = \frac{e}{2}$$

is universal, by the asymptotics (6.8) and (6.16) for κ_{per} and κ_{hom} . This proves the lemma. ■

Locally near $r = 0$, $\varepsilon = 0$, the 2-per solutions are confined to an exponentially thin wedge in parameter space. More specifically, we have the following.

6.4. Lemma. *Consider system (6.1) in a suitably small, fixed neighborhood of $r = 0$, $\varepsilon = 0$.*

Then there exist two C^M -curves

$$\varepsilon_2 = \underline{\kappa}(\varepsilon_1) \quad \text{and} \quad \varepsilon_2 = \bar{\kappa}(\varepsilon_1), \quad \varepsilon_1 > 0,$$

with $\underline{\kappa} < \bar{\kappa}$, such that (6.1) does not possess 2-per or 2-hom solutions for ε outside of the wedge

$$\{(\varepsilon_1, \varepsilon_2) \mid \underline{\kappa}(\varepsilon_1) < \varepsilon_2 < \bar{\kappa}(\varepsilon_1)\} \tag{6.21}$$

The curves $\underline{\kappa}$ and $\bar{\kappa}$, respectively, have the same exponential asymptotics as the curves $\frac{1}{2}\kappa_{\text{per}}$ and $2\kappa_{\text{hom}}$ [see lemmata 6.2, 6.3, (6.8), and (6.16)].

Proof. Suppose system (6.1) does possess a 2-per/2-hom solution $\mathbf{r} = (r_0, r_1)$, that is, $r_0 \neq r_1$ and

$$\begin{aligned} 0 &= -r_1 + \varepsilon_2 + a(\varepsilon)r_0^{1+\varepsilon_1} + h_0(r_0, r_1, \varepsilon) \\ 0 &= -r_0 + \varepsilon_2 + a(\varepsilon)r_1^{1+\varepsilon_1} + h_0(r_1, r_0, \varepsilon) \end{aligned} \tag{6.1}$$

Here we have used that, by the symmetry $r_0 \leftrightarrow r_1$, we know

$$h_1(r_0, r_1, \varepsilon) = h_0(r_1, r_0, \varepsilon)$$

at any solution (r_0, r_1) . We may assume

$$0 \leq r_0 < r_1$$

so that both error terms h_0, h_1 are of the order $O(r_1^{1+\omega})$. Taking the difference of the two equations (6.1) and dividing by $(r_1 - r_0) > 0$ we obtain an equation of the form

$$-1 = a(\varepsilon) \sigma_{\varepsilon_1}(r_0, r_1) + O(r_1^\omega) \tag{6.22}$$

where σ_{ε_1} denotes the slope

$$\sigma_{\varepsilon_1}(r_0, r_1) := \frac{r_1^{1+\varepsilon_1} - r_0^{1+\varepsilon_1}}{r_1 - r_0}$$

of the secant to the graph $r \mapsto r^{1+\varepsilon_1}$ at the two points $r = r_0, r_1$. Note that this graph is convex or linear or concave, depending on the sign of ε_1 . Thus σ_{ε_1} always lies in the interval with end points

$$r_1^{\varepsilon_1} \quad \text{and} \quad (1 + \varepsilon_1)r_1^{\varepsilon_1} \tag{6.23}$$

for $0 \leq r_0 < r_1$. The error term $O(r_1^\omega)$ is justified, since

$$\begin{aligned} & h_0(r_1, r_0, \varepsilon) - h_0(r_0, r_1, \varepsilon) \\ &= (r_1 - r_0) \int_0^1 (D_1 h_0(r_0 + \Theta(r_1 - r_0), r_1 - \Theta(r_1 - r_0), \varepsilon) \\ &\quad - D_2 h_0(r_0 + \Theta(r_1 - r_0), r_1 - \Theta(r_1 - r_0), \varepsilon)) d\Theta \end{aligned}$$

where the integral is $O(r_1^\omega)$, by differentiation of $O(|r|^{1+\omega})$.

It remains to construct $\underline{\kappa}(\varepsilon_1), \bar{\kappa}(\varepsilon_1)$ such that (6.22) cannot hold outside the specified wedge (6.21). Obviously, (6.22) cannot hold for $\varepsilon_1 \leq 0$, by estimate (6.23) for σ and because $a(0) < -1$. For $\varepsilon_1 > 0$, we can estimate ε_2 from the first equation in (6.1) as

$$(1 + O(r_1^\omega))r_1 \leq \varepsilon_2 \leq r_1(1 - a(\varepsilon)r_1^{\varepsilon_1} + O(r_1^\omega))$$

and, since $a(\varepsilon_1, \varepsilon_2)$ is Lipschitz in ε_2 , this implies an estimate

$$(1 + O(r_1^\omega))r_1 \leq \varepsilon_2 \leq r_1(1 - a(\varepsilon_1, 0)r_1^{\varepsilon_1} + O(r_1^\omega)) \tag{6.24}$$

Plugging (6.24) into (6.22) yields similarly

$$-1 = a(\varepsilon_1, 0) \sigma_{\varepsilon_1}(r_0, r_1) + O(r_1^\omega)$$

Let $\underline{C} \leq 0 \leq \bar{C}$ denote real constant such that

$$\underline{C}r_1^\omega < O(r_1^\omega) < \bar{C}r_1^\omega$$

above and in (6.24). Then (6.23) implies

$$(1 + \varepsilon_1) a(\varepsilon_1, 0)r_1^{\varepsilon_1} + \underline{C}r_1^\omega < -1 < a(\varepsilon_1, 0)r_1^{\varepsilon_1} + \bar{C}r_1^\omega$$

where the terms to the right and left of -1 are strictly decreasing with r_1 . Therefore, we conclude

$$r_1 \in (\underline{r}_1, \bar{r}_1)$$

where

$$-1 = a(\varepsilon_1, 0)\bar{r}_1^{\varepsilon_1} + \bar{C}\bar{r}_1^\omega$$

and

$$-1 = (1 + \varepsilon_1) a(\varepsilon_1, 0)\underline{r}_1^{\varepsilon_1} + \underline{C}\underline{r}_1^\omega$$

This defines $\underline{r}_1(\varepsilon_1), \bar{r}_1(\varepsilon_1)$. Note that $\underline{r}_1(\varepsilon_1)$ has the same exponential asymptotics (6.7) as $\bar{r}_1(\varepsilon_1)$, which is associated to period doubling. Similarly,

$\bar{r}_1(\varepsilon_1)$ has the “2-hom” asymptotics (6.15) of $\bar{r}(\varepsilon_1)$. These facts follow from lemma 5.1. Now we define

$$\begin{aligned} \underline{\kappa}(\varepsilon_1) &:= r_1(\varepsilon_1)(1 + \underline{C}r_1(\varepsilon_1)^\omega) \\ \bar{\kappa}(\varepsilon_1) &:= \bar{r}_1(\varepsilon_1)(1 - a(\varepsilon_1, 0) \bar{r}_1(\varepsilon_1)^{\varepsilon_1} + \bar{C}\bar{r}(\varepsilon_1)^\omega) \end{aligned}$$

Then (6.24) implies that ε_2 lies between $\underline{\kappa}$ and $\bar{\kappa}$:

$$\underline{\kappa}(\varepsilon_1) < \varepsilon_2 < \bar{\kappa}(\varepsilon_1)$$

