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Stochastic Resonance in Chaotic Systems

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The phenomenon of stochastic resonance (SR) is investigated for chaotic systems perturbed by white noise and a harmonic force. The bistable discrete map and the Lorenz system are considered as models. It is shown that SR in chaotic systems can be realized via both parameter variation (in the absence of noise) and by variation of the noise intensity with fixed values of the other parameters.

KEY WORDS: Chaos; noise; residence time; signal-to-noise ratio.

1. INTRODUCTION

Dynamical chaos has the remarkable property that allows one to investigate experimentally regimes of oscillation which are unstable in the Lyapunov sense. Moreover, real chaotic systems are, in principle, structurally unstable.[1] These properties of chaos dynamics do not correspond to the basic principles of classical oscillation theory, which are based on assumptions of structural stability and the stability of all realizable solutions.

Strictly speaking, strange attractors can be realized only in structurally stable, hyperbolic systems. Until now, however, in practice nobody could find such systems. Attractors of the Lorenz type are more similar to real strange attractors, since they satisfy the condition of hyperbolicity, but are not structurally stable.[2] Usually, regimes of dynamical chaos are observed in structurally unstable, quasihyperbolic systems (that is, systems with quasiattractors). The simultaneous coexistence of a number of regular and chaotic attractors is typical for quasiattractors. These attractors undergo an infinite number of different bifurcations as the system parameters are varied. As a result of these properties, systems with quasiattractors are extremely sensitive to external perturbations.[3–7] An external perturbation

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of such a system can generate a set of interesting effects due to the interactions of the attractors, including noise-induced phase transitions.[8–11] One of these effects is stochatic resonance (SR).[12–14]

The mechanism of SR has been investigated in detail for simple bistable systems with two stable and one unstable equilibrium points.[5,15–21] Noise forcing in these systems induces random switching between the two stable states, with a mean frequency which depends on the noise intensity and the height of the potential barrier.[22] The classical phenomenon of SR can be realized in such a bistable system driven simultaneously by noise and a weak harmonic signal. SR manifests itself as a maximum in the signal-to-noise ratio (SNR) for a specific noise intensity D, at which the coherence between modulation and switching frequencies takes place.

The problem of the interaction of two chaotic attractors induced by external noise and by the variation of control parameters can be considered. This interaction is also characterized by a certain switching frequency that depends on the noise intensity and parameter values. Resonance phenomena are generated under additional modulation and SR can be observed.

In this paper, we consider two examples of quasihyperbolic systems which demonstrate SR as a result of the interaction of chaotic attractors.

2. SR IN ONE-DIMENSIONAL CUBIC MAP

Consider the discrete system[23]

$$x_{n+1} = (a-1)\, x_n - a x_n^3 \tag{1}$$

For $0 < a < 2$ the map (1) has only one stable fixed point at the origin ($x_1 = 0$). A pitchfork bifurcation takes place at $a = 2$. There are two stable fixed points, $x_{2,3} = \pm c$, $c = [(a-2)/a]^{1/2}$, and one unstable fixed point at the origin for $2 < a < 3$. The cascade of period-doubling bifurcations is realized in the interval of $3 \leqslant a < 3.3$ and the map (1) demonstrates chaos for $a \geqslant 3.3$.

If $3.3 < a \leqslant 3.6$, there are two disjoint symmetric Feigenbaum-type attractors, whose basins of attraction are separated by the separatrix $x_1 = 0$. A stationary probability density for this case is represented in Fig. 1a and consists of two noncrossing functions. At $a \cong a^* = 3.598...$,[23] the attractors merge into one chaotic set, and a new chaotic attractor with the probability density shown in Fig. 1b exists for $a > a^*$. The bifurcation of attractor merging is followed by a phenomenon of intermittency of the "chaos–chaos" type.[23,24] The phase trajectory resides in the basins of partial attractors for a long time and makes random transitions from one

