KRAVCHUK ORTHOGONAL POLYNOMIALS

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A survey of the principal works of Academician M. P. Kravchuk and his students in the area of orthogonal polynomials of a discrete variable is presented. The value of these studies for the further development of the theory, for drawing generalization, and for the construction of different applications of this class of special functions is noted.

The name of the outstanding Ukrainian scientist and mathematician, M. P. Kravchuk, will live forever in the world's mathematical literature. He has been immortalized by having a class of orthogonal polynomials of a discrete variable which has been named after him.

The present article represents an attempt at a historical scientific review of the full range of Kravchuk's creativity relating to the discovery and study of what are known as Kravchuk polynomials. The results found by Kravchuk's school in this area are analyzed, and the subsequent course of certain studies in the area of orthogonal polynomials of a discrete variable and various applications of these polynomials are outlined.

It should be noted that the publications from the period 1929—1931, which contain the principal results concerning Kravchuk polynomials, have long since been extremely difficult to locate by readers who might wish to become acquainted with these results in the original.

In 1921—1929 Kravchuk held the post of professor and "chairman of the department of mathematics and variational statistics" at the Kiev State Agricultural Institute. It was there that the scientific journal Zapysky Kyivskogo Sil's'ko—Gospodars'kogo Instytutu, of which Kravchuk was a member of the editorial board, was published. It was in the issue of December 29, 1928 of this journal that Kravchuk's article, "On interpolation by means of orthogonal polynomials" [1], appeared. The article consisted of a short forward, four sections, and an abstract in French.

In the forward Kravchuk stated that "P. Chebyshov [Chebyshev] repeatedly analyzed the problem of parabolic interpolation using the method of least squares by means of orthogonal polynomials. The source of his studies was the theory of algebraic continued fractions. His general formulas are especially useful in practical computations in the special case in which all the given values of the interpolated function possess the same weight and are equidistant."

It is known that, in order to simplify the processing of the results of artillery fire, Chebyshev, using the method of least squares and constructing corresponding tables of artillery fire, introduced what are now known as Chebyshev polynomials of a discrete variable. In his article [1], Kravchuk emphasized that "he was briefly setting forth Chebyshev's discoveries independent of the theory of continued fractions and presenting a detailed analysis of the special case referred to above," noting that it was precisely such an approach that "led Chebyshev to a generalization of Legendre polynomials."

A second case occurs "when the weight of the given values of the interpolated function is replaced in accordance with the law of binomial probability distributions, [thereby] providing a generalization of Hermite's polynomials." It was just this case that led to new classical orthogonal polynomials of a discrete variable, what are now known as Kravchuk polynomials.

Kravchuk's article [1] contained six references to Chebyshev's studies [2, 3] and there can be no doubt that it was written under the influence of the studies completed by this great mathematician.

Let us analyze in more detail the principal results of [1]. In Section 1 may be found a general statement of the problem. Here is part of the statement: "Suppose we are given the values of an independent variable x: $x_0, x_1, \ldots, x_{n-1}$, all of which are distinct, and corresponding values of a function y: $y_0, y_1, \ldots, y_{n-1}$.

Kiev University. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 44, No. 7, pp. 880-888, July, 1992. Original article submitted March 16, 1992.

 0041-5995/92/4407-0792$12.50

"The coefficients A_m of the approximate equality

$$y \approx A_0\psi_0(x) + A_1\psi_1(x) + \dots + A_k\psi_k(x) \quad (k < n), \tag{1}$$

where $\psi_m(x)$ is a polynomial of degree m, are determined by means of the formula

$$J_k^2 = \sum_{i=0}^{n-1} p_i [y_i - A_0\psi_0(x_i) - A_1\psi_1(x_i) - \dots - A_k\psi_k(x_i)]^2 = \min, \tag{2}$$

and the function $\psi_m(x)$ by means of the orthogonality conditions

$$\sum_{i=0}^{n-1} p_i\psi_l(x_i)\,\psi_m(x_i) = 0 \quad (l \neq m) \tag{3}$$

and the normality conditions

$$\sum_{i=0}^{n-1} p_i\psi_m^2(x_i) = 1, \tag{4}$$

where $p_i > 0$ and $\sum_{i=0}^{n-1} p_i = 1$».

