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Global Existence Proof for Relativistic Boltzmann Equation

Marek Dudyński[1] **and Maria L. Ekiel-Jeżewska**[2]

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The existence and causality of solutions to the relativistic Boltzmann equation in L^1 and in L^1_{loc} are proved. The solutions are shown to satisfy physically natural *a priori* bounds, time-independent in L^1. The results rely upon new techniques developed for the nonrelativistic Boltzmann equation by DiPerna and Lions.

KEY WORDS: Relativistic Boltzmann equation; causality; global existence.

1. INTRODUCTION

A global existence proof for the Boltzmann equation has been recently given by DiPerna and Lions.[1] Its sophisticated construction has stimulated a new direction in mathematical kinetic theory.

In this paper one of the straightforward applications of the new techniques introduced by DiPerna and Lions is presented. Namely, the global existence proof for the relativistic version of the Boltzmann equation is given.

Due to the growing interest in its applications,[2] the relativistic Boltzmann equation has been recently analyzed and the structure of its linearized form has been established, including existence, uniqueness, causality proofs,[3–5] and the justification of the hydrodynamic approximation.[6] The basic structure of the relativistic Boltzmann equation is essentially the same as in the nonrelativistic case, with slight modifications due to the relativistic interactions. However, the relativistic bound on velocity has led to some qualitatively new results, such as causality of the equation

[1] Institute for Fundamental Technological Research, Polish Academy of Sciences, PL-00-049 Warsaw, Poland.

[2] Institute for Theoretical Physics, Polish Academy of Sciences, PL-02-668 Warsaw, Poland.

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(i.e., the dependence of its solutions on the initial data only inside the past interior of the light cone) and the analytic dispersion relation.

The nonlinear relativistic Boltzmann equation in $L^1(R^3 \times R^3)$ needs more delicate treatment. In the relativistic modification of the DiPerna and Lions global existence proof a new time-independent estimate for the solution norm is needed. The reason is that the nonrelativistic estimate is based on the nonrelativistic collision invariant $(\mathbf{x} - \mathbf{v}t)^2$, which has no relativistic analog.

Fortunately, the DiPerna and Lions techniques can be used to prove the causality of the relativistic Boltzmann equation. This leads to the desired estimate: for finite time intervals, solutions of the relativistic Boltzmann equation remain in $L^1_{\text{loc}}(R^3; L^1(R^3))$, providing the initial data have been in $L^1_{\text{loc}}(R^3; L^1(R^3))$. This local structure of the equation allows one to repeat the DiPerna and Lions global existence proof with only minor adjustments.

The paper is organized as follows.

In Section 2 the relativistic Boltzmann equation is introduced and assumptions on the relativistic cross sections are specified. In Section 3 the existence proof in $L^1_{\text{loc}}(R^3; L^1(R^3))$ is given. The result is used in Section 4 to construct the global existence proofs in $L^1(R^3 \times R^3)$, and in $L^1_{\text{loc}}(R^3; L^1(R^3))$ with initial data close to equilibrium.

2. THE RELATIVISTIC BOLTZMANN EQUATION

The relativistic Boltzmann equation (RBE) has the form[7,8] ($c = 1$; the signature is $+ - - -$)

$$\left(\partial_t + \frac{\mathbf{p}}{p_0} \cdot \nabla_{\mathbf{x}}\right) f = Q(f, f) \tag{2.1}$$

where

$$Q(f, f) = Q^+(f, f) - Q^-(f, f) \tag{2.2}$$

$$Q^+(f, f) = \frac{1}{p_0} \int_{R^3} \frac{d_3\mathbf{p}_*}{p_{*0}} \int_{S^2} d\Omega\, f(\mathbf{p}')\, f(\mathbf{p}'_*)\, B(g, \vartheta) \tag{2.3}$$

$$Q^-(f, f) = fL(f) = f(\mathbf{p}) \frac{1}{p_0} \int_{R^3} \frac{d_3\mathbf{p}_*}{p_{*0}} \int_{S^2} d\Omega\, f(\mathbf{p}_*)\, B(g, \vartheta) \tag{2.4}$$

