This estimate shows that the iterates of $\mathcal{J}$-symplectic matrics that are sufficiently close to W are bounded. We have not only proved stability of iterates of $\tilde{W}$, but also showed how to find an effective estimate for these iterates.

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NILPOTENT GROUPS OF FINITE ALGORITHMIC DIMENSION

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Problems of algorithmic dimension of algebraic systems have been a subject of attention of many authors (see [1, 11]). The main problem in this direction is that of an algebraic characterization of systems of various algorithmic dimensions. In this connection, an interesting question is that on possible algorithmic dimension of systems in standard classes. Goncharov has first found, in [1], examples of nonautostable algebraic systems of finite algorithmic dimensions, while in [3] he has constructed a solvable, of step 2, nonautostable group of finite algorithmic dimension and has shown that an Abelian group may be either autostable or of infinite algorithmic dimension. The question on possible algorithmic dimension of nilpotent groups remained open.

It is shown in this article that there exist nilpotent groups of step 2 of any algorithmic dimension. The appropriate construction was first proposed by S. S. Goncharov and then, on the basis of this construction, N. S. Romanovskii and A. V. Molokov independently constructed examples of torsion-free as well as periodic of period 4 or 4 (where p is a prime, p > 2) nilpotent groups of step 2 which have a given algorithmic dimension. These examples are presented below.

Essential definitions which we will use below can be found in books: [12, 13] in group theory, [14] in the theory of constructive models, [15] in the theory of recursive functions.

1. Preliminaries on Nilpotent Groups of Step 2

1.1. Recall that if G is a nilpotent group, then the collection of elements generating G modulo its commutator subgroup G' is a system of generators of the entire group G.

Let F be a free nilpotent group of step 2 with a basis $\{x_i \mid i \in I\}$, where I is an ordered set. Note that F' is a free Abelian group with a basis $\{[x_i, x_j] \mid i < j,\ i,\ j \in I\}$. We denote by $\mathscr{K}$ the class of groups of the form F/R, where $R \subseteq F'$. This class consists exactly of nilpotent groups of step 2 whose quotient group over the commutator subgroup is a free Abelian group. Let $G \in \mathscr{K}$ and let H be a subgroup of G, we define the number $r_G(H)$ as the rank of the Abelian group HG'/G'.

Let A and B be nilpotent grops of step 2. We denote by $A \circ B$ the 2-step nilpotent product of these groups, i.e., the group defined in the variety of nilpotent groups of step 2 by means of the union of the systems of generators and the defining relations of the groups A and B. The groups A and B can be embedded as subgroups in $A \circ B$ in a standard way. The validity of the following statement can be easily verified.

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LEMMA 1. Let $A, B \in \mathscr{K}$, $G = A \circ B$, $\{a_i \mid i \in I\}$ a basis of A module A', $\{b_j \mid j \in J\}$ a basis of B modulo B'. Then $\{a_i, b_j \mid i \in I, j \in J\}$ is a basis of G modulo G' and we have a decomposition G' = A' × B' × C, where C is a free Abelian group with a basis $\{[a_i, b_j] \mid i \in I, j \in J\}$.

1.2. Let $k \geqslant 1$ and let A be a free Abelian group of rank k with some fixed basis $\{a_1, \ldots, a_k\}$. Suppose that $B \in \mathscr{K}$ and B has a subgroup C which is a free nilpotent group of step 2 with a basis $\{c_j \mid j \in J\}$. Suppose that the last set may be completed by elements b_i $(i \in I)$ to a basis of B modulo B'. Consider the group $G = B \circ A/[a_1, C]$, we will also denote it by <B, (C, A)>. Obviously, B and A can be embedded in G as subgroups. By Lemma 1, G' can be written as a direct product of the subgroup B' and a free Abelian group D with a basis $\{[b_i, a_l], [c_j, a_m] \mid i \in I, j \in J, 1 \leqslant l \leqslant k, 2 \leqslant m \leqslant k\}$. Note that the centralizer of the element a_1 in G is equal to CAG'.

