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Negative-Temperature States and Large-Scale, Long-Lived Vortices in Two-Dimensional Turbulence

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We study Onsager's theory of large, coherent vortices in turbulent flows in the approximation of the point-vortex model for two-dimensional Euler hydrodynamics. In the limit of a large number of point vortices with the energy per *pair* of vortices held fixed, we prove that the entropy defined from the microcanonical distribution as a function of the (pair-specific) energy has its maximum at a finite value and thereafter decreases, yielding the negative-temperature states predicted by Onsager. We furthermore show that the equilibrium vorticity distribution maximizes an appropriate entropy functional subject to the constraint of fixed energy, and, under regularity assumptions, obeys the Joyce–Montgomery mean-field equation. We also prove that, under appropriate conditions, the vorticity distribution is the same as that for the canonical distribution, a form of equivalence of ensembles. We establish a large-fluctuation theory for the microcanonical distributions, which is based on a level-3 large-deviations theory for exchangeable distributions. We discuss some implications of that property for the ergodicity requirements to justify Onsager's theory, and also the theoretical foundations of a recent extension to continuous vorticity fields by R. Robert and J. Miller. Although the theory of two-dimensional vortices is of primary interest, our proofs actually apply to a very general class of mean-field models with long-range interactions in arbitrary dimensions.

KEY WORDS: Negative temperatures; coherent vortices; statistical mechanics approach to turbulence; maximum entropy principle; large deviations.

1. INTRODUCTION: THE ONSAGER THEORY

In a famous paper published in 1949 Onsager proposed a statistical theory of the formation of large-scale, long-lived vortex structures in turbulent flows.[1] We would like here to briefly review the elements of Onsager's

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theory with an emphasis on its dynamical foundations. Thereafter, we review also some of the theoretical developments and critical discussions which have followed Onsager's proposal. As the conclusion of this introductory section, we state precisely the theorems which are our own contribution to the literature of the subject.

It must be emphasized at the outset that Onsager's theory is not, apparently, directly relevant to the scale-invariant, turbulent cascade state which is often the subject of turbulence theory.[2] The latter universal statistical state is believed to attain in high-Reynolds-number turbulent flows for the "inertial subrange" of scales, far below the integral length scale imposed by the macroscopic flow boundaries and far above the inner or dissipation scale where viscosity effects become influential (see refs. 2 and 3). The regime considered by Onsager is also far above the scale set by viscous dissipation—so that the Euler equations of ideal hydrodynamics should be applicable—but in fact consists of the largest scales of the flow, of the macroscopic length scale. It may be misleading to use a terminology similar to that employed for homogeneous turbulence, since the situations we consider are actually rather different, but it seems appropriate to refer to this regime as the "inertial super-range" of scales. More simply and less prejudicially, we may refer to the "macroscopic range" of scales. In fact, as we shall see, the influence of boundaries or global flow constraints plays a vital role in the phenomena we discuss.

An essential limitation of Onsager's theory is that it applies to only quasi-two-dimensional flows in nature. Since there are some important situations, such as atmospheric or geostrophic turbulence, where a two-dimensional description seems valid,[(5)] the theory is not without practical interest. Two-dimensional turbulence is, of course, also useful as a theoretical toy and may be tested against numerical simulations.

Another limitation of Onsager's theory, as originally formulated, is that it applied only to situations where a *vortex model* of the two-dimensional Euler equations should be valid. The latter approximation applies when the fluid vorticity is dilute and concentrated into small "blobs." It has been known since Kirchhoff[(6)] that such vortex blobs obey approximately a Hamiltonian particle dynamics. To be specific, let us consider a bounded flow domain Λ. The system of Euler equations for the fluid velocity $\mathbf{v} = (v_1, v_2)$ in Λ is

$$\begin{aligned} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} &= -\nabla p \\ \nabla \cdot \mathbf{v} &= 0 \\ \mathbf{v} \cdot \hat{\mathbf{n}}|_{\partial\Lambda} &= 0 \\ \mathbf{v}(x, 0) &= \mathbf{v}_0(\mathbf{x}) \end{aligned} \tag{1.1}$$

[2] However, an interesting recent attempt along these lines is contained in ref. 4, which discusses vortex filaments as equilibrium models of the three-dimensional cascade state.

where p is the fluid pressure and $\hat{\mathbf{n}}$ is the normal to $\partial\Lambda$. It is assumed here that the density of the fluid is $\rho = 1$. An equivalent description is in terms of the (pseudoscalar) *vorticity* $\omega = (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{e}}_3$:

$$\begin{aligned} \partial_t \omega + \nabla \cdot (\mathbf{v}\omega) &= 0 \\ \omega(x, 0) &= \omega_0(x) \end{aligned} \tag{1.2}$$

$$\begin{aligned} \nabla \times \mathbf{v} &= \omega \hat{\mathbf{e}}_3 \\ \nabla \cdot \mathbf{v} &= 0 \\ \mathbf{v} \cdot \hat{\mathbf{n}}|_{\partial\Lambda} &= 0 \end{aligned} \tag{1.3}$$

The second group of equations may be solved by introducing the *stream function* ψ according to the relation $\mathbf{v} = \nabla \times (\psi \hat{\mathbf{e}}_3) = (\partial_2 \psi, -\partial_1 \psi)$. Then, the function ψ must obey

$$\begin{aligned} -\Delta\psi &= \omega \\ \psi|_{\partial\Lambda} &= 0 \end{aligned} \tag{1.4}$$

If $-V_\Lambda(x, y)$ is the Green's function of the Laplacian with 0-Dirichlet b.c. on $\partial\Lambda$, then the first equation above may be inverted as

$$\psi(x) = \int_\Lambda dy\, V_\Lambda(x, y)\, \omega(y) \tag{1.5}$$

Now, imagine a situation where $\omega = \sum_{i=1}^N \omega_i$, the ω_i have definite signs, small disjoint supports centered at $x_1, \ldots, x_N$, and total circulations $\int dx\, \omega_i(x) = R_i$, $i = 1, \ldots, N$. Then, it is known that the centers of the "blobs" move approximately according to a Hamiltonian evolution

$$R_i \frac{dx_{i1}}{dt} = +\frac{\partial H_\Lambda}{\partial x_{i2}} \tag{1.6}$$

$$R_i \frac{dx_{i2}}{dt} = -\frac{\partial H_\Lambda}{\partial x_{i1}} \tag{1.7}$$

where

$$H_\Lambda(x_1, \ldots, x_N) = \sum_i R_i^2 W_\Lambda(x_i) + \sum_{i<j} R_i R_j V_\Lambda(x_i, y_j) \tag{1.8}$$

The terms $W_\Lambda(x_i)$, which are absent for free space $\Lambda = \mathbb{R}^2$, represent the individual energies of each vortex with the image charges necessary to maintain the boundary conditions and with an infinite self-energy subtracted:

$$W_\Lambda(x) = \lim_{y \to x} \tfrac{1}{2}[V_\Lambda(x, y) - V_\infty(x, y)] \tag{1.9}$$

where

$$V_{\infty}(x, y) = -\frac{1}{2\pi} \log |x - y| \tag{1.10}$$

is the pair-potential in free space. In fact, it may be proved rigorously that the vorticity field calculated from the point-vortex model solution $x_i(t)$ for initial data $x_i(0) = x_i$,

$$\omega_t(x) \equiv \sum_i R_i \delta(x - x_i(t)) \tag{1.11}$$

is, for a finite time interval, the weak limit of the solution of (1.2) if the initial data ω_0 for the latter converge weakly to $\sum_i R_i \delta(x - x_i)$. (See Theorem 4.3 of ref. 7.) The vortex model breaks down as an approximation to Euler dynamics for blobs of finite radius in a close approach: e.g., a strong vortex may stretch a nearby weaker vortex into a long ribbon of vorticity under the shear of its velocity field.[8] However, such close encounters occur for initial conditions of vanishing Liouville measure as the core radius shrinks to zero (in fact, this may be used to construct a global dynamics for the point-vortex model as a limit of a model with "cores"; see ref. 9 and Theorem 2.1 of ref. 7). The vortex model will further fail to describe the evolution of vortex blobs in real fluids, governed by Navier–Stokes equations, as the effects of (kinematic) viscosity ν, such as diffusion of vorticity, manifest on a long time scale $\sim 1/\nu$.

The fundamental hypothesis of the Onsager theory was that, under a wide variety of circumstances, the long-time distribution of the point vortices should be governed by *equilibrium statistics*. The dynamical basis of this hypothesis lies in the assumptions (i) that the system of vortices shall be energetically isolated, and (ii) that, as a consequence of the ergodicity of the point-vortex dynamics, a microcanonical equilibrium distribution shall be achieved over the energy surface. The assumption (i) of energetic isolation can never be strictly true, as the vortex system shall always lose energy to molecular degrees of freedom by the effects of viscosity. However, there is a well-known tendency in two-dimensional turbulence for energy to flow to and reside in the largest scales, the so-called *inverse energy cascade* predicted by Kraichnan.[10] This is seen, for example, in the decay of an initial turbulent vorticity field studied by McWilliams[11] as the evolution of large coherent vortices from the chaotic surrounding flow (see also ref. 12). Such large vortex structures are only weakly dissipated by viscosity. The second assumption (ii), ergodicity of the point-vortex model, is a difficult dynamic problem and rather problematic. There are certainly indications that, in many circumstances, the vortex dynamics shall not have a single ergodic component, but, indeed, a region of finite Lebesgue measure where it is completely integrable (see ref. 13 and ref. 7, Section 2). However, as we shall see later, conditions much weaker than strict ergodicity would suffice to justify equilibrium statistics. A more serious question is the time scale of relaxation to equilibrium, since, for the validity of Onsager's proposal, the effective microcanonical distribution must be achieved, certainly, in less than the viscous diffusion time. Simulations of the point-vortex model itself show that configurations with widely scattered clusters of vortices may fail to equilibrate rapidly overall[14] (although individual clusters may achieve a "local equilibrium").

However, let us assume with Onsager that, in fact, the large-time statistics is a microcanonical equilibrium. From this we may see already that, for high energies, the configurations with a close clustering of vortices of like sign shall be statistically dominant. This is a consequence just of the energy constraint; cf. the expression (1.8) for the vortex Hamiltonian. Therefore, the formation of large compound vortices in two dimensions may be considered a simple analogue of the more complicated three-dimensional phenomenon of "folding" of stretched vortex tubes.[15]

On the other hand, another illuminating point of view may be obtained by going to an equivalent canonical description. It was a fundamental observation of Onsager that, as a consequence of the boundedness of the phase space, the state of maximum entropy for the vortex gas does not correspond to the maximum possible energy. Clearly, the state with maximum Gibbs entropy

$$S(\mu_N) = -\int_{\Lambda^N} dx_1 \cdots dx_N \, \mu_N(x_1,\dots, x_N) \log \mu_N(x_1,\dots, x_N) \tag{1.12}$$

is just the distribution

$$\mu_N(x_1,\dots, x_N) = \frac{1}{|\Lambda|^N} \tag{1.13}$$

[$|\Lambda| = \lambda(\Lambda)$, where λ is Lebesgue measure], for which the vortices are independently, uniformly distributed over Λ, and the maximum entropy value is $N \cdot \log |\Lambda|$. However, this distribution does not give the maximum energy. To keep things simple, consider the case where all $R_i = 1$. Then, the above state corresponds to a mean energy

$$E_{\text{crit}} = \frac{1}{2} N(N-1) \int_\Lambda \frac{dx}{|\Lambda|} \int_\Lambda \frac{dy}{|\Lambda|} V_\Lambda(x, y) + N \int_\Lambda \frac{dx}{|\Lambda|} W_\Lambda(x) \tag{1.14}$$

However, much larger energies are possible for states in which the vortices are all squeezed close together. This suggests that, if the entropy is written as a function of the mean energy $S(E)$, it shall be *decreasing* for $E > E_{\text{crit}}$, corresponding to a *negative absolute temperature.* For a negative-temperature canonical distribution, like-sign vortices shall statistically attract, which provides another explanation of the phenomenon of large vortex clusters. We would like to remark that Onsager made his original argument with the Boltzmann entropy defined from the microcanonical phase-space volume, whereas the Gibbs form of the entropy allows an easy evaluation of the critical energy.

It may be worth pointing out, parenthetically, that the above considerations also apply for clusters of vortices in essentially unbounded flow situations as a consequence of additional conserved quantities of the vortex dynamics. In fact, the *center of vorticity*

$$X = \frac{\sum_{i=1}^{N} R_i x_i}{\sum_{i=1}^{N} R_i} \tag{1.15}$$

and the *angular momentum* (internal, or relative to the center)

$$L^2 = \sum_{i=1}^{N} R_i^2 (x_i - X)^2 \tag{1.16}$$

are both conserved. Notice that the latter in particular restricts

$$|x_i - X| \leqslant \frac{L}{\max_i |R_i|} \tag{1.17}$$

so that the accessible phase space in the frame of the mean motion of the vortices is, in fact, bounded.

Now, Onsager did not in fact propose an asymptotic limit for the validity of his theory. Indeed, the study of the standard thermodynamic limit had only been just begun in 1949 by van Hove.[16] However, one expects a thermodynamic description to be valid in a limit $N \to \infty$. This question was taken up by Fröhlich and Ruelle in 1982.[17] They studied a neutral vortex gas ($\sum_i R_i = 0$) in the standard thermodynamic limit where $N \to \infty$ with $e = E/N$, $\varrho = N/|\Lambda|$ held fixed. (Because of the scaling properties of the vortex gas, one can also consider this limit to be one in which Λ is held fixed: see Section 5.4 of ref. 17.) The result of their work was that the entropy function $s(e, \varrho)$ obtained as the limit of the microcanonical entropy in the above standard form did *not* decrease over any interval, as suggested by Onsager's argument, and therefore the predicted negative-temperature states did not exist. Fröhlich and Ruelle admitted the possibility of alternative limits (Section 5.4), but they suggested that their result ruled out a nontrivial entropy function for any other limiting procedure. This argument is not conclusive, because the scaling relation (3.8) in their paper implies, with their main result, that the scaling we consider below gives a limiting behavior $+\infty - \infty$ and could, therefore, be nontrivial.

In fact, it seems clear that a different scaling is called for. In the first place, Onsager was attempting to explain the origin of nontrivial vortex structures on the scale of the flow domain Λ. If such a vortex structure is composed of a "cloud" of point vortices, then clearly it has an energy $\sim N^2$, where N is the number of point vortices in the cloud. Indeed, every pair of vortices at positions x_i, x_j in the cloud contributes an interaction energy $V(x_i, y_i)$ of order unity. Furthermore, the critical energy E_{crit} observed by Onsager to give the maximum entropy itself grows like N^2, as may be directly seen from the formula (1.14). Therefore, we infer that the negative-temperature description proposed by Onsager is likely to be valid when the number of vortices N is large and the energy per *pair* of vortices, $e = E/N^2$, is of order unity, and, in fact, greater then some critical value

$$e_{\text{crit}} \equiv \frac{1}{2} \int_\Lambda \frac{dx}{|\Lambda|} \int_\Lambda \frac{dy}{|\Lambda|} V_\Lambda(x, y) < +\infty \tag{1.18}$$

To our knowledge, the correct scaling of the energy as N^2 for Onsager's theory was first stated in 1977 by Lundgren and Pointin.[14] It may be worth remarking that if one approximates a continuous vorticity distribution by N point vortices of strength $1/N$ [cf. Eq. (1.27)], then the hydrodynamic energies considered are relatively high, i.e., $e > e_{\text{crit}}$, but finite-valued in the limit $N \to \infty$.

As a formal device, it is equivalent also to consider a standard limit with $e = E/N$ (and Λ) held fixed if one replaces the Hamiltonian (1.8) by

$$\tilde{H}_\Lambda(x_1,..., x_N) = \frac{1}{N} \sum_i W_\Lambda(x_i) + \frac{1}{N} \sum_{i<j} V_\Lambda(x_i, y_j) \tag{1.19}$$

In this formulation, the problem is clearly of *mean-field type*. The thermodynamic limit for the canonical distribution with this class of mean-field Hamiltonian was first established in 1982 by Messer and Spohn.[18] Their proof covers even the situation of negative-temperature canonical distributions, but was restricted to the situation where the potentials V_Λ and W_Λ are *bounded*. More recently, Caglioti *et al.*[19] and Kiessling[20] have extended that work to the point-vortex model with singular Coulomb potentials as described above. In these works also it was first pointed out that the formal mean-field scaling is the correct thermodynamic limiting procedure for Onsager's theory. However, these works do not discuss the origin of the negative temperatures, which would require consideration of the microcanonical distributions and a theorem on the equivalence of ensembles. We emphasize the physical primacy of the microcanonical distributions. There seems to be no other reasonable explanation of negative-temperature states: in particular, it is hard to conceive of "negative-temperature reservoirs" in nature.

For any symmetric distribution of the N-vortex system with a density $\mu^{(N)}(x_1,..., x_N)$, one may define the density of the rth marginal measure, or reduced distribution, $\varrho_r^{(N)}(x_1,..., x_r)$, as

$$\varrho_r^{(N)}(x_1,..., x_r) \equiv \int_{\Lambda^{N-r}} dx_{r+1} \cdots dx_N \, \mu^{(N)}(x_1,..., x_N) \tag{1.20}$$

When $\mu^{(N)}$ is an equilibrium distribution, microcanonical or canonical, it is easily shown that the $\varrho_r^{(N)}$ obey a stationary hierarchy of equations which couple $\varrho_r^{(N)}$ to $\varrho_{r+1}^{(N)}$. However, Joyce and Montgomery have argued that in the limit $N \to \infty$, for the canonical case, $\varrho_r^{(N)}(x_1,..., x_r) \to \prod_{i=1}^r \varrho(x_i)$, allowing them to close the hierarchy.[21] The limiting one-particle distribution ϱ is then seen to obey the *mean-field equation*

$$\varrho(x) = Z^{-1} \exp\left[-\beta \int_\Lambda dy \, V_\Lambda(x, y) \, \varrho(y) \right] \tag{1.21}$$

where β is the inverse temperature and Z is a normalization factor. The same equation was earlier derived (actually, a slight variant) by Joyce and Montgomery, as a heuristic application of a *maximum entropy principle.*[22] In other words, the probability distribution ϱ on Λ which maximizes the entropy

$$s(\varrho) = -\int_\Lambda dx \, \varrho(x) \log \varrho(x) \tag{1.22}$$

subject to the constraint that

$$\varepsilon(\varrho) = \tfrac{1}{2} \int_\Lambda dx \int_\Lambda dy \, V_\Lambda(x, y) \, \varrho(x) \, \varrho(y) \tag{1.23}$$

takes on a fixed value e was argued to satisfy (1.21) for an appropriate β and to give the "likely" vorticity distribution of the system. Notice, in fact, that if ϱ is interpreted as a vorticity field ω, then (1.21) may be written as

$$-\Delta\psi(x) = \omega(x) = Z^{-1} \exp[-\beta\psi(x)] \tag{1.24}$$

where ψ is the associated stream function given by (1.5). Furthermore, this vorticity field is therefore a stationary solution of the Euler equation (1.2), as is any field of the form $\omega = f(\psi)$ for some f.

