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DISCRETE GROUPS IN THREE-DIMENSIONAL LOBACHEVSKY SPACE GENERATED BY TWO ROTATIONS

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This paper will consider discrete groups of transformations in three-dimensional Lobachevsky space. A full classification of two-generator Fuchsian groups, i.e., discrete groups of orientation preserving transformations of the Lobachevsky plane, has already been given. This classification was completed by Purzitsky [1], and Matelski gave a new proof of the result using a geometric method [2]. The ideas of the latter proved useful in the study of the three-dimensional case. In this paper, we give necessary and sufficient conditions for the discreteness of a group of isometries of three-dimensional Lobachevsky space, generated by two elliptical elements with intersecting, mutually perpendicular axes.

We shall consider the group of orientation preserving isometries of the Lobachevsky space $\mathbf{H}^3$, which, as is known, is isomorphic to $PSL(2, \mathbf{C})$. All non-trivial elements of this group are divided into three types: elliptical, parabolic and loxodromic (see, for example, [3, p. 65]). Elliptical elements are determined by their axes (the set of stationary points) and by the angle and direction of rotation around these axes. An elliptical element of finite order $n \geqslant 2$ with angle of rotation $2\pi/n$ is said to be primitive.

We shall assume that two lines in the space $\mathbf{H}^3$ may either lie in one plane (and then they may intersect, by parallel, or diverge) or intersect. In the latter case, they have a common perpendicular. The angle between intersecting lines is the dihedral angle between the planes passing through the common perpendicular to each of the lines separately.

Then we have

THEOREM. If $A, B \in PSL(2, \mathbf{C})$ are primitive elliptical elements of order n and m, respectively $(n \leqslant m)$, with mutually perpendicular axes, then the group generated by these elements G = <A, B> is discrete if and only if one of the following conditions holds:

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1) $[A, B] = ABA^{-1}B^{-1}$ is a loxodromic element,

2) $[A, B]$ is a parabolic element,

3) $[A, B]$ is an elliptical element with dihedral angle $4\pi/\ell$ and one of the following is true:

a) the triple (n, m, ℓ) is one of:

$(3, 3, 5), (4, 5, 3), (5, 5, 3), (3, 6, 3), (3, 3, 6), (3, 4, 4), (4, 6, 3), (5, 6, 3), (6, 6, 3), (4, 4, 4)$,

b) ℓ is an integer and $1/\ell + 1/m < 1/2$,

c) $n = 2$, $m \geq 7$, $\ell = m/2$,

d) $n = m \geq 7$, $\ell = n/2$.

Suppose A and B are elements satisfying the conditions of the theorem, where a is the axis of A and b is the axis of B, and that e is the common perpendicular to the lines a and b. Let e_a denote the line obtained from the line e by rotation around the line a by an angle of π/n in the direction of rotation of the element A. Similarly, the line e_b is obtained from the line e by rotation around the line b by an angle of π/m in the direction of rotation of the element B. We use E, E_a, and E_b to denote elliptical elements of order two with axes e, e_a, and e_b (respectively). Then $A = E_aE$, $B = E_bE$ and $[A, B] = ABA^{-1}B^{-1} = (E_aE) \times (E_bE)(EE_a)(EE_b) = (E_aEE_b)^2 = (AE_b)^2$.

Let α denote the plane passing through a and e_a, β denote the plane passing through b and e_b, γ denote the plane passing through e, e_a and b, and δ denote the plane passing through e, e_b and a. Note that the plane γ and δ, α and γ, β and δ intersect at right angles, and that the planes α and δ intersect at an angle of π/n, whilst the planes β and γ intersect at an angle π/m. The planes α and β may intersect in a line (Fig. 1), be parallel (intersecting in a single point at infinity) or diverge.

Let R, R_a and R_b denote the reflections in the planes δ, α and β (respectively). Then $A = R_aR$, and $E_b = RR_b$. Thus $[A, B] = (AE_b)^2 = (R_aRRR_b)^2 = (R_aR_b)^2$.

The proof of the theorem follows from the following five lemmas.

<u>LEMMA 1.</u> Suppose the lines e_a and b intersect or are parallel. Then

1) $[A, B]$ is an elliptical element,

2) if $[A, B]$ has angle of rotation $4\pi/\ell$, then the group G = <A, B> is discrete if and only if the triple (n, m, ℓ) belongs to the set $S = \{(3, 3, 5); (4, 5, 3); (5, 5, 3); (3, 6, 3); (3, 3, 6); (3, 4, 4); (4, 6, 3); (5, 6, 3); (6, 6, 3); (4, 4, 4)\}$.