Moreover, $\underline{\kappa}$ has the same exponential asymptotics as $\frac{1}{2}\kappa_{\text{per}}$. Indeed,

$$\begin{aligned} \lim_{\varepsilon_1 \searrow 0} \underline{\kappa}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} &= \lim_{\varepsilon_1 \searrow 0} r_1(\varepsilon_1) \cdot |a(0)|^{1/\varepsilon_1} \\ &= \exp(-1 - D_{\varepsilon_1} a(0)/a(0)) \\ &= \lim_{\varepsilon_1 \searrow 0} \frac{1}{2} \kappa_{\text{per}}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} \end{aligned}$$

by lemma 6.2. Likewise,

$$\begin{aligned} \lim_{\varepsilon_1 \searrow 0} \bar{\kappa}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} &= \lim_{\varepsilon_1 \searrow 0} \bar{r}_1(\varepsilon_1) |a(0)|^{1/\varepsilon_1} \cdot (1 - a(\varepsilon_1, 0) \bar{r}_1(\varepsilon_1)^{\varepsilon_1} + \bar{C}\bar{r}(\varepsilon_1)^\omega) \\ &= 2 \exp(-D_{\varepsilon_1} a(0)/a(0)) \\ &= \lim_{\varepsilon_1 \searrow 0} 2\kappa_{\text{hom}}(\varepsilon_1) |a(0)|^{1/\varepsilon_1} \end{aligned}$$

by lemma 6.3. This proves the lemma. ■

To complete the proof of theorem B, it remains to detect a two-dimensional continuum \mathcal{C} of 2-per solutions which connects the curve of period doubling solutions to the curve of 2-hom orbits. We construct this continuum using a topological multiparameter result due to Alexander and Antman (1981).

We briefly describe the basic idea, fixing the setting along our way. Fix small enough bounds $\varepsilon_2^0 > 0$ and ρ^0 . Then choose $\varepsilon_1^0 > 0$ small enough, depending on these bounds, as detailed below. We consider the box

$$\mathcal{B} = \{(\varepsilon, \mathbf{r}) \mid 0 < \varepsilon_2 < \varepsilon_2^0, 0 < \varepsilon_1 < \varepsilon_1^0, 0 < \mathbf{r} < \rho^0\}$$

where the \mathbf{r} -inequality is understood componentwise, for r_0 and r_1 . Fix ε_1 , for a moment. The corresponding section $\mathcal{B}_{\varepsilon_1}$ of the box \mathcal{B} is drawn in Fig. 5. We see the 1-per branch in the $r_0 = r_1$ diagonal, the period-doubling

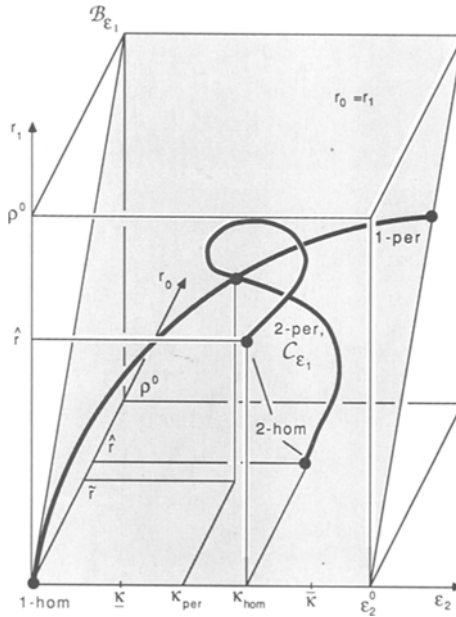


Fig. 5. Bifurcation within the box \mathcal{B}_{ϵ_1} for fixed small positive ϵ_1 .

point $\epsilon_2 = \kappa_{\text{per}}(\epsilon_1)$, $r_0 = r_1 = \tilde{r}(\epsilon_1)$ on that 1-per branch, and the two 2-hom points given by $\epsilon_2 = \kappa_{\text{hom}}(\epsilon_1)$ and $r_0 = 0, r_1 = \hat{r}(\epsilon_1)$ or $r_0 = \hat{r}(\epsilon_1), r_1 = 0$. By lemma 6.2, the sign of the determinant Δ of the \mathbf{r} -linearization of (6.1) changes at $\epsilon_2 = \kappa_{\text{per}}$ along the 1-per branch. By the classical global Rabinowitz (1971) theorem, this implies the existence of a global bifurcating continuum \mathcal{C}_{ϵ_1} of 2-per solutions ($r_0 \neq r_1$) from the 1-per branch. Since $\epsilon_2 = \kappa_{\text{per}}$ is the only bifurcation point, this continuum \mathcal{C}_{ϵ_1} extends to the boundary of the open box \mathcal{B}_{ϵ_1} . Since \mathcal{C}_{ϵ_1} consists of 2-per solutions, the values of ϵ_2 on \mathcal{C}_{ϵ_1} are confined to the wedge region $\underline{\kappa}(\epsilon_1) < \epsilon_2 < \bar{\kappa}(\epsilon_1)$. Of course, we assume here that ϵ_1^0 is chosen such that $\bar{\kappa}(\epsilon_1) < \epsilon_2^0$ for $0 < \epsilon_1 < \epsilon_1^0$. Since \mathcal{C}_{ϵ_1} extends to the boundary $\text{bdy } \mathcal{B}_{\epsilon_1}$, the closure $\text{clos } \mathcal{C}_{\epsilon_1}$, intersected with $\text{bdy } \mathcal{B}_{\epsilon_1}$, consists precisely of the two 2-hom points. The following lemma gives a 2-dimensional account of this 1-dimensional result, including the parameter ϵ_1 which was held fixed above. For the notion of topological dimension see, e.g., Hurewicz and Wallman (1948).

6.5. Lemma. Consider system (6.1) in the open box \mathcal{B} constructed above. Let $\mathcal{C}_{\text{per}}, \mathcal{C}_{\text{hom}}$ denote the period doubling and 2-hom curves, given, respectively, by

$$\begin{aligned} \mathcal{C}_{\text{per}} &= \{(\mathbf{r}, \varepsilon) \in \text{clos } \mathcal{B} \mid \varepsilon_2 = \kappa_{\text{per}}(\varepsilon_1), r_0 = r_1 = \tilde{r}(\varepsilon_1)\} \\ \mathcal{C}_{\text{hom}}^{\varepsilon_1} &= \{(\mathbf{r}, \varepsilon) \in \text{clos } \mathcal{B}_{\varepsilon_1} \mid \varepsilon_2 = \kappa_{\text{hom}}(\varepsilon_1), r_0 = 0, r_1 = \hat{r}(\varepsilon_1), \text{ or } r_0 = \hat{r}_1(\varepsilon_1), r_1 = 0\} \\ \mathcal{C}_{\text{hom}} &= \bigcup_{0 \leq \varepsilon_1 \leq \varepsilon_1^0} \mathcal{C}_{\text{hom}}^{\varepsilon_1} \end{aligned}$$

Then there exists a continuum $\mathcal{C} \subseteq \mathcal{B}$ of 2-per solutions of (6.1) (i.e., $r_0 \neq r_1$) with the following properties:

- (a) $\text{clos } \mathcal{C} \cap \{(\mathbf{r}, \varepsilon) \in \text{clos } \mathcal{B} \mid r_0 = r_1\} = \mathcal{C}_{\text{per}}$
- (b) $\text{clos } \mathcal{C} \cap \text{bdy } \mathcal{B}_{\varepsilon_1} = \mathcal{C}_{\text{hom}}^{\varepsilon_1}$, for $0 \leq \varepsilon_1 \leq \varepsilon_1^0$
- (c) \mathcal{C} has topological dimension at least 2 at every point.