and $p = (p_0, \mathbf{p})$, where $p_0 = (M^2 + \mathbf{p}^2)^{1/2}$ is the particle energy, M is the particle rest mass, $s^{1/2} = |p_* + p|$ is the total energy in the center-of-mass system, $2g = |p_* - p|$ is the relative momentum value in the center-of-mass system, $\cos\vartheta = 1 - 2(p - p_*)(p - p')(4M^2 - s)^{-1}$ defines the angle of scattering ϑ in the center of mass system $d\Omega = d(\cos\vartheta)\,d\phi$,

$$B(g, \vartheta) = \frac{gs^{1/2}}{2}\,\sigma(g, \vartheta)$$

and $\sigma(g, \vartheta)$ is the scattering cross section in the center-of-mass system.

In the proof we make the following assumption for the scattering cross section:

$$\text{(i)}\quad B \in L^1_{\text{loc}}(R^3 \times S^2) \tag{2.5}$$

$$\text{(ii)}\quad \frac{1}{p_0}\int_{B_R} \frac{d^3\mathbf{p}_*}{p_{*0}}\, A(g) \to 0 \qquad \text{as} \quad |\mathbf{p}| \to \infty, \qquad \text{for all} \quad R < \infty \tag{2.6}$$

where $A(g) = \int d\Omega\, B(g, \vartheta)$.

Conditions (i)–(ii) are the straightforward modification of the nonrelativistic assumptions (9) and (23) adopted by DiPerna and Lions,[1]

$$\text{(i)}\quad B \in L^1_{\text{loc}}(R^N \times S^{N-1}) \tag{2.7}$$

$$\text{(ii)}\quad (M^2 + |\mathbf{p}|^2)^{-1} \int_{|\mathbf{z} - \mathbf{p}| \leqslant R} d\mathbf{z}\, \tilde{A}(\mathbf{z}) \to 0 \qquad \text{as} \quad |\mathbf{p}| \to \infty,$$

$$\text{for all} \quad R < \infty \tag{2.8}$$

However, the relativistic assumptions (2.5)–(2.6) are more restrictive than the nonrelativistic ones (2.7)–(2.8). Namely, the nonrelativistic condition (2.8) includes so-called hard interactions. However, it is easy to check that for $A(g) \propto g^\alpha$, $\alpha \geqslant 0$, we have[3]

$$C_1\, p_0^{\alpha/2 - 1} \leqslant \frac{1}{p_0}\int_{B_R} \frac{d^3\mathbf{p}_*}{p_{*0}}\, A(g) \leqslant C_2\, p_0^{\alpha/2 - 1} \tag{2.9}$$

Thus the relativistic assumption (2.6) gives $\alpha < 2$, which excludes relativistic hard interactions defined as[5]

$$A(g) \geqslant \text{const} \cdot g^2 \tag{2.10}$$

3. GLOBAL EXISTENCE OF SOLUTIONS

We are concerned in this section with the main result of this work, namely the global existence of solutions to the Cauchy problem for the

[3] The estimate is entirely different than the nonrelativistic one, since $g^2 = 2(p_{*0}p_0 - \mathbf{p}_* \cdot \mathbf{p} - M^2) \propto_{p_0 \to \infty} p_0$.

relativistic Boltzmann equation in $C([0, \infty); L^1_{loc}(R^3; L^1(R^3)))$. The proof relies heavily on DiPerna and Lions[1] and we assume that the reader is familiar with their paper, as we are going to show that locally in $\mathbf{x}$ the Cauchy problem for the relativistic equation can be cast as a problem virtually identical with the nonrelativistic situation.

Theorem 3.1. Let f_0 satisfy the following conditions:

(a) $f_0 \geqslant 0$ a.e. in $R^3 \times R^3$.

