TURAN-TYPE INEQUALITIES IN CERTAIN INTEGRAL METRICS

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Let $T_n(t)(P_n(x))$ be a nontrivial trigonometric (algebraic) polynomial of degree n, $n \in N$, with all real zeros which, in the case of the algebraic polynomial, lie in the interval $[-1, +1]$. In what follows, $\|T_n\|_p = \left\{\int\limits_0^{2\pi} |T_n(t)|^p\, dt\right\}^{1/p}$, $\|P_n\|_p = \left\{\int\limits_{-1}^{+1} |P_n(x)|^p\, dx\right\}^{1/p}$, $1 \leqslant p < +\infty$, $\|T_n\|_\infty = \|T_n\|_C = \max\limits_{0 \leqslant t \leqslant 2\pi} |T_n(t)|$, $\|P_n\|_\infty = \|P_n\|_C = \max\limits_{-1 \leqslant x \leqslant +1} |P_n(x)|$. We note that under the assumptions we have made, $T_n(t) = C \prod\limits_{i=1}^{2n} \sin\frac{t-t_i}{2}$, where c, $t_i \in \mathbb{R}$, $i = \overline{1, 2n}$, $C \neq 0$, $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_{2n} < t_1 + 2\pi$.

Turan [1] has proved the inequality $\|P_n'\|_C \geqslant \frac{\sqrt{n}}{6}\|P_n\|_C$, where the constant $\frac{\sqrt{n}}{6}$ is not precise; however, Erod [2] has obtained precise values of the constant in the Turan inequality:

$$\frac{\|P_n'\|_C}{\|P_n\|_C} \geqslant \begin{cases} \frac{n}{2}, \ n = 2, 3, \\ \frac{n}{\sqrt{n-1}}\left(1 - \frac{1}{n-1}\right)^{(n-2)/2}, \ n = 4\ 6.\ 8, \ldots, \\ \frac{n^2}{(n-1)\sqrt{n+1}}\left(1 - \frac{\sqrt{n+1}}{n-1}\right)^{(n-3)/2}\left(1 + \frac{1}{\sqrt{n+1}}\right)^{(n-1)/2}, \\ \qquad n = 5, 7, 9, \ldots . \end{cases}$$

and V. F. Babenko and S. A. Pichugov [3] have established that

$$\|T_n'\|_C \geqslant \sqrt{\frac{n}{2}}\left(1 - \frac{1}{2n}\right)^{n-1/2} \|T_n\|_C,$$

where the equality holds for the polynomials $T_n(t) = C\left(\sin\frac{t-\gamma}{2}\right)^{2n}$, $\forall \gamma \in \mathbb{R}$, $C \neq 0$

In the present note, we show that, for $1 \leq p < +\infty$

$$\inf_{\|T_n\|_C=1} \|T_n'\|_p = \left\|\left[\left(\sin\frac{t-\gamma}{2}\right)^{2n}\right]'\right\|_p = n\left\{2B\left(\frac{(2n-1)p+1}{2}, \frac{n+1}{2}\right)\right\}^{1/p}. \quad (1)$$

The inequality (1) is a consequence of the following theorem.

THEOREM. For any continuous increasing function $\chi(u)$, $u \geqslant 0$, $\chi(0) = 0$, which is convex downwards, we have that

$$\min_{\|T_n\|_C=1} \int\limits_0^{2\pi} \chi(|T_n'(t)|)\, dt = \int\limits_0^{2\pi} \chi\left(\left|\left[\left(\sin\frac{t-\gamma}{2}\right)^{2n}\right]'\right|\right) dt \ \ \forall \gamma \in \mathbb{R}, \quad (2)$$

where $T_n(t)$ is a trigonometric polynomial of degree n with all real zeros.

In the first part of the proof of the theorem we will follow the argument in [4], where instead of the greatest lower bound in (2), a least upper bound is obtained. Thus we fix n = 1, 2, ..., and let $0 \leqslant \xi \leqslant n$. Letting $0 \leqslant u \leqslant n$, we construct the function $\varphi_\xi(u) = 0$, $0 \leqslant u < \xi$, $\varphi_\xi(u) = u$, $\xi \leqslant u \leqslant n$.