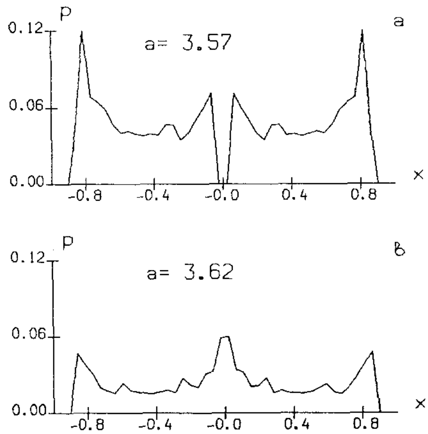

Fig. 1. Probability density distributions $p(x)$ for attractors of the dynamical system (1) at different values of the parameter a.

region to the other. The mean residence time τ_1 in each of the attractors (duration of "laminar" phase) satisfies the universal critical relation

$$\tau_1 \sim (a - a^*)^{-\gamma}, \qquad \gamma = 0.5 \tag{2}$$

At the moment of attractor merging, the power spectrum has the shape typical for $1/f^{\alpha}$ noise.[25,26] The effect of intermittency of the "chaos–chaos" type can be obtained as a result of additive noise perturbation. With this, the character of dependence (2) is maintained, and the critical index γ becomes a function of noise intensity, $\gamma = \gamma(D)$.

Let us add to map (1) periodic modulation and noise:

$$x_{n+1} = (a - 1)\, x_n - a x_n^3 + \varepsilon \sin(2\pi f_0 n) + \xi(n) \tag{3}$$

where ε and f_0 are the amplitude and frequency of the modulation, respectively, and $\langle \xi(n) \rangle = 0$, $\langle \xi(n)\, \xi(n+k) \rangle = 2D\delta(k)$. Let us investigate the dynamics of system (3) on the "two-state" level by replacing the coordinate $x(n)$ with $+1$ if $x(n) > 0$ and -1 if $x(n) < 0$.

Under the assumption that $\dot{x} = x_{n+1} - x_n$, we can transform the discrete model (1) into the differential equaton $\dot{x} = (a-2)x - ax^3$ and calculate the potential $U(x)$:

$$U(x) = -\frac{a-2}{2} x^2 + a \frac{x^4}{4} \tag{4}$$

This allows one to define a Kramers rate

$$r_0 = -\frac{a-2}{\pi\sqrt{2}}\exp\left[-\frac{(a-2)^2}{4aD}\right] \tag{5}$$

and to derive an expression for the SNR within the adiabatic approximation:

$$\mathrm{SNR} = -\frac{(a-2)^2\,\varepsilon^2}{aD^2}\exp\left[-\frac{(a-2)^2}{4aD}\right] \tag{6}$$

Consider the dynamics of (3) when the two symmetrical chaotic attractors coexist at $a = 3.4$. Noise addition (provided that $\varepsilon = 0$) makes the probability density $p(x)$ smoother and induces a switching among the attractors.

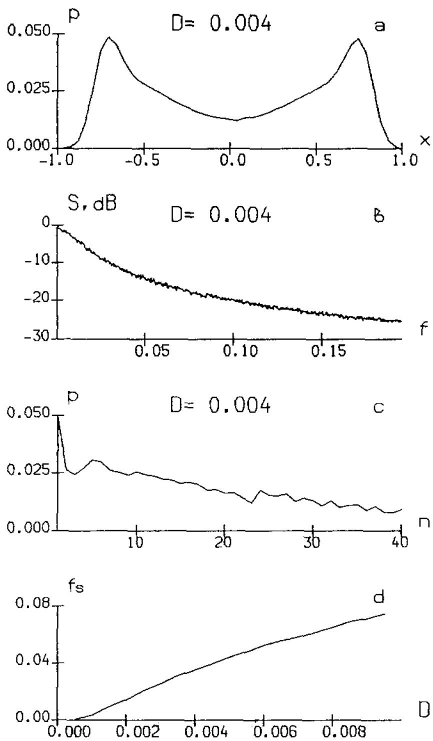

Fig. 2. Characteristics of system (3) at $a = 3.4$, $\varepsilon = 0$: (a) probability density distribution $p(x)$, (b) power spectrum $S(f)$, (c) probability density distribution of residence time $p(n)$, and (d) plot of mean switching frequency f_s versus noise intensity D.