(b) For any compact set $\Omega \subset R^3$

$$\int_\Omega d_3\mathbf{x} \int_{R^3} d_3\mathbf{p}\,(1 + p_0 + |\ln f_0|)\, f_0 < \infty$$

then there exists $f \in C([0, \infty); L^1_{loc}(R^3; L^1(R^3)))$ satisfying

(i) $f|_{t=0} = f_0$

(ii) $$(1+f)^{-1}\, Q^-(f, f) \in L^\infty(0, T; L^1_{loc}(R^3 \times R^3))$$
$$(1+f)^{-1}\, Q^+(f, f) \in L^1(0, T; L^1_{loc}(R^3 \times R^3))$$
$$L(f) \in L^\infty(0, T; L^1_{loc}(R^3 \times R^3))$$

for all $T < \infty$, which is a mild or equivalently a renormalized solution of the RBE (2.1) with cross sections obeying (2.5)–(2.6). In addition, f is causal and it satisfies

$$f \geqslant 0 \qquad \text{a.e. in } R^3 \times R^3$$

$$\sup_{t \in [0, T]} \int_\Omega d_3\mathbf{x} \int_{R^3} d_3\mathbf{p}\,(1 + p_0 + |\ln f|)\, f < \infty$$

Proof. Our proof hinges on the crucial observation that the inherent causal structure of the relativistic Boltzmann equation allows us to construct its approximate solutions locally in $\mathbf{x}$ for finite times. To show this, we consider a compact set $\Omega \subset R^3$ and $t \in [0, T]$ with $T < \infty$. We introduce a set $\Omega_t \subset R^3$ defined as follows:

$$\Omega_t = \{\mathbf{x} \in R^3;\ \exists \mathbf{y} \in \Omega \text{ such that } |\mathbf{x} - \mathbf{y}| \leqslant ct\} \tag{3.1}$$

Our aim is to construct a solution to the RBE in $S = \Omega \times R^3$. Following DiPerna and Lions, with truncation and regularization of the initial data, we can obtain $f_0^n \in D'(R^3 \times R^3)$ such that $f_0^n \geqslant 0$. With a convenient approximation of the collision kernel we arrive at the problem of solving the approximate equation

$$\left(\partial_t + \frac{\mathbf{p}}{p_0}\cdot\nabla_{\mathbf{x}}\right) f^n = Q_n^*(f^n, f^n) \qquad \text{in} \quad (0, T)\times R^3\times R^3 \tag{3.2}$$

with initial conditions $f^n|_{t=0} = f_0^n$, where Q_n^* acts on any function ϕ as

$$Q_n^* = \left(1 + n^{-1}\int_{R^3} |\phi|\, d_3\mathbf{p}\right)^{-1} Q_n \tag{3.3}$$

$$Q_n(\phi, \phi) = \frac{1}{p_0}\int_{R^3} \frac{d_3\mathbf{p}_*}{p_{*0}} \int_{S^2} d\Omega\, (\phi'\phi'_* - \phi\phi_*)\, B_n(g, \vartheta) \tag{3.4}$$

and $B_n \in L^\infty \cap L^1(R^3\times S^2)$.

We first observe that Q_n^* is bounded in $L^1\cap L^\infty$. Thus, the standard iteration procedure shows that for each n a unique nonnegative solution to Eq. (3.2) exists which is causal, i.e., the solution on $S = \Omega\times R^3$ at time t depends only on the initial data specified on the set $S_t = \Omega_t\times R^3$, for all bounded Ω. In particular, the solution on the set S for $t\in[0, T]$ depends only on the initial data on the set $S_T = \Omega_T\times R^3$. Thus, as long as we confine our attention to the solution in $[0, T]\times S$, it is sufficient to assume the following initial data:

$$\hat{f}_0 = \begin{cases} f_0; & (\mathbf{x}, \mathbf{p})\in S_T \\ 0; & (\mathbf{x}, \mathbf{p})\notin S_T \end{cases} \tag{3.5}$$