<u>Proof.</u> Since $n \leq m$ and the lines e_a and b intersect or are parallel, the lines e_b and a also intersect or are parallel. In this case, the planes α, β, γ, δ are the faces of a tetrahedron, some vertices of which may lie at infinity. This tetrahedron is said to be <u>fundamental</u>.

Let c be the line of intersection of the planes α and β and π/ℓ the dihedral angle of the fundamental tetrahedron with edge c. Then it is easy to see that R_aR_b is an elliptical element with axis c and angle or rotation $2\pi/\ell$. Thus the element $[A, B] = (R_aR_b)^2$ is a rotation by an angle $4\pi/\ell$ around the line c. This proves point 1) of Lemma 1.

We now prove point 2) of Lemma 1. Suppose the group G is discrete. Then ℓ must be rational. We consider two separate cases: ℓ is an integer and $\ell = p/q$ is an irreducible fraction.

Suppose ℓ is an integer. Then all the dihedral angles of the fundamental tetrahedron are of the form π/r for some integer r. The classification of such tetrahedra is known and is given in [4], whence we deduce that the triple $(n, m, l) \in S$.

Suppose $\ell = p/q$ is an irreducible fraction. It is not hard to see that in the fundamental tetrahedron, the dihedral angle with edge c must be acute, $\pi/\ell < \pi/2$ or $p/q > 2$. We shall show moreover that $p \leq 6$.

Consider the groups G, $\tilde{G}$ = <G, E> and $\tilde{\tilde{G}}$ = <$\tilde{G}$, R>, where E and R are as previously defined. The group G is either equal to the group $\tilde{G}$ or is a subgroup of index two in it. Actually, as a geometric consequence of the obvious relationships $EAE^{-1} = A^{-1}$ and $EBE^{-1} = B^{-1}$,

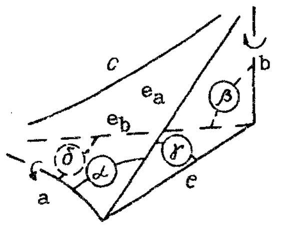

Fig. 1

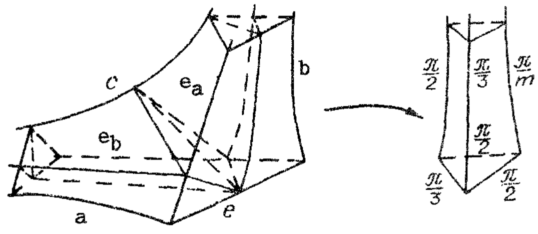

Fig. 2

we have $\tilde{G} = G \cup GE$. If $E \in G$, then G = $\tilde{G}$ but if $E \notin G$, then $G \lhd \tilde{G}$. Using similar arguments, and taking into account the fact that the reflection R alters the orientation, whence $R \notin \tilde{G}$, we deduce that $\tilde{G}$ is a subgroup of index two in the group $\tilde{\tilde{G}}$. Thus the groups G, $\tilde{G}$, and $\tilde{\tilde{G}}$ are simultaneously discrete or not discrete.

All the reflections in the faces of the fundamental tetrahedron belong to the group $\tilde{\tilde{G}}$. Since one of its angles is equal to $q(\pi/p)$, this dihedral angle is divided into q parts (and the fundamental tetrahedron is divided into q tetrahedra) by the additional planes of the reflections of $\tilde{\tilde{G}}$, passing through the edge c, where the angle between adjacent planes is π/p. If in any one of the q tetrahedra, not all the dihedral angles are of the form π/r for some integer r, then it is possible to continue to subdivide this tetrahedron in a similar way. If the group $\tilde{\tilde{G}}$ is discrete, the process of subdivision must terminate, and after a finite number of steps we will obtain a tetrahedron, all of whose dihedral angles are of the form π/r. All such tetrahedra were calculated in [4], and have dihedral angles not less than $\pi/6$. Thus $\pi/p \geqslant \pi/6$ and $p \leqslant 6$.

Whence we have an irreducible fraction $p/q > 2$ with $p \leqslant 6$. With this restriction, p/q is not an integer for p = 5, q = 2 only. Easy calculations show that there are no tetrahedra of the form T[2, 2, n; 2, 5/2, m] (where n and m are integers) in the Lobachevsky space. (We recall that if π/λ_1, π/λ_2, π/λ_3 are the dihedral angles of a tetrahedron whose edges lie in one plane, and π/μ_i are the angles subtended by the angles π/λ_i, i = 1, 2, 3, then such a tetrahedron is denoted by $T[\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3]$.)