Corresponding results were rigorously obtained in the above-cited works on the mean-field limit for the canonical distribution. In particular, it was shown that

$$\varrho_r^{(N)}(x_1,..., x_r) \to \int \nu(d\varrho)\, \varrho(x_1) \cdots \varrho(x_r) \qquad \text{(weakly)} \tag{1.25}$$

where ν is a probability measure on the space of distributions over Λ. Moreover, ν was shown to be concentrated on the L^∞ solutions of the mean-field equation (1.21) which minimize the *free energy*

$$\phi_\beta(\varrho) = \beta\varepsilon(\varrho) - s(\varrho) \tag{1.26}$$

(Actually, ϕ_β is β times the usual free energy f_β.) If there is a *unique* solution of this variational problem, then the factorization predicted by Joyce and Montgomery indeed occurs. Furthermore, in that case, there is a *law of large numbers* for the empirical vorticity distribution,

$$\hat{\omega}_N(x) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) \tag{1.27}$$

according to which, for any bounded, continuous function f on Λ and any $\delta > 0$, and for the free energy minimizer ϱ^*,

$$\left| \int_\Lambda dx\, \hat{\omega}_N(x)\, f(x) - \int_\Lambda dx\, \varrho^*(x)\, f(x) \right| < \delta \tag{1.28}$$

with probability going to one as $N \to \infty$. This renders precise the proposal that ϱ^* is "most likely".

Let us now state our own results.

In the first place, we study the mean-field thermodynamics of the vortex gas starting from a microcanonical distribution on an "energy shell." To be precise, we consider the microcanonical-type distributions

$$\mu_{[e_-,e_+]}^{(N)}(dx_1 \cdots dx_N) = \frac{1}{\Omega_{N,[e_-,e_+]}} \chi\{e_- \leqslant \tilde{H}^{(N)}/N \leqslant e_+\}\, \lambda^N(dx_1 \cdots dx_N) \tag{1.29}$$

where

$$\Omega_{N,[e_-,e_+]} = \lambda^N\left(\left\{e_- \leqslant \frac{\tilde{H}^{(N)}}{N} \leqslant e_+\right\}\right) \tag{1.30}$$

and $\tilde{H}^{(N)}$ is the mean-field Hamiltonian defined in (1.19) (with a technical modification to be discussed below). Defining as usual a *specific entropy* $s_{N,\theta}(e)$ as

$$s_{N,\theta}(e) = \frac{1}{N} \log \Omega_{N,[e-\theta/2, e+\theta/2]} \tag{1.31}$$

we show that the limit

$$s(e) = \lim_{\theta \to 0} \lim_{N \to \infty} s_{N,\theta(e)} \tag{1.32}$$

exists for all e and defines an upper semicontinuous function. In fact,

$$s(e) = \sup\{s(\varrho) \colon \varrho \in \mathscr{P}^1(\Lambda), \varepsilon(\varrho) = e\} \tag{1.33}$$

where $\mathscr{P}^1(\Lambda)$ is the set of (Borel) probability measures on Λ and $s(\varrho)$, $\varepsilon(\varrho)$ are the obvious extensions of the previously defined functions to $\mathscr{P}^1(\Lambda)$. Furthermore, we show, at least in the most favorable cases, that $s(e)$ exhibits the behavior predicted by Onsager: its maximum is achieved at e_{crit} given by (1.18) and thereafter it is a decreasing function of e. The schematic behavior of $s(e)$ is sketched in Fig. 1.

Note $s(e) = -\infty$ for $e \leqslant 0$. Also, as pictured, $s(e)$ is concave over its entire range and has asymptotic slope -8π as $e \to +\infty$. The graph is for

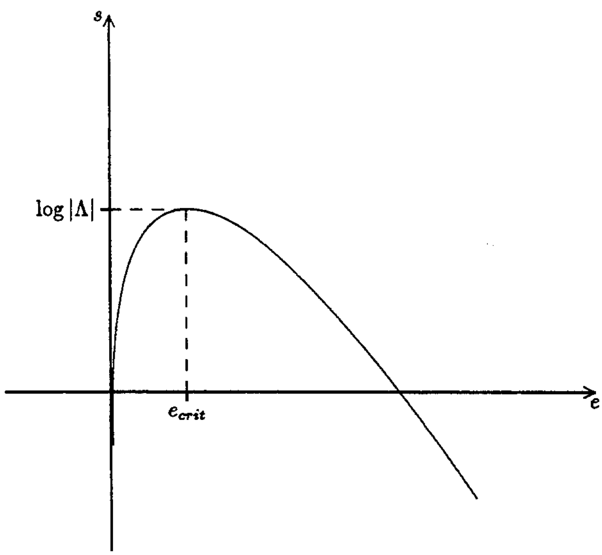

Fig. 1

the case where Λ is a circular domain. However, the latter two properties, particularly the concavity, are proved only under particular conditions. In general, we can prove concavity of $s(e)$ only for $e \leqslant e_{\text{crit}}$. If one defines

$$\phi(\beta) = \inf\{\phi_\beta(\varrho) : \varrho \in \mathscr{P}^1(\Lambda)\} \tag{1.34}$$

which was shown in previous work to be (β times) the free energy, then we show that its Legendre transform

$$\bar{s}(e) = \inf_\beta [\beta e - \phi(\beta)] \tag{1.35}$$

is the closed, concave hull of $s(e)$.

Further results are a precise form of a *maximum entropy principle* and an associated statement on the *equivalence* of microcanonical and canonical ensembles. To be precise, we show for the microcanonical distributions $\mu^{(N)}_{[e_-, e_+]}$ that the same factorization of reduced distributions occurs in the limit $N \to \infty$ as for the canonical case, namely,

$$\varrho_r^{(N)} \to \int \nu(d\varrho)\, \varrho^{\otimes r} \qquad \text{weakly} \tag{1.36}$$

but that now ν is concentrated on a set

$$\mathscr{S}_e = \{\varrho \in \mathscr{P}^1 : \varepsilon(\varrho) = e,\, s(\varrho) = s(e)\} \tag{1.37}$$

where $e \in [e_-, e_+]$ is the point where the maximum of s over that interval is achieved (this point will be unique if s is strictly concave there). Furthermore, if s is differentiable at e, then

$$\mathscr{S}_e = \{\varrho \in \mathscr{P}^1 : \phi_{\beta(e)}(\varrho) = \phi(\beta(e))\} \tag{1.38}$$

where $\beta(e) = s'(e)$ and the support of ν is contained in the support of the measure for the corresponding canonical distribution at inverse temperature $\beta(e)$. If $\phi_{\beta(e)}(\varrho) = \phi(\beta(e))$ has a unique solution, then the two ensembles completely agree for the equilibrium vorticity distribution. We also discuss the situation when strict convexity or differentiability of s does not attain.

Lastly, we establish a *large-fluctuation theory* for the empirical vorticity distribution (1.27) with respect to the microcanonical distribution on the phase space of N vortices. This result quantifies the previous law of large numbers by stating, essentially, that for any $\delta > 0$

$$\mu^{(N)}_{[e_-, e_+]}\left(\left\{\left|\int_\Lambda dx\, \hat{\omega}_N(x) f(x) - \int_\Lambda dx\, \varrho^*(x) \cdot f(x)\right| \geqslant \delta\right\}\right) \leqslant e^{-N \cdot \Delta} \tag{1.39}$$

for some $\Delta > 0$. In fact, Δ is the difference between the maximum entropy, $s(e) = s(\varrho^*)$, and the maximum of $s(\varrho)$ under the additional constraint $|\varrho(f) - \varrho^*(f)| \geqslant \delta$. This result is proved by means of a *large-deviations theorem* for exchangeable distributions, which somewhat generalizes well-known theorems of Sanov type. The latter also yields independent proofs of the main results on the convergence of entropy and correlation functions. However, we establish these results mainly for their foundational significance in understanding the statistical and dynamical basis of Onsager's theory.

For convenience we have so far discussed the case of univalent vortices with all circulations $R_i \equiv 1$, and, likewise, the proofs we give below are for that special situation. The restriction especially to vortices all of the same sign is rather severe as it rules out many interesting cases, e.g., the commonly simulated example of the overall neutral vortex gas composed of equally many vortices with $R_i = +1$ and $R_j = -1$ in a periodic domain. (However, it does correspond well to certain experimental situations, such as the final equilibrium states of initial shear layers.[23]) At the cost of some complication of the arguments, our proofs can be generalized to the situations of both positive and negative signs of vortices, and the results of the analysis are similar to those for the single-sign case. The vorticity fields are then best considered as *signed measures* (see, e.g., ref. 7), whose "charge" densities in the equilibrium distribution obey a modified form of the mean-field equation in (1.21):

$$\rho(x) = \frac{1}{Z_+} \exp\left[-\beta \int V_\Lambda(x, y)\, \rho(y)\, dy \right] - \frac{1}{Z_-} \times \exp\left[+\beta \int V_\Lambda(x, y)\, \rho(y)\, dy \right] \tag{1.40}$$

This distribution is characterized as the one which maximizes the entropy functional

$$s(\rho) = -\int \rho_+(x) \log \rho_+(x)\, dx - \int \rho_-(x) \log \rho_-(x)\, dx \tag{1.41}$$

subject to the constraint of fixed energy (here ρ_+, ρ_- are the densities of the positive and negative vortices, giving the total vorticity $\rho = \rho_+ + \rho_-$). In fact, this was the situation considered originally by Joyce and Montgomery[22,21] and Eq. (1.40) seems to describe well some recent numerical simulations.[24] We have omitted treatment of this more general situation partly because, in our opinion, the Young measure methods discussed in Section 4 provide a more general and convenient description of signed vorticity distributions. In that setting the above mean-field equation is recovered in a "dilute-vorticity" limit.[25]

We must mention a technical limitation of our proof is that it works only for *bounded* potentials V_Λ and W_Λ. Therefore, we must "cut off" the singularities which occur in $V_\Lambda(x, y)$ for $|x-y| \to 0$ and in $W_\Lambda(x)$ for $\text{dist}(x, \partial\Lambda) \to 0$. To be specific, we may introduce regularized potentials V_δ, W_δ—along the lines of those defined in Chapter 2 of ref. 7 for a circular domain Λ—which agree with V_Λ, W_Λ outside a "core" of radius δ and which have the property to converge pointwise, monotonically to V_Λ, W_Λ as $\delta \to 0^+$. Then, strictly speaking, our results for the thermodynamic limit $N \to \infty$ hold only for a fixed $\delta > 0$. On the other hand, we may take δ as small as we please, e.g., an atomic radius! In particular, as we argue in detail in Section 4, the equilibrium predictions hold with good accuracy and high probability for a *fixed N* sufficiently large, independent of δ.

The physical limitation of Onsager's theory to the conditions where the vortex model is applicable is, however, quite restrictive, as Onsager himself recognized. On the other hand, the full Euler equations given in (1.2) are, in fact, an infinite-dimensional Hamiltonian system.[26,27] This suggests that a similar statistical theory might be based directly on (1.2) rather than on the special point-vortex approximation. Unfortunately, the infinite-dimensional character of the dynamics has prevented the construction of suitable equilibrium measures which might describe the long-time statistics of the empirical flow field. Recently, however, an extension of Onsager's theory to situations with a continuous distribution of vorticity has been made independently by Miller[25,28] and Robert.[29] Their methods appear rather different, but, in fact, the theories are completely equivalent. Their generalization of Onsager's theory removes the restriction to situations of dilute vorticity, although it is also limited, fundamentally, to times less than the viscous diffusion time. Both Miller and Robert emphasize the need to consider all the conserved first integrals of the Euler flow defined by (1.2). In our opinion, however, the dynamical foundations and the physical conditions for the validity of the Miller–Robert theory are not yet completely clarified. For that reason, we have restricted most of our discussion to the original point-vortex model of Onsager. However, in Section 4 we give an essentially self-contained review and critical discussion of the extended theory.

Let us summarize now the content of the remainder of this paper. In Section 2 we establish the thermodynamic limit of the microcanonical entropy and correlation functions, proving also an associated maximum entropy principle. In Section 3 we establish a form of equivalence of ensembles, valid under condition of concavity of the entropy, and validity of the Joyce–Montgomery mean-field equations. In Section 4 we discuss the large-fluctuation theory and the statistical foundations of both the original Onsager theory and the extended Robert–Miller theory. Finally, in the Appendix we give the proof of the large-deviations results.

2. CONVERGENCE OF THE ENTROPY AND THE MICROCANONICAL CORRELATION FUNCTIONS IN THE MEAN-FIELD THERMODYNAMIC LIMIT

As discussed in the Introduction, we wish to establish the limits

$$\lim_{\Delta \downarrow \{e\}} \lim_{N \to \infty} \frac{1}{N} \log \Omega_{N,\Delta} = s(e) \tag{2.1}$$

defining the thermodynamic entropy as a function of (pair-specific) energy, and in the appropriate sense, limits

$$\lim_{N \to \infty} \varrho_r^{(N)} = \varrho_r \tag{2.2}$$

of the correlation functions defined from the microcanonical distribution

$$\mu_\Delta^{(N)}(\cdot) \equiv \hat{\lambda}^N \left(\cdot \left| \frac{\tilde{H}^{(N)}}{N} \in \Delta \right. \right) \tag{2.3}$$

$\hat{\lambda}(\cdot) \equiv \lambda(\cdot)/\lambda(\Lambda)$ for a mean-field Hamiltonian of the form in (1.19):

$$\tilde{H}(N)(x_1,..., x_N) = \frac{1}{N} \sum_{i<j} V(x_i, y_i) + \frac{1}{N} \sum_i W(x_i) \tag{2.4}$$

We may here assume that V is nonnegative, continuous in $\Lambda \times \Lambda$, vanishing for one of its arguments in the boundary $\partial\Lambda$, and, therefore, bounded above by some constant $2B < +\infty$. Likewise, W is assumed continuous on Λ and bounded as $\|W\|_\infty < B$. The constant B diverges as the cutoff δ, discussed in the Introduction, goes to zero.

We now state the fundamental theorem from which these results shall directly follow. Let us define, for any energy interval Δ (open, closed, or half-open/half-closed)

$$s(\Delta) \equiv \sup\{s(\varrho): \varrho \in \mathscr{P}^1, \varepsilon(\varrho) \in \Delta\} \tag{2.5}$$

with $\mathscr{P}^1$ the Borel probability measures on Λ, and $s(\varrho)$, $\varepsilon(\varrho)$ defined as in Eqs. (1.22), (1.23) of the Introduction. Then, we have the following result.

Theorem 2.1. The following conditions hold:

(i) $\overline{\lim}_{N \to \infty} (1/N) \log \Omega_{N,\Delta} \leqslant s(\bar{\Delta})$, where $\bar{\Delta}$ is the closure of Δ.

(ii) $\varliminf_{N\to\infty} (1/N)\log \Omega_{N,\Delta} \geqslant s(\mathring{\Delta})$, where $\mathring{\Delta}$ is the interior of Δ.

(iii) If $s(\bar{\Delta}) = s(\mathring{\Delta})$, then any weak limit point of the sequence $\mu_\Delta^{(N)}$ is of the form

$$\mu = \int \nu_\mu(d\varrho)\, \varrho^{\mathbb{N}} \tag{2.6}$$

with ν_μ a Borel probability measure on $\mathscr{P}^1$ which is supported on the set

$$\mathscr{S}_{\bar{\Delta}} = \{\varrho \in \mathscr{P}^1 \colon \varepsilon(\varrho) \in \bar{\Delta},\, s(\varrho) = s(\bar{\Delta})\} \tag{2.7}$$

We choose a subsequence $\mu_\Delta^{(N_k)}$ such that $\mu_\Delta^{(N_k)} \to \mu$ (such a subsequence exists by compactness). Clearly, $\mu \in \mathscr{P}_\sigma$, i.e., μ is a symmetric distribution on $\Lambda^{\mathbb{N}}$. Hence, by the theorem of the Finetti[30] cited in the Appendix, an integral decomposition of the form (2.6) holds. We characterize now its support:

Lemma 2.2. ν_μ is supported on the set $\mathscr{E}_{\bar{\Delta}} = \{\varrho \in \mathscr{P}^1 \colon \varepsilon(\varrho) \in \bar{\Delta}\}$.

Proof. We consider the random variable $\tilde{H}^{(N_k)}/N_k$ distributed with respect to $\mu_\Delta^{(N_k)}$. It is easy to check that by weak convergence (along the subsequence) for any $n \in \mathbb{N}$,

$$\lim_{k\to\infty} \mu_\Delta^{(N_k)}\left(\left(\frac{\tilde{H}^{(N_k)}}{N_k}\right)^n\right) = \int \nu_\mu(\varrho)\, \varepsilon(\varrho)^n \tag{2.8}$$

where we have used that V is bounded, continuous. Since $|\tilde{H}^{(N_k)}/N_k| \leqslant B$, the powers form a convergence-determining class of functions and $\tilde{H}^{(N_k)}/N_k$ converge weakly to the random variable $\varepsilon(\varrho)$ distributed with respect to ν_μ (see ref. 31, Theorem 8.48 and Proposition 8.49). Since the distribution of $\tilde{H}^{(N_k)}/N_k$ with respect to $\mu_\Delta^{(N_k)}$ has support in $\bar{\Delta}$, so must the distribution of $\varepsilon(\varrho)$ with respect to ν_μ. ∎

Proof of the Theorem. We follow here a modified form of the strategy used by Messer and Spohn[18] for the canonical distribution. (We thank M. Pulvirenti for insisting on the possibility of doing so: our original proof of the theorem proceeded by means of the large-deviations theory of Section 4.)