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We split the interval $[0, n]$ into m equal parts by means of the points $\frac{kn}{m}$, $k=\overline{0, m}$, and we put $\Phi_m(u)=\sum_{k=1}^{m-1}\alpha_k\varphi_{\frac{kn}{m}}(u)$, where α_k is from $\Phi_m\left(\frac{kn}{m}\right)=\chi\left(\frac{kn}{m}\right)$, $k=\overline{1, m-1}$. It is easy to see that $\Phi_m(u)\to\chi(u)$ uniformly on $[0, n]$ as $m\to+\infty$. Relation (2) will be proved if we prove that, for all sufficiently large, m

$$\min_{\|T_n\|_C=1}\int_0^{2\pi}\Phi_m(|T_n'(t)|)\,dt=\int_0^{2\pi}\Phi_m\left(\left|\left[\left(\sin\frac{t-\gamma}{2}\right)^{2n}\right]'\right|\right)dt\quad\forall\gamma\in R. \tag{3}$$

Since the coefficients α_k are nonnegative, we have that

$$\min_{\|T_n\|_C=1}\int_0^{2\pi}\Phi_m(|T_n'(t)|)\,dt=\min_{\|T_n\|_C=1}\sum_{k=1}^{m-1}\alpha_k\int_0^{2\pi}\varphi_{\frac{kn}{m}}(|T_n'(t)|)\,dt\geqslant\sum_{k=1}^{m-1}\alpha_k\min_{\|T_n\|_C=1}\int_0^{2\pi}\varphi_{\frac{kn}{m}}(|T_n'(t)|)\,dt,$$

and to prove the theorem it is sufficient to prove that this last sum is the same as the right-hand side of (3). This fact is a consequence of the following result.

<u>LEMMA.</u> For $0\leqslant\xi\leqslant n$

$$\min_{\|T_n\|_C=1}\int_0^{2\pi}\varphi_\xi(|T_n'(t)|)\,dt=\int_0^{2\pi}\varphi_\xi\left(\left|\left[\left(\sin\frac{t-\gamma}{2}\right)^{2n}\right]'\right|\right)dt\quad\forall\gamma\in\mathbb{R}. \tag{4}$$

<u>Proof.</u> Let $E_\xi(T_n)=\{t\mid t\in\mathbb{R},\ |T_n'(t)|\geqslant\xi\}$, where $0\leqslant\xi\leqslant\|T_n'\|_C$. For $\xi>\|T_n'\|_C$ the set $E_\xi(T_n)$ is empty, and in this case we put mes $(E_\xi(T_n))=0$. If we put $e_\xi(T_n)=E_\xi(T_n)\cap[0, 2\pi]$, and take into account the notation introduced above, we get

$$\int_0^{2\pi}\varphi_\xi(|T_n'(t)|)\,dt=\int_{e_\xi(T_n)}|T_n'(t)|\,dt=\bigvee_{e_\xi(T_n)}T_n. \tag{5}$$

In what follows, we let $\tilde{T}_n(t)=\left(\sin\frac{t-\gamma}{2}\right)^{2n}$ $\forall\gamma\in\mathbb{R}$. From (5) it follows that (4) is equivalent to the inequality

$$\bigvee_{e_\xi(T_n)}T_n\geqslant\bigvee_{e_\xi(\tilde{T}_n)}\tilde{T}_n, \tag{6}$$

where $\|T_n\|_C=1$, $0\leqslant\xi\leqslant n$.

The remainder of the argument is essentially different from that in [4]. Suppose that $\|\tilde{T}_n'\|_C\leqslant\xi\leqslant n$. By virtue of the inequality [3] $\|T_n'\|_C\geqslant\|\tilde{T}_n'\|_C$ $(\|T_n\|_C=1)$ we get that mes $(e_\xi(T_n))\geqslant 0=$ mes $(e_\xi(\tilde{T}_n))$, i.e., $\bigvee_{e_\xi(T_n)}T_n\geqslant 0=\bigvee_{e_\xi(\tilde{T}_n)}\tilde{T}_n$. In this case, (6) is trivial. Suppose that $\xi=0$. Then (6) is also satisfied by virtue of the obvious relation $(e_\xi(T_n)=[0, 2\pi])$ $\bigvee_0^{2\pi}T_n\geqslant 2=\bigvee_0^{2\pi}\tilde{T}_n$, $\|T_n\|_C=1$.