The main quantitative characteristics of the dynamics (3) without modulation ($\varepsilon = 0$) are represented in Fig. 2 and reflect typical properties of bistable systems in the presence of noise.

The periodic stimulation $\varepsilon = 0.05$, $f_0 = 0.125$ results in a sharp peak in the power spectrum at the frequency f_0 (Fig. 3a). The probability distribution of the residence time $p(n)$ has the specific shape of a sequence of strong Gaussian-like peaks of exponentially decreasing amplitude (Fig. 3b). The peaks are centered at odd integer multiples of the modulation half-period. The plot of SNR(D) (Fig. 3c) is also typical for systems demonstrating SR. Maximum signal-to-noise ratio corresponds to the noise intensity $D = D_0$, with the mean transition frequency f_s near $f_0/3$ (compare Fig. 2d and Fig. 3c). Though conditions of the adiabatic approximation are not fulfilled here, the dependence SNR(D) is in good agreement with the theoretical predictions.[15] The solid curve in Fig. 3c corresponds to the approximation of SNR(D) via the expression

$$\mathrm{SNR}(D) = \frac{g}{(D+D_0)^2} \exp\left(-\frac{w}{D+D_0}\right) \tag{7}$$

where $D_0 = 0.0036$, $g = 0.003865$, and $w = 0.013428$.

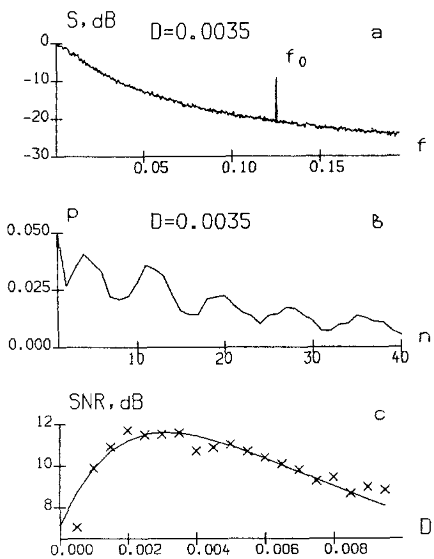

Fig. 3. (a) Power spectrum $S(f)$ and (b) probability density distribution of residence time $p(n)$ in the system (3) for $\varepsilon = 0.05$, $f_0 = 0.125$, $a = 3.4$. (c) Signal-to-noise ratio (SNR) versus noise intensity D: crosses, results of numerical calculation; solid curve, approximation by expression (7).

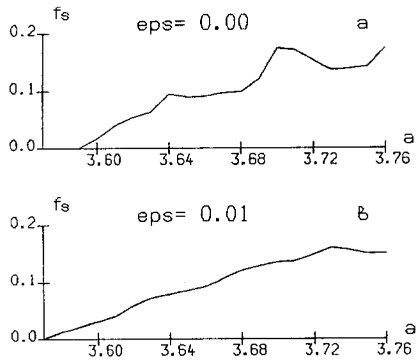

Fig. 4. Mean switching frequency in the system (3) at $D=0$ and (a) $\varepsilon=0$ and (b) $\varepsilon=0.01$ versus parameter a.

Note that computation on the level of a complete dynamics of the map (3) has not shown any resonance effects in the SNR(D) dependence.