Traditional methods used to find the function $\psi_m(x)$ and the coefficients A_m, together with an analysis of (1) and the mean-square error, are given in Section 2. The results are stated in the form of determinants.

In Section 3 Kravchuk gave a detailed analysis of the simple case in which $p_0 = p_1 = \dots = p_{n-1} = 1/n$ and all the intervals $\Delta x_i = x_{i+1} - x_i$ $(i = 0, 1, \dots, n-1)$ are also equal, moreover, $x_0 = 0$ and $\Delta x_i = 1$. He proves that

$$\psi_m(x) = c_m\Delta^m[x(x-1)\dots(x-m+1)(x-n)(x-n-1)\dots(x-n-m+1)], \tag{5}$$

where c_m are found from the relation

$$1 = \frac{1}{n}\, c_m^2\, \frac{(m!)^2\, n\,(n^2-1)(n^2-4)\dots(n^2-m^2)}{2m+1}. \tag{6}$$

"Therefore," continues Kravchuk, "in this case (1) has the following form":

$$y \approx \sum_{m=0}^{k} \frac{(2m+1)\sum_{i=0}^{n-1}(i+1)\dots(i+m)(n-1+i)\dots(n-m-i)}{(m!)^2 n\,(n^2-1)(n^2-4)\dots(n^2-m^2)} \times \tag{7}$$
$$\times\,\Delta^m y_i \Delta^m[x(x-1)\dots(x-m+1)(x-n)(x-n-1)\dots(x-n-m+1)].$$

The mean-square error of this approximation was written out.

Note that (5) and (6) completely determine Chebyshev polynomials of a discrete variable. Further, "introducing quantities $x/n = t$ and $1/n = dt$ $(n \to \infty)$," Kravchuk deduced from them the following "expression: $\frac{\sqrt{2m+1}}{m!}\frac{d^m}{dt^m}t^m(1-t)^m$, that is, Legendre's polynomial."

The principal novel result is given in Section 4. Kravchuk remarked the following:

"Once again, let $x_0 = 0$ and $\Delta x_i = 1$, though we also set

$$p_x = p(x) = \binom{n-1}{x} p^x q^{n-1-x} = \frac{(n-1)(n-2)\dots(n-x)}{1\cdot 2\dots x}\, p^x q^{n-1-x}, \tag{8}$$

where $p > 0$, $q > 0$, and $p + q = 1$.

Moreover, note that $\binom{k}{l} = 0$ for $\ell < 0$, $\ell > k$.

This case is of particular importance for mathematical statistics."

It is established that

$$\psi_m(x) = c_m \Delta^m \left[\binom{n-m-1}{x-m} p^{x-m} q^{n-x-1}\right] : \left[\binom{n-1}{x} p^x q^{n-1-x}\right], \tag{9}$$

where

$$c_m^2 = \frac{(n-1)(n-2)\ldots(n-m)}{m!}(pq)^m. \tag{10}$$

"Therefore," continues Kravchuk, "the interpolation formula (1) in this case assumes the form

$$y \approx \sum_{m=0}^{k}\left[\binom{n-1}{m}(-pq)^m \sum_{i=0}^{n-1}\binom{n-m-1}{i-m} p^{i-m} q^{n-i-1} \Delta^m y_{i-m}\right] \times \\ \times \frac{\Delta^m\left[\binom{n-m-1}{x-m} p^{x-m} q^{n-1-x}\right]}{\binom{n-1}{x} p^x q^{n-1-x}}, \tag{11}$$

and its mean-root-square error is given as

$$J_k = \sqrt{\sum_{i=0}^{n-1}\binom{n-1}{i} p^i q^{n-1-i} y_i^2 \sum_{i=0}^{k}\binom{n-1}{m}(pq)^m \times \\ \times \left[\sum_{i=0}^{n-1}\binom{n-m-1}{i-m} p^{i-m} q^{n-i-1} \Delta^m y_{i-m}\right]^2}. \tag{12}$$

[Further,] introducing in place of the variable x the variable t by means of the equality $x - np = t\sqrt{npg}$ and setting $dt = 1/\sqrt{npq}$ $(n \to \infty)$, [it was found,] by taking limits, that

$$p = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}, \quad \psi_m = \frac{e^{\frac{t^2}{2}}}{\sqrt{m!}} \frac{d^m}{dt^m} e^{-\frac{t^2}{2}},$$

whence the polynomial (9) is a generalization of what are known as Hermite's polynomials."