Suppose now that $0<\xi<\|\tilde{T}_n'\|_C$. Here we use an approach which is used in [3] for similar problems. We will consider the polynomial $T_n(t)$ as a function of the variables C, t, and t_i, $i=\overline{1, 2n}$. In the space $\mathbb{R}^{2n+4}$ of the variables C, x, y, z, t_i, $i=\overline{1, 2n}$, we consider the set Q defined by the inequalities $t_{2n}-2\pi<y<t_1\leqslant t_2\leqslant\ldots\leqslant t_{2n}<t_1+2\pi$, $t_{2n-1}-2\pi<z<x<y$, $-\infty<C<+\infty$. We put m, $k_i\in N$, $i=\overline{1, m}$, $1\leqslant k_1,\ldots,k_m$, $m\leqslant 2n$, $\sum_{i=1}^{m}k_i=2n$. The sets $Q_m(k_1,\ldots,k_m)\subset Q$ defined by the equalities $t_1=t_2=\ldots=t_{k_1}$, $t_{k_1+1}=\ldots=t_{k_1+k_2}$, $\ldots$, $t_{k_1+\cdots+k_{m-1}+1}=\ldots=t_{2n}$ may be regarded as "faces" of the set Q. We note that $Q=\bigcup_{m=1}^{2n}\bigcup_{\{k_1,\ldots,k_m\}}Q_m(k_1,\ldots,k_m)$, and $Q_m(k_1,\ldots,k_m)$ is a region in $\mathbb{R}^{m+4}$ in which the parameters C, x, y, z, $\bar{t}_1=t_1=\ldots=t_{k_1}$, $\ldots$, $\bar{t}_m=t_{k_1+\cdots+k_{m-1}+1}=\ldots=t_{2n}$ vary. On the set Q we consider the auxiliary extremal problem:

$$T'_n(x) = \xi,\ T'_n(y) = 0,\ T_n(y) = 1,\ T''_n(z) = 0,\ T_n(x) \to \inf. \tag{7}$$

It can be shown that the greatest lower bound in (7) is achieved. Taking into account the smoothness of the constraints and considering the problem (7) on each region $Q_m(k_1, \ldots, k_m)$ separately, with the help of Lagrange multipliers ([5], pp. 47-48) we get that all of the stationary points of (7) lie in $Q_1(2n)$. Hence we can conclude immediately that the greatest lower bound in (7) is $\tilde{T}_n(x)$, where $\tilde{T}'_n(x) = \xi,\ x > z,\ \tilde{T}''_n(z) = 0$.

The auxiliary extremal problem

$$T'_n(x) = \xi,\ T'_n(y) = 0,\ T_n(y) = 1,\ T''_n(z) = 0,\ T_n(x) \to \sup \tag{8}$$

on the set $\{t_{2n} - 2\pi < y < t_1 \leqslant t_2 \leqslant \ldots \leqslant t_{2n} < t_1 + 2\pi,\ t_{2n-1} - 2\pi < x < z < y,\ -\infty < C < +\infty\}$ can be solved in a manner similar to that just applied to problem (7). The solution to problem (8) will be $\tilde{T}_n(x)$, where $\tilde{T}'_n(x) = \xi$, $x < z$, $\tilde{T}''_n(z) = 0$, i.e., the polynomial $\tilde{T}_n(t)$ is extremal for both problem (7) and problem (8). The case where x lies to the right of y is treated similarly.

In [3], the relation

$$\max_{y_1 \leqslant t \leqslant y} |T'_n(t)| \geqslant \|\tilde{T}'_n\|_C \tag{9}$$

is proved, where $T'_n(y) = T'_n(y_1) = 0,\ T_n(y) = 1,\ t_{2n-1} - 2\pi \leqslant y_1 < y < t_1$. Hence it is not difficult to obtain the inequality

$$\max_{y \leqslant t \leqslant y_1} |T'_n(t)| \geqslant \|\tilde{T}'_n\|_C, \tag{10}$$

where $T'_n(y) = T'_n(y_1) = 0,\ T_n(y) = 1,\ t_{2n} - 2\pi < y < y_1 \leqslant t_2$. The inequalities (9) and (10) show that, for $0 < \xi < \|\tilde{T}'_n\|_C$, $\|T_n\|_C = 1$, the set $e_\xi(T_n)$ consists of not fewer than two intervals which lie to the right and left of y,

Thus if [a, b] is the interval of the set $E_\xi(T_n)$ which lies on the left, say, and closest to the point y, $T_n(y) = 1$, which is a local maximum of the polynomial $T_n(t)$ satisfying the constraints of (7) and (8), then the solutions of these problems allow us to write

$$T_n(a) \leqslant \tilde{T}_n(a_0),\ T_n(b) \geqslant \tilde{T}_n(b_0),\ a_0 = a(\tilde{T}_n),\ b_0 = b(\tilde{T}_n). \tag{11}$$

From (9) and (10) it follows that every polynomial $T_n(t)$, $\|T_n\|_C = 1$, satisfies the constraints of the problems (7) and (8) if the zeros are suitably indexed. Taking (11) into account, we get

$$\tilde{T}_n(a_0) = \max_{\|T_n\|_C = 1} T_n(a), \quad \tilde{T}_n(b_0) = \min_{\|T_n\|_C = 1} T_n(b). \tag{12}$$

Here we have used the fact that $\|\tilde{T}_n\|_C = 1$. From (12) it follows that, for any $T_n(t)$, $\|T_n\|_C = 1$, we have

$$T_n(b) - T_n(a) \geqslant \min_{\|T_n\|_C = 1} T_n(b) - \max_{\|T_n\|_C = 1} T_n(a) = \tilde{T}_n(b_0) - \tilde{T}_n(a_0) > 0,$$

where $a = a(T_n)$ and $b = b(T_n)$. The case where the interval $[a, b] \subset E_\xi(T_n)$ lies to the right of y is treated similarly.