Thus, for appropriately regularized initial data $\hat{f}_0^n$, solutions to Eq. (3.2) satisfy

$$\operatorname{supp} f^n(t) \subseteq S_{2T} = (\Omega_T)_T\times R^3 = \Omega_{2T}\times R^3 \qquad \text{for} \quad t\in[0, T] \tag{3.6}$$

Consequently, we confine our analysis to $L^1(S_{2T})$ and we can finally choose such a regularization of the initial data that

$$\int_{S_T} d_3\mathbf{x}\, d_3\mathbf{p}\, |f_0 - f_0^n|(1 + |\mathbf{x}|^2 + p_0) \xrightarrow[n\to\infty]{} 0 \tag{3.7}$$

$$\int_{S_T} d_3\mathbf{x}\, d_3\mathbf{p}\, \hat{f}_0^n\, |\ln \hat{f}_0^n| < C \qquad \text{independent of } n \tag{3.8}$$

and we finally set

$$\tilde{f}_0^n = \hat{f}_0^n + n^{-1}\exp(-|\mathbf{x}|^2 - p_0)\, 1_{\Omega_T} \tag{3.9}$$

We solve now Eq. (3.2) with initial data $\tilde{f}_0^n$, and exactly the same reasoning as in ref. 1 shows the existence of a unique nonnegative solution f^n satisfying the following conditions:

$$\iint_{R^3 \times R^3} d_3\mathbf{x}\, d_3\mathbf{p}\, |D^\alpha f^n|(1+|\mathbf{x}|^k + p_0^k) \leqslant C_n(T, m, k) \tag{3.10}$$

$$D^\alpha f^n \in L^\infty((0, T) \times R^3 \times R^3) \tag{3.11}$$

$t \in [0, T]$; $T < \infty$, $k \geqslant 0$, $m \geqslant 0$, and D^α denote any derivatives up to order m. Combining inequalities (3.10) and the fact that for all n, $\operatorname{supp} f^n \subseteq S_{2T}$ with properties of the relativistic collision term, we can easily show that for $T < \infty$ these f^n satisfy

$$f^n, \nabla_{\mathbf{x}} f^n, \nabla_{\mathbf{p}} f^n, \partial_t f^n \in L^\infty \cap L^1((0, T) \times R^3 \times R^3) \tag{3.12}$$

$$Q^+(f^n, f^n), Q^-(f^n, f^n) \in L^1_{\text{loc}}((0, T) \times R^3 \times R^3) \tag{3.13}$$

$$\sup_{t \in [0, T]} \iint_{R^3 \times R^3} d_3\mathbf{x}\, d_3\mathbf{p}\, f^n(1 + |\mathbf{x}|^2 + p_0 + |\ln f^n|) \leqslant C_T \tag{3.14}$$

$$\int_0^T \iiiint_{R^3 \times R^3 \times R^3 \times S^2} dt\, d_3\mathbf{x}\, d_3\mathbf{p}\, d_3\mathbf{p}_*\, d\Omega\, B(f^{n\prime} f_*^{n\prime} - f^n f_*^n) \ln \frac{f^{n\prime} f_*^{n\prime}}{f^n f_*^n} \leqslant C_T \tag{3.15}$$

Having a sequence of approximate solutions satisfying the properties (3.12)–(3.15), it requires only minor adjustments of the proof to check that the main theorem of DiPerna and Lions is true also in the relativistic case.

Lemma 3.1 (Theorem VII.1 in ref. 1). Assume that B satisfies Eqs. (2.5) and (2.6). Then the following assertions hold for a sequence f^n satisfying (3.12)–(3.15):

(i) For all $\psi \in L^\infty((0, T) \times R^3 \times R^3)$,

$$\int_{R^3} d_3\mathbf{p}\, f^n(t, \mathbf{x}, \mathbf{p})\, \psi(t, \mathbf{x}, \mathbf{p}) \xrightarrow[n]{} \int_{R^3} d_3\mathbf{p}\, f(t, \mathbf{x}, \mathbf{p})\, \psi(t, \mathbf{x}, \mathbf{p})$$

in $L^p(0, T; L^1(R^3))$ for all $p < \infty$, and

$$L(f^n) \xrightarrow[n]{} L(f)$$

in $L^1((0, T) \times R^3 \times B_R)$ for all $R < \infty$.