(a) *Upper bound.* Let $\mu_n^{(N_k)}$ be the marginal of $\mu_\Delta^{(N_k)}$ onto $\{1,..., n\}$. By subadditivity of entropy for any N and n

$$\frac{1}{N_k} S_{N_k}(\mu_\Delta^{(N_k)}) \leqslant \frac{1}{N_k}\left[\!\left[\frac{N_k}{n}\right]\!\right] S_n(\mu_n^{(N_k)}) + \frac{1}{N_k} S_{N_k - n[\![N_k/n]\!]}\,(\mu^{(N_k)}_{N_k - n[\![N_k/n]\!]}) \tag{2.9}$$

where $[\![\cdot]\!]$ denotes integer part. By Jensens's inequality,

$$S_{N_k - n[\![N_k/n]\!]}(\mu^{(N_k)}_{N_k - n[\![N_k/n]\!]}) \leqslant \left(N_k - n\left[\!\!\left[\frac{N_k}{n}\right]\!\!\right]\right) \log |\Lambda| \\ \leqslant n \log |\Lambda| \tag{2.10}$$

Thus, by the upper semicontinuity of S_n,

$$\overline{\lim_{k\to\infty}} \frac{1}{N_k} S_{N_k}(\mu^{(N_k)}_\Delta) \leqslant \frac{1}{n} S_n(\mu_n) \tag{2.11}$$

for every n. Taking the limit (or the infimum) in n, we see that

$$\overline{\lim_{k\to\infty}} \frac{1}{N_k} S_{N_k}(\mu^{(N_k)}_\Delta) \leqslant h(\mu) \equiv \int \nu_\mu(d\varrho)\, s(\varrho) \tag{2.12}$$

(b) *Lower bound.* For any energy interval Δ, define the set $\mathscr{E}^{>}_{\Delta} \equiv \{\varrho \in \mathscr{P}^1 \colon \varepsilon(\varrho) \in \Delta,\ s(\varrho) > -\infty\}$. We claim that if one takes any open interval $G \subseteq \Delta$ and any ν supported on $\mathscr{E}^{>}_{G}$, then the following estimate from below is satisfied:

$$\underline{\lim}_{N\to\infty} \frac{1}{N} S_N(\mu^{(N)}_\Delta) \geqslant \int \nu(d\varrho)\, s(\varrho) \tag{2.13}$$

We may as well assume that $\int \nu(d\varrho)\, s(\varrho) > -\infty$, since the estimate is clearly true under the opposite assumption. Define the probability measure $\sigma^{(N)}_G(\cdot)$ on $(\Lambda^N, \mathscr{B}^N)$ by

$$\sigma^{(N)}_G \equiv \frac{1}{Z_N} \int \nu(d\varrho)\, \varrho^N \left(\left\{\frac{\tilde{H}^{(N)}}{N} \in G\right\} \cap (\cdot)\right) \tag{2.14}$$

with the normalizing factor

$$Z_N \equiv \int \nu(d\varrho)\, \varrho^N \left(\left\{\frac{\tilde{H}^{(N)}}{N} \in G\right\}\right) \tag{2.15}$$

We shall see below that $Z_N > 0$ for sufficiently large N, so that $\sigma^{(N)}_G$ is well defined. Observe that $\sigma^{(N)}_G \ll \lambda^N$, with the density for λ^N-a.e. $(x_1,..., x_N)$

$$\frac{\sigma^{(N)}_G(dx_1 \cdots dx_N)}{\lambda(dx_1) \cdots \lambda(dx_N)} = \frac{1}{Z_N} \chi_{\{\tilde{H}^{(N)}/N \in G\}}(x_1,..., x_N) \\ \times \int \nu(d\varrho)\, \varrho(x_1) \cdots \varrho(x_N) \tag{2.16}$$

as seen by applying the Tonelli theorem (we have taken the liberty to denote also $d\varrho/d\lambda$ by the same symbol ϱ, which should not create any confusion). Now, the maximum entropy characterization of μ_Δ^N states that

$$S_N(\mu_\Delta^{(N)}) = \sup\left\{S_N(\tilde{\mu}_N): \tilde{\mu}_N \in \mathscr{P}(\Lambda^N, \mathscr{B}^N), \tilde{\mu}_N\left(\left\{\frac{\tilde{H}^{(N)}}{N} \in \Delta\right\}\right) = 1\right\} \tag{2.17}$$

In particular,

$$S_N(\mu_\Delta^{(N)}) \geqslant S_N(\sigma_G^{(N)}) \tag{2.18}$$

On the other hand,

$$\begin{aligned} S_N(\sigma_G^{(N)}) &= \log Z_N - \frac{1}{Z_N}\int_{\{\tilde{H}^{(N)}/N \in G\}} d\lambda^N \\ &\quad \times \left[\int \nu(d\varrho)\,\varrho^N\right] \log\left[\int \nu(d\varrho)\,\varrho^N\right] \\ &\geqslant \log Z_N - \frac{1}{Z_N}\int \nu(d\varrho) \int_{\{\tilde{H}^{(N)}/N \in G\}} d\lambda^N\,\varrho^N \\ &\quad \times \left(\sum_{j=1}^N \log \varrho(x_j)\right) \end{aligned} \tag{2.19}$$

the latter by Jensen's inequality. From a generalized form of the strong law of large numbers [see Eq. (A11) in the Appendix]

$$\lim_{N\to\infty} \frac{\tilde{H}^{(N)}}{N} = \frac{1}{2}\varrho^{\mathbb{N}}(V) = \varepsilon(\varrho) \qquad \varrho^{\mathbb{N}}\text{-a.s.} \tag{2.20}$$

Since G is open and $\varepsilon(\varrho) \in G$, $\tilde{H}^{(N)}/N \in G$ for sufficiently large N, and

$$\lim_{N\to\infty} \varrho^{\mathbb{N}}\left(\left\{\frac{\tilde{H}^{(N)}}{N} \in G\right\}\right) = 1 \tag{2.21}$$

by dominated convergence. A second application of dominated convergence shows that

$$\lim_{N\to\infty} Z_N = \lim_{N\to\infty} \int \nu(d\varrho)\,\varrho^{\mathbb{R}}\left(\left\{\frac{\tilde{H}^{(N)}}{N} \in G\right\}\right) = 1 \tag{2.22}$$

Because of the inequalities $-\infty < s(\varrho) < \log |\Lambda|$ and the fact that $x \log x$ is bounded below, we see that for any $\varrho \in \mathscr{E}_G^{>}$, $|\log \varrho| \in L^1(\Lambda, \varrho)$. Hence, by the usual strong law of large numbers,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \log \varrho(x_j) = -s(\varrho) \qquad \varrho^{\mathbb{N}}\text{-a.s.} \tag{2.23}$$

Using that $-\infty < \int \nu(d\varrho)\, s(\varrho) < \log |\Lambda|$ and that

$$0 \leqslant \int \nu(d\varrho) \int \varrho(dx)[\log \varrho(x)]^- \leqslant 1/e$$

(with $x^{\pm} \equiv |x| \pm x/2$), we can again make the argument that

$$\int \nu(d\varrho) \int \varrho(dx)\, |\log \varrho(x)| < +\infty \tag{2.24}$$

Then, writing

$$\begin{aligned}
&\varrho^N \left(\chi_{\{\tilde{H}^{(N)}/N \in G\}} \left(-\frac{1}{N} \sum_{j=1}^{N} \log \varrho(x_j) \right) \right) \\
&\quad = s(\varrho)\, \varrho^N \left(\left\{ \frac{\tilde{H}^{(N)}}{N} \in G \right\} \right) \\
&\qquad + \varrho^N \left(X_{\{\tilde{H}^{(N)}/N \in G\}} \left(-\frac{1}{N} \sum_{j=1}^{N} \log \varrho(x_j) - s(\varrho) \right) \right)
\end{aligned} \tag{2.25}$$

we can apply dominated convergence once more to infer that

$$\lim_{N\to\infty} \int \nu(d\varrho)\, \varrho^N \left(\chi_{\{\tilde{H}^{(N)}/N \in G\}} \left(\frac{1}{N} \sum_{j=1}^{N} \log \varrho(x_j) \right) \right) = -\int \nu(d\varrho)\, s(\varrho) \tag{2.26}$$

Therefore, finally,

$$\varliminf_{N\to\infty} \frac{1}{N} S_N(\mu_\Lambda^{(N)}) \geqslant \int \nu_\mu(d\varrho)\, s(\varrho) \tag{2.27}$$

(c) We now infer the main statements of the theorem. For every subsequence $(\mu_\Lambda^{(N_k)})$ along which $(1/N_k)\, S_{N_k}(\mu_\Lambda^{(N_k)})$ converges, we may clearly choose another subsequence such that $\mu_\Lambda^{(N_k)}$ also weakly converges. If $\mathrm{acc}\mu_\Lambda^{(N)}$ is the set of weak accumulation points of $(\mu_\Lambda^{(N)})$, then the reasoning of (a) clearly establishes that

$$\varlimsup_{N\to\infty} \frac{1}{N} S_N(\mu_\Lambda^{(N)}) \leqslant \sup_{\mu \in \mathrm{acc}\mu_\Lambda^{(N)}} \int \nu_\mu(d\varrho)\, s(\varrho) \tag{2.28}$$

However, by the lemma, we see that the latter is bounded above by $s(\bar{\Delta})$, giving (i). To obtain (ii), take $G = \mathring{\Delta}$ in (b), so that

$$\varliminf_{N \to \infty} \frac{1}{N} S_N(\mu_\Delta^{(N)}) \geqslant \sup_{\nu: \nu(\mathscr{E}_{\mathring{\Delta}}^{>}) = 1} \int \nu(d\varrho)\, s(\varrho) \tag{2.29}$$

Specializing to ν which are delta distributions δ_ϱ with

$$\varrho \in \mathscr{E}_{\mathring{\Delta}}^{>}$$

one obtains the lower bound by $s(\mathring{\Delta})$. Under the assumption $s(\bar{\Delta}) = s(\mathring{\Delta})$, the previous estimates show that

$$\lim_{N \to \infty} \frac{1}{N} S_N(\mu_\Delta^{(N)}) = s(\Delta) = \int \nu_\mu(d\varrho)\, s(\varrho) \tag{2.30}$$

for *any* $\mu \in \operatorname{acc} \mu_\Delta^{(N)}$. We have already seen from the lemma that ν_μ is supported on $\mathscr{E}_{\bar{\Delta}}$. Since $\int \nu_\mu(d\varrho)\, s(\varrho) = s(\bar{\Delta})$, ν_μ must in fact be supported on $\mathscr{S}_{\bar{\Delta}}$, for, in the opposite case, $\int \nu_\mu(d\varrho)\, s(\varrho) < s(\bar{\Delta})$. ■

In defining $s(\Delta)$, we have used the convention that $\sup \varnothing = -\infty$. Let us note that if $s(\bar{\Delta}) > -\infty$, then $\mathscr{S}_{\bar{\Delta}} \neq \varnothing$. This follows since, in the weak topology, $\mathscr{P}^1$ is compact, s is u.s.c., and ε is continuous. To see the latter, one should note that for any sequence (ϱ_n) in $\mathscr{P}^1$, $\varrho_n \xrightarrow{w} \varrho$, the sequence of functions $\psi_n(x) \equiv \int V(x, y)\, \varrho_n(dy)$ is an equicontinuous family by uniform continuity of V on $\Lambda \times \Lambda$ and therefore $\psi_n(x) \to \psi(x) \equiv \int V(x, y)\, \varrho(dy)$ uniformly on Λ.

We now return to the physically-motivated problems mentioned at the beginning of this section. Toward addressing the first, let us define the following function:

$$s(e) = \sup\{s(\varrho): \varrho \in \mathscr{P}^1, \varepsilon(\varrho) = e\} \tag{2.31}$$

with the same convention $\sup \phi = -\infty$. Let us note that by exactly the argument used for $s(\Delta)$, the supremum in (2.31) is always attained. Therefore, for each $c \leqslant \log |\Lambda|$,

$$\{e: s(e) \geqslant c\} = \varepsilon(\{\varrho \in \mathscr{P}^1: s(\varrho) \geqslant c\}) \tag{2.32}$$

From this identity it follows further that $s(e)$ is u.s.c. since its level sets are compact, as a consequence of the continuity of ε and the compactness of level sets of $s(\varrho)$ on $\mathscr{P}^1$. These properties are enough to deduce the following result:

Proposition 2.3. For any sequence of intervals $\Delta \downarrow \{e\}$ and with $s(e)$ defined as above,

$$\lim_{\Delta \downarrow \{e\}} \lim_{N \to \infty} \frac{1}{N} \log \Omega_{N,\Delta} = s(e) \tag{2.33}$$

Proof. Combining the upper and lower limit statements of Theorem 2.1, we see that

$$s(\mathring{\Delta}) \leqslant \varliminf_{N \to \infty} \frac{1}{N} \log \Omega_{N,\Delta} \leqslant \varlimsup_{N \to \infty} \frac{1}{N} \log \Omega_{N,\Delta} \leqslant s(\bar{\Delta}) \tag{2.34}$$

However, just as a matter of working out the definitions,

$$s(\Delta) = \sup\{s(e) : e \in \Delta\} \tag{2.35}$$

for any Δ. But then, as a consequence of the u.s.c. of $s(e)$,

$$\lim_{\Delta \downarrow \{e\}} s(\mathring{\Delta}) = \lim_{\Delta \downarrow \{e\}} s(\bar{\Delta}) = s(e) \tag{2.36}$$

which implies the result. ■

We will establish now some of the basic properties of the entropy $s(e)$. Define dom $s = \{e : s(e) > -\infty\}$. From (2.31) it follows that

$$\operatorname{dom} s = \varepsilon(\{\varrho \in \mathscr{P}^1 : s(\varrho) > -\infty\}) \tag{2.37}$$

so that, clearly, dom $s \subseteq [0, B]$. In fact, $0 \notin$ dom s, since we assume that $V(x, y) > 0$ for $x, y \notin \partial\Lambda$. Thus, $\varepsilon(\varrho) = 0$ iff supp $\varrho \subseteq \partial\Lambda$, but then surely $s(\varrho) = -\infty$. Hence, $s(0) = -\infty$. B may or may not belong to dom s, depending upon the exact way in which V is cut off: for example, if $V(x, y) = 2B$ for a set of (x, y) of positive λ^2-measure, then $B \in$ dom s, but, typically, $B \notin$ dom s. On the other hand, using the continuity of V, one can construct by elementary means for each $e \in (0, B)$ a $\varrho \in \mathscr{P}^1$ with $\varepsilon(\varrho) = e$ and $s(\varrho) > -\infty$. Therefore, it always holds that $(0, B) \subseteq$ dom s.

Let us next observe that s has the basic property conjectured by Onsager: namely, s takes on its maximum value at e_{crit} given by

$$e_{\text{crit}} = \frac{1}{2\,|\Lambda|^2} \int dx \int dy\, V(x, y) \tag{2.38}$$

Indeed, $s(\varrho)$ takes on its maximum $\log |\Lambda|$ for the unique distribution in $\mathscr{P}^1$ defined by $\lambda(\cdot)/|\Lambda| = \varrho_{\text{crit}}(\cdot)$, with $|\Lambda| = \lambda(\Lambda)$. Since $\varepsilon(\varrho_{\text{crit}}) = e_{\text{crit}}$, it is just a consequence of the definition of s that $s(e_{\text{crit}}) = \log |\Lambda|$ and $s(e) < \log |\Lambda|$

for $e \neq e_{\text{crit}}$. If, as in favorable cases, s is also (closed) *concave*, than it must be decreasing for $e > e_{\text{crit}}$, as was also suggested by Onsager. In particular, $\beta(e) \equiv s'(e) < 0$, where s is differentiable for $e > e_{\text{crit}}$, and, at the possible countable number of points where s is nondifferentiable, the superdifferential $\partial s(e) = [\beta_-(e), \beta_+(e)] \subseteq (-\infty, 0)$ [with $\beta_\pm(e) = D_\pm s(e)$ the left and right derivatives].

The function $s(e)$ defined in (2.31) is not, however, *a priori* concave. A very similarly defined function

$$\bar{s}(e) = \sup\{h(\mu) \colon \mu \in \mathscr{P}_\sigma, \bar{\varepsilon}(\mu) = e\} \tag{2.39}$$

on the other hand, *is* concave. Here, for each $\mu \in \mathscr{P}_\sigma$,

$$\begin{aligned} \bar{\varepsilon}(\mu) &= \tfrac{1}{2} \int \mu_2(dx, dy)\, V(x, y) \\ &= \int \nu_\mu(d\varrho)\, \varepsilon(\varrho) \end{aligned} \tag{2.40}$$

The concavity of $\bar{s}$ is a consequence of the fact that $\bar{\varepsilon}$ is an *affine* function on $\mathscr{P}_\sigma$, so that, for each $e_1, e_2 \in \mathbb{R}$, $0 \leqslant \lambda \leqslant 1$,

$$\lambda \bar{\varepsilon}^{-1}(\{e_1\}) + (1-\lambda)\, \bar{\varepsilon}^{-1}(\{e_2\}) \subseteq \bar{\varepsilon}^{-1}(\{\lambda e_1 + (1-\lambda)\, e_2\}) \tag{2.41}$$

Thus,

$$\begin{aligned} \bar{s}(\lambda e_1 + (1-\lambda)\, e_2) &\geqslant \sup h(\lambda \bar{\varepsilon}^{-1}(\{e_1\}) + (1-\lambda)\, \bar{\varepsilon}^{-1}(\{e_2\})) \\ &= \sup\{h(\lambda\mu_1 + (1-\lambda)\, \mu_2) \colon \mu_i \in \mathscr{P}_\sigma, \bar{\varepsilon}(\mu_i) \\ &= e_i, i = 1, 2\} \\ &= \lambda \cdot \sup\{h(\mu) \colon \mu \in \mathscr{P}_\sigma, \bar{\varepsilon}(\mu) = e_1\} \\ &\quad + (1-\lambda) \sup\{h(\mu) \colon \mu \in \mathscr{P}_\sigma, \bar{\varepsilon}(\mu) = e_2\} \\ &= \lambda \cdot \bar{s}(e_1) + (1-\lambda)\, \bar{s}(e_2) \end{aligned} \tag{2.42}$$

since h is affine. Since $\bar{\varepsilon}$ as defined in (2.40) is clearly weakly continuous on $\mathscr{P}_\sigma$, $\bar{s}(e)$ is u.s.c. by exactly the same argument given above for $s(e)$, but in the context of $\mathscr{P}_\sigma$ rather than $\mathscr{P}^1$. That is, $\bar{s}$ is a closed, concave function. Other properties of $\bar{s}$ are similar to those of s: dom $\bar{s}$ = dom s, $\bar{s}(e_{\text{crit}}) = \log |\Lambda|$, and $\bar{s}(e) < \log |\Lambda|$ for $e \neq e_{\text{crit}}$. These follow by easy adaptations of the arguments given for $s(e)$.