As a result, we get that, for any

$$\bigvee_{e_\xi(T_n)} T_n \geqslant |T_n(a) - T_n(b)| + |T_n(c) - T_n(d)| \geqslant 2|\tilde{T}_n(a_0) - \tilde{T}_n(b_0)| = \bigvee_{e_\xi(\tilde{T}_n)} \tilde{T}_n, \tag{13}$$

where [a, b]([c, d]) is the closest interval in the set $E_\xi(T_n)$ to the left (right) of y, $\|T_n\|_C = T_n(y)$, where $0 < \xi < \|\tilde{T}'_n\|_C$. The relation (13) is the same as (6). This proves the lemma, and therefore the theorem.

COROLLARY 1. For any trigonometric polynomial $T_n(t)$ with all real zeros, we have the precise inequality $(1 \leqslant p < +\infty)$

$$\left\{2B\left(\frac{(2n-1)p+1}{2},\ \frac{p+1}{2}\right)\right\}^{1/p} \cdot n \cdot \|T_n\|_C \leqslant \|T'_n\|_p.$$

The equals sign holds for the polynomials $T_n(t) = C\cdot\left(\sin\frac{t-\gamma}{2}\right)^{2n}$ $\forall\gamma$, $C\in\mathbb{R}$, $C\neq 0$.

Corollary 1 is obtained from the theorem for $\chi(u)=u^p$, $1\leqslant p<+\infty$.

COROLLARY 2. For any algebraic polynomial $P_n(x)$ with all real zeros in the interval $[-1, +1]$, we have the inequality $(1\leqslant p<+\infty)$

$$\left\{B\left(\frac{(2n-1)p+1}{2}, \frac{p+1}{2}\right)\right\}^{1/p} n\|P_n\|_C \leqslant \|p_n'(x)(1-x^2)^{\frac{p-1}{2\cdot p}}\|_p.$$

In [6] it is shown that $\max\limits_{\|T_n\|_C=1}\|T_n\|_q=\|\cos n\cdot t\|_q$, $1\leqslant q<+\infty$. Hence we get this result from Corollary 1.

COROLLARY 3. For any trigonometric polynomial $T_n(t)$ with all real zeros we have the inequality $(1\leqslant p,\ q<+\infty)$

$$\frac{\left\{2B\left(\frac{(2n-1)\cdot p+1}{2}, \frac{p+1}{2}\right)\right\}^{1/p}}{\left\{2B\left(\frac{1}{2}, \frac{q+1}{2}\right)\right\}^{1/q}} n\|T_n\|_q<\|T_n'\|_p.$$

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QUASIINPUT PROCESS IN AN M/G/1/∞ SYSTEM

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1. Introduction. We consider a classical M/G/1/∞ queue with delays. Let $\xi_i(\eta_i)$ denote the time at which the i-th customer enters into (respectively leaves) service. The series η_i of points on the time axis constitutes the so-called departure process. This has been studied in great detail (see for example [1]), whereas investigation of the related series $\{\xi_i\}_{i\geqslant 1}$ of times of entry into service is comparatively recent. The series $\{\xi_i\}_{i\geqslant 1}$ is the input process for the service mechanism and therefore it is naturally referred to as the quasiinput process. The study of the quasiinput process is all the more important since often we may observe only the on/off times of the service mechanism, i.e., the quasiinput and the departure processes.

This process was first considered in [2], where certain of its properties in the stationary case were studied. In this paper we continue the investigation of the quasiinput process and we strengthen results derived in [2].

Notation: λ, the traffic arrival rate; $\beta(s)$, the Laplace—Stieltjes transform of the service time distribution function $B(x)$; $\beta_k=(-1)^k\beta^{(k)}(0)$, $\rho=\lambda\beta_1$, $S_i=\eta_i-\xi_i$ the service time of the i-th customer; $I_i=\xi_i-\eta_{i-1}$ the idle interval before the i-th customer enters service; N_i, the length of the queue immediately prior to the instant η_i; ν_i, the number of customers entering the system in the time interval (ξ_i, η_i).

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