LEMMA 2. If H is an Abelian subgroup of G and $r_G(H) \geqslant 3$, then H is contained either in BG' or in AG'.

Proof. Suppose the contrary, let H be a counter-example. Without loss of generality, we may assume that $H \supseteq G'$. Note that each element of the group G can be uniquely written, modulo the commutator subgroup G', as a standard product $a_1^{\alpha_1} \ldots a_k^{\alpha_k} \cdot b \cdot c$, where b is some ordered product of powers of the elements b_i, c an ordered product of powers of the elements c_j. We choose a canonical element h_1 in H such that $h_1 = a_1^{\alpha_1} \ldots a_p^{\alpha_p} \cdot b \cdot c$, where p is maximal relative to the condition $\alpha_p \neq 0$ and the number α_p is minimal in absolute value among all such possible numbers. Then we may assume that H is generated modulo G' by the canonical elements h_1, $h_2, \ldots$, and the decompositions for $h_2, h_3, \ldots$ involve no elements a_l, $l \geqslant p$. Let $h_2 = a_1^{\alpha_1'} \ldots a_s^{\alpha_s} \cdot b' \cdot c'$, $s < p$.

Suppose first that p > 1. Then, if b' ≠ 1 or c' ≠ 1, then $[h_1, h_2] = u \cdot v$, where $u \in B'$, $v \in D$, and we can assert that the element v is distinct from 1 because its expression relative to the basis of the group D would contain a commutator of the form $[b_i, a_p]$ or $[c_j, a_p]$. This contradicts the commutativity of H. Therefore, $h_2, h_3, \ldots \in A$. Since $r_G(H) \geqslant 3$, at least one of the elements $h_2, h_3, \ldots$ depends on some a_l, where $\ell > 1$. Suppose it is h_2, i.e., $s > 1$, $\alpha_s' \neq 0$. Since $h_1 \notin AG'$, either b ≠ 1 or c ≠ 1. In this case again we can show that $[h_1, h_2] \neq 1$. Again, we arrive at a contradiction to the commutativity of H.

Suppose p = 1, i.e., $h_1 = a_1^{\alpha_1} \cdot b \cdot c$, $\alpha_1 \neq 0$, and the elements $h_2, h_3, \ldots$ lie in B. If $h_2 = b' \cdot c'$ and b' ≠ 1, then the expression of the commutator $[h_1, h_2]$ contains a factor of the form $[a_1, b_i]^{\alpha}$, so $[h_1, h_2] \neq 1$. Therefore, $h_2, h_3, \ldots \in C$. But then $r_G(H \cap C) \geqslant 2$. In view of the fact that C is a free nilpotent group of step 2, this implies that H ∩ C is a non-Abelian group. The obtained contradiction proves the lemma.

1.3. Let $B \in \mathscr{K}$ and let $C_1, \ldots, C_s$ be subgroups of B which are free nilpotent groups of step 2 and the union of their bases can be complemented to a basis of B modulo B'. Consider distinct natural numbers $k_1, \ldots, k_{2s}$ $(k_i \geqslant 3)$ and also, for each i, a free Abelian group A_i with a fixed basis $\{a_{1,i}, \ldots, a_{k_i,i}\}$. We form the group $G = \langle B, (C_1, A_1), (C_1, A_2), \ldots, (C_s, A_{2s-1}), (C_s, A_{2s}) \rangle$.

LEMMA 3. Suppose that B has no maximal Abelian subgroups which, modulo B', have rank equal to one of the numbers k_i. Then:

1) The subgroup C_iG' consists exactly of elements which centralize $a_{1,2i-1}$ and $a_{1,2i}$;

2) if φ is an automorphism of the group G, then $a_{1,i}\varphi \equiv a_{1,i}^{\pm 1} \bmod G'$,

3) the subgroups A_iG' and C_iG' are characteristic in G.

Proof. The first statement is obvious because the centralizer of the element $a_{1,2i-1}$ in G is equal to $C_iA_{2i-1}G'$ while the centralizer of the element $a_{1,2i}$ is equal to $C_iA_{2i}G'$.

