Proposition 3.3. A homogeneous locally compact space with intrinsic metric is a manifold if and only if it is locally contractible. In that case, the group of all its motions, with the compact-open topology, is a Lie group.

Proof. It is clear that a manifold is locally contractible. Conversely, we stated at the beginning of the proof of Theorem 2.1 that the neutral connected component G of the group of all motions Γ of a locally compact space with intrinsic metric acts transitively, continuously and effectively on M. In addition, G is locally compact and satisfies the second axiom of countability. If M is locally contractible, then by results of [13] G is a Lie group. The space M, as it is homeomorphic to a quotient space G/H of G by a compact subgroup locally compact transformation group of M, it follows that Γ is a Lie group [12]. Finally, by Theorem 1.1, we may assume that Γ is endowed with the compact-open topology. This completes the proof.

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A NONLINEAR DIFFUSION EQUATION WITH A SOURCE TERM OF GENERAL FORM

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1. Introduction

A large number of applications and an interesting mathematical statement of the problem have given rise to diverse studies of wave solutions of parabolic equations. Problems treated therein involve existence of a wave, its stability, evolution of a solution into a wave solution, and several others. Fundamental results relating to the theory of waves, described by parabolic equations, available to date, are presented in [1] (which contains a rather complete bibliography).

In the present paper we examine conditions for solutions to evolve into a wave relative to form and velocity in the case of sources of sufficiently general form.

We consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u), \quad u(x, 0) = f(x), \tag{1}$$

where $F(u) \in C^2[0, 1]$, $F(0) = F(1) = 0$. We assume that f(x) is a monotonic, piecewise-continuous function with a finite number of points of discontinuity, $0 \leq f(x) \leq 1$. It is well known [2]

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that a solution u(x, t) of problem (1) exists which for $t > 0$ is continuous, together with second partial derivatives with respect to x and first partial derivatives with respect to t, and which for $t = 0$ agrees with the initial condition f(x) at points of its continuity. In addition, $0 \leqslant u(x, t) \leqslant 1$.

We present here results obtained earlier. We first introduce certain definitions and notation.

Let u_1, u_2 be such that $0 \leqslant u_1 < u_2 \leqslant 1$ and $F(u_1) = F(u_2) = 0$. By $w_c(x;\ u_1,\ u_2)$ we shall mean a solution of the equation

$$w'' + cw' + F(w) = 0,\ w(0) = (u_1 + u_2)/2, \tag{2}$$

which satisfies the conditions $w_c(+\infty;\ u_1,\ u_2) = u_1,\ \ w_c(-\infty;\ u_1,\ u_2) = u_2,\ w_c' < 0$. Function w_c is referred to as a wave with velocity c.

Equation (2) reduces to a system of ordinary differential equations

$$\begin{aligned} w' &= p, \\ p' &= -cp - F(w). \end{aligned} \tag{3}$$

In the phase plane (w, p), to the wave $w_c(x;\ u_1,\ u_2)$ there corresponds the trajectory joining the singular point $(u_2, 0)$ and $(u_1, 0)$. We introduce the function $\tau_c(w;\ u_1,\ u_2)$, which puts abscissas of points of this trajectory into correspondence with their ordinates. It is easy to show that τ_c is unique and continuous.

We shall say that function $\tau_c(w;\ a,\ b)$ exists for certain $a,\ b \in [0,\ 1],\ a < b$, if there exists an arc of a trajectory of system (3), which is imbedded in the halfstrip $0 \leqslant u \leqslant 1,\ p \leqslant 0$ and joins points (b, 0) and $(a, 0)$ (not necessarily singular). The function $\tau_c(w;\ a,\ b)$ is defined as above.

To a solution u(x, t) of problem (1) for $t > 0$ we can make correspond a function $\varphi(t,\ w;\ f) = u'(x,\ t)$, where $w = u(x, t)$. Then $\varphi(t,\ w;\ w_c(\cdot;\ u_1,\ u_2)) = \tau_c(w;\ u_1,\ u_2)$.

In proceeding, we shall use the following notation: if $a \in [0,\ 1],\ F(a) = 0$, then $I(a)$ is a domain of attraction of point a relative to the equation $\partial u/\partial t = F(u)$, i.e., $I(a)$ is a connected set of points of the interval [0, 1] which contains the point a and is such that when $u \in I(a),\ u \gtrless a$ we have the inequality $F(u) \lesseqgtr 0$; $F(u) \in g^{\pm}(a)$, if in some right half-neighborhood of point a the function F(u) is positive (negative); $F(u) \in g^0(a)$, if in any right half-neighborhood of point a we can find zeros of function F(u) distinct from a.

To define $l^{\pm}(a),\ l^0(a)$ we consider a left half-neighborhood of point a.

Definition. Let $m(t, \omega)$ be a solution of the equation

$$u(x,\ t) = \omega. \tag{4}$$

A solution u(x, t) of problem (1) comes out in the form of a wave $w_c(x;\ u_1,\ u_2)$ if, uniformly with respect to $x \in (-\infty,\ +\infty)$, as $t \to \infty$

$$u(x + m(t,\ (u_1 + u_2)/2),\ t) \to w_c(x;\ u_1,\ u_2).$$

If $\dot{m}(t,\ (u_1 + u_2)/2) \to c$ as $t \to \infty$, we say that we have emergence with respect to velocity. (Here, and in what follows, a dot above a letter means differentiation with respect to time.)

In the definition of emergence with respect to form and velocity the value $(u_1 + u_2)/2$ can be replaced by any other value from the interval (u_1, u_2). This is immaterial as far as having a solution emerge into the wave form.

Wave solutions of a nonlinear diffusion equation were considered for the first time in the basic paper [2]. A study was made therein of emergence of a solution into a wave for th case of a positive source and an initial condition of a particular form. These results underwent substantial development in [3-8] in which more general sources and initial conditions were considered.

The following results were obtained in [4, 5] in which sources with alternating signs were considered.

Let the function F(u) satisfy one of the following conditions:

1) $F'(0) < 0,\ F'(1) < 0$;

2) $F(u) \leqslant 0$ for $u \in (0,\ a)$, $F(u) > 0$ for $u \in (a,\ 1)$ for some $a \in (0,\ 1)$ and $\int_0^1 F(u)\,du > 0$;

3) $F(u) < 0$ for $u \in (0, a)$, $F(u) \geqslant 0$ for $u \in (a, 1)$ and $\int_0^1 F(u)\,du < 0$;

4) $F(u) < 0$ for $u \in (0, a)$, $F(u) \geqslant 0$ for $u \in (a, 1)$.

Then, if the wave exists (existence is proved in Cases 2-4), it is unique and emerges as a wave in form and velocity for an arbitrary monotonic initial condition $f(x)$, $f(+\infty) \in I(0)$, $f(-\infty) \in I(1)$.

A stronger statement relative to emergence into a wave was made in [4] for Case 1, namely: instead of emergence with respect to form we have the convergence $m(t, 1/2) - ct \to h$, where h is a constant and where the initial condition is not assumed to be monotonic.