Thus, if the group G is discrete, the number ℓ is not a proper fraction.

Conversely, suppose $(n, m, l) \in S$.

If ℓ is even, then there exists an integer k such that $[A, B]^k$ is a rotation by an angle $2\pi/\ell$. Set $C = [A, B]^k$. Then $E_a = CB^{-1}$, $E = E_a A = CB^{-1}A$, whence $E \in G$ and $G = \langle E, E_a, B \rangle$. The conditions of Poincaré's theorem hold (see [5]) for the generators E, E_a and B and for the double tetrahedron obtained from the fundamental tetrahedron and its reflection in the face γ, and thus we deduce that the group G is discrete.

If ℓ is odd, then the group G is discrete by Poincaré's theorem, applied to the generators A and B and the quadruple tetrahedron obtained from the double tetrahedron and its reflection in the face δ. (Note that in this case $E \notin G$.) The proof of Lemma 1 is complete.

LEMMA 2. Suppose n > 2. Then the following conditions are equivalent.

1) The lines a and b diverge and e_b and a intersect or are parallel,

2) [A, B] is an elliptical element with angle of rotation $4\pi/\ell$ and the following inequalities hold

$$1/l + 1/n \geqslant 1/2, \quad 1/l + 1/m < 1/2. \tag{1}$$

In this case, the group G = <A, B> is discrete if and only if ℓ is an integer.

Proof. Suppose that n > 2, that the lines e_a and b diverge, and that e_b and a intersect or are parallel. In this case, the planes α and β intersect in a line which we shall denote by c. As in the previous lemma, it can be shown that [A, B] is an elliptical element with axis c and angle of rotation $4\pi/\ell$, where π/ℓ is the angle between the planes α and β. We note that the planes α, β, γ, δ form an infinite triangular prism with base δ and lateral force α, β and γ. As noted by Kaplinskaya [6], there exists a plane ε perpendicular to all the lateral faces. From a theorem of Andreev [7, 8], it follows that inequalities (1) hold for all infinite triangular prisms with faces α, β, γ, δ, ε. Conversely, if [A, B] is an elliptical element and the inequalities (1) hold, then the planes α, β, γ, δ form an infinite triangular prism, and so the lines e_a and b diverge and the lines e_b and a intersect or are parallel.

Suppose the group G is discrete. Then ℓ is rational. We show that ℓ cannot be a proper fraction.

Suppose $\ell = p/q$ is an irreducible fraction. Since G is discrete, the group $\tilde{G} = \langle G, E\rangle$ is also discrete. Since $E_a, B \in \tilde{G}$ and the plane ε is invariant under E_a and B, by considering the restrictions of E_a and B to the plane, we deduce that the group generated by these elements must be Fuchsian. Using the results of [2], it follows that q = 2 and $p = m \geqslant 7$.

Consider the vertex V formed by the intersection of the edges a and e_b. If V lies at infinity, then by considering the restrictions of the elements A and E_b of $\tilde{G}$ to the orisphere corresponding to the vertex V, we deduce that the group they generate is isomorphic to a group of transformations of the Euclidean plane and is furthermore discrete (since $\tilde{G}$ is discrete). The classification of discrete groups of transformations of the Euclidean plane is known [9], of such orientation-preserving groups, only cyclic groups contain elliptical elements of order $p \geqslant 7$. In our case, the axes of A and E_b intersect the orisphere at a number of different points, consequently the group they generate is noncyclic. Thus, V does not lie at infinity.

If the vertex V is proper, then any non-Euclidean sphere with center at the vertex V is invariant under A and E_b; thus these elements generate a discrete group of transformations of a two-dimensional sphere. All such groups are known [9], and no such group contains two elliptical elements with different axes, and orders n and p with n > 2 and $p \geqslant 7$. Thus the vertex V cannot be proper.

Thus ℓ cannot be a proper fraction if the group G is discrete.

The sufficiency of the discreteness of G is proved accordingly as ℓ is even or odd, as in Lemma 1.

LEMMA 3. Suppose the lines e_a and b diverge and e_b and a intersect, and moreover that n = 2. Then

1) [A, B] is an elliptical element with angle of rotation $4\pi/\ell$,

2) the group G is discrete if and only if the following conditions hold

a) ℓ is an integer and $1/\ell + 1/m < 1/2$,

b) $m \geqslant 7$, $l = m/2$.

Proof. It is easy to show that if the planes α and β intersect in the line c, and the angle between the planes is π/ℓ, then [A, B] is a rotation by an angle of $4\pi/\ell$ around the axis c.