By Lemma 2, G has a unique maximal Abelian subgroup which, modulo G', has rank k_i, it is A_iG'. Thus, A_iG' is a characteristic subgroup. The cyclic subgroup generated by the element $a_{1,i}$ multiplied by G' can be characterized as the collection of elements of A_iG' whose centralizers are greater than the set A_iG'. Obviously, this connection must be automorphically admissible. Hence $a_{1,i}\varphi \equiv a_{1,i}^{\pm 1} \bmod G'$. Then it follows from Subsec. 1 that each subgroup C_iG' is characteristic. The lemma is proved.

1.4. The following two statements can be easily verified.

LEMMA 4. Consider a group G defined in the variety of nilpotent groups of step 2 by means of generators a, b, x_n $(n \in N)$ and defining relations $[a, x_n] = [b, x_{n+1}]$ $(n \in N)$. Then, if $x \in \mathrm{gr}(x_n \mid n \in N)$ and $[a, x_n] = [b, x]$, then $x \equiv x_{n+1} \bmod G'$.

LEMMA 5. Let G be a group defined in the variety of nilpotent groups of step 2 by means of generators c, d, y_n, u_n $(n \in N)$ and defining relations $[y_n, u_n] = 1$, $[y_n, c] = [y_{n+1}, c]$, $[u_n, d] = [u_{n+1}, d]$ $(n \in N)$. Then, if $y \in \mathrm{gr}(y_n \mid n \in N)$, $u \in \mathrm{gr}(u_n \mid n \in N)$, $y \neq 1$, $u \neq 1$, and $[y, u] = 1$, then for some n we have $y \in \mathrm{gr}(y_n)$, $u \in \mathrm{gr}(u_n)$; if, furthermore, $[y, c] = [y_0, c]$, $[u, d] = [u_0, d]$, then $y \equiv y_n$, $u \equiv u_n \bmod G'$.

1.5. Let $\mathcal{K}_2$ denote the class of nilpotent groups of step 2 in which the quotient group over the commutator subgroup has period 2 (the commutator subgroup in such a group also has period 2), let $\mathcal{K}_p$ be the class of nilpotent groups of step 2 and of period p (p is a prime, p > 2). These classes contain analogues of constructions and statements studied above for the class $\mathcal{K}$.

2. Main Theorem

Proposition. For each infinite computable family S of recursively enumerable sets there exists a nilpotent group G of step 2 such that there is an algorithm of constructing a constructivization ν_γ of the group G for each one-valued computable numeration γ of the family S; here ν_γ is not autoequivalent to $\nu_{\gamma'}$ if and only if γ and γ' are nonequivalent numerations, and for each constructization ν of the gorup G there exists a numeration γ such that ν and ν_γ are autoequivalent.

Proof. Consider a family S of recursively enumerable sets and a computable one-valued numeration γ of the family S*, where $S^* = \{s^* \mid s \in S\}$ and $s^* = \{c(n, k) \mid n \in s,\ k \in N\}$. We define a strictly computable family of finite sets $\{\gamma^t(n) \mid n, t \in N\}$ such that $\bigcup_{t \geq 0} \gamma^t(n) = \gamma(n)$, $\gamma^0(n) = \varnothing$ and $|\gamma^{t+1}(n) \setminus \gamma^t(n)| \leq 1$ for all $n, t \in N$.

Remark. If γ is a one-valued computable numeration of S, then $\gamma^*(n) \rightleftharpoons \{c(k, l) \mid k \in \gamma(n)\}$ is a one-valued computable numeration of S*, if γ is a one-valued computable numeration of S*, then $\gamma^0(n) = \{k \mid \gamma(n) \in c(k, 0)\}$ is a one-valued computable numeration of S.

By this remark, we have a one-to-one correspondence between one-valued computable numerations of the families S and S*, and $\gamma_1 \leq \gamma_2$, if and only if $\gamma_1^* \leq \gamma_2^*$. Hence, S and S* possess the same number of nonequivlent one-valued computable numerations. Henceforth, we will work with the family S and its computable one-valued numeration γ assuming that S has the following property: if $c(n, l) \in s$ in S, then for each natural number m the element c(n, m) also lies in s. Without loss of generality, we may assume that every natural number lies in one of the elements of S. If this is not so, then we could add the set N to the family S, and $S \cup \{N\}$ would possess the same properties.