In these papers the notion of minimal decomposition was introduced, which means that an interval [0, 1] can be represented as the union of a finite number of intervals $[a_k, b_k]$, $a_{k+1} = b_k$, on each of which a wave $w_{c_k}(x; a_k, b_k)$, $c_k > c_{k+1}$ exists and satisfies one of the conditions 1-4 (interval [0, 1] in these conditions is replaced by $[a_k, b_k]$). If the minimal decomposition consists of more than one interval, the wave $w_c(x; 0, 1)$ does not exist and we have emergence of solutions into a system of waves, i.e., as $t \to \infty$, uniformly with respect to $u \in [u_+(t), u_-(t)]$ $(u_\pm(t) = u(\pm\infty, t))$,

$$\varphi(t, u; f) \to R_0(u) \equiv \tau_{c_k}(u; a_k, b_k) \quad \text{for} \quad a_k \leqslant u \leqslant b_k$$

or, uniformly on each finite interval with respect to x,

$$u(x + m(t, (a_k + b_k)/2), t) \to w_{c_k}(x; a_k, b_k).$$

The case in which the source F(u) is positive in the interval (0, 1) was studied in considerable detail in [2, 6, 7]. Here there is a semiaxis of velocities $c \geqslant c_0$, for which there exist waves $w_c(x; 0, 1)$, and conditions can be given on the function f(x) for which a solution emerges onto a definite wave.

If function F(u) is positive, not on the hole interval (0, 1) but only in some neighborhood of the point 0, it follows that the wave $w_c(x; 0, 1)$, if it exists for some (positive, to be specific) value of the velocity, also exists for the half-interval of velocities $[c_*, c^*)$, $c_* \geqslant 2\sqrt{\alpha}$ [1]. [Here, and in what follows, $\alpha = F'(0)$, $\beta = F'(1)$.] In this case emergence conditions were obtained only on the wave $w_{c_*}(x; 0, 1)$, assuming that $c_* > 2\sqrt{\alpha}$, $\alpha > 0$, $\beta < 0$ [8]. As is well-known [6], for a positive source the asymptotic behaviors of solutions for the cases $c = 2\sqrt{\alpha}$ and $c > 2\sqrt{\alpha}$ are distinct. Apparently, this is also true for sources positive only in a neighborhood of 0.

In the present paper we study conditions for emergence into a wave minus any restrictions on the function F(u) except for an indicated smoothness condition. We show that for an arbitrary monotonic initial condition $f(x)$, $f(+\infty) \in I(0)$, $f(-\infty) \in I(1)$, there is emergence into a wave $w_c(x; 0, 1)$ if it exists and is unique (Theorem 7). If a wave exists and is not unique (for positive velocity this is the case for sources positive in a neighborhood of 0), we indicate conditions for the behavior of initial functions at infinity for which a solution emerges into the minimal wave $w_{c_*}(x; 0, 1)$ (Theorem 6). We introduce the notion of a minimal system of waves, which can be defined for arbitrary functions F(u) and generalizes the notion of minimal decomposition. If the source is not positive close to 0 and is not negative close to 1, emergence into a minimal system of waves occurs from arbitrary initial conditions (Theorem 9). Otherwise it is necessary to introduce additional restrictions on the behavior of the initial conditions at infinity (Theorem 10). A study of the emergence of solutions onto non-minimal waves and systems of waves will be published separately.

We clarify the idea of the proof for these assertions by an example of emergence into the wave $w_c(x; 0, 1)$ for the case in which it is unique (a detailed exposition is given in Sec. 4).

We give a proof of the convergence

$$\varphi(t, u; f) \to \tau_c(u; 0, 1)$$

uniform with respect to u, from which, as is well-known, we have emergence into a wave with respect to form.

The inequality $\varphi(t, u; f) \geqslant \tau_c(u; 0, 1) - \varepsilon$, $u_+(t) \leqslant u \leqslant u_-(t)$, for arbitrary $\varepsilon > 0$ and t sufficiently large, follows from the relations [2, 5, 6]

$$\varphi(t,\ u;\ f) > \varphi(t,\ u;\ \chi),\quad \varphi(t,\ u;\ \chi) \uparrow \tau_c(u;\ 0,\ 1),$$

where $\chi(x) = \begin{cases} 1, & x \leqslant 0, \\ 0, & x > 0. \end{cases}$ To prove the inequality

$$\varphi(t,\ u;\ f) \leqslant \tau_c(u;\ 0,\ 1) + \varepsilon,\quad u_+(t) \leqslant u \leqslant u_-(t) \tag{5}$$

we examine the asymptotic behavior of solutions of the boundary problem

$$\begin{cases} \dot{\varphi} = \varphi^2\varphi'' - F\varphi' + F'\varphi, \\ \varphi(t,\ u_i) = b_i,\quad i = 1,\ 2,\quad \varphi(0,\ u) = \varphi_0(u) \end{cases} \tag{6}$$

in the domain $t \geqslant 0$, $u_1 \leqslant u \leqslant u_2$, where b_i, i = 1, 2 are negative constants and $\varphi_0(u)$ is a sufficiently smooth function negative on the interval $[u_1,\ u_2]$. We show that if a trajectory of system (3) exists which passes through the points $(u_2,\ b_2)$ and $(u_1,\ b_1)$, then $\tilde{\varphi}(t,\ u)$ is a solution of problem (6) which converges uniformly with respect to u to the function $\tau_c(u)$, which makes abscissas of points of this trajectory correspond to their ordinates (Sec. 2).

We consider a function $\tau_c(u;\ v_1,\ v_2)$ such that $\tau_c(u;\ 0,\ 1) < \tau_c(u;\ v_1,\ v_2) < \tau_c(u;\ 0,\ 1) + \varepsilon/2$, $0 < v_1 \leqslant u \leqslant v_2 < 1$. It is easy to show existence of the corresponding trajectory. If $b_i = \tau_c(u_i,\ v_1,\ v_2)$, $i = 1,\ 2$, then the solution $\tilde{\varphi}(t,\ u)$ of problem (6) tends towards the function $\tau_c(u;\ v_1,\ v_2)$. If, in addition, we can choose the function $\varphi_0(u)$ such that for $t \geqslant 0$

$$\varphi(t,\ u;\ f) < \tilde{\varphi}(t,\ u),\quad u_1 \leqslant u \leqslant u_2, \tag{7}$$

and the quantities $|u_i - v_i|$ are small, inequality (5) will have been proved.

Inequality (7) can be obtained from a theorem on positiveness of a solution [9] for the difference $\tilde{\varphi}(t,\ u) - \varphi(t,\ u;\ f)$, since the function $\varphi(t, u; f)$ is a solution of the boundary problem [4, 6]

$$\begin{cases} \dot{\varphi} = \varphi^2\varphi'' - F\varphi' + F'\varphi, \\ \varphi(t,\ u_i) = \varphi(t,\ u_i;\ f),\quad \varphi(0,\ u) = \varphi(0,\ u;\ f). \end{cases}$$

In this regard, we require satisfaction of the inequalities

$$\varphi(0,\ u;\ f) < \varphi_0(u),\quad u_1 \leqslant u \leqslant u_2, \tag{8}$$

$$\varphi(t,\ u_i;\ f) < b_i = \tau_c(u_i;\ v_1,\ v_2),\quad i = 1,\ 2. \tag{9}$$

Inequality (8) is easily guaranteed with a choice of the function $\varphi_0(u)$. Justification of inequality (9) is substantially more involved. We clarify this point with the simple case $F(u) \in g^-(0)$, $F(u) \in l^+(1)$. Here we can assume that $F(v_1) < 0$, $F(v_2) > 0$.