SR in system (3) in the absence of noise ($D=0$). As is seen from Fig. 1b, chaotic attractors merge to one attractor at $a \geqslant a^*$, and the switching effect is achieved due to the intrinsic deterministic dynamics of

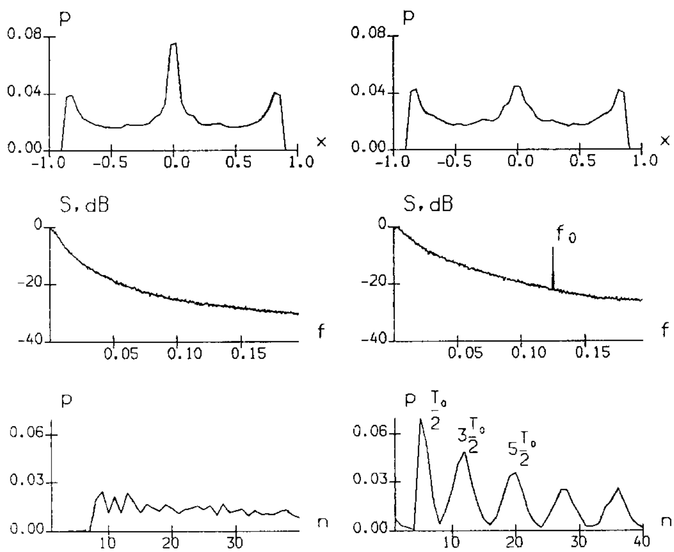

Fig. 5. Probability density distribution $p(x)$, power spectrum $S(f)$, and probability distribution of residence time $p(n)$ for the attractors of (3) at $D=0$, $a=3.6$ without modulation (on the left) and with modulation $\varepsilon=0.01$, $f_0=0.125$ (on the right).

the system. The chaotic structure of the attractor causes the transitions, which are realized randomly in time with the frequency f_s determined by the value of the parameter a, as shown in Fig. 4.

By analogy with classic SR phenomena, we can suggest that a coherent interaction of transition and modulation frequencies occurs when the dependence SNR(a) demonstrates the maximum without noise forcing. Figure 5 shows computational results for $p(x)$, $S(f)$, and $p(n)$ at $\varepsilon = 0$ (on the left) and for modulation with amplitude $\varepsilon = 0.01$ (on the right) for system (3) at $a = 3.6$, $D = 0$. In the absence of modulation ($\varepsilon = 0$), the probabiliy distribution $p(n)$ is similar to a uniform one. With modulation $\varepsilon = 0.01$ and $f_0 = 0.125$ the shape of the probability distribution $p(n)$ becomes typical for SR. Calculation of the dependence SNR(a) confirms this proposition. The curves SNR(a) are shown in Fig. 6. They were calculated both on the level of "two-state" dynamics (Fig. 6a) and for the complete dynamics (Fig. 6b). SR is observed in both cases here, unlike the results for $a = 3.4$. Moreover, there are three maxima on the curve SNR(a) at three values of the parameter a (see Fig. 6), corresponding to resonances $f_s : f_0 = 1:3$, $1:1$, and $4:3$. Note that the total power of the process $x(n)$ is almost constant here.

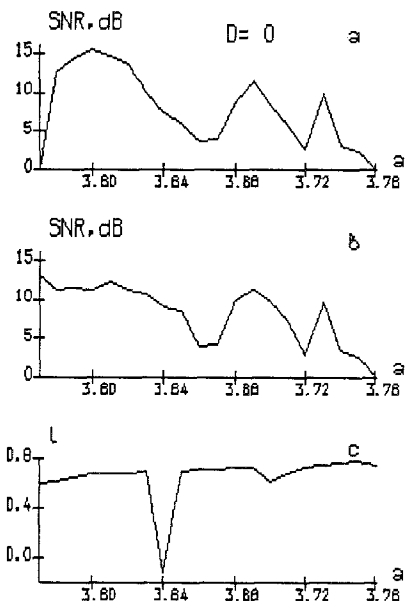

Fig. 6. Calculation of SNR(a) (a) on the level of "two-state" dynamics and (b) for a complete dynamics in system (3) at $\varepsilon = 0.01$, $f_0 = 0.125$, $D = 0$. (c) Lyapunov exponent l versus parameter a at $\varepsilon = 0.0$.