Let $A \subseteq N^3$ be such that

$$A = \{c^3(n, t, k) \mid \gamma^{t+1}(n) \setminus \gamma^t(n) = \{k\},\ n, t, k \in N\}.$$

This set is recursively enumerable, nonempty and infinite. So there exists a recursive function f_γ: N → N such that f_γ maps N bijectively onto $c^3(A)$, where c^3: N → N is a one-to-one numeration of all triples of natural numbers [15].

Now, we will construct the required group G = G_γ. Suppose a group B is given in the variety of nilpotent groups of step 2 by means of generators $a, b, c, d, x_n, y_n, z_n, u_n, v_n$ $(n \in N)$ and defining relations

$$[x_n, a] = [x_{n+1}, b],\ [y_n, u_n] = 1,\ [z_n, v_n] = 1,\ [y_n, c] = [y_{n+1}, c],$$
$$[z_n, d] = [z_{n+1}, d],\ [u_n, d] = [u_{n+1}, d],\ [v_n, c] = [v_{n+1}, c],$$

here $n \in N$, $[x_k, y_s] = 1$, and $[y_s, z_n] = 1$ if for some t we have $f_\gamma(s) = c^3(n, t, k)$.

LEMMA 6. If H is an Abelian subgroup of B, then $r_B(H) \leq 2$.

Proof. Assume the contrary. Then there exists a number q such that the group B_q given by the generators $a, b, c, d, x_n, y_n, z_n, u_n, v_n$ $(1 \leq n \leq q)$ and defining relations

$$[x_n, a] = 1,\ [x_n, b] = 1,\ [x_n, y_1] = 1, \ldots, [x_n, y_q] = 1,$$
$$[y_n, u_1] = 1, \ldots, [y_n, u_q],\ [y_n, z_1] = 1, \ldots, [y_n, z_q] = 1,$$
$$[z_n, v_1] = 1, \ldots, [z_n, v_q] = 1,\ [y_n, c] = 1,\ [v_n, c] = 1,$$
$$[z_n, d] = 1,\ [u_n, d] = 1,$$

where $1 \leqslant n \leqslant q$, has an Abelian subgroup E such that $r_{B_q}(E) \geqslant 3$. The group B_q can be obtained by a sequential procedure starting with a free nilpotent group of step 2 with a basis $\{x_1, \ldots, x_q\}$, adding, at each step, one generator and one system of defining relations in the same order as that listed above. This enables us to apply Lemma 2 by means of which we conclude that an Abelian subgroup E with the condition $r_{B_q}(E) \geqslant 3$ could not exist. The lemma is proved.

Consider the following subgroups of B which are free nilpotent groups of step q: A = gr(a), B = gr(b), C = gr(c), D = gr(d), X_0 = gr(x_0), X = gr$(x_n | n \in N)$, Y = gr$(y_n | n \in N)$, Z = gr$(z_n | n \in N)$, U = gr$(u_n | n \in N)$, V = gr$(v_n | n \in N)$. We assign to these groups pairs of natural numbers $(k_1, k_2), \ldots, (k_{19}, k_{20})$, respectively, where $k_i \geqslant 3$, $k_i \neq k_j$ for i ≠ j. For each k_i we take a free Abelian group A_i with a fixed basis $\{a_{1,i}, \ldots, a_{k_i,i}\}$ and, using the construction described in 1.2 and 1.3, define the group

$$G_\gamma = \langle B, (A, A_1), (A, A_2), \ldots, (V, A_{19}), (V, A_{20}) \rangle.$$

A constructivization of the group just defined is given as follows. We have essentially described the group G_γ by means of the set of generators $M = \{a, b, c, d, x_n, y_n, z_n, u_n, v_n, a_{ij} \mid n \in N, 1 \leqslant i \leqslant k_j, 1 \leqslant j \leqslant 20\}$ and some system of defining relations. We consider its standard representation as a quotient group F/R_γ, where F is a free nilpotent group of step 2 with the basis M. Let ν be the Gödel numeration of elements of the group F. We define the constructivization ν_γ of the group G_γ putting $\nu_\gamma(n) \rightleftharpoons \nu(n) R_\gamma$.