In fact, $\bar{s}$ is a majorant of s: for all $e \in \mathbb{R}$,

$$\bar{s}(e) \geqslant s(e) \tag{2.43}$$

This follows simply from the definition of $\bar{s}$, specializing $\mu \in \mathscr{P}_\sigma$ there to $\mu = \varrho^{\mathbb{N}}$, $\varrho \in \mathscr{P}^1$. (In fact, we see below that $\bar{s}$ is the closed, concave hull of s, i.e., the least closed, concave majorant of s.) For $e \leqslant e_{\text{crit}}$, more can be said. From what we have said above, $\bar{s}$ is actually (strictly) *increasing* for $e < e_{\text{crit}}$. Furthermore, for every $e \in \operatorname{dom} \bar{s}$, the set

$$\tilde{\mathscr{S}}_e \equiv \{\mu \in \mathscr{P}_\sigma : \tilde{\varepsilon}(\mu) = e, h(\mu) = \bar{s}(e)\} \neq \phi \tag{2.44}$$

using—for the weak topology—the compactness of $\mathscr{P}_\sigma$, u.s.c. of h, and the continuity of $\tilde{\varepsilon}$. Hence, for any $\mu \in \tilde{\mathscr{S}}_e$ with $e < e_{\text{crit}}$,

$$\begin{aligned}
\bar{s}(e) = h(\mu) &= \int \nu_\mu(d\varrho)\, s(\varrho) \\
&\leqslant s(\pi_1(\mu)) \qquad \text{with} \quad \pi_1(\mu) = \int \nu_\mu(d\varrho)\, \varrho \\
&\qquad [\text{by Jensens's inequality, using concavity of } s(\varrho)] \\
&\leqslant s(\varepsilon(\pi_1(\mu))) \\
&\leqslant \bar{s}(\varepsilon(\pi_1(\mu))) \leqslant \bar{s}(e)
\end{aligned} \tag{2.45}$$

since $e = \varepsilon(\mu) = \int \nu_\mu(d\varrho)\, \varepsilon(\varrho) \geqslant \varepsilon(\pi_1(\mu))$ by convexity of $\varepsilon(\varrho)$. Therefore, $\bar{s}(\varepsilon(\pi_1(\mu))) = \bar{s}(e)$, which can only be true if $\varepsilon(\pi_1(\mu)) = e$, and thus, by the chain of inequalities,

$$s(e) = \bar{s}(e) \qquad (e \leqslant e_{\text{crit}}) \tag{2.46}$$

An immediate consequence is that $s(e)$ is indeed concave for $e \leqslant e_{\text{crit}}$. However, we can give no argument, in general, that $s(e)$ is concave for $e \geqslant e_{\text{crit}}$, and, indeed, for certain domain geometries Λ, violations of concavity are found.[32]

A natural question which arises is the relation of $s(e)$ to the free energy $\phi(\beta)$ (actually, β times free energy) defined from the canonical partition function

$$Z_{N,\beta} \equiv \int dx_1 \cdots dx_N\, \varepsilon^{-\beta \tilde{H}^{(N)}(x_1,\dots,x_N)} \tag{2.47}$$

as

$$\phi(\beta) \equiv -\lim_{N \to \infty} \frac{1}{N} \log Z_{N,\beta} \tag{2.48}$$

If we define, for each $\beta \in \mathbb{R}$, $\phi_\beta : \mathscr{P}^1 \to \mathbb{R}$ by

$$\phi_\beta(\varrho) = \beta\varepsilon(\varrho) - s(\varrho) \tag{2.49}$$

and $\tilde{\phi}_\beta: \mathscr{P}_\sigma \to \mathbb{R}$ by

$$\tilde{\phi}_\beta(\mu) = \beta\tilde{\varepsilon}(\mu) - h(\mu) = \int v_\mu(d\varrho)\, \phi_\beta(\varrho) \tag{2.50}$$

then it follows from the work of Messer and Spohn[18] that the limit in (2.48) exists and equals

$$\begin{aligned}\phi(\beta) &= \inf\{\phi_\beta(\varrho)\}: \varrho \in \mathscr{P}^1\} \\ &= \inf\{\tilde{\phi}_\beta(\mu): \mu \in \mathscr{P}_\sigma\}\end{aligned} \tag{2.51}$$

We see then at once that

$$\begin{aligned}\phi(\beta) &= \inf_{e>0} [\beta e - s(e)] \\ &= \inf_{e>0} [\beta e - \bar{s}(e)]\end{aligned} \tag{2.52}$$

so that ϕ is a Legendre transform of both s and $\bar{s}$. Because $\bar{s}$, in particular, is closed, concave, it is the conjugate function to $\phi(\beta)$:

$$\bar{s}(e) = \inf_\beta [\beta e - \phi(\beta)] \tag{2.53}$$

Therefore we can also infer that $\bar{s}$ is indeed the closed, concave hull of s.

In the particular case that $\bar{s}(e)$ is *strictly concave*, we can infer that

$$\bar{s}(e) = s(e) \tag{2.54}$$

for *all* $e \in \mathbb{R}$, not merely for $e \leqslant e_{\text{crit}}$; from which, trivially, $s(e)$ is also strictly concave. This is equivalent to the requirement that $\phi(\beta)$ be essentially smooth. For our case of a bounded (cutoff) potential, this is just the requirement that $\phi(\beta)$ be differentiable, which may be expected to hold in many situations. For the case $\delta = 0$ with Λ a circular domain, $\phi(\beta)$ is an explicitly known essentially smooth function of β, with $\phi(\beta) = +\infty$ for $\beta < -8\pi$ (see refs. 19 and 20). Hence, in that case, $\lim_{e \to +\infty} s'(e) = -8\pi$. For our modification, with $\delta > 0$ but extremely small, we may expect that $s(e)$ for $e_{\text{crit}} \ll e \ll B$ shall be nearly linear with slope -8π, but for $e \lesssim B$ shall turn downward and $\lim_{e \to B^-} s'(e) = -\infty$.

We have now verified the main expected features of s as it appears in Fig. 1. However, in addition to the possible nonconcavity we have cautioned may occur for $e \geqslant e_{\text{crit}}$, we also do not see how to rule out that s might have cusps, where it is nondifferentiable, or flat portions, where it is affine, for certain domains Λ.

Let us turn, finally, to a brief discussion of the limiting behavior of *correlation functions* defined by the microcanonical distributions $\mu_\Delta^{(N)}$. From the correlation functions, defined generally as in (1.20) of the Introduction, we can also define correlation measures $d\mu_r^{(N)} \equiv \varrho_r^{(N)}\, d\lambda^r$, or, directly, as marginals of $\mu_\Delta^{(N)}$ on $(\Lambda^r, \mathscr{B}^r)$. The following is a sample corollary of the fundamental Theorem 2.1:

Proposition 2.4. If $s(\mathring{\Delta}) = s(\bar{\Delta}) > -\infty$, then at least along a subsequence (N_k), all of the correlation measures $\mu_r^{(N)} \to \mu_r$ weakly, for each $r \in \mathbb{N}$, to the rth marginal of some $\mu \in \mathscr{P}_\sigma$. Furthermore, $\mu_r \ll \lambda^r$ with densities ϱ_r given by

$$\varrho_r(x_1,..., x_r) = \int \nu(d\varrho)\, \varrho(x_1) \cdots \varrho(x_r) \qquad \lambda^r\text{-a.e.} \tag{2.55}$$

with ν a Borel probability measure supported on $\mathscr{S}_{\bar{\Delta}}$.

The above proposition is, in fact, equivalent to part (iii) of Theorem 2.1, since, to verify weak convergence of $\mu_\Delta^{(N)} \to \mu \in \mathscr{P}_\sigma$, it is enough to show that $\mu_\Delta^{(N)}(f) \to \mu(f)$ for every continuous, *local* function f, and this is just what is implied by the weak convergence of all the correlation measures. The statement of the support properties of ν here and in the corresponding part of Theorem 2.1 is what is often referred to as the *maximum entropy principle.*[33,34] In fact, this makes rigorous part of the content of the "maximum entropy principle" heuristically employed by Joyce and Montgomery,[22] according to which also $\varrho \in \mathscr{S}_{\bar{\Delta}}$ are the "most probable" vorticity distributions over Λ.

If $\mathscr{S}_{\bar{\Delta}} = \{\varrho\}$, a singleton, then the latter statement is justified by the following remark: for $\hat{\omega}_N$ the *empirical* vorticity distribution defined in the Introduction and for any $f \in C(\Lambda)$ it follows that

$$\lim_{N \to \infty} \mu_\Delta^{(N)}((\hat{\omega}(f) - \varrho(f))^2) = 0 \tag{2.56}$$

As an immediate consequence of the Chebyshev inequality, the random variable $\hat{\omega}_N(f)$ distributed under $\mu_\Delta^{(N)}$ converges in probability to $\varrho(f)$ for each $f \in C(\Lambda)$. Therefore, the distribution ϱ is indeed overwhelmingly likely. In Section 4 we discuss even sharper statements of this form.

3. EQUIVALENCE OF ENSEMBLES AND THE JOYCE–MONTGOMERY MEAN-FIELD EQUATIONS

We discuss now a version of the equivalence of ensembles which is relevant for our problem. We have just seen that, if $s(\mathring{\Delta}) = s(\bar{\Delta})$, then the limits of $\mu_\Delta^{(N)}$ are of the form $\mu = \int \nu_\mu(d\varrho)\, \varrho^{\mathbb{N}}$, where ν_μ is supported on the set $\mathscr{S}_{\bar{\Delta}} = \{\varrho \in \mathscr{P}^1 : \varepsilon(\varrho) \in \bar{\Delta},\ s(\varrho) = s(\bar{\Delta})\}$. Let us define for any single point $e \in \mathbb{R}$ a corresponding set

$$\mathscr{S}_e = \{\varrho \in \mathscr{P}^1 : \varepsilon(\varrho) = e,\ s(\varrho) = s(e)\} \tag{3.1}$$

For the canonical distributions an analogous result on limiting distributions and support properties was established by Messer and Spohn.[18] In terms of the functionals defined in the previous section, they showed that, for each $\beta \in \mathbb{R}$, the canonical distributions at inverse temperature β, $\mu_\beta^{(N)}$, defined by

$$\mu_\beta^{(N)}(\cdot) \equiv \frac{1}{Z_{N,\beta}} \int dx_1 \cdots dx_N\, e^{-\beta \tilde{H}^{(N)}(x_1,\dots,x_N)}(\cdot) \tag{3.2}$$

have as their weak accumulation points measures of the form $\mu = \int \nu_\mu(d\varrho)\, \varrho^{\mathbb{N}}$, where ν_μ is supported on the set

$$\mathscr{F}_\beta = \{\varrho \in \mathscr{P}^1 : \phi_\beta(\varrho) = \phi(\beta)\} \tag{3.3}$$

of free energy minimizers.

We now prove the fundamental result which relates the supporting sets of the accumulation points for the two ensembles. We must emphasize that our argument depends crucially on the *concavity* of the entropy s at e, i.e., $s(e) = \bar{s}(e)$, and that we cannot make any statement on equivalence without that assumption. In the following argument, we freely identify s and $\bar{s}$ at the point e.

Theorem 3.1. For each $e \in \operatorname{dom} s$, define $\beta_-(e) = (D_+ s)(e)$, $\beta_+(e) = (D_- s)(e)$, corresponding to the right and left deviatives of s at e. Then, if s is concave at e,

$$\mathscr{S}_e \subseteq \bigcup_{\beta \in [\beta_-(e), \beta_+(e)]} \mathscr{F}_\beta \tag{3.4}$$

Proof. (1) The argument is a standard one in mathematical programming and calculus of variations based on the duality of convex functions (e.g., see ref. 35). However, we give the argument in detail, for the sake of completeness. It is natural to consider separately the cases $e \leqslant e_{\text{crit}}$ and $e \geqslant e_{\text{crit}}$ corresponding, respectively, to regions of positive and negative temperatures. Define, for every $e \in \operatorname{dom} s$, the *Lagrangian function* L_e on $\mathscr{P}^1 \times \mathbb{R}$, by

$$L_e(\varrho, \beta) = -s(\varrho) + \beta(\varepsilon(\varrho) - e) = \phi_\beta(\varrho) - \beta e \tag{3.5}$$

Now show that for $e \in \operatorname{dom} s$ and $e \leqslant e_{\text{crit}}$

$$-s(e) = \sup_{\beta \geqslant 0} \inf_{\varrho \in \mathscr{P}^1} L_e(\varrho, \beta) = \inf_{\varrho \in \mathscr{P}^1} \sup_{\beta \geqslant 0} L_e(\varrho, \beta) \tag{3.6}$$

and likewise for $e \in \operatorname{dom} s$ and $e \geqslant e_{\text{crit}}$,

$$-s(e) = \sup_{\beta \leqslant 0} \inf_{\varrho \in \mathscr{P}^1} L_e(\varrho, \beta) = \inf_{\varrho \in \mathscr{P}^1} \sup_{\beta \leqslant 0} L_e(\varrho, \beta) \tag{3.7}$$

We consider first in detail the case $e \leqslant e_{\text{crit}}$. Observe that

$$\inf_{\varrho \in \mathscr{P}^1} L_e(\varrho, \beta) = \inf_{\varrho \in \mathscr{P}^1} \phi_\beta(\varrho) - \beta e = \phi(\beta) - \beta e \tag{3.8}$$

Since $s(e)$ is the concave conjugate of $\phi(\beta)$, it follows that, for $e \leqslant e_{\text{crit}}$,

$$-s(e) = \sup_{\beta \geqslant 0} [\phi(\beta) - \beta e] \tag{3.9}$$

and, in fact, the supremum is achieved for $\beta^* \in [\beta_-(e), \beta_+(e)]$. This gives the first equality in (3.6). On the other hand,

$$\begin{aligned} \sup_{\beta \geqslant 0} L_e(\varrho, \beta) &= -\mathrm{s}(\varrho) + \sup \beta(\varepsilon(\varrho) - e) \\ &= \begin{cases} -s(\varrho) & \text{if} \quad \varepsilon(\varrho) \leqslant e \\ +\infty & \text{if} \quad \varepsilon(\varrho) > e \end{cases} \end{aligned} \tag{3.10}$$

Then, since $s(e)$ is nondecreasing for $e \leqslant e_{\text{crit}}$, it follows that

$$\inf_{\varrho \in \mathscr{P}^1} \sup_{\beta \geqslant 0} L_e(\varrho, \beta) = -\sup_{\varrho \in \mathscr{P}^1 : \varepsilon(\varrho) \leqslant e} s(\varrho) = -s(e) \tag{3.11}$$

and, as we already observed, the supremum is achieved for some $\varrho^* \in \mathscr{P}^1$. In particular, we have the second equality in (3.6).

The changes required in the argument for $e \geqslant e_{\text{crit}}$ should be completely obvious.

(2) We now show that $\varrho^* \in \mathscr{S}_e$ iff there exists a $\beta^* \in [\beta_-(e), \beta_+(e)]$ such that (ϱ^*, β^*) is a *saddle point* of $L_e(\varrho, \beta)$, that is,

$$L_e(\varrho^*, \beta) \leqslant L_e(\varrho^*, \beta^*) \leqslant L_e(\varrho, \beta^*) \tag{3.12}$$

for all $\varrho \in \mathscr{P}^1$ and $\beta \geqslant 0$ (in the case $e \leqslant e_{\text{crit}}$) or $\beta \leqslant 0$ (in the case $e \geqslant e_{\text{crit}}$).

Let us prove the direct statement for the case $e \leqslant e_{\text{crit}}$. We have seen above that

$$\sup_{\beta \geqslant 0} L_e(\varrho^*, \beta) = -s(e) = \inf_{\varrho \in \mathscr{P}^1} L_e(\varrho, \beta^*) \tag{3.13}$$

Clearly, it suffices to show that $L_e(\varrho^*, \beta^*) = -s(e)$. However, this is also obvious since, from the above, $L_e(\varrho^*, \beta^*) \leqslant -s(e) \leqslant L_e(\varrho^*, \beta^*)$. The argument for $e \geqslant e_{\text{crit}}$ is essentially identical.

Conversely, suppose (ϱ^*, β^*) is a saddle point of $L_e(\varrho, \beta)$. Taking the supremum over β and the infimum over ϱ in (3.13), we obtain

$$-s(\varrho^*) \leqslant L_e(\varrho^*, \beta^*) \leqslant \phi(\beta^*) - \beta^* e \tag{3.14}$$

and $\varepsilon(\varrho^*) \leqslant e$ (for $e \leqslant e_{\text{crit}}$) or $\varepsilon(\varrho^*) \geqslant e$ (for $e \geqslant e_{\text{crit}}$), since $L_e(\varrho^*, \beta^*)$ is finite by assumption. But then $s(\varrho^*) \leqslant s(e)$ and also $-\phi(\beta^*) + \beta^* e \geqslant s(e)$. Therefore,

$$-s(e) = -s(\varrho^*) = L_e(\varrho^*, \beta^*) = \phi(\beta^*) - \beta^* e \tag{3.15}$$

Since $s(e)$ is increasing for $e \leqslant e_{\text{crit}}$ and decreasing for $e \geqslant e_{\text{crit}}$, we conclude that, indeed, $\varrho^* \in \mathscr{S}_e$.