3. SR IN THE LORENZ MODEL

The Lorenz model is quite suitable for investigations of the influence of noise on the dynamics of chaotic systems. This model allows a correct introduction of Langevin sources into the dynamical equations.[27] For the Lorenz model with noise, an ergodicity has been proved.[28] Moreover, there are regions of the Lorenz attractor and of quasiattractors in the Lorenz model, i.e., a phase transition "Lorenz attractor–quasiattractor" can take place (see Fig. 7).[29] Consider the parameter values corresponding to quasihyperbolic attractors. Just here, effects of the interactions of attractors with different structures can be observed. Particularly, effects of "chaos–chaos" intermittency induced by noise can be realized.[6]

Let us investigate the noise-induced transition in the region of the existence of the quasiattractor for parameter values $\sigma = 10$, $r = 210$, and $b = 8/3$. The stochastic differential equatons of the Lorenz system are

$$\begin{aligned} dx/dt &= -\sigma x + \sigma y + \xi_1(t), \qquad & dy/dt &= -y + rx - xz + \xi_2(t) \\ dz/dt &= -bz + xy + \xi_3(t), \qquad & \langle \xi_i(t)\, \xi_j(t+\tau) \rangle &= \delta_{ij} D \delta(\tau) \end{aligned} \tag{8}$$

Contours of constant two-dimensional stationary probability density $p(x, y)$ obtained by using numerical integration of system (8) are plotted in the absence of noise $D = 0$ in Figs. 8a and 8b and in the presence of noise with intensity $D = 1.0$ in Fig. 8c. In the absence of noise, two symmetric attractors are realized in the system for different initial conditions (Figs. 8a and 8b). When $D \neq 0$, the noise induces a merging of the two symmetrical

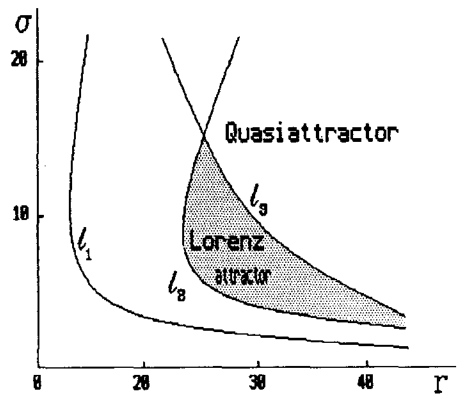

Fig. 7. Bifurcation diagram of the Lorenz system in the (r, σ) plane at $b = 8/3$: l_1, line of separatrix loop; l_2, line of birth of Lorenz attractor; l_3, boundary between regions of Lorenz attractor and quasiattractor.

attractors into one chaotic set. For weak noise intensities, the phase trajectory is retained on each of the attractors for a long time and makes quick transitions between them. So, "chaos–chaos" intermittency induced by external noise is realized in the system. To confirm the conclusion about the existence of intermittency, we have calculated the power spectrum $S(\omega)$ of the process $x(t)$. Figure 9a shows the power spectrum calculated in the absence of noise. Its low-frequency part is represented in Fig. 9b. In the

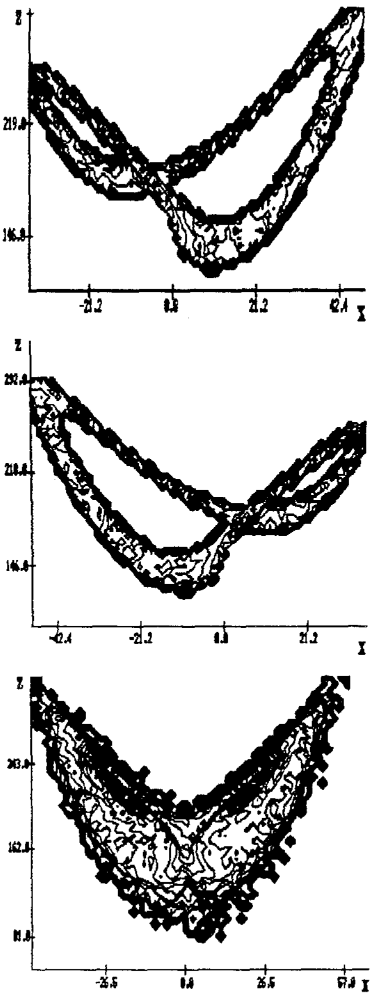

Fig. 8. Contours of constant two-dimensional stationary probability density $p(x, z)$: (a) $D=0$, $x_0=40.0$, $y_0=1.0$, $z_0=300.0$, (b) $D=0$, $x_0=-40.0$, $y_0=1.0$, $z_0=300.0$, (c) $D=1.0$, $x_0=40.0$, $y_0=1.0$, $z_0=300.0$.