<u>LEMMA 7.</u> If γ_1 and γ_2 are two one-valued computable numerations of the family S, then the groups G_{γ_1} and G_{γ_2} are isomorphic.

<u>Proof.</u> Consider a one-to-one map h of the set N onto N such that $\gamma_1(n) = \gamma_2(h(n))$. For each pair (l, n) such that $l \in \gamma_1(n)$ there exists a triple (n, t, l) such that $l \in \gamma_1^{t+1}(n) \setminus \gamma_1^t(n)$. Since $l \in \gamma_1(n) = \gamma_2(h(n))$, there exists a triple $(h(n), t', l)$ for which $l \in \gamma_2^{t'+1}(h(n)) \setminus \gamma_2^{t'}(h(n))$. We define a function ψ putting $\psi(s) = s'$ if the equality $f_{\gamma_1}(s) = c^3(n, t, l)$ implies $f_{\gamma_2}(s') = c^3(h(n), t', l)$. This function is a one-to-one map of N onto N. The required isomorphism $G_{\gamma_1} \to G_{\gamma_2}$ is defined on the generators as follows:

$$a \to a, \quad b \to b, \quad c \to c, \quad d \to d, \quad x_n \to x_n, \quad y_n \to y_{\psi(n)},$$
$$z_n \to z_{h(n)}, \quad u_n \to u_{\psi(n)}, \; v_n \to v_{h(n)}, \quad a_{ij} \to a_{ij}$$
$$(n \in N, \quad 1 \leqslant i \leqslant k_j, \quad 1 \leqslant j \leqslant 20).$$

The lemma is proved.

Note that in the proof of the lemma the isomorphism $G_{\gamma_1} \to G_{\gamma_2}$ is constructed effectively relative to the map h. So we have

<u>LEMMA 8.</u> If γ_1 and γ_2 are equivalent one-valued computable numerations of the family S, then ν_{γ_1} and ν_{γ_2} are autoequivalent.

<u>LEMMA 9.</u> For each computable numeration γ of the family S the numerated group (G, ν_γ) is constructive.

<u>Proof.</u> It suffices to verify that the equality relation is decidable on the numbers of elements of the group G. This is equivalent to the decidability of the problem of equality of elements $\nu_\gamma(n)$ to the identity. The latter does hold because for each subgroup H generated by a finite subset of the set M of given generators of the group G one can effectively construct a finite system of defining relations, and it is well known that the equality problem is decidable in finitely generated nilpotent groups. The lemma is proved.

Each group G (including ours) that can be written in the form F/R, where F is a free nilpotent group of step 2 with a basis M and $R \subseteq F'$ satisfies

<u>LEMMA 10.</u> Each map M → G equal to the identity modulo G' can be extended to an automorphism of the group G.

<u>LEMMA 11.</u> If ν is a constructivization of the group G, then there exist recursive functions $\sigma_x, \sigma_y, \sigma_z, \sigma_u, \sigma_v$ and permutations p and q of the set N such that

$$\nu(\sigma_x(n)) \equiv x_n, \; \nu(\sigma_y(n)) \equiv y_{p(n)}, \; \nu(\sigma_z(n)) \equiv z_{q(n)},$$
$$\nu(\sigma_u(n)) \equiv u_{p(n)}, \; \nu(\sigma_v(n)) \equiv v_{q(n)} \bmod G'.$$

<u>Proof.</u> Let $\bar{m}$ denote the ν-number of an element m in M. Then we put $\sigma_x(0) = \bar{x}_0$, $\sigma_x(n+1) = \mu l([\nu(l), a_{1,11}] = 1 \,\&\, [\nu(l), a_{1,12}] = 1 \,\&\, [\nu(l), b] = [\nu(\sigma_x(n)), a])$.