Let \mathcal{C}^+ denote the one point compactification of \mathcal{C} , that is, \mathcal{C}^+ is $\text{clos } \mathcal{C}$ with points on $\text{bdy } \mathcal{B} \cup \mathcal{C}_{\text{per}} \cup \mathcal{C}_{\text{hom}}$ identified to a single point. Then we also have

- (d) there exists a continuous and essential (i.e., non-contractible) map from $\text{clos } \mathcal{C}^+$ to the standard 2-sphere S^2 .

Proof. The proof is basically an application of Alexander and Antman (1981, theorem 2.2). The 1-per sheet

$$\varepsilon_2 = \varepsilon_2(r, \varepsilon_1), \quad r_0 = r_1 = r > 0, \quad \varepsilon_1 > 0$$

which can also be parametrized over $\varepsilon = (\varepsilon_1, \varepsilon_2)$, plays the role of the trivial solution. Just above lemma 6.3, we have already checked the basic assumption of Alexander and Antman (1981): a change of (Brouwer) degree, viz., of sign Δ , as we cross the bifurcation curve \mathcal{C}_{per} in the 1-per sheet. As a result, we obtain a continuum \mathcal{C} of 2-per solutions such that (6.25c,d) both hold. [By Alexander and Antman (1981), \mathcal{C} is called $S_{1.}$]

To prove (6.25a), we first claim that $\text{clos } \mathcal{C}$ intersects the 1-per sheet at most in the bifurcation curve \mathcal{C}_{per} . Indeed, the linearization is nonsingular outside of \mathcal{C}_{per} ($\Delta \neq 0$) and the implicit function theorem applies, isolating the 1-per sheet from 2-per solutions. Also, $\text{clos } \mathcal{C}$ cannot contain points in

$$\{(\mathbf{r}, \varepsilon) \in \text{bdy } \mathcal{B} \mid r_0 = r_1\} \setminus \mathcal{C}_{\text{per}} \tag{6.26}$$

Indeed, $\text{clos } \mathcal{C}$ is confined to the wedge region

$$\{(\mathbf{r}, \varepsilon) \mid \underline{\kappa}(\varepsilon_1) \leq \varepsilon_2 \leq \bar{\kappa}(\varepsilon_1)\} \tag{6.27}$$

In this wedge, nonzero solutions with $r_0 = r_1$ and with $r_0 \in \{0, \delta^0\}$ or $\varepsilon_1 \in \{0, \varepsilon_1^0\}$ or $\varepsilon_2 \in \{0, \varepsilon_2^0\}$ do not exist. This proves our claim.

Next we claim that $\text{clos } \mathcal{C}$ contains \mathcal{C}_{per} . Indeed, as we have shown above, the Brouwer degree, *viz.*, $\text{sign } \Delta$, changes across $\varepsilon_2 = \kappa_{\text{per}}(\varepsilon_1)$ for any fixed $\varepsilon_1 > 0$. Therefore, $\text{clos } \mathcal{C} \supseteq \mathcal{C}_{\text{per}}$ by Alexander and Antman (1981, theorem 2.2). This proves (6.25a).

To prove (6.25b), we first claim

$$\text{clos } \mathcal{C} \cap \text{bdy } \mathcal{B}_{\varepsilon_1} \subseteq \mathcal{C}_{\text{hom}}^{\varepsilon_1}$$

Indeed, the left-hand side is contained in the wedge region (6.27). But $\text{clos } \mathcal{C} \cap \text{bdy } \mathcal{B}_{\varepsilon_1}$, in that region, can consist only of 2-hom points and, if $\varepsilon_1 = 0$, possibly zero. All these points are in $\mathcal{C}_{\text{hom}}^{\varepsilon_1}$.

To complete the proof of (6.25b), we also claim

$$\text{clos } \mathcal{C} \cap \text{bdy } \mathcal{B}_{\varepsilon_1} \supseteq \mathcal{C}_{\text{hom}}^{\varepsilon_1}$$

Indeed, consider any ε_1 -section $\mathcal{C}_{\varepsilon_1}$ of \mathcal{C} , $\varepsilon_1 > 0$. By Alexander and Antman (1981, p. 349), the section $\mathcal{C}_{\varepsilon_1}$ is not contained in any compact subset of the box section $\mathcal{B}_{\varepsilon_1}$, because $\text{clos } \mathcal{C}_{\varepsilon_1}$ cannot contain any point on the 1-per sheet except for the unique bifurcation point on \mathcal{C}_{per} with $\varepsilon = (\varepsilon_1, \kappa_{\text{per}}(\varepsilon_1))$. Therefore,

$$\text{clos } \mathcal{C}_{\varepsilon_1} \cap \text{bdy } \mathcal{B} \neq \emptyset$$

In other words, $\text{clos } \mathcal{C}_{\varepsilon_1}$ contains at least one of the two points

$$\varepsilon_2 = \kappa_{\text{hom}}(\varepsilon_1), \quad r_0 = 0, \quad r_1 = \hat{r}_1(\varepsilon_1)$$

or

$$r_0 = \hat{r}_1(\varepsilon_1), \quad r_1 = 0$$

which constitute \mathcal{C}_{hom} . By the symmetry $r_0 \leftrightarrow r_1$ of (6.1), we may then assume that $\text{clos } \mathcal{C}_{\varepsilon_1}$ also contains the other point. This proves our claim.

A minor point remains to be settled. Augmenting $\mathcal{C}_{\varepsilon_1}$ by its symmetric counterpart, as above, could have destroyed the existence of an essential map (6.25d), in principle. We claim this does not happen. Indeed \mathcal{C}^+ consists of three disjoint sets:

- \mathcal{C}_0^+ : the collapsed point,
- $\mathcal{C}_{<}^+$: points in \mathcal{C} with $r_0 < r_1$,
- $\mathcal{C}_{>}^+$: points in \mathcal{C} with $r_0 > r_1$.

Evidently, we can omit either $\mathcal{C}_{<}^+$ or $\mathcal{C}_{>}^+$ from \mathcal{C}^+ and still find an essential map into S^2 ; say

$$\mathcal{C}_0^+ \cup \mathcal{C}_{<}^+ \rightarrow S^2$$

Extending this map to $\mathcal{C}_>^+$ such that orbits under $r_0 \leftrightarrow r_1$ get mapped to the same point, we find a symmetric continuum \mathcal{C} for which (6.25d) still holds.

This completes the proof of lemma 6.5 and the proof of theorem B. ■

7. DISCUSSION

We put our main theorems A, B in a variety of perspectives. First we compare and contrast our results, as they stand, with earlier work by other authors. Then we reemphasize a central weakness of our approach via Ljapunov–Schmidt reduction: we have lost control over N -hom/ N -per orbits with $N > 2$. We indicate how this defect can be remedied by constructing a suitable center manifold. Because our discussion of Poincaré return maps has caused us a lot of technical troubles, we also explain why another idea, which is based on exponential dichotomies, does not work as well for homoclinic bifurcations. Hard-core applications of our results are very scarce, so far. We blame this deplorable fact on the lack of numerical pathfollowing schemes for homoclinic orbits. After pointing out analytical difficulties with our results in infinite-dimensional settings, we return to the pathfollowing question from a global, theoretical point of view. We sketch an idea for an emerging index theory for paths of homoclinic orbits which hopefully should bridge the gap between local “birth” of homoclinic orbits at B -points and, on the other side, chaotic dynamics of the Shilnikov type. We summarize some known ingredients, but much still remains to be done.