Let v_{21}, v_{22} be such that $v_{21} < v_2 < v_{22}$, $F(v_{21}) = 0$, $F(u) > 0$ for $v_{21} < u \leqslant v_{22}$. For values of c_2 sufficiently large, there exists a function $\tau_{c_2}(u;\ v_{21},\ v_{22})$, for which $\varphi(0,\ u;\ f) < \tau_{c_2}(u;\ v_{21},\ v_{22})$, $v_{21} \leqslant u \leqslant v_{22}$. By virtue of comparison theorems on the phase plane [7] for $t \geqslant 0$

$$\varphi(t,\ u;\ f) < \tau_{c_2}(u;\ v_{21},\ v_{22}),\quad v_{21} \leqslant u \leqslant v_{22}.$$

Therefore, for values of u_2 sufficiently close to v_2 inequality (9) is satisfied (Fig. 1).

Analogous reasoning may be carried out for the point u_1.

We remark that in these discussions it was assumed that f(x) is a smooth function and $f'(x) < 0$. If f(x) is piecewise-continuous and not strictly monotonic, we can then take as initial condition the function $u(x,\ t_0)$ for arbitrary $t_0 > 0$. Here, and in what follows, we use the so-called theorems of comparison on the phase plane. We now present one of such theorems [7].

Comparison Theorem. Let $x_0(t)$ be a function continuous for $t \geqslant 0$ or $x_0(t) \equiv -\infty$; let $f_i(x)$, i = 1, 2 be monotonic smooth functions, where $f_1(x)$ is defined for $-\infty < x < +\infty$, and $f_2(x)$ is defined for $x_0(t) \leqslant x < +\infty$, $u_0(t) = u(x_0(t),\ t;\ f_2)$, $J_1(t) = [u(+\infty,\ t;\ f_1),\ u(-\infty,\ t;\ f_1)]$, $J_2(t) = [u(+\infty,\ t;\ f_2),\ u_0(t)]$.

Let the following conditions be satisfied:

1) $\varphi(0,\ u;\ f_1) < \varphi(0,\ u;\ f_2)$, $u \in J_1(0) \cap J_2(0)$,

2) if $u_0(t) \in J_1(t)$, then $\varphi(t,\ u_0(t);\ f_1) < \varphi(t,\ u_0(t);\ f_2)$. Then for all $u \in J_1(t) \cap J_2(t)$ we have $\varphi(t,\ u;\ f_1) \leqslant \varphi(t,\ u;\ f_2)$.

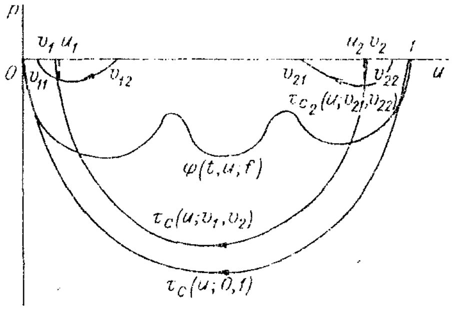

Fig. 1

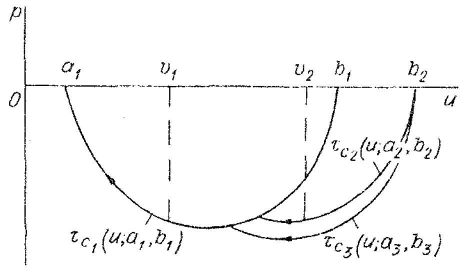

Fig. 2

2. Behavior of Waves as $x \to \infty$

In this section we consider solutions w(x) of the equation $w'' + cw' + F(w) = 0$ in the class of functions which, along with their second derivatives, are continuous. We assume that $w(x) > 0$ for $x \geqslant N$ for some N, and that $w(x) \to 0$ as $x \to +\infty$. By a change of variables, all the results of this section can be carried over to the case $w(x) \to 1$ for $x \to -\infty$.

The propositions presented below are, for the most part, uncomplicated and the proofs for some of them will be omitted. For completeness we also present some well-known results [1-8].

LEMMA 1. There exists a value of x_0 such that $w'(x) < 0$ for $x \geqslant x_0$ and $w'(x) \to 0$ as $x \to +\infty$.

It is easy to deduce from Eq. (2) that

$$w''/w + c(w'/w) \to -F'(0) = -\alpha \tag{10}$$

as $x \to +\infty$. Differentiating Eq. (2) with respect to x, we obtain

$$w''' + cw'' + F'(w)w' = 0$$

[the derivative w''' is defined and continuous by virtue of the continuity of $F'(w)$], from whence, as $x \to +\infty$, we have

$$w'''/w' + c(w''/w') \to -\alpha. \tag{11}$$

We introduce the functions $\varphi_1(x) = w'(x)/w(x)$, $\varphi_2(x) = w''(x)/w'(x)$. Functions $\varphi_i(x)$, $i = 1, 2$ are continuous and differentiable for $x \geqslant x_1 = \max(x_0, N)$;

$$\varphi_1'(x) = \frac{w''w - (w')^2}{w^2} = \frac{w''}{w} - \varphi_1^2, \quad \varphi_2'(x) = \frac{w'''w' - (w'')^2}{(w')^2} = \frac{w'''}{w'} - \varphi_2^2.$$

From this and from relations (10) and (11), as $x \to +\infty$, we arrive at the relationship

$$\varphi_i'(x) + c\varphi_i(x) + \varphi_i^2(x) \to -\alpha, \quad i = 1,2. \tag{12}$$

LEMMA 2. The functions $|\varphi_i(x)|$ are bounded for $x \geqslant x_0$.

THEOREM 1. As $x \to +\infty$ $\varphi_i(x) \to \lambda$, where λ is a solution of the equation

$$\lambda^2 + c\lambda + \alpha = 0. \tag{13}$$

Proof. If the limit $\lim\limits_{x \to +\infty} \varphi_i(x)$ does not exist, it is then possible to select sequences $\{x_n\}$, $\{y_n\}$, for which $x_n, y_n \to +\infty$, $\varphi_i(x_n) \to \lambda_1$, $\varphi_i(y_n) \to \lambda_2$ as $n \to \infty$, $\lambda_1 < \lambda_2$. Let λ_0 satisfy the inequality $\lambda_1 < \lambda_0 < \lambda_2$ and let it not be a solution of Eq. (13). If, for example, $\lambda_0^2 + c\lambda_0 + \alpha > 0$, then, selecting a sequence $\{z_n\}$ such that $z_n \to +\infty$, $\varphi_i(z_n) = \lambda_0$, $\varphi_i'(z_n) \geqslant 0$, we obtain

$$\varphi_i'(z_n) + c\varphi_i(z_n) + \varphi_i^2(z_n) \geqslant c\lambda_0 + \lambda_0^2 > -\alpha,$$

which contradicts relation (12).