We define the binary function $(\sigma_y, \sigma_u)(n)=(\sigma_y(n), \sigma_u(n))$ as follows: $(\sigma_y, \sigma_u)(0)=(\bar{y}_0, \bar{u}_0)$, $(\sigma_y, \sigma_u)(n+1)=(\mu l, \mu r)([\nu(l), a_{1,13}]=1 \,\&\, [\nu(l), a_{1,14}]=1 \,\&\, [\nu(r), a_{1,17}]=1 \,\&\, [\nu(r), a_{1,18}]=1 \,\&\, [\nu(l), c]=[y_0, c] \,\&\, [\nu(r), d]=[u_0, d] \,\&\, \underset{k\leqslant n}{\&}\, [\nu(l), \nu(\sigma_y(k))]\neq 1)$. The binary function (σ_z, σ_v) is defined similarly. The fact that the constructed functions satisfy the required conditions follows from Lemmas 3-5. The lemma is proved.

Note that the subgroup generated in the group G by the elements x_n, y_n, z_n $(n\in N)$ is defined, in terms of the given generators by the defining relations $[x_k, y_s] = 1$ and $[y_s, z_n] = 1$ for some t we have $f_\gamma(s) = c^3(n, t, k)$. This implies

LEMMA 12. For given k and n there exists s such that $[x_k, y_s] = 1$ and $[y_s, z_n] = 1$ if and only if $k\in\gamma(n)$.

LEMMA 13. If ν is a constructivization of the group G_γ, then there exists a one-valued computable numeration γ' of the family S such that ν and $\nu_{\gamma'}$ are autoequivalent.

Proof. Consider the functions $\sigma_x, \sigma_y, \sigma_z, \sigma_u, \sigma_v$ and the permutations p and q constructed in Lemma 11. We define a family S' of recursively enumerable sets and its computable numeration γ' putting

$$\gamma'(n)=\{k \mid \exists s([\nu(\sigma_x(k)), \nu(\sigma_y(s))]=1 \,\&\, [\nu(\sigma_y(s)), \nu(\sigma_z(n))]=1)\},$$
$$S'=\{\gamma'(n) \mid n\in N\}.$$

Recall that $\nu(\sigma_x(k))\equiv x_k$, $\nu(\sigma_y(s))\equiv y_{p(s)}$, $\nu(\sigma_z(n))\equiv z_{q(n)} \bmod G'$. So $\gamma'(n)=\{k \mid \exists s([x_k, y_{p(s)}]=1 \,\&\, [y_{p(s)}, z_{q(n)}]=1)\}$. Since p and q are permutations of the set N, Lemma 12 immediately implies that $\gamma'(n)=\gamma(q(n))$. But then S' = S and γ' is a computable numeration of the family S. We will define from γ' a constructivization $\nu_{\gamma'}$ of the group $G_{\gamma'}$.

Consider the following map $M \to G_{\gamma'}$:

$$x_n\to\nu(\sigma_x(n)),\quad y_n\to\nu(\sigma_y(n)),\quad z_n\to\nu(\sigma_z(n)),$$
$$u_n\to\nu(\sigma_u(n)),\quad v_n\to\nu(\sigma_v(n))\qquad(n\in N),$$
$$m\to m \text{ for the remaining elements } m\in M.$$

Obviously, this map defines an isomorphism $\psi\colon G_\gamma\to G_{\gamma'}$. The recursiveness of the functions $\sigma_x, \sigma_y, \sigma_z, \sigma_u, \sigma_v$ implies that there exists a recursive function ξ such that $\psi(\nu_{\gamma'}(n))=\nu(\xi(n))$. The lemma is proved.

LEMMA 14. Suppose that constructive groups $(G_{\gamma_1}, \nu_{\gamma_1})$ and $(G_{\gamma_2}, \nu_{\gamma_2})$ are recursively isomorphic. Then γ_1 and γ_2 are equivalent numerations.