(3) To complete the argument, we now show that if (ϱ^*, β^*) is a saddle point of $L_e(\varrho, \beta)$, then ϱ^* minimizes ϕ_{β^*} for some $\beta^* \in [\beta_-(e), \beta_+(e)]$. From what was shown above, the condition that (ϱ^*, β^*) be a saddle point of L_e is equivalent to the pair of inequalities

$$\phi_\beta(\varrho^*) - \beta e \leqslant -s(e) \leqslant \phi_{\beta^*}(\varrho) - \beta^* e \tag{3.16}$$

for the relevant range of β and for all $\varrho \in \mathscr{P}^1$. Now, as we have essentially observed,

$$\begin{aligned}
\forall \varrho \in \mathscr{P}^1, \qquad & \beta^* e - s(e) \leqslant \phi_{\beta^*}(\varrho) \\
& \Leftrightarrow \beta^* e - s(e) \leqslant \inf_{\varrho \in \mathscr{P}^1} \phi_{\beta^*}(\varrho) = \phi(\beta^*) = \inf_e \,[\beta^* e - s(e)] \\
& \Leftrightarrow \beta^* e - s(e) = \phi(\beta^*) \\
& \Leftrightarrow \beta^* \in [\beta_-(e), \beta_+(e)]
\end{aligned} \tag{3.17}$$

Furthermore, specializing the first inequality to $\beta = \beta^*$, we obtain

$$\phi_{\beta^*}(\varrho^*) \leqslant \beta^* e - s(e) = \phi(\beta^*) \tag{3.18}$$

Thus, ϱ^* minimizes ϕ_{β^*}. ■

It is worth observing that there is a conditional converse of the above statement. Namely, if s is *strictly concave* at e, then for every minimizer ϱ^* of ϕ_{β^*} for some $\beta^* \in [\beta_-(e), \beta_+(e)]$, (ϱ^*, β^*) is a saddle point of $L_e(\varrho, \beta)$. Indeed, since $\beta^* \in [\beta_-(e), \beta_+(e)]$, we have already the second inequality

$-s(e) \leqslant \phi_{\beta^*}(\varrho) - \beta^* e$ by the above argument. To establish the first inequality, we argue as follows: set $\varepsilon(\varrho^*) = e^*$, so that $s(\varrho^*) \leqslant s(e^*)$. Hence,

$$\beta^* e^* - s(e^*) \leqslant \beta^* \varepsilon(\varrho^*) - s(\varrho^*) = \phi_{\beta^*}(\varrho^*) = \phi(\beta^*) \tag{3.19}$$

However, since $\phi(\beta^*) = \inf_e [\beta^* e - s(e)]$, in fact,

$$\beta^* e^* - s(e^*) = \beta^* e^* - s(\varrho^*) = \phi(\beta^*) \tag{3.20}$$

Therefore, $e^* \in (\partial\phi)(\beta^*)$, the superdifferential of the concave function ϕ at β^*. Since $\beta^* \in [\beta_-(e), \beta_+(e)] = (\partial s)(e)$ and, by assumption, s is strictly concave, $(\partial\phi)(\beta^*) = \{e\}$, i.e.,

$$\varepsilon(\varrho^*) = e^* = e \tag{3.21}$$

Hence, using the second half of (3.20), we see that

$$s(\varrho^*) = \beta^* e - \phi(\beta^*) = s(e) \tag{3.22}$$

However, we have seen in the proof of the above theorem that $\varepsilon(\varrho^*) \leqslant e$ (for $e \leqslant e_{\text{crit}}$) or $\varepsilon(\varrho^*) \geqslant e$ (for $e \geqslant e_{\text{crit}}$) and $s(\varrho^*) = s(e)$ are precisely equivalent to the validity of the inequality

$$\phi_\beta(\varrho^*) - \beta e \leqslant -s(e) \tag{3.23}$$

for all relevant β [i.e., $\beta \geqslant 0$ or $\beta \leqslant 0$, respectively; see (1) of the proof]. We conclude that (ϱ^*, β^*) is indeed a saddle point of L_e.

Therefore, under the additional assumption that s is strictly concave at e, we have the precise *equality*

$$\mathscr{S}_e = \bigcup_{\beta \in [\beta_-(e), \beta_+(e)]} \mathscr{F}_\beta \tag{3.24}$$

On the other hand, if s is linear over some segment including e, the inclusion will in general be strict. We may note, similarly, that if $\phi(\beta)$ is strictly concave (which we expect: see below), then s is essentially smooth and $\partial s(e) = \{\beta(e)\}$, where $\beta(e) = s'(e)$. Hence, under the assumption that ϕ is strictly concave

$$\mathscr{S}_e \subseteq \mathscr{F}_{\beta(e)} \tag{3.25}$$

and under *both* assumptions that $s(e)$ is essentially smooth *and* strictly concave, identity holds,

$$\mathscr{S}_e = \mathscr{F}_{\beta(e)} \tag{3.26}$$

The implication we wish to draw from the theorem is the following: the support $\mathscr{S}_e$ of the limiting microcanonical measures is contained in the support $\mathscr{F}_\beta$ of the limit of some canonical measure for a temperature $\beta \in [\beta_-(e), \beta_+(e)]$. In the most favorable case that $s(e)$ is essentially smooth and strictly concave, there is identity,

$$\mathscr{S}_e = \mathscr{F}_{\beta(e)} \tag{3.27}$$

However, even then there may be in general a nonidentity

$$\mathrm{acc}_{N\uparrow\infty}\, \mu^{(N)}_{[e_-,e_+]} \neq \mathrm{acc}_{N\uparrow\infty}\, \mu^{(N)}_{\beta(e)} \tag{3.28}$$

$\{$with $s(e) = \sup s([e_-, e_+])\}$. In the very special but important situation that s is essentially smooth and $\mathscr{F}_{\beta(e)} = \{\varrho_{\beta(e)}\}$, a singleton, then also $\mathscr{S}_e = \{\varrho_{\beta(e)}\}$, and for $N \to \infty$

$$\mu^{(N)}_{\beta(e)} \xrightarrow{W} \varrho^{\mathbb{N}}_{\beta(e)} \tag{3.29}$$

and

$$\mu^{(N)}_{[e_-,e_+]} \xrightarrow{W} \varrho^{\mathbb{N}}_{\beta(e)} \tag{3.30}$$

also. According to the remark at the end of Section 2, the empirical vorticity distribution under these assumptions distributed according to $\mu^{(N)}_{\beta(e)}$ or $\mu^{(N)}_{[e_-,e_+]}$ converges in probability to $\varrho_{\beta(e)}$. In that case, we have a complete equivalence of the two ensembles.

For the canonical distribution, there is actually the further information obtained from explicit correlation bounds[18-20] that for $\mu \in \mathrm{acc}_{N\uparrow\infty}\, \mu^{(N)}_\beta$

$$\nu_\mu(\mathscr{F}_\beta \cap L^\infty) = 1 \tag{3.31}$$

Now, it is not hard to verify that $s(\varrho)$, $\varepsilon(\varrho)$ are C^1-functions on the Banach space L^∞ with differentials

$$ds(\varrho)\cdot h = -\int_\Lambda dx\,[\log \varrho(x) + 1]\, h(x) \tag{3.32}$$

and

$$d\varepsilon(\varrho)\cdot h = \int_\Lambda dx \int_\Lambda dy\; V(x, y)\, \varrho(x)\, h(y) \tag{3.33}$$

defined for all $h \in L^\infty(\Lambda)$ and for ρ in an L^∞-neighborhood of any

$\rho^* \in L^\infty_+ \equiv \{\rho \in L^\infty: L^\infty: \text{ess.inf}\, \rho > 0\}$. Therefore also $\phi_\beta(\varrho) = \beta\varepsilon(\varrho) - s(\varrho)$ is differentiable, and any solution of the variational problem $\inf\{\phi_\beta(\varrho): \varrho \in \mathscr{P}^1 \cap L^\infty_+\}$ is also a solution of the variational equation

$$d\phi_\beta(\varrho) = \lambda^* \tag{3.34}$$

where λ^* is a Lagrange multiplier to incorporate the normalization constraint

$$\int dx\, \varrho(x) = 1 \tag{3.35}$$

By using the explicit forms of the differentials ds, $d\varepsilon$, it is easily seen that the variational equation is equivalent to the *mean-field equation*

$$\varrho(x) = \frac{\exp[-\beta \int dy\, V(x, y)\, \varrho(y)]}{\int dx \exp[-\beta \int dy\, V(x, y)\, \varrho(y)]} \tag{3.36}$$

such as was first derived by Joyce and Montgomery[22] (actually, as noted in the Introduction, their equation was for a slightly different situation). It is here seen to be obeyed by the limiting canonical distributions under the mild extra condition that ess.inf $\rho > 0$ [which from Eq. (3.36) is, indeed, found true *a posteriori*]. On the other hand, it should be noted that there may be solutions of the mean-field equation which are not solutions of the variational problem. It has been pointed out by Caglioti[32] that strict concavity of $\phi(\beta)$ follows if the mean-field equation is obeyed, since no ρ may satisfy Eq. (3.36) for two distinct β's.

While we expect that for the microcanonical case also, if $s([e_-, e_+]) > -\infty$ and if $\mu \in \text{acc}_{N\uparrow\infty}\, \mu^{(N)}_{[e_-, e_+]}$, then $\nu_\mu(L^\infty) = 1$, we do not see how to obtain the correlation bounds to verify this. Of course, if s is differentiable and $\mathscr{F}_{\beta(e)} = \{\varrho_{\beta(e)}\}$, then also $\mathscr{S}_e = \{\varrho_{\beta(e)}\} \subseteq L^\infty$, so the result follow. If, in any case, we have *a* priori that $\varrho \in \mathscr{S}_e \cap L^\infty_+$, then the method of Lagrange multipliers implies, since $\psi(x) = \int_\Lambda dy\, V(x, y)\, \varrho(y) \neq 0$, that there exists a $\beta^* \in \mathbb{R}$ such that

$$ds(\varrho) = \beta^*\, d\varepsilon(\varrho) + \lambda^* \tag{3.37}$$

which is identical to the above variational equation and the equivalent mean-field equation, for $\beta = \beta^*$, derived from the canonical distribution. It then follows easily in conjunction with Theorem 3.1 and the assumption $\psi \neq 0$ that $\beta^* \in [\beta_-(e), \beta_+(e)]$.

4. LARGE FLUCTUATIONS AND THE STATISTICAL FOUNDATIONS OF ONSAGER'S EQUILIBRIUM THEORY

We here discuss the topic of large equilibrium fluctuations of the empirical vorticity in the mean-field limit. Then we draw some conclusions for the ergodicity requirements necessary to justify Onsager's equilibrium assumption, emphasizing their extreme weakness in practice. These results and observations have, of course, a general validity in equilibrium statistical mechanics, and our point here is just that they carry over to the mean-field situation. The proofs of the statements given here are mostly relegated to the Appendix, for readers interested in the technical aspects, and we only give precise statements and a discussion of the physical implications. We finally make a critical discussion of the recent Robert–Miller extension of the Onsager theory to allow continuous distributions of vorticity. Although we believe this theory is an important advance, we are not satisfied with the attempts so far to provide it a dynamical foundation, which, in fact, do not give a clear understanding of the *physical conditions* necessary for its validity. In fact, it was largely the Robert–Miller work which motivated this section, just because the dynamical picture is much clearer for the vortex model and a rather complete discussion of the theoretical foundations can be made. The disadvantage of our restriction to the vortex model is that it applies, literally, in relatively few situations.

The principal result of this section concerns the asymptotics of fluctuation probabilities for the *empirical vorticity*

$$\hat{\omega}_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \in \mathscr{P}^1 \tag{4.1}$$

considered as a random variable distributed under the microcanonical distribution $\mu_\Delta^{(N)}$. It is physically just a statement of the Einstein–Boltzmann fluctuation principle, and technically an upper-bound large-deviations estimate. In a form which suffices for our purposes we have:

Theorem 4.1. If Δ is an interval with $s(\mathring{\Delta}) = s(\bar{\Delta})$ and $\mathscr{A}$ any Borel subset of $\mathscr{P}^1$, then

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mu_\Delta^{(N)}(\hat{\omega}_N \in \mathscr{A}) \leqslant \sup_{\varrho \in \bar{\mathscr{A}}} \Delta s(\varrho \mid \bar{\Delta}) \tag{4.2}$$

where $\Delta s(\varrho \mid \bar{\Delta}) = s(\varrho \mid \bar{\Delta}) - s(\bar{\Delta}) \leqslant 0$, and

$$s(\varrho \mid \bar{\Delta}) = \begin{cases} s(\varrho) & \text{if } \ \varepsilon(\varrho) \in \bar{\Delta} \\ -\infty & \text{if } \ \varepsilon(\varrho) \notin \bar{\Delta} \end{cases} \tag{4.3}$$

We here offer only a brief discussion of the proof. Recalling the definition of the m.c. distribution as

$$\mu_{\Delta}^{(N)}(\cdot) \equiv \hat{\lambda}^N(\cdot \mid \tilde{H}_N/N \in \Delta) \tag{4.4}$$

we see that estimates of the following form would suffice:

$$\varlimsup_{N \to \infty} \frac{1}{N} \log \hat{\lambda}^N(\{\hat{\omega}_N \in \mathscr{A}\} \cap \{\tilde{H}^{(N)}/N \in \Delta\}) \leqslant \sup_{\varrho \in \overline{\mathscr{A} \cap \varepsilon^{-1}(\Delta)}} (s(\varrho) - \log |\Lambda|) \tag{4.5}$$

and

$$\varliminf_{N \to \infty} \frac{1}{N} \log \hat{\lambda}^N\left(\left\{\frac{\tilde{H}^{(N)}}{N} \in \Delta\right\}\right) \geqslant \sup_{\varrho \in \varepsilon^{-1}(\mathring{\Delta})} (s(\varrho) - \log |\Lambda|) \tag{4.6}$$

Since it is easy to check that

$$\left\| \frac{\tilde{H}^{(N)}}{N} - \varepsilon(\hat{\omega}_N) \right\|_{\infty} \leqslant \frac{2B}{N} \tag{4.7}$$

these are essentially large-deviations estimates of the standard form, but for the product measures $\hat{\lambda}^N$. In fact, the required results are nearly identical to the so-called *Sanov property*:

$$\varlimsup_{N \to \infty} \frac{1}{N} \log \hat{\lambda}^N(\hat{\omega}_N \in \mathscr{A}) \leqslant \sup_{\varrho \in \bar{\mathscr{A}}} (s(\varrho) - \log |\Lambda|) \tag{4.8}$$

and

$$\varliminf_{N \to \infty} \frac{1}{N} \log \hat{\lambda}^N(\hat{\omega}_N \in \mathscr{A}) \geqslant \sup_{\varrho \in \mathring{\mathscr{A}}} (s(\varrho) - \log |\Lambda|) \tag{4.9}$$

for each $\mathscr{A} \in \mathscr{B}(\mathscr{P}^1)$, which is discussed, for example, in refs. 34 and 36. Because we require a slight variation, for which we did not find a convenient source in the literature, we outline proofs in the Appendix. In fact, our argument there follows very closely a proof of Georgii of *level-3* large deviations for Gibbsian random fields on integer lattices $\mathbb{Z}^d$.[33] By taking advantage of the affine structure at level 3 we are able to prove technically somewhat stronger results than those obtained by a level-2 formulation.

The immediate implication of the theorem is the following: let $\mathscr{K}$ be a closed subset of $\mathscr{P}^1$ such that $\mathscr{K} \cap \mathscr{S}_{\bar{\Delta}} = \varnothing$. Then,

$$\Delta_{\mathscr{K}} \equiv -\sup_{\varrho \in \mathscr{K}} \Delta s(\varrho \mid \bar{\Delta}) > 0 \tag{4.10}$$

and, for any small $\eta > 0$ and all sufficiently large N,

$$\mu_{\Delta}^{(N)}(\hat{\omega}_N \in \mathscr{K}) \leqslant e^{-N(\Delta_{\mathscr{K}} - \eta)} \tag{4.11}$$

That is, the probability for the empirical vorticity to lie in the set $\mathscr{K}$, which excludes the set $\mathscr{S}_{\bar{\Delta}}$ of entropy maximizers, decays exponentially in the number of vortices N. This is physically the sharpest form of the *maximum entropy principle*, since it states that the probability to observe anything other than the equilibrium (maximum entropy) configuration is exponentially small.

We have already observed in the Introduction that serious doubts may be raised about the (strict) ergodicity of the point-vortex model. However, we see now that results considerably weaker than strict ergodicity will suffice to provide equilibrium statistics. The above result shows, after all, that the region of the energy shell in which nonequilibrium behavior is manifested has a microcanonical measure (essentially reduced Liouville measure) which decays exponentially in the number of vortices. Therefore it would suffice, for example, if the vortex dynamics had a very large but fixed number of ergodic components as $N \to \infty$ which had fixed, finite fractions of the total measure of the energy shell, or even if the number of ergodic components went to infinity if only their individual measure fractions went to zero slower than exponentially in N. In any such case, the long-time probability starting in any ergodic component to observe a nonequilibrium vorticity distribution would be vanishingly small. The rate of approach to equilibrium in any ergodic component could be a far more serious constraint on the applicability of Onsager's theory.

The large-deviations result also justifies the physical relevance of the cutoff models with δ positive but (extremely) small. One might worry that $N \to \infty$ followed by $\delta \to 0$ is the "wrong" order of limits. In particular, for the limit $N \to \infty$ first there is in fact an infinite number of point vortices within a core radius δ of any fixed vortex and the condition for the δ-core dynamics to agree with the point-vortex dynamics, and therefore also the real dynamics of fluid blobs, will be violated. Notice, however, that $\Delta s_\delta(\varrho \mid \bar{\Delta})$ in the large-deviations result is just a macroscopic quantity and continuous in δ as $\delta \downarrow 0$. In particular, the minimum number N of vortices to ensure a smaller probability than ε of a nonequilibrium event $\mathscr{K}$,

$$N_\delta(\varepsilon, \mathscr{K}) \equiv \frac{\log(1/\varepsilon)}{-\sup_{\varrho \in \mathscr{K}} \Delta s_\delta(\varrho \mid \bar{\Delta})} \tag{4.12}$$

is essentially independent of δ for $0 \leqslant \delta < \delta_0$. Thus, we may first select the "deviation" $\mathscr{K}$ and the "likelihood" ε, and then choose δ so that

$$\delta^2 N_\delta(\varepsilon, \mathscr{K}) \ll 1 \tag{4.13}$$

This guarantees that the number of particles in the typical final equilibrium configuration which are within a core radius δ of any fixed particle shall be essentially zero. Considerations of this sort essentially show that the result with bounded potentials suffices to justify the use of the mean-field equations.