The polynomials (9), where c_m are specified by means of (10), also constitute a system of the well-known Kravchuk orthogonal polynomials satisfying the conditions

$$\sum_{i=0}^{n-1}\binom{n-1}{x} p^x q^{n-1-x} \psi_l(x)\,\psi_m(x) = \begin{cases} 0 & (l \neq m) \\ 1 & (l = m) \end{cases}. \tag{13}$$

In 1926—1929, one of the most prestigious mathematical journals in the world, the Comptes rendus of the Paris Academy of Sciences, published eleven studies by Kravchuk, prominent among which was an article written in French, entitled "On a generalization of Hermite's polynomials" [4]. It was submitted on September 23, 1929 to the leading French mathematician, Emile Borel, then director of the Institut Henri Poincaré and member of the Paris Academy of Sciences.

In a concise presentation without any proofs the article presents the principal results concerning the polynomial (9) established in [1]. Interestingly, alongside formula (9) Kravchuk presents yet another formula:

$$\psi_m(x) = \psi_m(x, n; p, q) = \\ = \sqrt{\binom{n-1}{m}^{-1}(pq)^{-m}} \sum_{i=0}^{m}(-1)^i \binom{n-x-1}{m-i}\binom{x}{i} p^{m-i} q^i. \tag{14}$$

In his analysis of the limiting behavior of the polynomials (14) as $n \to \infty$, Kravchuk suggested another substitution of the variable x, introducing in place of x a new variable t by means of the formula $x = p(n-1) + t\sqrt{2qp(n-1)}$. As before, when limits are taken, Hermite's polynomial appears:

$$\text{const} \cdot e^{t^2} \frac{d^m}{dt^m} e^{-t^2}.$$

By comparison with [1], the new result is a formula that defines the limiting behavior of the polynomials (14) as $n \to \infty$ if $p(n-1) = a$. It is found that this limit passage results in the polynomials

$$\text{const} \cdot \frac{x!}{a^x} \Delta^m \left[\frac{a^{x-m}}{(x-m)!} \right]. \tag{15}$$

It is remarked upon in passing that the polynomials (15) are also related to classical orthogonal polynomials of a discrete variable. They were first introduced in 1905 by the famous Swiss mathematician C. Charlier [5] and were given the name Charlier polynomials. In his study [1] Kravchuk did not refer to this limiting case at all. Here these polynomials were presented without any reference whatsoever to original sources. Going ahead of ourselves a bit, we would like to point out that in his study [6], Kravchuk had already given a detailed analysis of "what are known as the Poisson expansion of the probabilities $e^{-a}a^x/x!$" as the limiting case in Bernoulli sampling and proved that "when this expression is interpreted as a characteristic function," the polynomials (15) will themselves be orthonormalized relative to it. Kravchuk, moreover, referred to their "obvious form," which had been pointed out by Jozdan in his monograph [7], who, in particular, had himself undertaken an investigation of orthogonal polynomials of a discrete variable [8].

In [4] formulas were also established for computing what are known as incomplete generalized and factorial moments of order k of the binomial probability distribution $P(x, n; p, q) = \binom{n-1}{x} p^x q^{n-1-x}$. An interesting relation associated with the expansion of this distribution in the polynomials (14) is presented.

In 1929 Kravchuk concentrated his efforts in the area of mathematical research and the management of science in the All-Ukraine Academy of Sciences, where he served as scientific secretary of the department of mathematics and physics. His extraordinarily active creativity is underscored, however, by the fact that in 1930 he gave four technical talks at sessions of the department of mathematics and physics of the All-Ukraine Academy of Sciences.