Proof. Let ψ be a recursive isomorphism of the first group into the second one and let ξ be a recursive function such that $\psi\nu_{\gamma_1}=\nu_{\gamma_2}\xi$. By Lemma 3, the subgroups A, B, C, D, X_0, X, Y, Z, U, V are characteristic in G. Hence $\psi(m)\equiv m^{\pm1} \bmod G'$, where $m\in\{a, b, c, d, x_0\}$. Suppose, by induction, that $\psi(x_n)\equiv x_n^{\pm1} \bmod G'$. Then the relations of the group G imply that $[x_n, a]\equiv[\psi(x_{n+1})^{\pm1}, b]$. By Lemma 4, we deduce that $\psi(x_{n+1})\equiv x_{n+1}^{\pm1} \bmod G'$.

Since $\psi(z_n)\in ZG'$, $\psi(v_n)\in VG'$ and $[\psi(z_n), \psi(v_n)]=1$, Lemma 5 implies that $\psi(z_n)\equiv z_{q(n)}^{\pm1}$, $\psi(v_n)\equiv v_{q(n)}^{\pm1} \bmod G'$ for some permutation q of the set N. The recursiveness of the isomorphism ψ implies the recursiveness of the permutation q. Since $\gamma_2(n) = \gamma_1(q(n))$, γ_1 and γ_2 are equivalent numerations. The lemma is proved.

Remark. If the definition of the group G is modified by adding the relation $m^p = 1$, where p is a prime and $m\in M$, then we obtain a group in the class $\mathcal{K}_p$ (cf. 1.5) which also satisfies the main Lemmas 7-9, 13, 14.

THEOREM. For each n $(1\leqslant n\leqslant\omega)$ the classes $\mathcal{K}$ and $\mathcal{K}_p$ contain groups of algorithmic dimension n.

Indeed, consider a family S_n of recursively enumerable sets having exactly n nonequivalent one-valued computable numerations. Such families have been constructed in [1, 2]. From this family we construct the group G in the appropriate class, and it is as required.

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RELATIVELY STANDARD ELEMENTS IN NELSON'S INTERNAL SET THEORY

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All discussions in the present article are carried out within the framework of the axiomatic system for nonstandard analysis – Nelson's internal set theory (IST) [1].

As we know, the existence of actual infinitely large and infinitesimal numbers in nonstandard analysis enables us to give simpler formulations for the classical notions of analysis. For example, for a standard function $f\colon \mathbf{R}\to\mathbf{R}$ and standard numbers $a,\ b\in\mathbf{R}$

$$\lim_{x\to a} f(x) = b \Leftrightarrow \forall\alpha\sim 0\ \ f(a+\alpha)-b\sim 0 \tag{1}$$

($\gamma \sim 0$ means that γ is infinitesimal). Here, as in other equivalence of this kind, it is essential that f, a, and b are standard. There arises the problems of existence of simple nonstandard criteria in the general case where arbitrary nonstandard elements occur in a given definition. A situation, where this is necessary, arises quite often. The simplest example is obtained when we try to give a nonstandard definition of $\lim_{x\to 0}\lim_{y\to 0} f(x,y)=A$, even in the case of standard f and A. Indeed, from (1) we get the equivalent condition $\forall\alpha\sim 0 \lim_{y\to 0} f(\alpha,y)\sim A$. However f(α, y) is already a nonstandard function for α ~ 0 and equivalence (1) is not applicable to it. This example suggests that it is necessary to introduce infinitesimals of <u>orders substantially higher than a given α</u>, i.e., numbers that remain infinitesimal even if α is assumed to be finite. This view was put forth in the seminars at the Moscow State University on the 70th birth anniversary of A. G. Dragalinyi. For its realization an extension of IST that is noncontradictory with respect to ZFC has been introduced in [2] by the addition of a countable family of (not definable in IST) predicates $\mathrm{St}_k(x)$ (x is standard of degree 1/k), with the help of which we can give a simple criterion so that $\lim_{x\to a} f(x)=b$ for a standard f and nonstandard a, and b. Let us also observe that for a partial solution of this problem we can apparently use the construction of the double nonstandard enlargement from [3], although this problem has not been touched upon there.

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