We turn now to a discussion of the Robert–Miller theory, which we shall first review from the point of view of Robert.[29] A basic tool of his theory is the description of vorticity fields in terms of *Young measures*, i.e., a space $\mathfrak{M}_m(\Lambda)$ of measurable mappings $x \mapsto \nu_x$ from Λ to the set $\mathscr{P}^1([-m, m])$ of Borel probability measures on $[-m, m]$. (Here, measurability is with respect to the Borel σ-algebra for the weak topology on $\mathscr{P}^1([-m, m])$.) The bounded interval $[-m, m]$ is to be considered as a set of possible vorticity levels. Young measures generalize in a natural way the notion of a measurable mapping from Λ to $[-m, m]$: for each $\omega \in L_m^\infty(\Lambda) = \{\omega \in L^\infty(\Lambda)\colon \|\omega\|_\infty \leqslant m\}$ we may define a *trivial Young measure* δ_ω via

$$\delta_\omega : x \mapsto \delta_{\omega(x)} \quad \text{a.e.} \tag{4.14}$$

The space $\mathfrak{M}_m(\Lambda)$ may be supplied with the vague topology of positive Radon measures on $\Lambda \times [-m, m]$ by the identification of the Young measures with such a measure ν by the formula

$$\nu(f) = \int_\Lambda dx \langle \nu_x, f(x, \cdot) \rangle \, dx, \qquad f \in C(\Lambda \times [-m, m]) \tag{4.15}$$

Any $\nu \in \mathfrak{M}_m(\Lambda)$ may be approximated in the vague sense by a sequence of δ_ω's (with ω even restricted to step functions) in which the spatial distribution of vorticity in an arbitrarily small neighborhood of a.e. $x \in \Lambda$ (i.e., the fraction of the area corresponding to each level set of vorticity) converges weakly to ν_x. From this we obtain the proper intuitive interpretation of ν_x as describing the local distribution of vorticity values. In fact, the usual use of Young measures in PDEs is to describe the small-scale spatial oscillations of weak solutions. In Robert's theory, Young measures enter again in that respect. Even though in two dimensions global classical solutions exist to the Euler system (4.1) for classical initial data, their *infinite-time limits* may be measure-valued even for classical initial data, since the dynamics mixes the vorticity to increasingly fine scales. Hence, $\mathfrak{M}_m(\Omega)$ becomes a natural space to consider in a theory of long-time statistical equilibria.

If $\omega_0 \in L_m^\infty(\Lambda)$ has energy $\varepsilon(\omega_0)$ and vorticity distribution π_0, defined as

$$\pi_0(f) = \int_\Lambda dx\, f(\omega_0(x)) \tag{4.16}$$

for all $f \in C([-m, m])$, and if its Euler flow trajectory, $\omega(t) \equiv \Phi_t \omega_0 \in L_m^\infty(\Lambda)$ for each $t \in \mathbb{R}$, converges at least along a subsequence $\omega(t_n) \to \nu \in \mathfrak{M}_m(\Lambda)$ in the vague sense, then it is easy to check that

$$\pi_0 = \int_\Lambda dx\, \nu_x \tag{4.17}$$

and

$$\varepsilon(\omega_0) = \varepsilon(\bar{\nu}) \tag{4.18}$$

where $\bar{\nu} \in L_m^\infty(\Lambda)$ is defined as

$$\bar{\nu}(x) = \int_{[-m,m]} \lambda \nu_x(d\lambda) \qquad \text{a.e.} \quad x \in \Lambda \tag{4.19}$$

Thus, the long-time limits retain the full information about the initial constants of the motion of the Euler flow Φ_t defined by (4.2). In fact, Robert shows that there is a natural extension of Φ_t from $L_m^\infty(\Lambda)$ to a flow $\bar{\Phi}_t$ on $\mathfrak{M}_m(\Lambda)$, actually unique subject to the requirement that it be continuous on $\mathfrak{M}_m(\Lambda)$, which is defined as

$$(\bar{\Phi}_t \nu)_x = \nu_{\phi_t^{-1}(x)} \tag{4.20}$$

where ϕ_t is the Lagrangian flow map associated to the initial vorticity $\bar{\nu}$. According to this evolution, the local spatial fluctuations of vorticity are "frozen" and transported by the mean velocity.

There is also a notion of entropy associated to the Young measures analogous to the entropy $s(\varrho)$ we have defined for $\varrho \in \mathscr{P}^1(\Lambda)$. For $\pi_0 \in \mathscr{P}^1([-m, m])$, a given reference probability measure, and $\pi = dx \otimes \pi_0$ the "uniform" Young measure, with the same distribution at each point, one may define the Kullback entropy

$$\mathscr{K}_\pi(\nu) = \begin{cases} -\int \log\left(\dfrac{d\nu}{d\pi}\right) d\nu & \text{if} \quad \nu \ll \pi \\ -\infty & \text{if} \quad \nu \not\ll \pi \end{cases} \tag{4.21}$$

through the identification of Young measures with positive Radon measures on $\Lambda \times [-m, m]$. So defined, $\mathscr{K}_\pi$ is $\leqslant \log |\Lambda|$ and u.s.c. in the vague topology on $\mathfrak{M}_m(\Lambda)$. It is also easy to see from the definitions that $\mathscr{K}_\pi(\bar{\Phi}_t \nu) = \mathscr{K}_\pi(\nu)$.

In terms of this entropy, one can formulate a heuristic *maximum entropy principle*, similar to that enunciated by Joyce and Montgomery, according to which, for a given initial datum $\omega_0 \in L_m^\infty(\Lambda)$, the "most likely" long-time limit $\nu^* \in \mathfrak{M}_m(\Lambda)$ is the solution of the variational problem

$$\mathscr{H}_\pi(\nu^*) = \max_{\nu \in \mathscr{E}} \mathscr{H}_\pi(\nu) \tag{4.22}$$

$$\mathscr{E} = \left\{ \nu \in \mathfrak{M}_m(\Lambda) \colon \varepsilon(\bar{\nu}) = \varepsilon(\omega_0), \int_\Lambda dx\, \nu_x = \pi_{\omega_0} \right\} \tag{4.23}$$

In particular, the long-time behavior should depend only on the initial values of the constants of the motion and not upon the specific initial datum ω_0. If $\mathscr{E}^*$ is the maximizing subset of $\mathscr{E}$, then, by the invariance of $\mathscr{H}_\pi$, $\bar{\Phi}_t(\mathscr{E}^*) = \mathscr{E}^*$, and, in particular, $\bar{\Phi}_t(\nu^*) = \nu^*$ if $\mathscr{E}^* = \{\nu^*\}$. Therefore, certainly under the assumption of uniqueness, ν^* is stationary under the (extended) Euler flow $\bar{\Phi}_t$, and is a quite reasonable candidate for the long-time limit. In fact, under the assumptions that $\mathscr{H}_\pi(\nu^*) > -\infty$ and $\varrho^*(x, \lambda) \equiv \nu^*(dx\, d\lambda)/\pi(dx\, d\lambda)$ is in $L^\infty(\Lambda \times [-m, m])$, one may apply the Lagrange principle to show that a *mean-field equation* is obeyed:

$$\varrho^*(x, \lambda) = \frac{1}{Z(x)} \exp[-\alpha(\lambda) - \beta\lambda\psi^*(x)] \tag{4.24}$$

where $\alpha(\lambda)$ is a function on $[-m, m]$ and β a real number chosen so that $\varrho^*\pi \in \mathscr{E}$, ψ^* is the stream function associated to

$$\omega^*(x) = \int_{[-m,m]} \lambda\varrho^*(x, \lambda)\, \pi_0(d\lambda) \tag{4.25}$$

and

$$Z(x) = \int e^{-\alpha(\lambda) - \beta\lambda\psi^*(x)} \pi_0(d\lambda) \tag{4.26}$$

is a normalizing integral.

Robert attempts to justify the assumption that $\mathscr{E}^*$ is the "most likely" behavior in terms of a "concentration theorem." The exact formulation of his notion of conditional concentration is rather complex, and we do not attempt to explicate it here precisely. However, we observe that it is dependent upon a certain sequence of *a priori* distributions on $\mathfrak{M}_m(\Lambda)$. If $\mathfrak{X}$ is a finite measurable partition of Λ, $\mathfrak{X} = \{\Delta^i \colon i = 1, \ldots, N(\mathfrak{X})\}$, such that $|\Delta^i| = |\Delta^j|$ for all i, j, and $\delta(\mathfrak{X}) = \sup_i (\mathrm{diam}\, \Delta^i)$ is the *diameter* of the partition, then for $(\lambda_1, \ldots, \lambda_{N(\mathfrak{X})}) \in [-m, m]^{N(\mathfrak{X})}$, define a trivial Young measure

$$\hat{\nu}_{\mathfrak{X}}(\lambda_1, \ldots, \lambda_{N(\mathfrak{X})}) \equiv \delta_{(\sum_{i=1}^{N(\mathfrak{X})} \lambda_i \chi_{\Delta^i})} \tag{4.27}$$

associated to the step function

$$\sum_{i=1}^{N(\mathfrak{X})} \lambda_i \chi_{\Delta^i}$$

piecewise constant over the partition $\mathfrak{X}$. One may regard $\hat{v}_{\mathfrak{X}}$ as a random variable distributed with respect to the *a priori* distribution $\mathscr{P}_{\mathfrak{X}} = \pi_0^{N(\mathfrak{X})}$ over $[-m, m]^{N(\mathfrak{X})}$. The meaning of conditional concentration is, essentially, that $\mathscr{E}^*$ is the overwhelmingly most likely subset of $\mathscr{E}$ with respect to $\mathscr{P}_{\mathfrak{X}}$ as $\delta(\mathfrak{X}) \downarrow 0$. In fact, Robert asserts (without proof) the following large-deviations estimates, which seem to us a clearer expression of the same fact:

$$\overline{\lim_{\delta(\mathfrak{X}) \downarrow 0}} \frac{1}{N(\mathfrak{X})} \log \mathscr{P}_{\mathfrak{X}}(\hat{v}_{\mathfrak{X}} \in \mathscr{A}) \leqslant \sup_{v \in \mathscr{A} \lesssim} \mathscr{K}_\pi(v) \tag{4.28}$$

and

$$\underline{\lim_{\delta(\mathfrak{X}) \downarrow 0}} \frac{1}{N(\mathfrak{X})} \log \mathscr{P}_{\mathfrak{X}}(\hat{v}_{\mathfrak{X}} \in \mathscr{A}) \geqslant \sup_{v \in \mathscr{A}} \mathscr{K}_\pi(v) \tag{4.29}$$

for any Borel subset of $\mathfrak{M}_m(\Lambda)$. These are obviously another extended version of the familiar Sanov results.

While we believe that Robert has indeed correctly identified what will be, in many situations, the long-time limits, we do not think that the theory has been justified in a sufficiently clear way. In particular, the physical conditions for its validity are not precisely spelled out. It is usual in statistical mechanics that its predictions become valid in a certain limit where a physical parameter becomes large or small, such as our limit $N \to \infty$. It is not clear from Robert's presentation what this physical limit should be. Furthermore, we do not think that Robert's Young measure method should be regarded as a *replacement* for the statistical mechanics analysis in terms of stationary measures. We have already seen that the probability measures v_x, $x \in \Lambda$, should be interpreted as the *spatial* distribution of vorticity. It is well defined for a given empirical flow field (to a certain degree of resolution) and has nothing to do whatsoever with the statistical fluctuations which shall occur over time or over an ensemble of similarly prepared samples. Such statistical fluctuations will certainly occur for a *finite* value of the physical parameter, and, indeed, it is observed in numerical simulations that final equilibria have to a good approximation a permanent form but are subject to continuous slight change in shape and size (e.g., see ref. 37). Although the overwhelmingly most probable behavior is of the greatest interest, the fluctuations also occur and should be described by an appropriate stationary measure.

The large-deviation estimates of Robert are actually suggestive of an alternative of this sort. Corresponding to a given partition $\mathfrak{X}$ of the flow domain Λ, one might define a *microcanonical distribution*

$$\mu_{\mathfrak{X},\mathscr{E}}(\cdot) \equiv \mathscr{P}_{\mathfrak{X}}(\cdot \mid \hat{v}_{\mathfrak{X}} \in \mathscr{E}) \tag{4.30}$$

By employing the appropriate large-deviations theory, one could justify (following the argument in the Appendix) the thermodynamic limit of the entropy, the maximum entropy principle, and the large-fluctuation theory, such as we earlier discussed for the point-vortex model. An upper-bound large-deviations estimate for $\mu_{\mathfrak{X},\mathscr{E}}$ is an *a priori* reasonable statement that $\mathscr{E}^*$ is "most" of $\mathscr{E}$ (with respect to $\mu_{\mathfrak{X},\mathscr{E}}$) as $\delta(\mathfrak{X}) \to 0$. The chief difficulty with this proposal is that $\mu_{\mathfrak{X},\mathscr{E}}$ is *not* a stationary measure for the Euler flow $\bar{\Phi}_t$ when considered as a distribution over $\mathfrak{M}_m(\Lambda)$. It certainly is not obviously the correct candidate to describe the actual statistical fluctuations (although one may believe that it probably suffices in practice.)

However, the approach of Miller is essentially equivalent to the above prescription in a canonical version.[25,28] His starting point is a formal expression for the Gibbs distribution as a functional measure:

$$\frac{1}{Z(\beta, \pi_0)} \prod_{x \in \Lambda} d\omega(x)\, \delta[\pi_\omega - \pi_0]\, e^{-\beta H(\omega)} \tag{4.31}$$

with the partition function

$$Z(\beta, \pi_0) = \int \prod_{x \in \Lambda} d\omega(x)\, \delta[\pi_\omega - \pi_0]\, e^{-\beta H(\omega)} \tag{4.32}$$

which is *formally* invariant under the Euler flow Φ_t [here, $H(\omega) = \varepsilon(\omega)$]. To render the above formal expression meaningful, Miller introduces a square-lattice partition of the domain, and restricts the integral to the piecewise-constant vorticity fields. Furthermore, looking at β, which scales with a as $\beta_a \equiv \beta/a^2$, he applies heuristically the method of steepest descent to argue that

$$\begin{aligned} &-\lim_{a \to 0} a^2 \log Z(\beta, \pi_0) \\ &\quad = \inf\left\{\beta\varepsilon(\bar{v}) - \mathscr{K}_{\pi_0}(v) \colon v \in \mathfrak{M}_m(\Lambda), \int dx\, v_x = \pi_0\right\} \end{aligned} \tag{4.33}$$

in the Young measure notation of Robert. This is clearly a canonical form of the limit result previously suggested for the microcanonical measure $\mu_{\mathfrak{X},\mathscr{E}}$, with $\delta(\mathfrak{X}) = a$ and $N(\mathfrak{X}) \sim |\Lambda|/a^2$. Therefore, it has the same limitation that the lattice-measure for finite a is not stationary, whereas the continuum measure is only a formal object.

In his thesis[28] Miller has made some further suggestions which, we believe, illuminate the situation. (An extended account of his work has just appeared in ref. 38.) We would like here to paraphase his discussion and expand upon it somewhat.

First, Miller is much clearer that a *physical limit* is required for the validity of the theory, whereas this is very much implicit in Robert's approach. Indeed, Miller implies that the constant a should be interpreted as a (spatial) *dissipation scale*, where viscous dissipation becomes significant, analogous to the Kolmogorov scale η of homogeneous turbulence. It is important to understand that the theory depends upon a large separation of scales, $L \gg a$, between the macroscopic dimension L of the flow domain and the dissipation scale a. This corresponds also to a large number of degrees of freedom in the macroscopic (or superinertial) range of the hydrodynamic equations. It is also important to understand that *small viscosity* plays a double role in justifying the theory: first, it implies that there will be a sufficiently long-time period (somewhat less than the viscous diffusion time) and, second, a sufficiently large range of spatial scales over which ideal Euler hydrodynamics will be valid.

Furthermore, Miller has made a suggestion which we believe has promise to provide a better dynamical understanding of the extended Robert–Miller theory. There have been recent investigations of finite-mode analogs of two-dimensional Euler equations based on $SU(N)$ Lie algebras (e.g., see ref. 39). These algebraic models have $O(N^2)$ modes and $O(N)$ conserved equations. Formally, both the dynamics and the conserved integrals converge to those of the two-dimensional Euler equations as $N \to \infty$. A sufficiently good convergence result, analogous to that of Hald[40] for simple Fourier truncations, might provide an appropriate dynamical basis for the theory. One can conceive of a final justification along the following lines: First, a convergence proof that, for small viscosity, $O(N^2)$ Fourier modes of the hydrodynamic equations (in the superinertial range) shall have their evolution well-approximated over a long time interval by the $SU(N)$ model equation with N large. This reduces the problem to a dynamical analysis of the $SU(N)$ model. In particular, there is the (very difficult) problem to establish that the $SU(N)$ dynamics achieves an "effective" microcanonical distribution within the time interval of its validity. Granting that—the usual ergodic assumption of statistical mechanics—one must analyze the large-N limit of the $O(N^2)$-mode microcanonical distributions. This presumably reproduces the results of Miller and Robert. Such a convergence result could certainly illuminate a point which both Miller and Robert have emphasized: the importance of considering all the conserved first integrals of the Euler flow. At present, there is only limited evidence of the importance of the additional conserved quantities,[23] but it is certainly a key issue of the Robert–Miller theory. Finally, the $O(N^2)$-mode microcanonical distribution would be a theoretically well-founded measure to describe the statistical fluctuations which occur for finite N (i.e., finite viscosity) over the time interval of quasistationarity of the vortex structure.

Note. After the completion of this work we received the thesis of Caglioti.[32] In that thesis more extensive studies have been made than here of the thermodynamic potentials and equilibrium vorticity distributions. Also, another proof of the mean-field limit for entropy and microcanonical correlation functions is given, which, however, is restricted to energies where the entropy is concave and thus equivalence of ensembles holds.