Postponing further speculation, for a little while, let us return to the homoclinic side-switching bifurcation. According to theorem A, this bifurcation is associated to nontwisted homoclinic orbits and can therefore occur for vector fields in the plane. Indeed, Sanders and Cushman (1986) have found resonant side-switching to occur in the (averaged) equation for a Josephson junction

$$\begin{aligned}\dot{\phi} &= y \\ \dot{y} &= -\sin \phi + \varepsilon(a - (1 + \gamma \cos \phi) y)\end{aligned}\tag{7.1}$$

with $\phi \in S^1$, ε small positive, and with real parameters a, γ . They obtain resonant side-switching near $a = 16/3\pi$, $\gamma = 1$. In particular, the correct asymptotic expression for the per-fold, given in (1.16), was obtained by Sanders and Cushman (1986, 5.11). This includes the computation of a_0 and of the finite limit in (1.16). Their explicit computations are based on averaging of a perturbed pendulum equation, Picard–Fuchs equations, and associated Riccati equations. On the other hand, our much less explicit

approach carries over to vector fields which are far from a Hamiltonian structure and to vector fields in higher space dimensions. An earlier reference which is closely related to resonant side-switching is Leontovich (1951). There, it is stated in theorem 3 (putting $n=0$, $c_0=0$, $a_1 \neq 0$) that up to two periodic orbits can coexist for perturbations of a resonant homoclinic orbit in the plane. Still for planar C^2 vector fields, Nozdracheva (1982) essentially obtained theorem A. Tangency of the per-fold at $\varepsilon_1=0$ was noticed, but the asymptotic expression (1.16) was not derived.

The bifurcation of 2-hom orbits is a question which was first addressed by Yanagida (1987) in a traveling wave context as follows. Let

$$\zeta_t = D\zeta_{xx} + G(\alpha_2, \zeta), \quad x \in \mathbb{R}, \quad \alpha_2 \in \mathbb{R} \quad (7.2)$$

be a reaction diffusion system with nonnegative diagonal diffusion matrix D and with nonlinearity G . Then special solutions of the form $\zeta = \zeta(x + ct)$, called traveling waves, satisfy the ODE

$$0 = -c\dot{\zeta} + D\ddot{\zeta} + G(\alpha_2, \zeta) \quad (7.3)$$

which is of the form

$$\dot{z} = F(\alpha, z) \quad (1.1_\alpha)$$

with suitably defined z and with $\alpha = (\alpha_1, \alpha_2) := (c, \alpha_2)$. Specifically, Yanagida considers 2-hom orbits to an equilibrium with unstable dimension 1 ($n=1$, in our notation). This assumption enables him to treat the Poincaré map directly, avoiding Shilnikov variables like $(s, x^{\text{in}}, y^{\text{out}})$. Unfortunately, he completely linearizes F near the equilibrium $z=0$ and then omits terms which are of higher order, formally. Linearizations of class C^k , ($k \geq 1$ to preserve tangencies), require certain diophantine non-resonance conditions on the eigenvalues. For linearization of flows which do not contain parameters, see, e.g., Belitskii (1973), Sternberg (1957, 1958), Sell (1984, 1985). In particular, our system violates the Sternberg condition of order 3 as soon as $z \in \mathbb{R}$, due to resonance. For smooth normal forms in the plane at resonance see Bogdanov (1985). We point out that the linearization question is related to the exponential expansion (3.9), at least for $z \in \mathbb{R}^2$; see Deng (1989a, Section 6). Anyway, Yanagida derives the correct (truncated) reduced normal form equation for 2-hom orbits, concluding existence in the twisted and nonexistence in the non-twisted case. We have now confirmed these results, extending them to higher unstable dimension and deriving exponential asymptotics of the bifurcation curve κ_{hom} . As we mentioned in Section 2 above, Glendinning (1987) has computed both the asymptotics for κ_{hom} and κ_{per} for the truncated reduced

normal form. He claims that reduction to this truncated form can be achieved “via standard techniques.” In a way, we have shown that he is right: a posteriori, and modulo some subtleties described below. We note here already that even the discussion of the full normal form equation (4.13) is a nontrivial task, since a systematic theory of equations where parameters enter in the exponents, i.e., transcendently, does not exist. In particular, it is not clear how to neglect terms “of higher order.” Kokubu (1987) also notes the linearization flaw in Yanagida’s presentation, announcing an alternative proof. Accidentally, the crucial twist condition is missing in his statement of Yanagida’s result (Kokubu, 1987, theorem 3). Writing this Discussion, we obtain the preprint (Kokubu, 1988) which, in theorem D, gives a correct version of Yanagida’s result in three space dimensions ($n = 1, m = 2$). The proof avoids linearization and relies on the Shilnikov expansion given in proposition 3.2 above. The method of proof differs from ours. In particular, periodic orbits are not discussed and the homoclinic orbits are not obtained through a Ljapunov–Schmidt reduction. Instead, a transversality assumption is made. Specifically, the existence of an $(m + 1)$ -dimensional invariant manifold W^{su} is used, which is assumed to be tangent to $T_0 W^s \oplus \text{span}(e^+)$ at $z = 0$. A manifold W^{us} , tangent to $T_0 W^u \oplus \text{span}(e^-)$, can be defined analogously. Transversality is assumed for the intersections

$$W^u \cap W^{su}, \quad W^s \cap W^{us}$$

This assumption enters already into continuation of the primary 1-hom branch. Existence of W^{su}, W^{us} entails a loss of regularity. These manifolds can be assumed to exist of class C^k , if the spectrum of the linearization at $z = 0$ satisfies the gap condition

$$\min(\bar{\mu}_1, \bar{\nu}_1) > k \cdot \max(\mu_0, \nu_0)$$

in the notation of assumption (1.6); see Hirsch *et al.* (1977, Section 5), Vanderbauwhede (1989). Generically, this forces us down to C^1 -regularity throughout. In contrast, we obtain a 2-hom curve κ_{hom} which is smooth except at bifurcation. Kokubu (1988) notices tangency of κ_{hom} to the 1-hom curve but does not derive the exponential expansion

$$0 < \lim_{\varepsilon_1 \searrow 0} \kappa_{\text{hom}}(\varepsilon_1) |a_0|^{1/\varepsilon_1} < \infty$$

However, the main thrust of Kokubu (1988) is on heteroclinic phenomena. This explains why periodic orbits are not considered.

We emphasize that Yanagida has, at least formally, found two other generic mechanisms for nonresonant homoclinic doubling besides

the resonant case, assuming $\mu_0 > \nu_0$; see Yanagida (1987, theorem 3.3). Geometrically, these other two mechanisms work as follows. Suppose we follow a path of homoclinic orbits $z^*(t)$, varying the parameter α along a curve. Then,

$$\lim_{t \rightarrow +\infty} \dot{z}^*(t)/|\dot{z}^*(t)|$$

might switch from the principal stable eigenvector $-e^-$ to e^- itself, somewhere along this curve. In between, there will be a parameter value, *viz.*, a bifurcation point, where the above limit is a nonprincipal eigenvector. This is the first mechanism. Alternatively, the unstable manifold W^u can intersect W_{loc}^s at the point \mathbf{q} in an exceptional position, so that the strong λ -lemma 3.3 does not apply. In other words, the status of the homoclinic orbit $z^*(t)$ switches from twisted to nontwisted at some (bifurcation) point along the parameter curve. This is the second mechanism which leads to homoclinic doubling. Both mechanisms are discussed under the, hopefully unnecessary, assumption $n = 1$ for the unstable dimension. At this stage, it seems likely that all generic mechanisms for homoclinic doubling involving real principal eigenvalues have been found. Strictly speaking, however, we are still lacking rigorous necessary criteria for homoclinic bifurcations. For reasons of space, we cannot pursue these questions any further here.