Let us put $\lambda = \lim\limits_{x \to +\infty} \varphi_i(x)$. If λ does not satisfy Eq. (13), it then follows from relation (12) that the function $\varphi_i(x)$ is unbounded. This completes the proof of the theorem.

COROLLARY 1. One of the following two relations is satisfied as $x \to +\infty$:

$$w'/w \to -c/2 + \sqrt{c^2/4 - \alpha}, \quad w''/w' \to -c/2 + \sqrt{c^2/4 - \alpha}, \tag{14}$$

$$w'/w \to -c/2 - \sqrt{c^2/4 - \alpha}, \quad w''/w' \to -c/2 - \sqrt{c^2/4 - \alpha}, \tag{15}$$

where $\alpha = F'(0)$, $c^2/4 - \alpha \geqslant 0$.

COROLLARY 2. If $F(w) \in g^0(0)$, then as $x \to +\infty$

$$w'/w \to -c, \quad w''/w' \to -c.$$

Proof. There exists a sequence $\{x_n\}$ such that $x_n \to +\infty$ and $F(w(x_n)) = 0$. We have $w''(x_n)/w'(x_n) = -c - F(w(x_n))/w'(x_n) = -c$. Since the limit of $w''(x)/w'(x)$ as $x \to +\infty$ exists, it can then only be equal to $-c$. This completes proof of the corollary.

THEOREM 2. Let one of the following conditions be satisfied:

1) $F(w) \in g^-(0)$,

2) if $F(w) \in g^+(0)$ or $F(w) \in g^0(0)$, then $c > 2\sqrt{\alpha}$.

Then there exists exactly one solution w(x) satisfying relation (15).

THEOREM 3. Assume that the wave $w_c(x;\ 0,\ 1)$ exists. Then for $c \neq 0$

1) if $F(w) \in g^+(0)$ $(l^-(1))$, then $F(w) \in l^+(1)$ $(g^-(0))$ and the wave $w_c(x;\ 0,\ 1)$ exists for a half-interval (semiaxis) of velocities $[c_*, c^*)$, $c_* > 0$ $((c^*, c_*],\ c_* < 0)$,

2) if $F(w) \in g^0(0)$ $(l^0(1))$, then $F(w) \in l^+(1)$ $(g^-(0))$ and the wave $w_c(x;\ 0,\ 1)$ is unique, $c > 0$ $(c < 0)$;

3) if $F(w) \in g^-(0)$ and $F(w) \in l^+(1)$, the wave $w_c(x;\ 0,\ 1)$ is then unique.

When $c = 0$ a necessary and sufficient condition for existence of the wave $w_c(x;\ 0,\ 1)$ is the following:

$$\int_0^1 F(u)\,du = 0, \quad \int_w^1 F(u)\,du > 0 \ \text{ for } \ 0 < w < 1.$$

The wave $w_c(x;\ 0,\ 1)$ is unique.

3. Phase Plane Boundary Problems

We consider the boundary problem

$$\begin{gathered} \dot{\varphi} = \varphi^2\varphi'' - F\varphi' + F'\varphi, \\ \varphi(t,\ u_i) = b_i, \quad i = 1,\ 2, \quad \varphi(0,\ u) = \varphi_0(u) \end{gathered} \tag{16}$$

in the domain $u_1 \leqslant u \leqslant u_2$, $t \geqslant 0$. Here b_i are negative constants, $0 < u_1,\ u_2 < 1$, and $\varphi_0(u)$ is a function negative on the interval $[u_1,\ u_2]$. We denote the solution of problem (16) by $\tilde{\varphi}(t,\ u)$.

Existence of a solution of problem (16) requires justification since the coefficient of the second derivative may be degenerate.

THEOREM 4. Let $\varphi_0(u) \in C^3[u_1,\ u_2]$, and assume that compatibility conditions of zero and first orders are satisfied, i.e.,

$$\varphi_0(u_i) = b_i, \quad \varphi^2\varphi'' - F\varphi' + F'\varphi\,|_{u=v_i} = 0,\ i = 1,\ 2.$$

Then a unique solution of problem (16) exists which has for $u_1 \leqslant u \leqslant u_2$, $t \geqslant 0$ continuous first derivatives with respect to t and second derivatives with respect to u.

Proof of this theorem is based on the use of á priori estimates of the solution $\tilde{\varphi}(t,\ u)$ and on known results concerning existence of a solution in the nondegenerate case [9]. We shall not supply the proof here.

Suppose that for some c there exists an arc of a trajectory of system (3), lying in the halfplane $p \leqslant 0$ and joining the points $(u_2,\ b_2)$ and $(u_1,\ b_1)$. We denote by $\tau_c(u)$ a function which puts abscissas of points of this arc into correspondence with their ordinates.

THEOREM 5. Under the conditions of Theorem 4, as $t \to \infty$, uniformly with respect to $u \in [u_1,\ u_2]$, we have $\tilde{\varphi}(t,\ u) \to \tau_c(u)$.

Proof. By virtue of the nondegeneracy of the solution of problem (16), we can determine the function $x_0(t) = \int_{u_2}^{u_1} \frac{du}{\tilde{\varphi}(t,\ u)}$. We put $s(t) = \tilde{\varphi}'(t,\ u_2)\tilde{\varphi}(t,\ u_2)$, $a(t) = -(s(t) + F(u_2))/b_2$ The functions

$s(t)$, $a(t)$ are continuous. In the domain $0 \leqslant x \leqslant x_0(t)$, $t \geqslant 0$, we consider the boundary problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a(t)\frac{\partial u}{\partial x} + F(u), \qquad (17)$$

$$u(0, t) = u_2,\ u(x_0(t), t) = u_1,\ u(x, 0) = f(x),$$

where $f(x)$ is a solution of the equation $f' = \varphi_0(f)$, $f(0) = u_2$. We show that the solution $u(x, t)$ of problem (17) satisfies the equation

$$x = \int_{u_2}^{u(x,t)} \frac{d\tau}{\tilde{\varphi}(t, \tau)}. \qquad (18)$$

It is easy to verify that the function $u(x, t)$, defined in this way, satisfies the initial and boundary conditions. We verify satisfaction of the first equation in Eqs. (17). From Eq. (18)

$$u'(x, t) = \tilde{\varphi}(t, u),\ \dot{u}' = \dot{\tilde{\varphi}}(t, u) + \tilde{\varphi}'(t, u)\dot{u},$$

$$u''(x, t) = \tilde{\varphi}'(t, u)u'(x, t),\ \tilde{\varphi}'' = (u'''u' - (u'')^2)/(u')^3.$$

Substituting into Eq. (16), we obtain

$$\dot{u}' - (u''/u')\dot{u} = u''' - (u'')^2/u' - (u''/u')F + u'F'. \qquad (19)$$

Let us put

$$g(x, t) = \dot{u} - u'' - F(u). \qquad (20)$$

Then, by virtue of equation (19), $g'(x, t) - (u''/u')g(x, t) = 0$, whence $g(x, t) = k(t)u'(x, t)$, where $k(t)$ is some function. We have $g(0, t) = k(t)u'(0, t) = k(t)b_2$. Since $\dot{u}(0, t) = 0$, then from Eq. (20)

$$g(0, t) = -u''(0, t) - F(u(0, t)) = -\tilde{\varphi}'(t, u_2)\tilde{\varphi}(t, u_2) - F(u_2) = -s(t) - F(u_2).$$

Thus,

$$k(t) = -\frac{s(t) + F(u_2)}{b_2} = a(t),\ g(x, t) = a(t)u'(x, t)$$

and from Eq. (20) we have $\dot{u} = u'' + a(t)u' + F(u)$. The solution $u(x, t)$ of problem (17) is unique [9].