(ii) For all $T < \infty$, $R < \infty$, $\psi \in L^\infty((0, T) \times R^3 \times R^3)$ supported in $[0, T] \times B_R \times B_R$, and for all $\varepsilon > 0$, there exists a subsequence of f^n that we shall denote by f^n for simplicity and a Borel set $E \subset [0, T] \times B_R \times B_R$ such that meas $E^C \leqslant \varepsilon$ and

$$\int_{R^3} \psi Q^-(f^n, f^n)\, 1_E\, d_3\mathbf{p} \underset{n}{\longrightarrow} \int_{R^3} \psi Q^-(f, f)\, 1_E\, d_3\mathbf{p} \qquad \text{in} \quad L^1((0, T) \times B_R)$$

$$\int_{R^3} \psi Q^+(f^n, f^n)\, 1_E\, d_3\mathbf{p} \underset{n}{\longrightarrow} \int_{R^3} \psi Q^+(f, f)\, 1_E\, d_3\mathbf{p} \qquad \text{in} \quad L^1((0, T) \times B_R)$$

$$\int_{R^3} \psi Q^\pm(f^n, f^n)\, d_3\mathbf{p} \underset{n}{\longrightarrow} \int_{R^3} \psi Q^\pm(f, f)\, d_3\mathbf{p}$$

in measure on $(0, T) \times B_R$ for all $R < \infty$

In particular, $Q^\pm(f, f) \in L^1(R^3_{\mathbf{p}})$ a.e. $t > 0$, $\mathbf{x} \in R^3$.

(iii) $Q^+(f, f)(1+f)^{-1} \in L^1((0, T) \times R^3 \times B_R)$ $(\forall R, T < \infty)$ and $Q^-(f, f)(1+f)^{-1} \in L^\infty((0, \infty); L^1(R^3 \times B_R))$ $(\forall R < \infty)$.

(iv) f is a renormalized solution of the RBE; equivalently, f is a mild solution of RBE and $f \in C([0, \infty); L^1(R^3 \times R^3))$.

Lemma 3.1 and the causality of the approximate solutions f_n immediately give the causality of f itself.

Using now the causality property of the solution f, we see that $f(t)|_S$ is independent of the choice of initial data outside S_T for $t \in [0, T]$. The application of Lemma 3.1 leads directly to the assertions of Theorem 3.1 in $[0, T] \times \Omega \times R^3$. This consideration can be repeated for all $T < \infty$ and bounded sets Ω provided that the initial data satisfy condition (b) of Theorem 3.1. The conservation laws and the convexity of the function $t(\ln t)$ show that

$$\sup_{t \in [0, T]} \int_\Omega d_3\mathbf{x} \int_{R^3} d_3\mathbf{p}\, (1 + p_0 + |\ln f|)\, f < \infty \tag{3.16}$$

for any compact Ω and $T < \infty$. This ends the proof of Theorem 3.1. ∎

We note here that it is possible to change a bit the order of arguments in the above proofs. It is quite easy to repeat the original proof of DiPerna and Lions for a sequence of approximate solutions defined on a common compact support in $\mathbf{x}$, without referring to the boundedness of $\|\mathbf{x}^2 f\|_{L^1}$. Morgenstern's lemma[9] still applies for such a sequence, leading to its weak convergence to a solution.

Lemma 3.2 (Morgenstern). Let $f_n \in L^1(R^N)$, $f_n \equiv f_n(x)$, be a sequence of nonnegative functions satisfying

$$\int f_n(x)(1 + |x|^\kappa + \log f_n)\, dx < C$$

Then there exists a subsequence $\{f_{n_j}\}$ converging weakly to a function $f \in L^1(R^N)$ and

$$\lim_{j \to \infty} \int f_{n_j}(x)\, \phi(x)\, dx = \int f(x)\, \phi(x)\, dx$$

providing $(1 + |x|)^{-\kappa'} \phi(x) \in L^\infty(R^N)$, $0 \leqslant \kappa' < \kappa$.