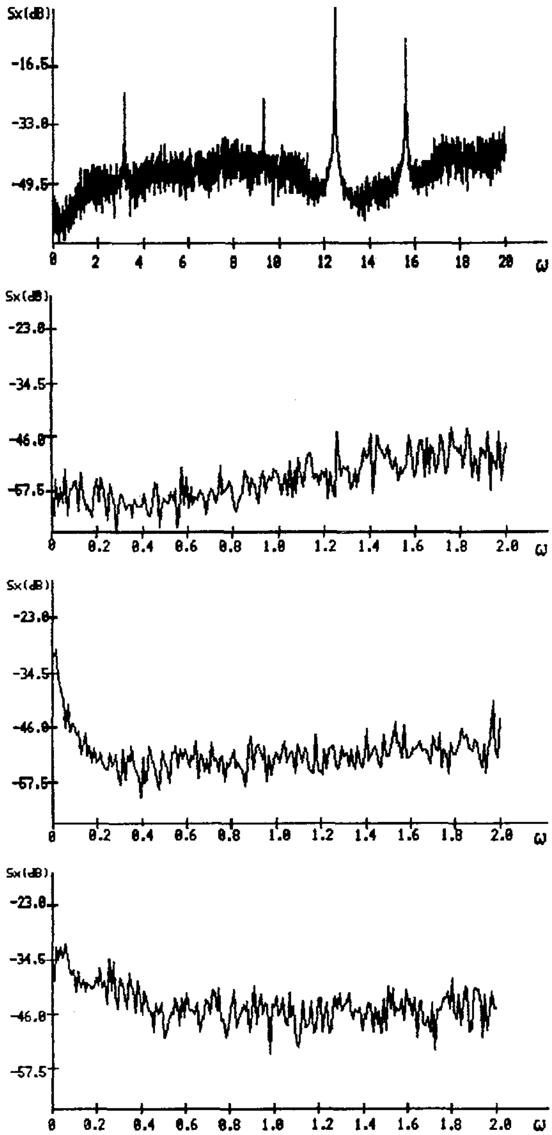

Fig. 9. Power spectrum $S_x(\omega)$ of system (8): (a) power spectrum at $D=0$, (b, c, d) low-frequency domain of spectrum at $D=0.0$, 0.2, and 0.5, respectively.

presence of additive noise, the power spectrum evolves to the low-frequency domain, which is verified by Fig. 9c ($D = 0.2$). The low-frequency component in the power spectrum is caused by the existence of two characteristic time scales. The first corresponds to a long residence of the phase trajectory on each of the symmetric attractors. The second corresponds to the transitions between them. The mean residence time T_s on each of the merged attractors is connected with the half-width of the low-frequency spectral component $\Delta\omega$ by the obvious relation $T_s = 2\alpha\pi/\Delta\omega$. If the noise intensity increases, the mean residence time of the phase trajectory on each of the merged attractors decreases: $T_s \sim \exp(\beta/D)$. This is demonstrated by a smoothing out of the low-frequency power spectrum domain (Fig. 9b). The variables α and β in the expressions for T_s are constants.