Suppose the group G is discrete. Then $\tilde{G} = \langle G, E\rangle$ is also discrete. From the conditions of the lemma, we deduce that the lines e_a and b are perpendicular to the plane δ. This shows that δ is invariant under the elements E_a and B, and thus the group $\langle E_a, B\rangle$ acting on δ must be Fuchsian. By virtue of the result in [2], we deduce the necessity of conditions a) and b) of the lemma.

The sufficiency of condition a) for the discreteness of the group G is proved as in Lemmas 1 and 2. Suppose condition b) holds, then it is easy to see that the group $\tilde{\tilde{G}} = \langle \tilde{G}, R\rangle$ is discrete. Consequently, its subgroup G is also discrete.

LEMMA 4. The following statements are equivalent

1) The lines e_b and a diverge and the planes α and β intersect in a line

2) [A, B] is an elliptical element with angle of rotation $4\pi/\ell$ and the inequalities

$$1/l + 1/n < 1/2, \quad 1/l + 1/m < 1/2. \tag{2}$$

Then, the following two conditions are necessary and sufficient for the group G to be discrete

a) ℓ is an integer,

b) $n = m \geqslant 7$, $l = m/2$.

Proof. Suppose that the lines e_b and a diverge and the planes α and β intersect in a line which we denote by c. As before, it can be shown that [A, B] is an elliptical element with axis c and angle of rotation $4\pi/\ell$, where π/ℓ is the angle between the planes α and β.

Since $n \leqslant m$ and the lines e_b and a diverge, e_a and b diverge. Moreover, n > 2 and m > 2. Then there exist planes ε and η such that ε is perpendicular to the planes α, β and γ, and η is perpendicular to the planes α, β and δ. By a theorem due to Andreev [7], the inequality (2) holds for the polyhedron in Lobachevsky space bounded by the planes α, β, γ, δ, ε, η. Conversely, if [A, B] is an elliptical element and inequality (2) holds, then the planes α and β intersect in a line and the lines e_b and a diverge.

Suppose now that the group G is discrete. Then ℓ cannot be irrational.

Suppose ℓ = p/q is an irreducible fraction. Since the group $\tilde{G}$ = <G, E> is discrete, and the elements E_a and B leave the plane ε invariant, then as before, using the results of [2], we obtain $m=2l$, $m \geqslant 7$. Similar arguments for the elements A and E_b which leave the plane η invariant, show that n = 2ℓ $n \geqslant 7$. Thus, if the group G is discrete and ℓ is rational, then condition b) holds.

We may now assume that condition b) holds. We consider the group $\tilde{\tilde{G}}$ = <G, E, R>. The planes of reflection of this group divide the initial polyhedron into six infinite triangular prisms (Fig. 2), each of which is a fundamental polyhedron for the group $\tilde{\tilde{G}}$. Thus the group $\tilde{\tilde{G}}$ is discrete. Consequently, G is discrete. The discreteness of the group G given condition a) is proved as in Lemmas 1 and 2. The proof of Lemma 4 is complete.

Lemmas 1-4 give necessary and sufficient conditions for the discreteness of the group G when the planes α and β intersect in a line.

LEMMA 5. Suppose the planes α and β are parallel or diverge. Then G is discrete.

Proof. If the planes α and β are parallel, then $[A, B]=(R_aR_b)^2$ is a parabolic element, whose fixed point (the point of intersection of α and β) lies at infinity. If α and β diverge, then [A, B] is a loxodromic element whose axis is the common perpendicular to these planes. In both cases, the group G is discrete, by Poincaré's theorem.

Lemmas 1-5 together with the previously stated theorem show that

COROLLARY. Let $A, B \in PSL(2, \mathbf{C})$ be primitive elliptical elements with intersecting, mutually perpendicular axes which generate a discrete group G = <A, B>. Then

1) If $\mathbf{H}^3/G$ is compact, then it is one of the three Lannerovsky groups of compact type [4]: T2 = T(2, 2, 3; 2, 5, 3), T3 = T(2, 2, 4; 2, 3, 5) or T4 = T(2, 2, 5; 2, 3, 5),
2) if $\mathbf{H}^3/G$ is finite, but not compact, then G is one of the following semi-Lannerovsky groups of noncompact type [4]: T^1 = T(3, 2, 2; 6, 2, 3), T^2 = T(2, 2, 3; 2, 6, 3), T^3 = T(3, 2, 2; 4, 2, 4); T^4 = T(4, 2, 2; 6, 2, 3), T^5 = T(5, 2, 2; 6, 2, 3), T^6 = T(6, 2, 2; 6, 2, 3) or T^7 = T(4, 2, 2; 4, 2, 4).