Thus, it is clear that the estimates depend only on the maximal linear dimension of the set Ω rather than on the particular location in the R^3 space. We have chosen an indirect approach to appeal directly to the DiPerna and Lions results, avoiding to repeat arguments which differ from theirs in details only. On the other hand, the other approach for the non-relativistic Boltzmann equation has no special value, as the solutions in this case are not causal.

4. THE CAUCHY PROBLEM IN L^1 AND FOR SYSTEMS CLOSE TO GLOBAL EQUILIBRIUM

We first note that in the proof of Theorem 3.1 bounds on the expression

$$\iint_S d_3\mathbf{x}\, d_3\mathbf{p}\, (1 + p_0 + |\ln f|)\, f$$

can be expressed in terms of the corresponding bounds on

$$\iint_{S_T} d_3\mathbf{x}\, d_3\mathbf{p}\, (1 + p_0 + |\ln f_0|)\, f_0$$

and they can grow with T. If we assume a uniform L^1 bound on initial data, i.e., $f_0 \geqslant 0$ a.e. in $R^3 \times R^3$ and

$$\int_{R^3} d_3\mathbf{x} \int_{R^3} d_3\mathbf{p}\, (1 + p_0 + |\ln f_0|)\, f_0 < C \tag{4.1}$$

then the following extension of Theorem 3.1 holds.

Theorem 4.1. Let f_0 satisfy (4.1).
Then there exists $f \in C([0, \infty); L^1(R^3 \times R^3))$ satisfying

$$f|_{t=0} = f_0$$
$$(1 + f)^{-1}\, Q^-(f, f) \in L^\infty(0, \infty; L^1(R^3 \times B_R))$$
$$(1 + f)^{-1}\, Q^+(f, f) \in L^1(0, T; L^1(R^3 \times B_R))$$
$$L(f) \in L^\infty(0, \infty; L^1(R^3 \times B_R))$$

for all $R, T < \infty$, which is a mild or equivalently a renormalized solution of RBE (2.1). In addition, the solution satisfies

$$\sup_{t \geqslant 0} \int_{R^3} d_3\mathbf{x} \int_{R^3} d_3\mathbf{p}\, (1 + p_0 + |\ln f|)\, f < C$$

Proof. For arbitrary $T < \infty$, we divide R^3 into boxes Ω_n with centers in $\mathbf{x}_n$ and length L each. L is chosen so large that

$$\Omega_{nT} \subset 2\Omega_n \tag{4.2}$$

where $2\Omega_n$ denotes a box with length $2L$ centered in $\mathbf{x}_n$. The Ω_{nT} was defined in the previous section. According to Theorem 3.1, for any set $S_n = \Omega_n \times R^3$ we can construct a solution, which we denote by $f_n^\Omega(t)$. It is easy to see that

$$\sum_{n=-\infty}^{\infty} \int_{\Omega_n} \int_{R^3} d_3\mathbf{x}\, d_3\mathbf{p}\, (1 + p_0 + |\ln f_n^\Omega|)\, f_n^\Omega$$
$$\leqslant C \int_{R^3} \int_{R^3} d_3\mathbf{x}\, d_3\mathbf{p}\, (1 + p_0 + |\ln f_0|)\, f_0 \tag{4.3}$$

In particular, due to the conservation laws, we have

$$\sum_{n=-\infty}^{\infty} \int_{\Omega_n} \int_{R^3} d_3\mathbf{x}\, d_3\mathbf{p}\, (1 + p_0)\, f_n^\Omega \leqslant 2 \int_{R^3} \int_{R^3} d_3\mathbf{x}\, d_3\mathbf{p}\, (1 + p_0)\, f_0 \tag{4.4}$$