It is a central weakness of our approach that we lose control over N -per/ N -hom orbits, $N \geq 3$, in our analysis of the reduced normal form

$$r_{j+1} = \varepsilon_2 + a(\varepsilon)r_j^{1+\varepsilon_1} + O(|\mathbf{r}|^{1+\omega}), \quad j(\bmod N) \quad (4.13)$$

for $\mathbf{r} = (r_0, \dots, r_{N-1})$. As we noticed in Section 2, the O -term is the reason why (4.13) does not describe an iteration of a single monotone scalar function, for which N -per/ N -hom orbits with $N \geq 3$ could not occur. This subtlety seems to have escaped the attention of some authors. We briefly describe two different angles of attack to the problem.

First, we can approach the problem algebraically. For example, consider the nontwisted case $a > 0$, *viz.*, $a > 1$. For $N = 2$, it can be shown that no additional solutions occur besides those with $r_0 = r_1 = r$, which are seen for $N = 1$ already. Indeed, suppose $0 \leq r_0 < r_1$. Then we can reconsider the proof of Lemma 6.4, this time for $a > 1$ rather than $a < -1$. We obtain a contradiction from (6.22), since both a and σ_{ε_1} are positive. For $N \geq 3$, however, these essentially scalar convexity arguments fail. In particular, we are not able to gain sufficient control over (4.13) to exclude the existence of N -per solutions algebraically.

Second, let us approach the problem more geometrically. Suppose there exists a two-dimensional center manifold attached to the homoclinic orbit $z^*(t)$, which describes bifurcations just as in the case of a periodic

orbit. In a Poincaré section, we then have a one-dimensional invertible return map. Thus N -per orbits with $N \geq 3$ cannot occur nearby, since they would have to lie on the center manifold. But does there actually exist such a center manifold W^c ? We sketch how W^c might be constructed, at least of class C^1 , following an idea which was pointed out to us by J. Guckenheimer. Basically, one would like to use the $(m + 1)$ -dimensional manifold W^{su} tangent to $T_0 W^s \oplus \text{span}(e^-)$, which we have mentioned above in the context of Kokubu (1988). Just assume that W^{su} and W^{us} intersect transversely along the orbit $\Gamma = \{z^*(t)\}$ at $\alpha = 0$ and define

$$W^c = W^{su} \cap W^{us}$$

obviously $\dim W^c = 2$. If W^{su} and W^{us} are constructed very carefully, then W^c will contain all orbits which remain in a tubular neighborhood U of the homoclinic orbit Γ . In particular, all N -hom and N -per orbits will lie in W^c , as required above. For a related approach to a simplified model of the Lorenz system, based on stable foliations, see Robinson (1988).

Constructing W^{su} , W^{us} for this purpose involves the following subtlety. The vector field F has to be modified such that W^{su} , e.g., can be constructed globally to consist of all those trajectories which do not escape to “infinity” at a rate faster than

$$e^{(v_0 + \varepsilon)t}, \quad t \rightarrow \infty$$

Usually, F is allowed to be modified everywhere outside a small neighborhood of the origin (Hirsch *et al.*, 1977, section 5A; Vanderbauwhede 1989). For our purposes, however, F must not be modified in a tubular neighborhood U of the entire homoclinic loop Γ . Therefore, constructing W^c will be a not so standard task, involving a global extension of the vector field to an appropriately chosen bundle over the homoclinic loop Γ . Once W^c is shown to exist, there remains a question about (linearized) stability of the bifurcating 1-per and 2-per solutions. We hope that a “principle of reduced stability” in the sense of Kielhöfer and Lauterbach (1983) holds: this would allow us to determine stability of the periodic solutions directly from the reduced normal form (4.13).

An alternative approach to pathfollowing of homoclinic orbits, based on exponential dichotomies, can be sketched as follows. We rewrite (1.1) as

$$\begin{aligned} \mathcal{F}(\alpha, z(\cdot)) &:= -\dot{z}(t) + F(\alpha, z(t)) = 0 \\ \mathcal{F} &: \mathbb{R}^2 \times Z_1 \rightarrow Z_0 \end{aligned} \tag{7.4}$$

where Z_ι , $\iota = 0, 1$, are suitably chosen Banach spaces like $BC^1(\mathbb{R}, \mathbb{R}^{m+n})$

with the uniform sup-norm, or like the Sobolev spaces $H^1(\mathbb{R}, \mathbb{R}^{m+n})$. It follows from Palmer (1984, lemma 4.2) that the linearization

$$D_z \mathcal{F}(0, z^*(\cdot))$$

at the homoclinic orbit $z^*(\cdot)$ is a Fredholm operator of Fredholm index zero. The results of Palmer are based on exponential dichotomies of certain families of projection operators along the orbit $z^*(t)$. Note that $\dot{z}^*(\cdot)$ is always in the kernel of $D_z \mathcal{F}(0, z^*(\cdot))$. Assuming surjectivity of the total derivative $D\mathcal{F}$ makes pathfollowing of homoclinic orbits amenable to an application of the implicit function theorem, and even to global continuation techniques. The formulation (7.4) has a fundamental drawback, however, when it comes to bifurcation problems. For illustration, consider resonant homoclinic doubling (theorem B). As $\varepsilon_1 \searrow 0$, $\varepsilon_2 = k_{\text{hom}}(\varepsilon_1)$, the 2-hom trajectory $\{\hat{z}(t) | t \in \mathbb{R}\}$ tends to the 1-hom trajectory $\{z^*(t) | t \in \mathbb{R}\}$ in phase space $z \in \mathbb{R}^{m+n}$. In contrast, $\hat{z}(\cdot)$ does not tend to $z^*(\cdot)$ in any of the function spaces Z_0 or Z_1 . In fact, $\hat{z}(\cdot)$ does not have a limit for $\varepsilon_1 \searrow 0$. A similar obstacle arises, when we try to capture periodic orbits limiting on a 1-hom at a blue sky catastrophe in the functional analytic framework of (7.4).

Hard-core applications of our results are scarce. Resonant side-switching, as we have mentioned above, was found by Sanders and Cushman (1986) to occur in the Josephson junction. Yanagida (1986) presents his result in the framework of a "generalized nerve equation introduced by FitzHugh," without being entirely specify about the underlying nonlinearities. Actually, the whole situation is somewhat reminiscent of period doubling, which is hardly ever detected analytically for a concrete given system. Rather, there are numerical pathfollowing routines for periodic orbits which find period doublings in a broad variety of applications. See, for example, Doedel and Kernevez (1985), Kubiček and Marek (1983), Seydel (1988). Efficient pathfollowing codes for homoclinic orbits in two parameters do not seem to exist. In AUTO (Doedel and Kernevez, 1985), homoclinic orbits are treated as periodic orbits of "large" period. The special structure of homoclinic orbits near stationary solutions is thus ignored. Concerning the numerical treatment of one-sided asymptotic boundary value problems, we mention de Hoog and Weiss (1979) and Lentini and Keller (1980). Note that, in the functional analytic framework of (7.4) above, a homoclinic orbit $z^*(t)$ solves the boundary value problem (7.4) with the two-sided asymptotic boundary condition $z^*(t) \rightarrow 0$ for $t \rightarrow \pm\infty$. Of course, this setting suffers from the above mentioned difficulties at bifurcations. The exponential asymptotics (1.16), (1.18–1.19), (5.9), (6.7), (6.15) pose an even more aggravating obstacle. For example,

consider resonant homoclinic doubling (theorem B). Then the 2-hom branch is close of order

$$|a_0|^{-1/\varepsilon_1}, \quad a_0 < -1 \quad (7.5)$$

both in parameter and in phase space, as we move away from the bifurcation point along the 1-hom branch by an arclength of $\varepsilon_1 > 0$. For moderate $a_0 = -2$, $\varepsilon_1 = 0.04$, the quantity (7.5) is still as small as $3 \cdot 10^{-8}$. Such effects are hard to detect in numerical case studies. For completeness, we mention that 2-hom orbits were detected analytically in a FitzHugh–Nagumo system with a complex pair of principal eigenvalues rather than real eigenvalues; see Evans *et al.* (1982) and Hastings (1982). This is related to a celebrated result by Shilnikov (1965), predicting shift dynamics near certain homoclinic orbits of flows. We return to this phenomenon at the end of this section.