Remark. The discrete group $T=T(\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3)$ generated by rotation around three edges of the fundamental tetrahedron $T[\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3]$, described in Lemma 1, has the following canonical representation

$$T=\left\{x, y, z\colon x^{\lambda_1}=y^{\lambda_2}=z^{\lambda_3}=(yz)^{\mu_1}=(xz)^{\mu_2}=(xy)^{\mu_3}=1\right\}.$$

It is natural to consider the group T as a three-dimensional analog of planar discrete groups generated by three rotations around the vertices of a triangle. From the proof of Lemma 1, it follows that the groups T2, T3, T4, T^1, T^4, T^5, T^6 are generated by two elliptical elements, and so their rank (minimal number of generators) is equal to two. A similar result for discrete planar groups was proved by Zieschang [10, pp. 170-171].

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MONADS OF PRO-ULTRAFILTERS AND EXTENSIONAL FILTERS

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UDC 517.11

In [1] we proposed an approach to the application of monadology, a branch of nonstandard analysis, to the study of cyclic filters, which arise in the context of Boolean-valued models. In this paper we characterize the monads of pro-ultrafilters and extensional filters and discuss some relevant properties of these objects. To save space, we shall use the notation and conventions described in detail in [1] without further explanation. We emphasize only that henceforth B will denote a fixed complete Boolean algebra and $V^{(B)}$ the corresponding separable Boolean-valued universe. The truth value of a formula φ of Zermelo—Fraenkel set theory will be denoted by $[\![\varphi]\!]$. When monadology is used, the neoclassical formulation is assumed. We shall generally adopt the hypothesis that the entourage is standard, without further mention.

1. Let X be a cyclic set (= descent of some B-set). As usual, the symbol μ_d will denote the operation producing the (discrete) monadic hull. In other words, $\mu_d(\varnothing) := \varnothing$, and if U is a nonempty set in X then $\mu_d(U)$ is the monad of the standardization of the external filter of supersets of U, i.e.,

$$x \in \mu_d(U) \leftrightarrow ((\forall^{st} V \subset X)\ U \subset V \rightarrow x \in U).$$

By analogy, we define the <u>cyclic monadic hull</u> μ_c as follows:

$$x \in \mu_c(U) \ \leftrightarrow (\forall^{st} V)\,(V = V {\uparrow\downarrow} \wedge V \subset X \wedge U \subset V \rightarrow x \in V).$$

Thus, if U is not empty, the cyclic monadic hull $\mu_c(U)$ is the monad of the cyclic closure of the standardization of the filter of supersets of U.

2. The cyclic monadic hull of a set is the cyclic closure of its monadic hull:

$$\mu_c(U) = \operatorname{mix}(\mu_d(U))$$

for every U.

◁ Let $U \neq \varnothing$ and let V be a standard set such that $V \supset \operatorname{mix}(\mu_d(U))$. By Theorem 2.3 of [1], there exists W in the filter $^*\{U_1 \subset X\colon U_1 \supset U\}$ such that $V \supset W{\uparrow\downarrow}$ and so $V \supset \mu_c(U)$. Thus $\mu_c(U) \subset \operatorname{mix}(\mu_d(U))$, since the set on the right is a monad. Conversely, if $V \supset \mu_c(U)$ and V is standard, then V contains the cyclic closure of a superset of U and thus V ⊃ U. Hence $V \supset \mu((^*\{W\colon W \supset U\}){\uparrow\downarrow})$ and it remains to appeal again to Theorem 2.3 of [1]. ▷

3. Cyclic filters in X that are maximal with respect to inclusion will be called <u>pro-ultrafilters</u> in X. An <u>essential point</u> in X is defined to be an element of the monad of a standard pro-ultrafilter. The external set of all essential points of X will be denoted by ^eX. It is useful to emphasize that the pro-ultrafilters in X are precisely the descents of the ultrafilters in the ascent $X{\uparrow}$ of X.

<u>4. Nonstandard Pro-Ultrafilter Criterion.</u> A filter is a pro-ultrafilter if and only if, first, its monad is cyclic, and, second, it is easily captured by a standard cyclic set.

◁ Let $\mathscr{F}$ be a filter. We have to prove that the following proposition is valid:

$$(\mathscr{F} \text{ is a pro-ultrafilter}) \leftrightarrow \mu(\mathscr{F}) = \operatorname{mix}(\mu(\mathscr{F})) \wedge (\forall^{st} V)\,(V = V{\uparrow\downarrow} \rightarrow \mu(\mathscr{F}) \subset V \vee \mu(\mathscr{F}) \subset V').$$

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