We show now that for arbitrary $\varepsilon > 0$ we can find a t_0 such that $\tau_c(u) - \varepsilon < \tilde{\varphi}(t, u) < \tau_c(u) + \varepsilon$, $u_1 \leqslant u \leqslant u_2$, for $t \geqslant t_0$. In this paper we prove only one of these inequalities. The other is proved in a similar way.

For values c_i, $i = 1, 2$, $c_1 \neq c_2$, $c_i \neq c$ close to c, we can select trajectories of system (3) so that the functions $\tau_{c_i}(u)$, $i = 1, 2$ are defined on the interval $[u_1, u_2]$ and $\tau_c(u) < \tau_{c_i}(u) < \tau_c(u) + \varepsilon$, $u_1 \leqslant u \leqslant u_2$. We show that $\tilde{\varphi}(t, u) \leqslant \max(\tau_{c_1}(u), \tau_{c_2}(u))$, $u_1 \leqslant u \leqslant u_2$ for $t \geqslant t_0$. To do this we consider functions $q_{c_i}(x)$, $i = 1, 2$, which are solutions of the equations

$$q'_{c_i} = \tau_{c_i}(q_{c_i}),\ q_{c_i}(x_i) = u_2,\ q_{c_i}(x_i + h_i) = u_1,$$

where x_i and h_i are constants, and the boundary problem

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + a(t)\frac{\partial v}{\partial x} + F(v),\ v(x, 0) = q_{c_i}(x),$$

$$v(x_i - b(t) + c_i t, t) = u_2,\quad v(x_i + h_i - b(t) + c_i t, t) = u_1,$$

where $b(t) = \int_0^t a(\tau)\,d\tau$, in the domain $t \geqslant 0$, $x_i - b(t) + c_i t \leqslant x \leqslant x_i + h_i - b(t) + c_i t$. As a solution of this boundary problem we have the function $v_i(x, t) = q_{c_i}(x + b(t) - c_i t)$.

Since $v_i'(x_i - b(t) + c_i t, t) = \tau_{c_i}(u_2) > \tau_c(u_2) = u'(0, t)$, $v_i'(x_i + h_i - b(t) + c_i t, t) = \tau_{c_i}(u) > \tau_c(u_1) = u'(x_0(t), t)$, then, for arbitrary values of x_i and h_i satisfying one of the following conditions:

$$x_i \geqslant x_0(0),\quad x_i + h_i \leqslant 0, \qquad (21)$$

Lemma 1 [7, Sec. 2] is applicable to the difference $v_i - u$, by virtue of which we can state the following: if the equation $u(x, t) = v_i(x, t)$ has for some t a solution $x = x^*$, then where both functions u and v_i are defined we have $u(x, t) \gtrless v_i(x, t)$ for $x \lesseqgtr x^*$, whence $u'(x^*, t) \leqslant v_i'(x^*, t)$.

It remains to show that for $t \geqslant t_0$, for arbitrary x^*, $0 \leqslant x^* \leqslant x_0(t)$, values of i and x_i can be found satisfying one of the conditions (21) and such that $x = x^*$ is a solution of the equation $u(x, t) = v_i(x, t)$ (in this case we say that <u>x^* possesses property 1</u>).

We define functions $x_{i1}(t) = x_0(0) + h_i - b(t) + c_i t$, $x_{i2}(t) = -h_i - b(t) + c_i t$. Then an arbitrary value x^*, $0 \leqslant x^* \leqslant x_0(t)$, satisfying one of the four conditions

$$x^* \geqslant x_{i1}(t),\ x^* \leqslant x_{i2}(t),\ i = 1,\ 2, \tag{22}$$

possesses property 1. Assuming for definiteness that $c_1 > c_2$, we find, for arbitrary $t \geqslant t_0 = \frac{h_1 + h_2 + x_0(0)}{c_1 - c_2}$, that $x_{12}(t) \geqslant x_{21}(t)$, whence $x_{11}(t) > x_{12}(t) \geqslant x_{21}(t) > x_{22}(t)$. Consequently, for arbitrary x^* one of the conditions (22) is satisfied. This completes the proof of the theorem.

4. Emergence onto the Wave $w_c(x;\ 0,\ 1)$

<u>THEOREM 6.</u> Let $F(u) \in g^+(0)$; assume that the wave $w_c(x;\ 0,\ 1)$ exists, and that the inequality

$$\overline{\lim_{x \to +\infty}} \frac{f'(x)}{f(x)} < -\left(\frac{c_*}{2} - \sqrt{\frac{c_*^2}{4} - \alpha}\right), \tag{23}$$

is satisfied, where c_* is the minimum velocity for which the wave $w_c(x;\ 0,\ 1)$ exists; and let $f(x)$ be a monotonic, piecewise-smooth function, $f(+\infty) = 0$, $f(-\infty) \in I(1)$. Then as $t \to \infty$, uniformly with respect to $x \in (-\infty, +\infty)$, $u\left(x + m\left(t, \frac{1}{2}\right), t\right) \to w_{c_*}(x; 0, 1)$, where $m(t,\ 1/2)$ is a solution of the equation $u(x,\ t) = 1/2$. Moreover, $m(t,\ 1/2) \to c_*$.

<u>Proof.</u> As was observed in Sec. 1, for arbitrary $t_0 > 0$, $u(x,\ t_0)$ is a smooth function, $u'(x,\ t_0) < 0$. Moreover, inequality (23) is satisfied for the function $u(x,\ t_0)$ (for $\alpha > 0$, - at least nonstrictly, which does not affect the proof). Therefore, with no loss of generality, we can assume that $f(x)$ is a smooth function, $f'(x) < 0$.

The proof is based on an application of Theorem 5 and proceeds, on the whole, as shown in Fig. 1. More specifically: we consider the trajectory $\tau_{c_0}(u; v_1, v_2)$, $c_0 > c_*$. Quantities $c_0 - c_*$, $1 - v_2$ are sufficiently small, $v_1 = 0$. Choosing a function $\varphi_0(u)$, satisfying inequality (8) and the conditions of Theorem 4 offers no complications. We need to select values u_1 and u_2 so that inequality (9) will hold. For this we construct functions $\tau_{c_i}(u; v_{i1}, v_{i2})$, $i = 1, 2$.

The function $\tau_{c_2}(u; v_{21}, v_{22})$ is constructed the same as in Sec. 1. There are some differences here for the positive source. In this case $v_{21} = 0$ and it is necessary to use condition (23).