On February 21, 1930 he completed a scientific paper, entitled "On orthogonal polynomials associated with trials of inverted and noninverted spheres." The article [6], published in 1931 in the journal Zapiskakh fiz.-mat. viddilu VUAN, gave an exhaustive and detailed explanation of this paper. It consisted of eight sections and a lengthy abstract in French.

Section 1 presents in abbreviated form the principal formulas established in [1, 4] for the polynomials (9) or (14). Limiting cases for these polynomials as well as recursive relations between three successive polynomials $\psi_m(x)$ are presented.

In Section 2 the explicit form of the polynomials (9) is subjected to a detailed analysis. For these polynomials, given in the form

$$\psi_m(x, n; p, q) = \frac{\sum_{k=0}^{m} a_k^{(m)}(n) [p(n-1)x]^k}{\sqrt{1 \cdot 2 \ldots m (pq)^m (n-1)(n-2) \ldots (n-m)}} \tag{16}$$

an algorithm is proved for finding the "coefficients $a_k^{(m)}(n)$." In addition to the representation (16), Kravchuk also emphasized his "interest in the inverse problem. It is clear that the following equality holds:

$$[p(n-1) - x]^m = \sum_{i=0}^{m} b_i^{(m)}(n) \theta_i, \tag{17}$$

where $\theta = \sqrt{1 \cdot 2 \cdot \ldots \cdot i (pq)^i (n-1)(n-2) \ldots (n-i)}\ \psi_i(x, n; p, q)$." He also established an algorithm for finding the numbers $b_i^{(m)}(n)$.

Certain characteristics of the binomial distribution P(x, n; p, q) that arise in a trial of inverted spheres (Bernoulli trials) are the subject of Sections 3 and 4. The formulas

$$R_m(x, n) = \sum_{i=0}^{x-1} P(i, n; p, q) [p(n-1) - x]^m \tag{18}$$

represent, respectively, the "incomplete m-th central moment";

$$\rho_m(x, n) = \sum_{i=0}^{x-1} P(i, n; p, q)\, \psi_m(i, n; p, q) \tag{19}$$

"incomplete m-th general moment";

$$S_{a|m}(n) = \sum_{x=0}^{n-1} \binom{x-a}{m} P(x, n; p, q) \tag{20}$$

"complete factorial moment relative to the point a"; and

$$t_{ab|mn}(u) = \sum_{x=0}^{n-1} \binom{x-a}{m}\binom{b-x}{n} P(x, u; p, q) \tag{21}$$

"complete (m, n)-th mixed factorial moment relative to the points a and b."

Moments such as (21) are used in Section 5 in the investigation of the expansion of the distribution P(x, n; p, q) in the polynomials (9).

An analysis of the Pearson distribution function, which arises in the trials of noninverted spheres,

$$Q(x; u; n, v) = \binom{u-1}{x}\binom{n-u}{v-x} : \binom{n-1}{v}, \tag{22}$$

is the subject of Section 6. A corresponding "general moment" of the type of (19) is studied.

The "arithmetic mean M" and "root-mean-square deviation σ of the distribution" (22) are computed in explicit form:

$$M = \frac{v}{n-1}(u-1),\ \sigma = \sqrt{\frac{v}{n-1}\left(1 - \frac{v}{n-1}\right)(u-1)\left(1 - \frac{u-2}{n-2}\right)}.$$

It is shown how "further central moments of this distribution" may be computed.

The following "expansion" is also of interest:

$$Q(x, u; n, v) = P(x, u; p, q) \sum_{m=0}^{u-1} \sqrt{\frac{\binom{u-1}{m}}{\binom{n-1}{m}}}\, \psi_m(v, n; p, q)\, \psi_m(x, u; p, q), \tag{23}$$

which, Kravchuk points out, "is true for x = 0, 1, ... , u − 1 and for arbitrary p."

A more "detailed analysis of the expansion" (23) is attempted in Section 7, which also studies "precisely to what extent the Pearson distribution function Q(x, u; n, v) may be reconstructed by means of the Bernoulli function P(x, u; p, q) and several of the leading polynomials in the series $\psi_0, \psi_1, \dots, \psi_{n-1}$."

The final section of the article is extraordinarily rich in results and potential generalizations of these results. Here Kravchuk investigated a new class of orthogonal polynomials of a discrete variable.