Equation (4.4) shows that the function f defined as

$$f(\mathbf{x}, \mathbf{p}, t) = \{ f_n^\Omega(\mathbf{x}, \mathbf{p}, t), (\mathbf{x}, \mathbf{p}) \in \Omega_n \times R^3, t \in [0, T] \} \tag{4.5}$$

belongs to $C([0, T]; L^1(R^3 \times R^3))$ for any $T < \infty$, as we can choose bounds describing the continuity property of functions f_n^Ω independently of n. All necessary estimates can be obtained from the inequality (4.4) and as it is independent of T, the assertions of Theorem 4.1 hold for $t \in (0, \infty)$. ∎

For systems close to global equilibrium or with defined asymptotic behavior at $|\mathbf{x}| \to \infty$ in the form $f_0(\mathbf{x}, \mathbf{p}) = CF_{\text{eq}}(\mathbf{p})$, where $F_{\text{eq}}(\mathbf{p})$ is Jütner equilibrium solution of RBE (2.1), we cannot obtain any estimate uniform in t. However, the causality of solutions allows for the following strong form of the local-in-time existence theorem:

Theorem 4.2. Let the initial data f_0 satisfy

$$\sup_{n} \int_{\Omega_n} \int_{R^3} d_3\mathbf{x}\, d_3\mathbf{p}\, (1 + p_0 + |\ln f_0|)\, f_0 < C$$

for any partition of R^3 made of equal boxes Ω_n.

Then for any $T<\infty$ there is an $f\in C((0,T];L^1_{\text{loc}}(R^3\times R^3))$ satisfying

$$f|_{t=0}=f_0$$
$$(1+f)^{-1}Q^-(f,f)\in L^\infty(0,T;L^1_{\text{loc}}(R^3\times R^3))$$
$$(1+f)^{-1}Q^+(f,f)\in L^1(0,T;L^1_{\text{loc}}(R^3\times R^3))$$
$$L(f)\in L^\infty(0,T;L^1_{\text{loc}}(R^3\times R^3))$$

for all $R<\infty$, which is a mild or equivalently a renormalized solution of RBE (2.1). For $t\in[0,T)$ this solution satisfies

$$\sup_n\int_{\Omega_n}d_3\mathbf{x}\int_{R^3}d_3\mathbf{p}\,(1+p_0+|\ln f|)\,f<C(T)$$

Proof. For a given $T<\infty$, we choose the partition Ω_n such that the condition (4.2) is fulfilled. Then the application of Theorem 3.1 leads to the existence of a solution f in $\Omega_n\times R^3$ for $t\in[0,T]$. Due to the fact that the relativistic Boltzmann equation is invariant with respect to translations in $\mathbf{x}$, we can solve the RBE in $\Omega_n\times R^3$, first translating it to the set $\Omega_0\times R^3$ and then translating it back to the original position. We see that all bounds connected with $\||\mathbf{x}|^2 f\|_{L^1(S_n)}$ can be expressed by bounds on $\|L^2 f_0\|_{L^1(S_{nT})}$. All other bounds are also dependent only on the L^1 bounds of the initial data on S_{nT} and can be directly expressed by

$$\sup_n\int_{\Omega_{nT}}\int_{R^3}d_3\mathbf{x}\,d_3\mathbf{p}\,(1+p_0+|\ln f_0|)\,f_0$$

Ω_{nT} grows with T, but due to our assumption, this supremum is finite for any $T<\infty$. ∎

Theorem 4.2 gives the existence of solutions to the RBE for systems close to global equilibrium where the hydrodynamic description of the system should be correct for the long-time limit as well as for initial data describing shock wave propagation. But in both cases we can so far guarantee existence only in finite (however arbitrarily large) times. In particular, we are not able to control the conservation laws and it is not possible at this stage to show an approach to equilibrium for these solutions. Similarly to the nonrelativistic Boltzmann equation, the uniqueness of this class of solutions remains an open problem.

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