One potential class of applications is concerned with infinite-dimensional dynamical systems. Specifically, we think of reaction diffusion systems, *viz.*, analytic semigroups, and of differential equations involving time delays. We caution the reader that it is not clear at present how to carry over the fundamental expansion

$$\begin{aligned} y^{\text{in}} &= r(\psi + R_y) \\ x^{\text{out}} &= r^{1/\nu_0}(\varphi + R_x) \end{aligned} \quad (3.9)$$

to such infinite-dimensional semiflows. In fact, the proof of (3.9) involves flows in both positive and negative time direction; see Deng (1989a). Therefore, we do not claim that our results hold, e.g., for reaction-diffusion equations. Techniques are being developed, though, to study homoclinic and heteroclinic orbits for infinite-dimensional semiflows in a more systematic fashion. We mention results on the blue sky catastrophe and an infinite-dimensional λ -lemma (Chow and Deng, 1989) and the results by Lin (1986) and Hale and Lin (1986) involving an exponential dichotomy setting as in (7.4) for functional differential equations. For inclination lemmas and bifurcations of homoclinic/heteroclinic orbits for functional differential equations, see also the fundamental papers (Walther, 1981, 1985–1989). For a detailed and explicit investigation of heteroclinic orbits of scalar reaction diffusion equations in one space dimension, see Brunovský and Fiedler (1988, 1989), Angenent and Fiedler (1988), and the references there. A scalar delay equation was treated by Fiedler and Mallet-Paret (1989), in a similar spirit. Representing an entirely different line of thought, we finally mention Kirchgässner (1982, 1983, 1988) and Fischer (1984), where homoclinic orbits of small amplitude are detected via a reduction to a finite-dimensional center manifold. These results essentially

deal with PDEs of elliptic type in unbounded domains like strips or cylinders. The homoclinic structure arises via appropriate boundary conditions at infinity in a setting which is an infinite dimensional version of (7.4), even though an elliptic system does not define a dynamical system, a priori. Time in (7.4) gets replaced by the unbounded space coordinate of the elliptic system. For the “nonautonomous” case, in this sense, see Kirchgässner (1983, 1988), Mielke (1986a, b) and the references there. Global continuation of homoclinic orbits is possible in principle; see, e.g., Kirchgässner (1988), Amick and Kirchgässner (1988), but suffers from the inherent drawbacks of the setting (7.4) which we have described above.

Returning to finite-dimensional questions, we mention the problem of nonautonomous perturbations. For example, consider

$$\dot{z}(t) = F(\alpha, z(t)) + \varepsilon h(t), \quad \alpha = (\alpha_1, \alpha_2)$$

for periodic forcing h with small amplitude $|\varepsilon|$. Fixing $\varepsilon \neq 0$ small, it is a difficult question to study the dynamic behavior of the corresponding two-parameter family of period map diffeomorphisms, as α varies in a neighborhood of a (nontwisted or twisted) homoclinic bifurcation associated to $\varepsilon = 0$. For an example concerning a homoclinic orbit hitting a stationary fold in the Josephson equation as in Fig. 8 below, see Schecter (1987b).

Hamiltonian systems and time-reversible systems are other classes of equations which we have excluded here. It is a common feature of these systems that any linearization at any (symmetric) equilibrium has symmetric spectrum, i.e.,

$$\mu \in \text{spec} \Leftrightarrow -\mu \in \text{spec}$$

In particular, any homoclinic orbit with real principal eigenvalues is automatically resonant in our sense (1.9). In this context, we remember that resonant side-switching was found by Sanders and Cushman (1986), perturbing the (Hamiltonian) mathematical pendulum. We are not aware of any results on resonant homoclinic doubling for Hamiltonian or reversible systems. For a discussion of certain Hamiltonian systems with periodic orbits, homoclinic orbits, and heteroclinic loops, see Hofer and Toland (1984).

After these excursions, let us now reconsider our original generic two-parameter problem

$$\dot{z} = F(\alpha, z), \quad \alpha \in \mathbb{R}^2 \tag{1.1_\alpha}$$

Broadening our view, in the remainder of this section, we address some questions about global pathfollowing of homoclinic orbits. As a starting

point, we resume our discussion of B -points, alias the Arnold–Bogdanov–Takens singularity, from section 1 [see (1.4) and Fig. 1]. At a B -point, as we recall, a path of homoclinic orbits starts (or terminates). As a goal, we would like to define an orientation of these paths in parameter space, globally, such that we can follow oriented paths “as far as possible” without ever cycling back onto a previously followed piece of the path. Instead of rendering this goal artificially precise, we now define an orientation. Then we will see what this does for us. Our attempt is much in the spirit of Mallet-Paret and Yorke (1980, 1982), Alligood *et al.* (1983), Chow *et al.* (1983), and Fiedler (1985, 1986).

As a prerequisite, we define an index $\Phi \in \{-1, 0, +1\}$ for a hyperbolic periodic orbit $\gamma = \{z(t) | 0 \leq t \leq p\}$ of minimal period $p > 0$. Hyperbolic means that γ does not possess Floquet multipliers on the unit circle, except the trivial one. Let σ^+, σ^- denote the number of real Floquet multipliers in the intervals $(1, \infty), (-\infty, -1)$, respectively, counting algebraic multiplicities. Then the orbit index Φ is defined as

$$\Phi := \frac{1}{2}((-1)^{\sigma^+} + (-1)^{\sigma^+ + \sigma^-}) \tag{7.6}$$

This orbit index Φ , defined first by Mallet-Paret and Yorke (1980), turns out to be a local homotopy invariant for periodic orbits of generic one-parameter vector fields.

Equipped with the orbit index Φ , we can now define an orientation of certain homoclinic paths in parameter space. We assume that periodic orbits γ near the homoclinic orbits occur only for parameters α on one side of the local path segment which we consider (see Fig. 6). We then orient the homoclinic path segment such that the periodic orbits γ are on the right if $\Phi = +1$. If $\Phi = -1$, then we want γ to be on the left. If $\Phi = 0$, then we refuse to define an orientation.

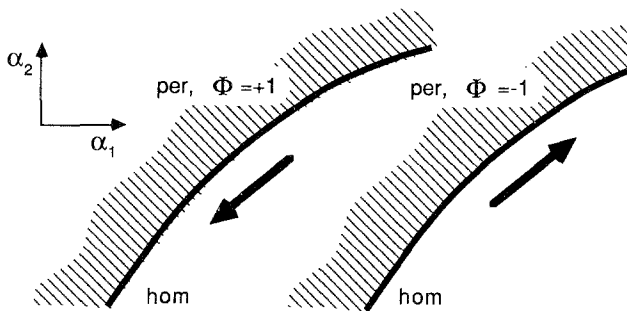


Fig. 6. Orienting homoclinic paths.