APPENDIX. MATHEMATICAL FRAMEWORK AND THE PROOF OF THE LARGE-DEVIATION ESTIMATES

Our setup is as follows: Λ is a compact subset of $\mathbb{R}^2$ (or, generally, $\mathbb{R}^d$), $\mathscr{B}$ its Borel σ-field, and $(\Omega, \mathscr{F}) = (\Lambda, \mathscr{B})^{\mathbb{N}}$ the product space. For each $k \in \mathbb{Z}^+$, define the *shift* $\theta_k: \Omega \to \Omega$ by

$$\theta_k(x_i: i \in \mathbb{N}) = (x_{i+k}: i \in \mathbb{N}) \tag{A1}$$

Clearly, $\theta = \{\theta_k: k \in \mathbb{Z}^+\}$ is a semigroup of endomorphisms of the measure space $(\Omega, \mathscr{F})$. If $\sigma^*: \mathbb{N} \to \mathbb{N}$ is a bijection fixing all but finitely many elements (i.e., a permutation of finitely many integers), then define $\sigma: \Omega \to \Omega$ as

$$\sigma(x_i: i \in \mathbb{N}) = (x_{\sigma^*(i)}: i \in \mathbb{N}) \tag{A2}$$

The set of such $\sigma: \Omega \to \Omega$ form a group S_∞ of automorphisms of $(\Omega, \mathscr{F})$. For each $N \in \mathbb{N}$, denote by S_N its subgroup of those $\sigma: \Omega \to \Omega$ such that $\sigma^*(i) = i$ for $i > N$, so that $S_\infty = \bigcup_{N \in \mathbb{N}} S_N$. Let $\mathscr{P}$ denote the set of all probability measures on $(\Omega, \mathscr{F})$, $\mathscr{P}_\theta$ the subset of $\mathscr{P}$ invariant under the semigroup θ, $\mathscr{P}_\sigma$ the subset of $\mathscr{P}$ invariant under the group S_∞, and $\mathscr{P}_\pi$ the subset of $\mathscr{P}$ consisting of elements of the form $\varrho^{\mathbb{N}}$ for some $\varrho \in \mathscr{P}^1 \equiv \mathscr{P}(\Lambda, \mathscr{B})$. We commonly refer to $\mathscr{P}_\theta$ as the *shift-invariant distributions*, to $\mathscr{P}_\sigma$ as the *symmetric* (or *exchangeable*) *distributions*, and to $\mathscr{P}_\pi$ as the *product distributions* on $(\Lambda, \mathscr{B})^{\mathbb{N}}$. It is a consequence of Theorem 3.2 of ref. 41 that $\mathscr{P}_\sigma \subseteq \mathscr{P}_\theta$, and, obviously, $\mathscr{P}_\pi \subseteq \mathscr{P}_\sigma$.

For each nonempty $S \subseteq \mathbb{N}$, we let $\Pi_S: \Omega \to \Lambda^S$ be the projection map defined by

$$\Pi_S(x_i: i \in \mathbb{N}) = (x_i: i \in S) \tag{A3}$$

and $\mathscr{F}_S = \Pi_S^{-1}\mathscr{B}^S$ the corresponding sub-σ-algebra of $\mathscr{F}$. For each $\mu \in \mathscr{P}$, we write $\mu_S = \mu \circ \Pi_S^{-1}$ for the marginal distribution of μ on $(\Lambda, \mathscr{B})^S$. In the particular case $S = \{1,..., N\}$, we write $\Pi_S = \Pi_N$, $\mathscr{F}_S = \mathscr{F}_N$, $\mu_S = \mu_N$. We say that $f: \Omega \to \mathbb{R}$ is *finite-body* or *local* if it is $\mathscr{F}_S$-measurable for some *finite* $S \subseteq \mathbb{N}$.

With respect to the product topology, $\Omega = \Lambda^{\mathbb{N}}$ is a compact metric space. If $C(\Omega)$ denotes the continuous functions on Ω, then the *weak topology* τ_W on $\mathscr{P}$ is the coarsest topology relative to which the evaluation maps $\mu \mapsto \mu(f)$ are continuous for each $f \in C(\Omega)$. It is equivalent to requiring continuity of the evaluation maps for $f \in \mathscr{L}^C \equiv \mathscr{L} \cap C(\Omega)$, since every continuous function on Ω is the uniform limit of continuous, local functions [see ref. 42, Remark 2.21(2)]. Therefore, we may take as an open base for τ_W all sets of the form

$$U(\mu) \equiv \{\nu \in \mathscr{P}: \max_{1 \leqslant i \leqslant n} |\nu(f_i) - \mu(f_i)| < \delta\} \tag{A4}$$

for $\mu \in \mathscr{P}, f_i \in \mathscr{L}^C$, $i = 1,..., n$, and $\delta > 0$. It is well known that $\mathscr{P}$ with respect to the τ_W-topology is itself compact and metrizable (e.g., ref. 43). Finally, it is easy to check that $\mathscr{P}_\theta$, $\mathscr{P}_\sigma$, and $\mathscr{P}_\pi$ are closed subsets of $\mathscr{P}$ and may be equipped with the subspace topologies. Each of the spaces $\mathscr{P}$, $\mathscr{P}_\theta$, $\mathscr{P}_\sigma$, and $\mathscr{P}_\pi$ are natural measure spaces in association with their Borel σ-algebras $\mathscr{B}(\mathscr{P})$, $\mathscr{B}(\mathscr{P}_\theta)$, etc.

We next recall the definition of the *specific entropy* or *mean entropy* of a measure $\mu \in \mathscr{P}_\theta$ as

$$h(\mu) = \lim_{S \uparrow \mathbb{N}} -\frac{1}{|S|} I(\mu_S; \lambda^S) \tag{A5}$$

where, for two measures α, β on the same measure space, $I(\alpha; \beta)$ denotes the relative entropy of α with respect to β (see ref. 42, Section 15.2). In fact, the last reference defines h and develops its properties for the case where $\mathbb{N}$ is replaced by $\mathbb{Z}^d$, and the shifts θ_k form a group of automorphisms rather than merely a semigroup of endomorphisms. However, it is easy to check that all the development there carries over to the case we consider. Hence, we may take as well known the following properties of h on $\mathscr{P}_\theta$:

(i) For all $\mu \in \mathscr{P}_\theta$,

$$h(\mu) = \inf_N \frac{1}{N} S_N(\mu_N) \in [-\infty, \log \lambda(\Lambda)]$$

(ii) h is u.s.c. and its level sets $\{h \geq c\}$, $c \leq \log \lambda(\Lambda)$, are compact and sequentially compact.

(iii) h is an affine function on $\mathscr{P}_\theta$.

In the statement of (i) we have used the more physical notation $S_N = -I(\cdot\,; \lambda^N)$. Since $\mathscr{P}_\sigma$ is a closed, convex subset of $\mathscr{P}_\theta$, the same properties hold for h when restricted to $\mathscr{P}_\sigma$, and, likewise, since $\mathscr{P}_\pi$ is closed in $\mathscr{P}_\theta$, (i) and (ii) hold when h is restricted there.

The structure of the space $\mathscr{P}_\sigma$ of symmetric or exchangeable distributions is important for our discussion. It is a well-known theorem of de Finetti[30] (see also Dynkin[44]) that, if $\mathscr{P}^1 = \mathscr{P}(\Lambda, \mathscr{B})$ and $\mathscr{E}(\mathscr{P}^1)$ is the evaluation σ-algebra of subsets of $\mathscr{P}^1$ generated by the sets $\{\varrho \in \mathscr{P}^1 : \varrho(B) \leq c\}$ with $B \in \mathscr{B}$ and $0 \leq c \leq 1$, then for every $\mu \in \mathscr{P}_\sigma$, there is a $\nu_\mu \in (\mathscr{P}^1, \mathscr{E}(\mathscr{P}^1))$ such that

$$\mu = \int \nu_\mu(d\varrho)\, \varrho^{\mathbb{N}} \tag{A6}$$

where identity is in the sense $\mu(f) = \int \nu_\mu(d\varrho)\, \varrho^{\mathbb{N}}(f)$ for $f \in \mathscr{L}$. Furthermore, the correspondence $\mu \mapsto \nu_\mu$ is bijective and affine. If $\mathscr{B}(\mathscr{P}^1)$ is the Borel σ-algebra for the weak topology on $\mathscr{P}^1$, it is easy to check that $\mathscr{E}(\mathscr{P}^1) = \mathscr{B}(\mathscr{P}^1)$ using the separability of $\mathscr{P}^1$ for the weak topology and a simple monotone class argument. Hence, ν_μ is a Borel probability measure on $\mathscr{P}^1$. We may note that, according to the theorem of Hewitt and Savage,[41] the extremal elements of $\mathscr{P}_\sigma$ are just the product measures,

$$\mathrm{ex}\mathscr{P}_\sigma = \{\varrho^{\mathbb{N}} : \varrho \in \mathscr{P}^1\} \tag{A7}$$

Therefore, the above representation is a usual type of extremal integral decomposition. It is a simple computation that

$$h(\varrho^{\mathbb{N}}) = s(\varrho) \tag{A8}$$

where $s(\varrho) = -I(\varrho; \lambda)$. As a consequence of Theorem 15.20 in ref. 42, it then follows that

$$h(\mu) = \int \nu_\mu(d\varrho)\, s(\varrho) \tag{A9}$$

for μ given by (A6) above.

We shall be concerned with *symmetric averages* or *symmetrizations* of local functions $f \in \mathscr{L}$, i.e., if f is $\mathscr{F}_N$-measurable, the function

$$P_N f \equiv \frac{1}{N!} \sum_{\sigma \in S_N} f \circ \sigma \tag{A10}$$

It is well known that for $\mu \in \mathrm{ex}\mathscr{P}_\sigma = \mathscr{P}_\pi$, $f \in \mathscr{L}$,

$$\lim_{N \to \infty} P_N f = \int f \, d\mu \qquad \mu\text{-a.s.} \tag{A11}$$

[e.g., see ref. 42, Example 7.16, Remark 7.13, and Theorem 7.12(a)]. Obviously, this is just a generalized form of Kolmogorov's strong law of large numbers. We are here concerned with large deviations from this ergodic behavior.

The collective asymptotic behavior of all such symmetric averages can be described conveniently by a single quantity, a *periodic empirical field*. For our purposes, given $N \in \mathbb{N}$ and a configuration $\omega \in \Lambda^N$, this is defined by

$$\varrho_N^\omega = \frac{1}{N!} \sum_{\sigma \in S_N} \delta_{(\sigma\omega)^{\mathrm{per}}} \in \mathscr{P} \tag{A12}$$

where $\omega^{\mathrm{per}} \in \Omega$ for $\omega \in \Lambda^N$ is the periodic continuation of ω defined as $(\omega^{\mathrm{per}})_i = x_{i (\mathrm{mod}\, N)}$. It is then immediately obvious that $\varrho_N: \omega \mapsto \varrho_N^\omega$ is a measurable function from Λ^N to $\mathscr{P}$. In fact, ϱ_N is measurable into $\mathscr{P}_\theta$:

Proposition A.1. $\varrho_N: \omega \mapsto \varrho_N^\omega$ is a measurable function from $(\Lambda^N, \mathscr{B}^N)$ into $(\mathscr{P}_\theta, \mathscr{B}(\mathscr{P}_\theta))$.

Proof. If C_N is the subgroup of S_N defined by the N *cyclic* permutations of $\{1,..., N\}$, then we may also write ϱ_N^ω as

$$\begin{aligned} \varrho_N^\omega &= \frac{1}{N} \sum_{\pi \in C_N} \frac{1}{N!} \sum_{\sigma \in S_N} \delta_{(\pi\sigma\omega)^{\mathrm{per}}} \\ &= \frac{1}{N} \sum_{i=0}^{N-1} \frac{1}{N!} \sum_{\sigma \in S_N} \delta_{\theta_i(\sigma\omega)^{\mathrm{per}}} \end{aligned} \tag{A13}$$

The latter expression clearly exhibits that $\varrho_N^\omega \in \mathscr{P}_\theta$. Moreover, by the separability of $\mathscr{P}_\theta$ for the weak topology and a simple monotone class argument, $\mathscr{B}(\mathscr{P}_\theta) = \mathscr{E}(\mathscr{P}_\theta)$, where $\mathscr{E}(\mathscr{P}_\theta)$ is the evaluation σ-algebra generated by the sets $\{\mu \in \mathscr{P}_\theta : \mu(B) \leqslant c\}$ with $B \in \mathscr{B}(\mathscr{P}_\theta)$ and $c \in [0, 1]$. From this remark, the $(\mathscr{P}_\theta, \mathscr{B}(\mathscr{P}_\theta))$-measurability is obvious. ■

For $f \in \mathscr{L}_N$, the $\mathscr{F}_N$-measurable subspace of $\mathscr{L}$, we define naturally a function $\varrho_N f: \Lambda^N \to \mathbb{R}$ as $\varrho_N f: \omega \mapsto \varrho_N^\omega(f) = \int f \, d\varrho_N^\omega$. Obviously, $\varrho_N f$ coincides with $P_N f$ previously defined.

However, ϱ_N^ω has the disadvantage that, although $\varrho_N^\omega \in \mathscr{P}_\theta$, nevertheless

$\varrho_N^\omega \notin \mathscr{P}_\sigma$. Therefore, it is natural to introduce also a *symmetric empirical field* $\tilde{\varrho}_N^\omega$, defined as

$$\varrho_N^\omega = \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}\right)^{\mathbb{N}} \in \mathscr{P}_\sigma \tag{A14}$$

Observe, as a matter of fact, that $\tilde{\varrho}_N^\omega \in \mathscr{P}_\pi$, which is important for the later development. This field may be used as a replacement for ϱ_N^ω as a consequence of the following proposition:

Proposition A.2. For each $f \in \mathscr{L}$,

$$\lim_{N \to \infty} \|\varrho_N f - \tilde{\varrho}_N f\|_\infty = 0 \tag{A15}$$

Proof. If $f \in \mathscr{L}_m$ for $m \leqslant N$,

$$\tilde{\varrho}_N^\omega(f) = \frac{1}{N^m} \sum_{\varphi \in M(m,N)} f(x_{\varphi(1)},..., x_{\varphi(m)}) \tag{A16}$$

where $M(m, N)$ is the set of all mappings from $\{1,..., m\}$ to $\{1,..., N\}$, and

$$\varrho_N^\omega(f) = \frac{(N-m)!}{m!} \sum_{\varphi \in M(m,N)} f(x_{\varphi(1)},..., x_{\varphi(m)}) \tag{A17}$$

where $M(m, N)$ is the set of all injective mappings in $M(m, N)$. Since therefore

$$|\tilde{\varrho}_N^\omega(f) - \varrho_N^\omega(f)| \leqslant 2\, \|f\|_\infty \left(1 - \frac{N!/(N-m)!}{N^m}\right) \tag{A18}$$

for all $\omega \in \Lambda^N$, the result follows. ∎

An important consequence of Proposition A.2 is that, for every $\omega \in \Omega$, $\tilde{\varrho}_N^\omega$ and ϱ_N^ω have the same accumulation points in $\mathscr{P}_\theta$ with the τ_W-topology. In particular, all of the accumulation points of ϱ_N^ω belong to $\mathscr{P}_\pi$.

Now we formulate the statement of the large-deviations theorem which we shall subsequently prove. Let $(\mu_N : N \in \mathbb{N})$ be a sequence of probability measures μ_N on $(\Lambda, \mathscr{B})^N$. The symmetric empirical measures $\tilde{\varrho}_N \in \mathscr{P}_\pi$ as random variables under μ_N are said to satisfy a *(level-3) large-deviations principle*, with rate function $I: \mathscr{P}_\pi \to [0, +\infty]$, if, for any measurable $C \subseteq \mathscr{P}_\pi$,

$$\overline{\lim_{N \to \infty}} \frac{1}{N} \log \mu_N(\tilde{\varrho}_N \in C) \leqslant -\inf I(\bar{C}) \tag{A19}$$

and

$$\underline{\lim}_{N\to\infty} \frac{1}{N} \log \mu_N(\tilde{\varrho}_N \in C) \geqslant -\inf I(\mathring{C}) \tag{A20}$$

where $\bar{C}$ (resp. $\mathring{C}$) is the closure (resp. the interior) of C in the τ_W-topology relativized to $\mathscr{P}_\pi$.

It is important to observe that the spaces $\mathscr{P}^1$ and $\mathscr{P}_\pi$ may be, essentially, identified. Indeed, if $\mathscr{P}_\pi$ is given the τ_W-subspace topology and $\mathscr{P}^1$ its own weak topology, it is easy to check that the bijection $\varrho \in \mathscr{P}^1 \mapsto \varrho^{\mathbb{N}} \in \mathscr{P}_\pi$ is actually a homeomorphism. Therefore, it is also, an isomorphism of the measurable spaces $(\mathscr{P}^1, \mathscr{B}(\mathscr{P}^1))$ and $(\mathscr{P}_\pi, \mathscr{B}(\mathscr{P}_\pi))$. In particular, we can see that there is no real distinction between the above level-3 principle and the corresponding *level-2* principle formulated in terms of the *empirical one-particle distribution*

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \in \mathscr{P}^1 \tag{A21}$$

for each $N \in \mathbb{N}$. In fact, in the level-2 formulation, results analogous to those we establish below, so-called Sanov properties, were proved before (e.g., see refs. 33, 34, and 36). However, working at level 3, we can prove somewhat stronger results: in particular, convexity restrictions on the conditioning set may be removed in the statement of the maximum entropy principle. (Observe that $\{\varrho \in \mathscr{P}^1: \varepsilon(\varrho) \in [e_-, e_+]\}$ is not convex, since $\varepsilon(\varrho)$ is not affine.)

It is convenient here for us to use an alternative, completely equivalent formulation of the large-deviations property (e.g., see ref. 33). We make the following definition:

Definition A.1. A *symmetric asymptotic empirical functional* (SAEF) $\{(F_N, F)\}$ is a family $(F_N: N \in \mathbb{N})$ of symmetric measurable functions $F_N: \Lambda^N \to (-\infty, +\infty]$, together with a functional $F: \mathscr{P}_\pi \to (-\infty, +\infty]$ which is bounded below and such that

$$\lim_{N\to\infty} \left\| \frac{1}{N} F_N - F(\tilde{\varrho}_N) \right\|_\infty = 0 \tag{A22}$$

The definition of the SAEF is not the most general for which the results below are valid, and in fact, Georgii has shown (Remark 1.4 of ref. 33) that the Theorems A.1 and A.2 below still hold under the weaker conditions that

$$\underline{\lim}_{N\to\infty} \inf_{\omega \in \Lambda^N} \left[\frac{1}{N} F_N(\omega) - \inf F(U(\tilde{\varrho}_N^\omega)) \right] \geqslant 0 \tag{A23}$$

and

$$\overline{\lim_{N\to\infty}} \sup_{\omega\in\Lambda^N} \left[\frac{1}{N}F_N(\omega) - \sup F(U(\tilde{\varrho}_N^{\omega}))\right] \leqslant 0 \tag{A24}$$

for every basis element

$$U(\nu) = \{\mu \in \mathscr{P}_\pi \colon \sup_{1\leqslant i\leqslant n} |\mu(f_i) - \nu(f_i)| < \delta\}$$

of ν for the τ_W-topology, with $f_i \in \mathscr{L}^C$, $1 = 1,..., n$, $\delta > 0$. We may refer to $\{(F_N, F)\}$ which satisfies the above weaker conditions as an SAEF *in the wide sense*. We remark that these weaker conditions are just a version in the context of lattice systems of those proved by Varadhan in a general large-deviations context to be sufficient for the Laplace theorem.[45]

For any $F: \mathscr{P}_\pi \to (-\infty, +\infty]$ we define F^{usc} (resp. F_{lsc}) to be the upper semicontinuous regularization (resp. lower semicontinuous regularization) of F on $\mathscr{P}_\pi$. Then, an equivalent formulation of the large-deviations property is just that, for every SAEF $\{(F_N, F)\}$,

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mu_N(e^{-F_N}) \leqslant -\inf[I + F_{\text{lsc}}] \tag{A25}$$

and

$$\underline{\lim_{N\to\infty}} \frac{1}{N} \log \mu_N(e^{-F_N}) \geqslant -\inf[I + F^{\text{usc}}] \tag{A26}$$

In particular, taking, for any measurable $C \subseteq \mathscr{P}_\pi$, $F(\mu) = 0$ if $\mu \in C$, $F(\mu) = +\infty$ if $\mu \notin C$, and $F_N(\omega) = NF(\tilde{\varrho}_N^{\omega})$, one obtains the previous statement.