In constructing the function $\tau_{c_1}(u; 0, v_{12})$ we need to distinguish two cases:

1) $\alpha > 0$; in addition, $c_0 > c_1 > c_*$ and $\overline{\lim_{u \to 0}} \frac{\varphi(0, u; f)}{u} < -\left(c_*/2 - \sqrt{c_*^2/4 - \alpha}\right) < -\left(c_1/2 - \sqrt{c_1^2/4 - \alpha}\right) = \tau_{c_1}'(0; 0, v_{12}) < \tau_{c_0}'(0; 0, v_2) = -\left(c_0/2 - \sqrt{c_0^2/4 - \alpha}\right)$, whence for v_{12} sufficiently small,

$$\varphi(0, u; f) < \tau_{c_1}(u; 0, v_{12}) \text{ for } 0 < u < v_{12}, \tag{24}$$

$$\tau_{c_1}(u; 0, v_{12}) < \tau_{c_0}(u; 0, v_2) \text{ in a neighborhood of } u = 0, \tag{25}$$

consequently, for $t \geqslant 0$

$$\varphi(t, u; f) \leqslant \tau_{c_1}(u; 0, v_{12}) \text{ for } 0 < u \leqslant v_{12}, \tag{26}$$

and for u_1 close to zero,

$$\varphi(t, u_1; f) \leqslant \tau_{c_1}(u_1; 0, v_{12}) < \tau_{c_0}(u_1; 0, v_2), \tag{27}$$

2) $\alpha = 0$, the function $\tau_{c_1}(u; 0, v_{12})$ corresponds to the minimal trajectory falling into the singular point $(0, 0)$, $\tau_{c_1}'(0; 0, v_{12}) = -c_1$, $\tau_{c_0}'(0; 0, v_{12}) = 0$. When $c_1 \to 0$ $v_{12} \to 0$, a value of c_1 sufficiently small can be found for which inequalities (24)-(27) are satisfied.

Emergence of a solution into a wave in form [5, 6] implies its emergence with respect to velocity. Actually, differentiating equation $u(m(t,\ 1/2),\ t) = 1/2$, we obtain $u' \cdot \dot{m} + \dot{u} = 0$, whence

$$\dot{m}\left(t, \frac{1}{2}\right) = -\frac{u''(m(t, 1/2), t) + F(u(m(t, 1/2), t))}{u'(m(t, 1/2), t)}.$$

From the uniform boundedness of the derivatives $u^{(i)}$, i = 1, 2, 3 and the convergence $u(x + m(t, 1/2), t) \to w_{c_*}(x; 0, 1)$ it follows that $u^{(i)}(m(t, 1/2), t) \to w_{c_*}^{(i)}(0; 0, 1)$, whence as $t \to \infty$

$$\dot{m}\left(t, \frac{1}{2}\right) \to -\frac{w''_{c_*}(0; 0, 1) + F(w_{c_*}(0; 0, 1))}{w'_{c_*}(0; 0, 1)} = c_*.$$

This completes the proof of the theorem.

Remark. By a change of variables the theorem can be extended to the case $F(u) \in l^-(1)$. Inequality (23) must be replaced by the following:

$$\lim_{x \to -\infty} \frac{-f'(x)}{1 - f(x)} > -\left(\frac{c_*}{2} + \sqrt{\frac{c_*^2}{4} - \beta}\right), \quad c_* < 0, \quad \beta = F'(1).$$

The theorem we present below is fundamental to this section. In proving it we employ a lemma whose validity follows from known results on estimating the minimal velocity for a positive source (see, for example, [6]).

LEMMA 3. If $F(a) = F(b) = 0$, $F(u) > 0$ for $a < u < b$, then $2\sqrt{F'(a)} \leqslant c(a, b) \leqslant 2\sqrt{\max_{a \leqslant u \leqslant b} F'(u)}$, where $c(a, b)$ is the $w_c(x; a, b)$.

THEOREM 7. If the wave $w_c(x; 0, 1)$ exists and is unique, then for an arbitrary monotonic piecewise-continuous function $f(x)$, $f(+\infty) \in I(0)$, $f(-\infty) \in I(1)$, as $t \to \infty$, uniformly with respect to $x \in (-\infty, +\infty)$,

$$u(x + m(t, 1/2), t) \to w_c(x; 0, 1).$$

Moreover, $\dot{m}(t, 1/2) \to c$.

Proof. We consider two cases: 1) c > 0, 2) c = 0. Case 3) c < 0 is obtained from the first case by a change of variables. The proof is carried out as in the example of Sec. 1 and is reduced to the construction of the functions $\tau_c(u; v_1, v_2)$ and $\tau_{c_i}(u; v_{i1}, v_{i2})$.

1. We distinguish two possibilities here: $F(u) \in g^-(0)$ and $F(u) \in g^0(0)$. The first was considered in Sec. 1. We consider the second.

1.1. In an arbitrary neighborhood of point u = 0 we can find points of positiveness of the function F(u).

We can assume that $v_2 \in I(1)$ and that v_1 is so close to zero that $\max_{0 \leqslant u \leqslant v_1} |F'(u)| < c^2/4$. Then for arbitrary a, b, $0 < a < b < v_1$, satisfying the conditions of Lemma 3, $c(a, b) < c$. We denote by $\tau_c(u; a, v_3)$ the function corresponding to the trajectory falling into the singular point $(a, 0)$ along the minimal direction.

We show that $v_3 > v_2$. Actually, if this is not so, then $v_3 \leqslant v_1$, $F(v_3) \geqslant 0$, $F(u) \in l^+(v_3)$. Let (a_1, b_1) be the largest interval of positiveness of function F(u) such that $v_3 \in [a_1, b_1]$. It then follows from the condition $c(a_1, b_1) < c$ that the trajectory leaving the point $(v_3, 0)$ (by virtue of the uniqueness of such a trajectory, to it there corresponds the function $\tau_c(u; a, v_3)$), falls into the singular point $(a, 0)$ along a non-minimal direction, which contradicts the definition of $\tau_c(u; a, v_3)$.

Thus, $\tau_c(u; a, v_3) < \tau_c(u; v_1, v_2)$ for $u \in [v_1, v_2]$. As in the previous theorem, $\varphi(t, u; f) \leqslant \tau_c(u; a, v_3) + \varepsilon/2 < \tau_c(u; v_1, v_2) + \varepsilon/2 < \tau_c(u; 0, 1) + \varepsilon$ for $t \geqslant t_0(\varepsilon)$.

1.2. In some neighborhood of the point u = 0, $F(u) \leqslant 0$, and in every neighborhood there is a point of negativeness of function F(u).

We can assume that $v_2 \in I(1)$ and that v_1 belongs to a neighborhood of the point u = 0 in which function F(u) is non-positive. Let a be such that $0 < a < v_1$, $F(a) < 0$. Consider now the function $\tau_c(u; a, v_3)$. It is easy to show that $v_3 > v_2$. Indeed, if this were not so, then $v_3 \leqslant v_1$ and $F(u) \in l^+(v_3)$, i.e., for $u \leqslant v_1$ points of positiveness of function F(u) can be found. Thus, for $u \in [v_1, v_2]$ $\tau_c(u; a, v_3) < \tau_c(u; v_1, v_2)$, and all the reasoning used in Sec. 1 is applicable to the function $\tau_c(u; a, v_3)$.