How compatible is this definition with the local bifurcation picture at a B -point? Note that $\sigma^- = 0$ and hence $\Phi \neq 0$, near a B -point. Therefore, the homoclinic path can be oriented. Following its orientation, the homoclinic path can either emanate from the B -point, or it can terminate there. These two cases fit nicely with the index B of B -points, which was defined by Fiedler (1986) for the purpose of global Hopf bifurcation. Specifically, $B = +1$ when a homoclinic path emanates and $B = -1$ at termination. The index B is defined in local terms and is easily accessible numerically. See the code BALCON (Fiedler and Kunkel (1987a, b), which is based on ALCON (Deufhard *et al.*, 1987).

What happens to the orientation, far from B -points? Consider resonant side-switching, first. By theorem A, Fig. 3a, there are two 1-per solutions at parameters α in region II. Their respective orbit indices Φ are opposite in sign (or are both zero), by homotopy invariance near the fold curve $\varepsilon_2 = \kappa(\varepsilon_1)$. One of the periodic orbits continues to a blue sky catastrophe at the right 1-hom branch $\varepsilon_1 > 0, \varepsilon_2 = 0$. The other orbit continues to the left 1-hom branch $\varepsilon_1 < 0, \varepsilon_2 = 0$. Since the 1-per orbits have opposite index Φ and approach the 1-hom branch from opposite sides, the orientation of the 1-hom branch is defined consistently on both sides of the bifurcation point $\varepsilon = 0$.

As a second example, consider resonant homoclinic doubling (theorem B and Fig. 3b). To determine all orientations, fix ε_2 small positive and vary only ε_1 . Crossing the κ_{per} -region in the 1-per sheet from left to right, we see that the number σ^- of Floquet-multipliers in $(-\infty, -1)$ changes by one. Thus $\Phi = 0$ on one side, and $\Phi \neq 0$ on the other side. Suppose $\Phi = +1$, for definiteness, as in Fig. 7. For simplicity, also suppose

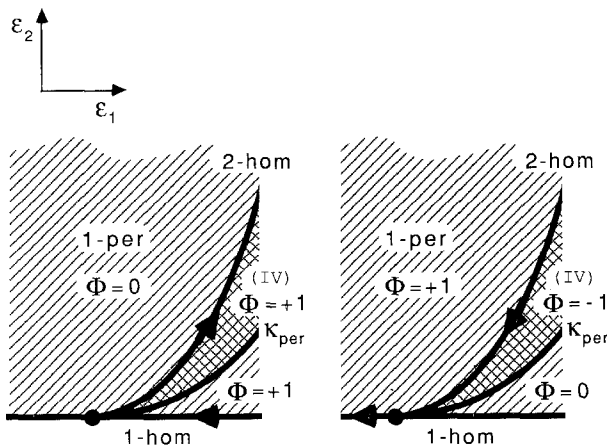


Fig. 7. Two cases of resonant homoclinic doubling.

that the 2-per sheet is confined to the region IV and is parametrized over ε . Then Φ on the 2-per sheet can be computed, by homotopy invariance of Φ , near the period doubling curve $\varepsilon_2 = \kappa_{\text{per}}(\varepsilon_1)$.

If no further period doublings occur on the 2-per sheet (leading to 4-per solutions) then Φ extends, constantly, to the 2-hom branch. This defines an orientation of the 2-hom branch, as in Fig. 7. We conclude that, in each case, precisely two of the three homoclinic half-branches are oriented, and this orientation extends consistently through the bifurcation point. These arguments can be extended to cover the case when the 2-per sheet is neither confined to the region IV nor parametrized over ε . However, due to the difficulties which we have described above, we cannot exclude the possibility of further period doublings from the 2-per sheet which would lead to 4-per orbits. In fact, we are lacking control over N -per orbits with $N \geq 3$. This difficulty does not arise for resonant side-switching because 2-per orbits do not exist, in that case.

Of course, the above remarks do not constitute a global bifurcation result for paths of homoclinic orbits. Other bifurcations can occur. We briefly mention some of the known results. But we do not tie these results in with our index theory, this time.

We have already mentioned above that non-resonant mechanisms for homoclinic doubling were found by Yanagida (1987). Now let us consider the base point A of a homoclinic orbit $z^*(t)$, that is, the stationary point to which $z^*(t)$ tends for $t \rightarrow \pm\infty$. In the previous sections we have discussed at length what happens when the principal eigenvalues at A become resonant. Now suppose one of the principal eigenvalues becomes zero. Generically, this means that the homoclinic path hits a fold line in parameter space while the base point A hits the fold. This situation was studied by Lukyanov (1982) and Schecter (1987a) for $z \in \mathbb{R}^2$. The bifurcation diagram is sketched in Fig. 8. It was noted by Schecter that the hom-curve hits the fold curve tangentially rather than transversely. Corresponding results for higher dimension of z are now available; see Chow and Lin

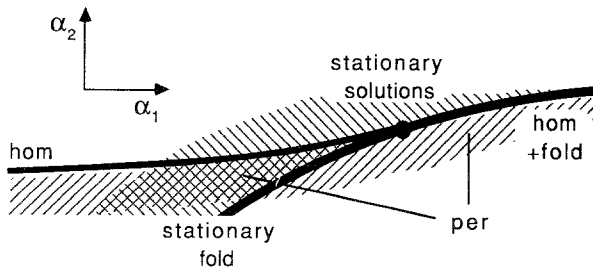


Fig. 8. A homoclinic path hitting a stationary fold.

(1988), Deng (1990b). Also in higher space dimension, but in one-parameter systems, the generation of (unique) periodic orbits from homoclinic orbits with base point on a fold was studied by Shilnikov (1962, 1966). Shilnikov (1969) and Afraimovich and Shilnikov (1974) also have observed that a fold point A , with $p \geq 2$ distinct homoclinic orbits attached to it, generates nearby dynamics which is equivalent to shift dynamics on p symbols. Evidently, this situation can arise during a pathfollowing process when p path segments successively hit a stationary fold, each time as in Fig. 8, and then coexist. In other words, the very intriguing results by Shilnikov (1969) and Afraimovich and Shilnikov (1974) indicate how pathfollowing of homoclinic orbits can lead to regions with chaotic, i.e., shift-type, dynamics near a stationary fold.

At this stage, it seems natural to ask what happens when the base point approaches a Hopf bifurcation along the homoclinic path. Long before that can happen, however, another celebrated result by Shilnikov takes effect. Namely, we may assume that both principal eigenvalues are in fact conjugate complex pairs, or that one of them is real, say $\nu_0 > 0$, while the other "one" is a conjugate complex pair, say $-\mu_0 \pm i\omega_0$, such that

$$0 < \mu_0 < \nu_0 \quad (7.7)$$

In both cases, nearby shift dynamics on a countable alphabet was discovered, generically. See Shilnikov (1965, 1967, 1970). A discussion of some of these results is given by Deng (1989b), with an emphasis on exponential expansions of the Shilnikov variables like (3.9). For a C^1 -version of (7.7) with $z \in \mathbb{R}^3$ see also Tresser (1984). For applications of these ideas in the FitzHugh–Nagumo equations see Evans *et al.* (1982) and Hastings (1982), again. Homoclinic tangencies of return maps occur on a dense set along homoclinic paths where (7.7) holds; see Ovsyannikov and Shilnikov (1987). The transition into a region where (7.7) holds was studied for two-parameter vector fields in \mathbb{R}^3 by Belyakov (1980, 1984). He detects, e.g., countably many paths of 2-hom and, in one of the cases, of 3-hom orbits accumulating at the transition point. See also Rodriguez (1986), for the 1-hom branch.