As indicated earlier, the characteristic function of (13) is the expression P(x, n; p, q). The Kravchuk polynomials (9) are "orthogonal and normal relative to this characteristic function. . . . For example, let us consider as the characteristic function the expression Q(x, u; n, v) in place of P(x, n; p, q) and find for it an appropriate orthogonal polynomial."

It is turns out that such "polynomials are given in the form of the following expressions:

$$\varphi_m(x, u; n, v) = c_m \Delta^m Q(x-m, u-m; n-2m, v-m) : Q(x, u; n, v) \quad (m = 0, 1, \dots, u-1), \tag{24}$$

where c_m is a positive constant," moreover,

$$c_m^2 = \frac{1}{m!\,(n-1)(n-2)\dots(n-m)}\left[\frac{v(n-v-1)(u-1)(n-u)}{(n-1)(n-2)}\right]^m. \tag{25}$$

Note that, by means of these new polynomials, "those problems for a Pearson distribution may be solved for which the corresponding function $\psi_m(x)$ of a Bernoulli distribution may be solved."

Unfortunately, unlike either (9) or (14), the polynomial (24) did not enter the world's mathematical literature as one more class (Kravchuk class) of orthogonal polynomials of a discrete variable. Only in 1949 did the German mathematician W. Hahn, in the course of an investigation [9] of more general orthogonal polynomials that comprehended all the known classical systems as limiting cases, once again investigate the polynomials (24) in particular.

The mathematical literature of Ukraine in the 1930s quite properly assigned full credit in this line of research to Kravchuk's studies. The polynomials (9) came to be called Kravchuk—Bernoulli polynomials, and the polynomials (24), Kravchuk—Pearson polynomials. By tradition, the classical orthogonal polynomials of a discrete variable have each been named after the first letter of the name of the mathematician who discovered the particular polynomial. For this reason, the polynomial (9) was referred to as $K_m^{(B)}(x)$, or $K_m^{(B)}(x, n; p, q)$, and the polynomial (24) by $K_m^{(P)}(x)$ or $K_m^{(P)}(x, u; n, v)$.

One of the most well-known students of Kravchuk's, the future (after 1938) Professor O. S. Smohorzhevs'kyy, gave, in several articles [10—12], a detailed investigation of the polynomials $K_m^{(B)}(x)$ and $K_m^{(P)}(x)$. Various relations and estimates, the relationship between hypergeometric series and the generalized hypergeometric series, hypergeometric curves — this represents only a partial list of topics involving these polynomials which Smohorzhev'skyy has studied.

Of especial note is the article [13], in which a new class of orthogonal polynomials of a discrete variable that, in addition, represents a generalization of Kravchuk polynomials was studied.

Another student of Kravchuk's, S. M. Kulyk, actively undertook mathematical research on the same topic. In his studies [14, 15] he established new recursive relations for Kravchuk polynomials, and investigated how they differed from and resembled Jacobi's polynomials. In one article [16] he established difference equations for the polynomials that had been investigated in [13], which for the first time were called generalized Kravchuk polynomials. Corresponding results for the polynomials (9) and (24) were obtained as special cases. Kulyk published a lengthy (nearly 50 pages long) article [17] that summarized all the results achieved by Kravchuk's school concerning orthogonal polynomials of a discrete variable.

We should also mention the interesting student work of O. K. Lebedyntseva [18], in which questions concerning the linear representation of the polynomials (9) were studied by means of Hermite's polynomials and formulas were obtained for successively determining the coefficients of this representation.

While serving as head of the department of mathematical statistics of the Institute of Mathematics of the Academy of Sciences of the Ukrainian RSR during the years 1934—1938, Kravchuk undertook in [19] an "investigation on orthogonal polynomials associated with different discrete probability distributions (Kravchuk, Smohorzhev'skyy, Kulyk)" as one of the important and promising directions for the activity of the department.

However, tragic events suddenly altered Kravchuk's life. In August of 1937 he was unjustly accused of harboring bourgeois nationalist tendencies, and, late in February 1938, he was arrested. The article [19] was the last of Academician Kravchuk's mathematical studies. The report of the Institute of Mathematics for the twentieth anniversary of the Great October Revolution [20], published in the Institute's own journal and sent to the printer on November 1, 1937, had already omitted mention of Kravchuk and the work of his department.