1.3. On some interval $[0, v_0]$, $v_0 > 0$, we have $F(u) \equiv 0$ and $F(u) \not\equiv 0$ on an arbitrary large interval.

The proof is rather involved, and we limit ourselves to a simple special case in which $F(u) > 0$ in a right half-neighborhood of point v_0. This item differs from the preceding one only in the construction of the function $\tau_{c_1}(u; v_{11}, v_{12})$. We can assume that $v_1 \in (0, v_0)$, and for an arbitrary v_{11}, $0 < v_{11} < v_1$, $\tau_{c_1}(u; v_{11}, v_{12}) \to 0$ as $c \to 0$, uniformly with respect to u, whereby $v_{12} \to v_0$. It follows from this that for c_1 sufficiently small, $\varphi(0, u; f) < \tau_{c_1}(u; v_{11}, v_{12})$, $v_{11} \leqslant u \leqslant v_{12}$, and for arbitrary u_1 close to v_1, $u_1 > v_1$,

$$\varphi(t, u_1; f) < \tau_c(u_1; v_1, v_2), \quad t \geqslant 0.$$

2. If $F(v_1) < 0$, $F(v_2) > 0$, the functions $\tau_{c_i}(u; v_{i1}, v_{i2})$ are then constructed as in Sec. 1.

We show that the general case can be reduced to that indicated. For this it is sufficient to construct a function $\tau_c(u; a, b)$ such that $\tau_c(u; a, b) < \tau_c(u; v_1, v_2)$ for $u \in [v_1, v_2]$. Here the values of a and b must satisfy the conditions $0 < a < v_1 < v_2 < b < 1$, $F(a) < 0$, $F(b) > 0$. If we assume that for an arbitrary function $\tau_c(u; a_1, b_1)$ such that $v_2 < b_1 < 1$, $F(b_1) > 0$, that the inequality $a_1 > v_2$, holds, then by virtue of the equation $\int_{a_1}^{b_1} F(u)\,du = 0$ we will have $\int_{v_2}^{1} F(u)\,du \leqslant 0$, which contradicts Theorem 3. Thus we can assume that $a_1 \leqslant v_1$ and $F(a_1) \leqslant 0$. If $F(a_1) = 0$, we use the inequality $\int_{a_1}^{v} F(u)\,du < 0$ $\forall v \in (a_1, b_1)$, from which it follows that in an arbitrary right half-neighborhood of point a_1 there is a point of negativeness of function F(u). Choosing a sufficiently close to a_1, $F(a) < 0$, and putting $b = b_1$, we obtain the required function $\tau_c(u; a, b)$.

Convergence $\dot{m}(t, 1/2) \to c$ as $t \to \infty$ may be proved as was done in Theorem 6. This completes the proof of Theorem 7.

5. Emergence into a Minimal System of Waves

Definition. By a system of waves we mean a function R(u), given on the interval [0, 1] and such that the condition $R(u) \leq 0$ is satisfied for $0 \leqslant u \leqslant 1$, $R(0) = R(1) = 0$. If $R(u) < 0$ for $u_1 < u < u_2$ and $R(u_1) = R(u_2) = 0$, then for some c there exists a function $\tau_c \times (u; u_1, u_2)$ for which $R(u) \equiv \tau_c(u; u_1, u_2)$, $u_1 \leqslant u \leqslant u_2$. In this case we say that the function $\tau_c(u; u_1, u_2)$ emerges into the system of waves R(u).

A system of waves $R_0(u)$ is said to be minimal if for an arbitrary second system of waves R(u) we have $R(u) \geqslant R_0(u)$, $0 \leqslant u \leqslant 1$.

THEOREM 8. For an arbitrary source F(u) there exists a minimal system of waves $R_0(u)$.

Proof. We put

$$R_0(u_0) = \inf\{p \mid \text{for some c there exists a function } \tau_c(u; a, b),\ 0 \leqslant a \leqslant b \leqslant 1, \text{ such that } p = \tau_c(u_0; a, b)\}.$$

It follows from the theorem concerning continuous dependence of the solution on the parameter (c) and on the initial condition that for an arbitrary fixed value of u_0 there exists a function $\tau_{c_0}(u; a_0, b_0)$ such that $R_0(u_0) = \tau_{c_0}(u_0; a_0, b_0)$, $0 \leqslant a_0 \leqslant b_0 \leqslant 1$.

Based on an analysis of the direction field for system (3) we can show that the function $R_0(u)$ is bounded.

We show that $R_0(u)$ is a system of waves, i.e., if $R_0(u) < 0$ for $u_1 < u < u_2$, $R_0(u_1) = R_0(u_2) = 0$, then there exists a function $\tau_c(u; u_1, u_2)$ for which $R_0(u) \equiv \tau_c(u; u_1, u_2)$, $u_1 \leqslant u \leqslant u_2$. To do this, we consider an arbitrary point $v_1 \in (u_1, u_2)$ and a function $\tau_{c_1}(u; a_1, b_1)$, $0 \leqslant a_1 \leqslant b_1 \leqslant 1$, $R_0(v_1) = \tau_{c_1}(v_1; a_1, b_1)$. If $R_0(u) \not\equiv \tau_{c_1}(u; a_1, b_1)$ for $a_1 \leqslant u \leqslant b_1$, we can then find $v_2 \in (a_1, b_1)$ for which $R_0(v_2) < \tau_{c_1}(v_2; a_1, b_1)$. For definiteness we assume that $v_2 > v_1$. We consider the function $\tau_{c_2}(u; a_2, b_2)$, $0 \leqslant a_2 \leqslant b_2 \leqslant 1$, $R_0(v_2) = \tau_{c_2}(v_2; a_2, b_2)$. Since $\tau_{c_1}(v_1; a_1, b_1) \leqslant \tau_{c_2}(v_1; a_2, b_2)$, $\tau_{c_1}(v_2; a_1, b_1) > \tau_{c_2}(v_2; a_2, b_2)$, the equation $\tau_{c_1}(u; a_1, b_1) = \tau_{c_2}(u; a_2, b_2)$ then has for $u \in [v_1, v_2]$ a solution, i.e., the corresponding trajectories intersect (Fig. 2). If $c_3 < c_2$, $c_2 - c_3$ is small, there then exists a function $\tau_{c_3}(u; a_3, b_3)$, $a \leqslant a_3 \leqslant b_3 = b_2$, such that $\tau_{c_3}(v_2; a_3, b_3) < \tau_{c_2}(v_2; a_2, b_2) = R_0(v_2)$, which contradics the definition of $R_0(u)$. Thus, $R_0(u) \equiv \tau_{c_1}(u; a_1, b_1)$, $a_1 \leqslant u \leqslant b_1$; consequently, $a_1 = u_1$, $b_1 = u_2$.

Minimality of the system of waves $R_0(u)$ follows from its definition. This completes the proof of the theorem.