When (7.7) is violated, that is,

$$0 < \nu_0 < \mu_0$$

then a 2-shift still occurs in \mathbb{R}^3 , provided that two distinct homoclinic orbits are limiting onto the same base point A , thus forming a figure "8." The two symbols correspond to excursions along the upper/lower loop of the figure 8. In particular, such a situation can arise for systems with a (reflection) \mathbb{Z}_2 -symmetry; see Holmes (1980), Tresser (1984), Gambaudo

et al. (1984). The case $\nu_0 > \mu_0 > 0$ was also discussed by Tresser (1984) and Glendinning (1984). Repeated period doublings near pairs of homoclinic orbits with real principal eigenvalues and coinciding principal directions

$$\lim_{t \rightarrow +\infty} \dot{z}^*(t)/|\dot{z}^*(t)|$$

due to \mathbb{Z}_2 -symmetry, are observed by Arneodo *et al.* (1981), Kuramoto and Koga (1982), Coulet *et al.* (1984), and Lyubimov and Zaks (1983). For the Lorenz system, see in particular Sparrow (1982). Again, complicated dynamics abound.

Toward the end of our long excursion into global pathfollowing, let us mention the largely open problem of bifurcation from heteroclinic cycles. From a pathfollowing point of view, heteroclinic cycles arise as follows. Let **A** denote the base point of a homoclinic orbit Γ , as before. Following a path in parameter space, some part of Γ might approach another stationary point **B**. In the limit, we have a heteroclinic cycle: one trajectory runs from **A** to **B** and another one returns from **B** to **A**. As references for bifurcations from heteroclinic cycles one may consult Reyn (1980) for $z \in \mathbb{R}^2$, Bykov (1980) and Tresser (1984) for $z \in \mathbb{R}^3$, when conjugate complex eigenvalues occur, and Rinzel and Terman (1982), Chow *et al.* (1986, 1990), Deng (1990a), and Kokubu (1987–1988) when the principal eigenvalues are real. In all these cases, it is assumed that both **A** and **B** are hyperbolic with unstable dimension given by $u_{\mathbf{A}} = u_{\mathbf{B}} = 1$. In some cases, Bykov (1980) and Tresser (1984) find shift-dynamics of the Shilnikov type. Deng (1990a), on the other hand, detects a phenomenon where, in parameter space, paths $\kappa_0, \kappa_1, \dots$ of heteroclinic solutions bifurcate from the heteroclinic cycle point such that the following holds. For $\alpha \in \kappa_N$, the heteroclinic orbit cycles through N complete loops near the heteroclinic cycle and only afterwards converges to **A**, **B** for $t \rightarrow \pm\infty$ without cycling further. The basic assumption there is again a certain twist condition. Deng (1990a) also sketches how this bifurcation can occur in planar vector fields. In general, heteroclinic cycles break apart under perturbations of the vector field; generically they occur as a phenomenon of codimension at least two. When enough symmetry is present, however, heteroclinic cycles can become structurally stable. Essentially, heteroclinic orbits can then come to lie in certain linear subspaces which have a symmetry-invariance and are therefore flow invariant. Of course, this influences the local bifurcation diagrams as well. For interesting examples see Armbruster *et al.* (1988) and Guckenheimer and Holmes (1988). But even when symmetry is not present, there remains an enormous number of largely unresolved cases of generic two-parameter bifurcations from heteroclinic cycles. These cases arise when the unstable dimensions $u_{\mathbf{A}}$ or $u_{\mathbf{B}}$ exceed one, possibly differing

from each other, with real or complex principal stable or unstable eigenvalues of **A** and **B**. For an intricate example of heteroclinic loops in the \mathbb{Z}_2 -symmetric Lorenz system, see Glendinning and Sparrow (1986) and also Sparrow (1982).

Each of the above cases, separately, will create a challenge to our attempt of sketching an emerging theory for global pathfollowing of homoclinic orbits in two-parameter flows. On this long way still ahead, it is our firm hope that paths of homoclinic orbits will turn into a systematic guiding thread from easily detected *B*-points up to regions with complicated, shift-type dynamics.

8. APPENDIX: GENERICITY ASSUMPTIONS

In this Appendix, we summarize the nondegeneracy assumptions which enter into our main theorems A and B.

Our first assumption was

$$\text{codim}(T_{\mathbf{p}}W^u + T_{\mathbf{p}}W^s) = 1 \tag{1.10a}$$

so that $T_{\mathbf{p}}W^s \cap T_{\mathbf{p}}W^u$ is given by the span of the tangent $z^*(0)$ of the homoclinic orbit at $z^*(0) = \mathbf{p}$, $\alpha = 0$. Then we have assumed

$$\mathbf{p} \notin W^{ss}, \quad \mathbf{p} \notin W^{uu} \tag{1.10b}$$

so that $z^*(t)$ approaches $z = 0$ tangentially to the principal eigendirections e^{\mp} as $t \rightarrow \pm\infty$. With the notation

$$T_{z^*(t)} := T_{z^*(t)}W^s + T_{z^*(t)}W^u$$

we have then assumed that $T_{z^*(t)}$, say, avoids certain exceptional positions. In that case, the strong λ -lemma (proposition 3.3) implies

$$\begin{aligned} \lim_{t \rightarrow -\infty} T_{z^*(t)} &= T_0W^{ss} \oplus T_0W^u \\ \lim_{t \rightarrow +\infty} T_{z^*(t)} &= T_0W^s \oplus T_0W^{uu} \end{aligned} \tag{1.13}$$

This assumption has enabled us to discern twisted and nontwisted homoclinic orbits. Moreover, this assumption made our Ljapunov–Schmidt reduction work, via the projection P_j^0 defined in (4.6).

So far, our genericity assumptions are concerned only with the vector field $F = F(\alpha, z)$ for fixed parameter $\alpha = 0$. These assumptions led to the reduced bifurcation equation

$$0 = \Phi_j(\mathbf{r}, \alpha) = -c_0(\alpha) + c_1(\alpha)r_{j+1} - c_2(\alpha)r_j^{1/\nu_0(\alpha)} + O(|\mathbf{r}|^{1+\omega}), \quad j \pmod N \tag{4.8}$$

with appropriate coefficients $c_0(\alpha)$, $c_1(\alpha)$, $c_2(\alpha)$, given explicitly by (4.9). The remaining genericity assumptions enable us to rewrite (4.8) as the reduced normal form

$$r_{j+1} = \varepsilon_2 + a(\varepsilon)r_j^{1+\varepsilon_1} + O(|\mathbf{r}|^{1+\omega}), \quad j \pmod{N} \tag{4.13}$$

with $|a(0)| \neq 0, 1$, by a parameter transformation

$$\alpha \rightarrow \varepsilon = \varepsilon(\alpha)$$

This diffeomorphism is given explicitly in (4.14a, b). Denoting the principal unstable eigenvalue by $v_0 = v_0(\alpha)$, with the principal stable eigenvalue normalized to be $-\mu_0(\alpha) = -1$, the relevant nondegeneracy assumptions were

- (a) $D_x c_0(0)$ and $D_x v_0(0)$ are linearly independent
 - (b) $c_1(0) > 0$
 - (c) $\left| \frac{c_2(\alpha)}{c_1(\alpha)} \right| \neq 0, 1$
- (4.15)

Genericity of these three assumptions was discussed in the proof of corollary 4.3. In particular positivity of $c_1(0)$ was generic, due to a sign convention.

With (1.10a, b), (1.13), and (4.15a–c), our list of genericity assumptions is complete.

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