The studies of Ukrainian mathematicians in this field were, in fact, halted. There is only the interesting article by S. Kulyk [21], which was received at the editorial office without, as was the custom, any references to Kravchuk's works, moreover in February 1941. Information on these points may be found in a postwar study by S. Kulik [Kulyk] [22] in the area of orthogonal polynomials of a discrete variable.

Fortunately, the works of foreign mathematicians in this area successfully continued the traditions of the Ukrainian school of mathematics. Already in 1934 the German mathematician J. Meixner investigated a new class of orthogonal polynomials of a discrete variable [23] which have since been referred to as Meixner polynomials. In his article [24] Meixner extended his studies.

Note, too, the dissertation of the English scientist H. T. Gonin [25] and the interesting study by the American mathematician M. J. Gottlieb [26], in which orthogonal polynomials associated with what are known as geometric probability distributions are constructed.

A classical monograph by a professor of mathematics at Stanford University, G. Szegö [27], written in 1937 will forever be the final word in the subject. Section 2.82 (pages 34—36) is entitled "Krawtchouk [Kravchuk]'s Polynomials."

General interest in orthogonal polynomials of a discrete variable increased significantly in the late 1950s in connection with the appearance of fundamental results in the area of probabilistic birth—and—death—processes [28]. Kravchuk polynomials were one of the research tools applied in [29].

The interest of the author of the present survey in the subject of orthogonal polynomials of a discrete variable was aroused in 1964 by M. Y. Yadrenko, Corresponding Member of the Academy of Sciences of Ukraine. One result of my university studies was a short article [30] in which a class of orthogonal polynomials that resembled Meixner polynomials was constructed, and birth—and—death processes associated with the polynomials that had been constructed were investigated.

The general approaches of mathematical and functional analysis to the investigation of orthogonal polynomials of a discrete variable and, in particular, Kravchuk polynomials are typical of a series of studies [31, 32]. For purposes of generalization, so-called Kravchuk q-polynomials were studied in [33].

Several studies [34—40] have been devoted to generalizations of classical orthogonal polynomials of a discrete variable in the case of two or more variables and to the identification of different properties and certain generalizations of these polynomials. Kravchuk polynomials are referred to in nearly all these studies. The most recent study in this area is the article [41].

In the area of stochastic processes, the concept of a Kravchuk process and the associated Kravchuk set and "Kravchuk moment" for the transition matrices of these processes was introduced in [42, 43].

It was found [44, 45] that Kravchuk polynomials are extraordinarily important in the study of representations of rotation groups in three-dimensional space and possess a wide range of physical applications [46]. An important tool in quantum mechanics, generalized spherical functions, may be easily expressed in terms of Kravchuk polynomials. Using Kravchuk polynomials it is also possible to work with spherical functions directly, based on the theory of Kravchuk polynomials discussed earlier.

Note, too, the studies authored by the Ukrainian scientist A. U. Klimyk and his students, a survey of which appears in the present issue (following article).

The world mathematical literature devoted to the subject of orthogonal polynomials includes references to the classical orthogonal polynomials of a discrete variable of Chebyshev, Charlier, Kravchuk, Meixner, and Hahn. For a long time there was no definitive proof of the hypothesis that these orthogonal polynomials all belong to a certain broad class of special functions. Note in this connection two important studies [48, 49].

Recently, thanks to the monographs of A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov [50—53] an approach that makes it possible to investigate all these polynomials as the solutions of a particular hypergeometric difference equation has been studied from a unified standpoint.

The classical orthogonal polynomials of a discrete variable constitute an important class of special functions that arise in numerous problems in applied and computational mathematics, probability theory, mathematical statistics, theoretical physics, and technology. These branches of human knowledge are now undergoing intensive study, and the topic of Kravchuk polynomials and generalizations of Kravchuk polynomials, understood as an integral and constituent element of the study of orthogonal polynomials of a discrete variable, have been successfully employed in all these fields.

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