COROLLARY. If the function $\tau_c(u; u_1, u_2)$ emerges into a minimal wave system $R_0(u)$, it follows that $F(u_i) = 0$, $i = 1, 2$.

Proof. Suppose, for example, that $F(u_1) > 0$. We denote by (a, b) an interval of positiveness of the function $F(u)$ containing point u_1. Then for some c there exists a function $\tau_c(u; a, b)$ and the system $R_0(u)$ is not minimal since $\tau_c(u_1; a, b) < 0$. The case $F(u_1) < 0$ is handled similarly. This completes the proof of the corollary.

We present examples of minimal systems of waves.

1. If the function $\tau_c(u; 0, 1)$ exists and is unique, then $R_0(u) \equiv \tau_c(u; 0, 1)$, $0 \leqslant u \leqslant 1$; if the function $\tau_c(u; 0, 1)$ exists for the half-interval of velocities $[c_*, c^*)$, $c_* > 0$, then $R_0(u) \equiv \tau_{c_*}(u; 0, 1)$, $0 \leqslant u \leqslant 1$.
2. If a minimal decomposition (see Sec. 1) exists, then $R_0(u) \equiv \tau_{c_k}(u; a_k, b_k)$, $a_k \leqslant u \leqslant b_k$.
3. Let $F(u) > 0$ for $0 < u < a$; $F(u) = 0$, $a \leqslant u \leqslant b$; $F(u) < 0$, $b < u < 1$. Then $R_0(u) \equiv \tau_{c_1}(u; 0, a)$ for $0 \leqslant u \leqslant a$; $R_0(u) \equiv 0$, $a < u < b$; $R_0(u) \equiv \tau_{c_2}(u; b, 1)$, $b \leqslant u \leqslant 1$. Here c_i are velocities minimal in absolute value for which the trajectories τ_{c_i} exist.

There exist minimal systems of waves formed by an infinite set of trajectories.

The following theorems are devoted to the emergence of solutions into minimal systems of waves. We remark that an arbitrary function $F(u)$ satisfies one of the conditions of these theorems, i.e., they contain sources of arbitrary form.

THEOREM 9. Let $F(u) \notin g^+(0)$, $F(u) \notin l^-(1)$. Then for an arbitrary monotonic function $f(x)$, $f(+\infty) \in I(0)$, $f(-\infty) \in I(1)$, as $t \to \infty$, uniformly with respect to $u \in [u_+(t), u_-(t)]$ $(u_\pm(t) = u(\pm\infty, t))$,

$$\varphi(t, u; f) \to R_0(u).$$

If $R_0(u) \equiv \tau_c(u; a, b)$ for $a \leqslant u \leqslant b$, then as $t \to \infty$, uniformly with respect to x on an arbitrary finite interval,

$$u(x + m(t, (a+b)/2), t) \to w_c(x; a, b).$$

Moreover, $\dot{m}(t, (a+b)/2) \to c$.

THEOREM 10. Let $f(x)$ be a monotonic piecewise-smooth function, $f(+\infty) \in I(0)$, $f(-\infty) \in I(1)$, and let one of the following conditions be satisfied:

1) $F(u) \in g^+(0)$, $F(u) \notin l^-(1)$,

$$\overline{\lim_{x \to +\infty}} \frac{f'(x)}{f(x)} < \frac{\alpha}{R_0'(0)}, \tag{28}$$

2) $F(u) \notin g^+(0)$, $F(u) \in l^-(1)$,

$$\lim_{x \to -\infty} \frac{-f'(x)}{1 - f(x)} > \frac{\beta}{R_0'(1)}, \tag{29}$$

3) $F(u) \in g^+(0)$, $F(u) \in l^-(1)$ and inequalities (28) and (29) hold.

Then as $t \to \infty$, uniformly with respect to $u \in [u_+(t), u_-(t)]$,

$$\varphi(t, u; f) \to R_0(u),$$

uniformly on an arbitrary finite interval with respect to x,

$$u(x + m(t, (a+b)/2), t) \to w_c(x; a, b),$$

if $R_0(u) \equiv \tau_c(u; a, b)$ for $a \leqslant u \leqslant b$. Moreover, $\dot{m}(t, (a+b)/2) \to c$.

Theorems 9 and 10 are easily proved by applying Theorems 6 and 7; we do this schematically. We show that for an arbitrary $\varepsilon > 0$ there exists a $t_0(\varepsilon)$ such that $R_0(u) - \varepsilon \leqslant \varphi(t, u; f) \leqslant R_0(u) + \varepsilon$, $t \geqslant t_0(\varepsilon)$, $u_+ \leqslant u \leqslant u_-$. The inequality on the left is proved in the usual way:

$$\varphi(t, u; f) > \varphi(t, u; \chi), \ \varphi(t, u; \chi) \uparrow R_0(u), \quad \chi(x) = \begin{cases} 1, & x < 0, \\ 0, & x \geqslant 0. \end{cases}$$

To prove the inequalities on the right, we introduce the function

$$f_0(x) = \begin{cases} f(x), & \text{if} \quad a \leqslant f(x) \leqslant b, \\ a, & \text{if} \quad f(x) < a, \\ b, & \text{if} \quad f(x) > b. \end{cases}$$

Then $\varphi(t, u; f) \leqslant \varphi(t, u; f_0) \to \tau_c(u; a, b) \equiv R_0(u)$ for $a \leqslant u \leqslant b$. It follows from the uniform in u convergence $\varphi(t, u; f) \to \tau_c(u; a, b)$, $a \leqslant u \leqslant b$ that $u(x + m(t, (a+b)/2), t) \to w_c(x; a, b)$, uniformly on an arbitrary finite interval, semiaxis, or axis with respect to x, depending on whether or not a and b are, respectively, 0 and 1.

Emergence into a wave with respect to velocity is proved in the usual way (Theorem 6).

A more detailed exposition of these results appears in [10].

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QUESTION OF EQUIVALENCE OF THE CLASSICAL METHODS OF SUMMATION OF ORTHOGONAL SERIES

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Let $\{\varphi_k(x)\}_{k=0}^{\infty}$ be an orthonormal system in some measure space $(\Omega, \mathscr{A}, \mu)$, $\varphi_k \in L^2(\Omega)$. The series $\sum c_k\varphi_k$, where $\sum c_k^2 < \infty$, will below be called an L^2 series. Two regular methods of summing T_1 and T_2 are said to be equivalent in L^2 if, for all orthonormal systems $\{\varphi_k\}$, the T_1 summability a.e. on $G \subset \Omega$ $(\mu(G) > 0)$ of the series $\sum c_k\varphi_k$ in L^2 implies the T_2 summability a.e. on G of this series, and vice versa.

If a subsequence $S_{n_m}(x)$ of partial sums of the series converges, then we say this series is summable by the method $T(n_m)$.

The following results are well-known:

THEOREM A. The methods of Cesaro (C, α) $(\alpha > 0)$, Abel and $T(2^m)$ are equivalent in L^2.

THEOREM B. The methods of Euler (E, q) $(q > 0)$, Borel and $T(m^2)$ are equivalent in L^2.

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