Thus, transitions induced by noise take place in the region of quasi-attractors of the Lorenz system. The interaction of attractors with "chaos–chaos" intermittency is realized here. Note that this effect is maintained for multiplicative noise as well as for the case when sources $\xi_i(t)$ have a finite correlation time.[30]

Consider the simultaneous action of noise and an external harmonic force,, included in the first equation of system (8):

$$dx/dt = -\sigma x + \sigma y + \varepsilon \sin \Omega_0 t + \xi_1(t) \tag{9}$$

The frequency of the external excitation $\Omega_0 = 2\pi f_0 = 0.1$ was chosen in accordance with the mean residence time of the phase trajectories on each of the attractors. The existence of two characteristic time scales in the process enables us to suggest SR occurring under variation of the noise intensity D. The low-frequency domain of the power spectrum is shown in Fig. 10 for the system (9) with harmonic perturbation. The system (9) was integrated during a period $T_{\max} \sim 5 \times 10^3$, corresponding to 80 periods of harmonic excitation. The SNR(D) dependence demonstrating SR is shown in Fig. 11.

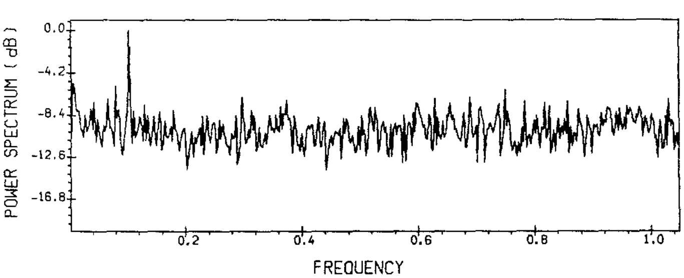

Fig. 10. Power spectrum of the system (8) with harmonic modulation $D = 0.2$, $\varepsilon = 2.0$, $\Omega_0 = 0.1$.

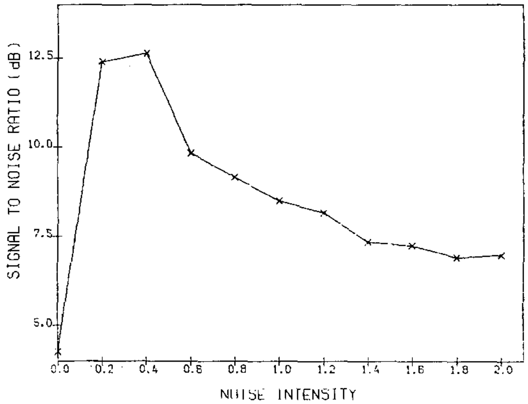

Fig. 11. Plot of signal-to-noise ratio versus noise intensity D.

The SR phenomenon was found in the Lorenz model at $r \gg 1$ by using a direct numerical experiment and can be explained theoretically. To do this, we have transformed the Lorenz equations (8) at $D=0$ into the form of a parametric inertial nonlinear oscillator in $\mathbb{R}^3$,[31,32]

$$\begin{aligned} \ddot{y} + \delta h \dot{y} + y^3 + (z-1)\,y &= 0 \\ \dot{z} &= \delta(\beta y^2 - \alpha z) \end{aligned} \tag{10}$$

where $\delta = (r-1)^{-0.5}$, $h = (1+\sigma)/\sqrt{\sigma}$, $\alpha = b/\sqrt{\sigma}$, and $\beta = (2\sigma - b)/\sqrt{\sigma}$. In our case, $r = 210 \gg 1$, therefore $\delta \ll 1$. So we can suppose $\dot{z} = 0$. Let us eliminate the variable z from (10) and obtain the equations of a noninertial oscillator in $\mathbb{R}^2$:

$$\ddot{y} + k\dot{y} + dy^3 - y = 0, \qquad k = \delta h, \qquad d = (\alpha + \beta)/\alpha \tag{11}$$

The system (11) describes a bistable dissipative oscillator investigated in detail by using the theory of SR.[17,33,34]

4. CONCLUSIONS

The results of numerical experiments presented in this paper yield the following conclusions.

1. In systems with quasiattractors, interaction of different types of attractors are realized. This leads to an intermittency effect. If only two attractors interact, then the system can be treated as bistable in a more general sense.

2. The interaction of attractors can be induced by both external noise and a variation of parameters responsible for the mean transition frequency.

3. The stochastic resonance can be realized in quasihyperbolic systems via both parameter variation (in the absence of noise) and variation of noise intensity under fixed values of the parameters.

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