We now state the main theorem. Denote by $\hat{\lambda}$ the normalized Lebesgue measure on $(\Lambda, \mathscr{B})$, i.e., $\hat{\lambda}(\cdot) = \lambda(\cdot)/|\Lambda|$ so that $\hat{\lambda} \in \mathscr{P}(\Lambda, \mathscr{B})$. Then, $\hat{\lambda}^N$ is a sequence of probability measures on $(\Lambda, \mathscr{B})^N$ for which the following holds:

Theorem A.1. $(\hat{\lambda}^N: N \in \mathbb{N})$ satisfies a level-3 large-deviations principle for $\tilde{\varrho}_N$ with rate function $I: \mathscr{P}_\pi \to [0, +\infty]$ given by $I(\mu) = \log|\Lambda| - h(\mu)$.

Proof of the upper bound. For the proof, let us only indicate the differences from the proof of the corresponding Theorem 1.2 in ref. 33. We may work in the compact, *convex* space $\mathscr{P}_\sigma$ and take $\{(F_N, F)\}$ an SAEF in the associated sense, i.e., $F: \mathscr{P}_\sigma \to [0, +\infty]$. Proving the upper bound in such a framework, one can return to the statement of our theorem by choosing $F = +\infty$ on $\mathscr{P}_\sigma/\mathscr{P}_\pi$. Just as in the proof in ref. 33, one introduces the Gibbs distribution μ_N^F on $(\Lambda, \mathscr{B})^N$, which, indeed, may be defined for any SAEF with $\lambda^N(\{F_N < +\infty\}) > 0$ by

$$\mu_N^F = [\lambda^N(e^{-F_N})]^{-1} e^{-F_N} \lambda^N \tag{A27}$$

Furthermore, one may define a measure $\tilde{\mu}_N^F \in \mathscr{P}_\sigma$ by

$$\tilde{\mu}_N^F = \frac{1}{N} \sum_{i=0}^{N-1} (\mu_N^F)^{\mathbb{N}} \circ \theta_i^{-1} \tag{A28}$$

which corresponds to that in (4.3) of ref. 33. Here, $(\mu_N^F)^{\mathbb{N}}$ stands for the probability measure on $(\Omega, \mathscr{F})$ relative to which the projections $(\Pi_{((i-1)N, iN)})_{i \in \mathbb{N}}$ are independent with identical distribution μ_N^F. Then, the Lemma 4.1 of ref. 33 still holds. Likewise, Lemma 4.2 still holds, with the following changes: (1) $\mathscr{L}^C$ must replace $\mathscr{L}$ in the definition of $\underline{F}$ and (2) $\tilde{\varrho}_N$ must replace ϱ_Λ.

We must finally verify that the measures $\tilde{\mu}_N^F$ and $\mu_N^F \tilde{\varrho}_N$ have the same accumulation points, as $N \to \infty$, as the analog of Lemma 4.3 in ref. 33. We state this as follows:

Lemma A.1. For all $f \in \mathscr{L}$,

$$\lim_{N \to \infty} [\tilde{\mu}_N^F(f) - \mu_N^F \tilde{\varrho}_N(f)] = 0 \tag{A29}$$

Proof. As a consequence of Proposition A.2, we may clearly replace $\tilde{\varrho}_N$ in the above statement by ϱ_N. Now it is very easy to check that, for any SAEF $\{(F_N, F)\}$,

$$\mu_N^F \varrho_N = \mu_N^F \circ (i_N^{\text{per}})^{-1} \tag{A30}$$

where $i_N^{\text{per}}: \Lambda^N \to \Omega$ is the injection $i_N^{\text{per}}(\omega) = \omega^{\text{per}}$. Indeed, for $f \in \mathscr{L}$,

$$\begin{aligned}(\mu_N^F \varrho_N)(f) &= \frac{1}{N!} \sum_{\sigma \in S_N} \int \mu_N^F(d\omega)\, f((\sigma\omega)^{\text{per}}) \\ &= \int \mu_N^F(d\omega)\, f(\omega^{\text{per}}) \end{aligned} \tag{A31}$$

where *the symmetry of F_N has been used.* But then it follows easily, as in Lemma 4.3 of ref. 33, that

$$\begin{aligned}|\mu_N^F(f) - \mu_N^F \varrho_N(f)| &\leqslant \\ &\leqslant \frac{2\, \|f\|_\infty}{N} |\{i \in \{0,..., N-1\}: \Delta + i \not\subseteq \{1,..., N\}\}| \end{aligned} \tag{A32}$$

where f is $\mathscr{F}_\Delta$-measurable. This gives the result. ■

The rest of the proof of the upper bound in ref. 33 goes through in our situation, with only some obvious alterations (e.g., $\mathscr{L}$ must be everywhere replaced by $\mathscr{L}^C$).

Proof of the lower bound. For this part of the proof, we must necessarily remain in the space $\mathscr{P}_\pi$. However, since *every* element of $\mathscr{P}_\pi$ is ergodic relative to θ, we may just follow exactly the proof in refs. 46 and 47 for the ergodic elements, to obtain the lower bound

$$\underline{\lim}_N \frac{1}{N} \log \hat{\lambda}^N(e^{-F_N}) \geqslant -[I(\nu) + F^{\text{usc}}(\nu)] \tag{A33}$$

pointwise for every $\nu \in \mathscr{P}_\pi$. Of course, this yields the result. ■

The Gibbs measures introduced in the course of the above proof have, obviously, more than just a technical interest. In fact, μ_N^H for an SAEF $\{(H_N, H)\}$ represents a very general version of a canonical Gibbs measure for a mean-field Hamiltonian H_N. As a consequence of the above theorem and a basic extension principle (e.g., see Theorem II.7.2 in ref. 48), the sequences (μ_N^H) have also a large-deviations property. We state this result as a corollary:

Corollary A.1. Let $\{(H_N, H)\}$ and $\{(F_N, F)\}$ be two SAEFs. Suppose also that H is continuous and $\inf[I+H] < +\infty$. Then, μ_N^H is essentially well defined and we have

$$\overline{\lim}_{N\to\infty} \frac{1}{N} \log \mu_N^H(e^{-F_N}) \leqslant -\inf[I^H + F_{\text{lsc}}] \tag{A34}$$

and

$$\underline{\lim}_{N\to\infty} \frac{1}{N} \log \mu_N^H(e^{-F_N}) \geqslant -\inf[I^H + F^{\text{usc}}] \tag{A35}$$

where the H-modified entropy function $I^H = I + H - \inf[I+H]$.

Proof. The proof follows that of Corollary 1.5 in ref. 33.

We now wish to state a limit theorem for the distributions of the type (μ_N^H) which is, properly, a "minimum free energy principle." By abuse of notation, we write μ_N^H for an arbitrary probability measure on $(\Omega, \mathscr{F})$ whose marginal distribution $(\Lambda, \mathscr{B})^N$ coincides with the Gibbs distribution (A28). Then, we have the following result.

Theorem A.2. Let $\{(H_N, H)\}$ be an SAEF satisfying

$$\inf[I + H_{\text{lsc}}] = \inf[I + H^{\text{usc}}] < +\infty \tag{A36}$$

Then, in the limit $N \to \infty$, the sequence (μ_N^H) admits at least one accumulation point $\mu \in \mathscr{P}_\sigma$, and each such μ has a representation $\mu = \int \nu\, w(d\nu)$ for a Borel probability measure on the compact set

$$M^H = \{\nu \in \mathscr{P}_\sigma : I(\nu) + H_{\mathrm{lsc}}(\nu) = \inf[I + H_{\mathrm{lsc}}]\} \tag{A37}$$

Proof. The proof of this theorem is essentially the same as that of Theorem 1.6 in ref. 33. In the same way as there, it is shown that

$$\phi \neq \mathrm{acc}_{N\uparrow\infty}\, \tilde{\mu}_N^H \subseteq \{I + \underline{H} = \min\} \tag{A38}$$

and $\{I + \underline{H} = \min\}$ is equal to the set of all mixtures of measures in $\{I + H_{\mathrm{lsc}} = \min\}$. Here, as before, $H: \mathscr{P}_\pi \to [0, +\infty]$ is extended to $\mathscr{P}_\sigma$ by the definition $H = +\infty$ on $\mathscr{P}_\sigma / \mathscr{P}_\pi$ and $\underline{H}$ is the lower convex envelope of H relative to $\mathscr{P}_\sigma$. The only thing that remains to be shown is that

$$\mathrm{acc}_{N\uparrow\infty}\, \tilde{\mu}_N^H = \mathrm{acc}_{N\uparrow\infty}\, \mu_N^H \tag{A39}$$

However, this was essentially already seen in the proof of Lemma A.1, for it was found there that

$$\mu_N^H \circ (i_N^{\mathrm{per}})^{-1} = \mu_N^H \varrho_N \tag{A40}$$

so that, for any $f \in \mathscr{L}_N$, with our abuse of notation,

$$\mu_N^H(f) = \mu_N^H \varrho_N(f) \tag{A41}$$

The latter equation obviously holds for any $f \in \mathscr{L}$ with N sufficiently large. Therefore, the result follows from the statement of Lemma A.1 and Proposition A.2. ■

Comment. In view of our earlier remark about the identity of $\mathscr{P}_\pi$ and $\mathscr{P}^1$, we may just as well represent the limit points in the above theorem as $\mu = \int \nu(d\varrho)\, \varrho^{\mathbb{N}}$ for a Borel probability measure ν on the compact set

$$M^H = \{\varrho \in \mathscr{P}^1 : h(\varrho) - s(\varrho) = \inf[h_{\mathrm{lsc}} - s]\} \tag{A42}$$

with $h(\varrho) \equiv H(\varrho^{\mathbb{N}})$.

As a corollary, we may deduce a similar result for generalized distributions of microcanonical type. Indeed, for any measurable $\tilde{C} \subseteq \mathscr{P}_\pi$ with $\lambda^N(\tilde{\varrho}_N \in \tilde{C}) > 0$, one can define a distribution $\mu_{\tilde{C}}^N$ on $(\Lambda, \mathscr{B})^N$ by

$$\mu_{\tilde{C}}^N(\cdot) \equiv \bar{\lambda}^N(\cdot \mid \tilde{\varrho}_N \in \tilde{C}) \tag{A43}$$

Then, the following holds.

Corollary A.2. (Maximum entropy principle.) if $\tilde{C}$ is an h-continuity set and $\overline{\text{co}}\tilde{C}$ its closed convex hull, then

$$\emptyset \neq \text{acc}_{N\uparrow\infty}\, \mu_{\tilde{C}}^{H}(\cdot) \subseteq \{\mu \in \overline{\text{co}}\tilde{C}: h(\mu) = \sup h(\tilde{C})\} \tag{A44}$$

Furthermore, each such accumulation point has an integral representation of the form $\mu = \int \nu(d\varrho)\, \varrho^{\mathbb{N}}$ for ν a Borel probability measure on the compact set

$$M^C = \{\varrho \in \mathscr{P}^1: s(\varrho) = \sup s(C)\} \tag{A45}$$

with $\tilde{C} = C^{\mathbb{N}}$.

The corollary follows directly from Theorem A.2 by taking $H(\mu) = 0$ if $\mu \in \tilde{C}$, $H(\mu) = +\infty$ if $\mu \notin \tilde{C}$, and $H_N(\omega) = NH(\tilde{\varrho}_N^{\omega})$.

The proof of Theorem 4.1 in Section 4 is an explicit application of the previous general results. We need the following lemma.

Lemma A.2. Define $\{(F_N, F)\}$ by

$$F_N(\omega) = \begin{cases} 0 & \text{if } \tilde{H}_N/N \in [e_-, e_+] \\ +\infty & \text{if } \tilde{H}_N/N \notin [e_-, e_+] \end{cases} \tag{A46}$$

and, likewise,

$$F(\mu) = \begin{cases} 0 & \text{if } \tilde{\varepsilon}(\mu) \in [e_-, e_+] \\ +\infty & \text{if } \tilde{\varepsilon}(\mu) \notin [e_-, e_+] \end{cases} \tag{A47}$$

Then, $\{(F_N, F)\}$ is an SAEF in the wide sense.

Remark. It is easy to check that $(\tilde{H}^{(N)}, \tilde{\varepsilon})$ is an SAEF. However, $\{(F_N, F)\}$ as defined above is *not*, because an SAEF must have the property that $\{F_N = +\infty\} = \{F(\tilde{\varrho}_N) = +\infty\}$ for N sufficiently large. This clearly does not hold.

Proof. Let us check the first condition (A23). Now, one can see that, given $U(\cdot) \equiv U(\cdot\,; f_1,..., F_n, \delta)$ defined as in (A4) with f_i, $i = 1,..., n$ and δ fixed, there is an $\eta > 0$ such that $\nu \in \tilde{\varepsilon}^{-1}([e_- - \eta, e_+ + \eta])$ implies that $U(\nu) \cap \tilde{\varepsilon}^{-1}([e_-, e_+]) \neq \emptyset$. Since obviously we can find a $\varrho > 0$ such that the open metric ball $B_\varrho(\nu) \subseteq U(\nu)$ for all $\nu \in \mathscr{P}_\pi$ [by uniform continuity of the evaluation maps $\mu \mapsto \mu(f)$, $f \in \mathscr{L}^C$], we can replace $U(\nu)$ in this statement by $B_\varrho(\nu)$. Define

$$K_\eta \equiv \tilde{\varepsilon}^{-1}([e_- - \eta, e_+ + \eta]) \tag{A48}$$

and

$$K \equiv \tilde{\varepsilon}^{-1}([e_-, e_+]) \tag{A49}$$

so that

$$\bigcap_{\eta > 0} K_\eta = K \tag{A50}$$

Because $\mathscr{P}_\pi$ is a compact, metric space for the weak topology and $\tilde{\varepsilon}$ is continuous, K_η, K are all *compact.* If $d(\mu, \nu)$ is the (Prohorov) metric on $\mathscr{P}_\pi$, define an u.s.c. function $d_K: \mathscr{P}_\pi \to \mathbb{R}^+$ by

$$d_K(\mu) = d(\mu, K) \equiv \inf_{\nu \in K} d(\mu, \nu) \tag{A51}$$

Then, define

$$\delta(\eta) \equiv \sup_{\mu \in K_\eta} d_K(\mu) < +\infty \tag{A52}$$

We show that

$$\lim_{\eta \to 0} \delta(\eta) = 0 \tag{A53}$$

Indeed, suppose the contrary, that there is a sequence $\eta_n \downarrow 0$ such that

$$\delta(\eta_n) \geqslant \delta > 0 \tag{A54}$$

for all n. Define for every η,

$$K_\eta(\delta) \equiv K_\eta \cap \{d_K \geqslant \delta\} \tag{A55}$$

a compact subset of K_η and decreasing for $\eta \downarrow 0$. By the assumption,

$$K_{\eta_n}(\delta) \neq \varnothing \tag{A56}$$

since the u.s.c. function d_K achieves its supremum on each K_{η_n}. However, since the $K_\eta(\delta)$ monotonically decrease as $\eta \downarrow 0$

$$K_\eta(\delta) \neq \varnothing \tag{A57}$$

for all η. Then,

$$\varnothing \neq \bigcap_{\eta > 0} K_\eta(\delta) = K \cap \{d_K \geqslant \delta\} \tag{A58}$$

However, clearly $K \cap \{d_K \geqslant \delta\} = \varnothing$, a contradiction. Thus, $\delta(\eta) \downarrow 0$ as $\eta \downarrow 0$, as claimed. To complete the argument, we now simply choose η sufficiently small that $\delta(\eta) < \varrho$. Then, clearly, for $v \in K_\eta$,

$$B_\varrho(v) \cap K \neq \varnothing \tag{A59}$$

since $d(v, \mu)$ is continuous in μ for v fixed and therefore achieves its infimum on K, which is less than ϱ.

The point of the above observation is the following: by choosing N sufficiently large that

$$\left\| \frac{1}{N} \tilde{H}^{(N)} - \tilde{\varepsilon}(\tilde{\varrho}_N) \right\|_\infty < \eta \tag{A60}$$

it follows that, if, for $\omega \in \Lambda^N$, $(1/N)\, \tilde{H}^{(N)}(\omega) \in [e_-, e_+]$, then also $\tilde{\varepsilon}(\tilde{\varrho}_N) \in [e_- - \eta, e_+ + \eta]$, so that by the above, $F(\mu) = 0$ for some $\mu \in U(\tilde{\varrho}_N^\omega)$, and $\inf F(U(\tilde{\varrho}_N^\omega)) = 0$. That is, $F_N(\omega) = 0$ implies $\inf F(U(\tilde{\varrho}_N^\omega)) = 0$ for all $\omega \in \Lambda^N$, for N sufficiently large. Obviously, this implies (A23).

The argument for the second estimate (A24) is virtually the same and may be omitted here. ■

Comments. (i) The above lemma actually holds if $\{(\tilde{H}^{(N)}, \tilde{\varepsilon})\}$ is replaced by *any* SAEF $\{(H_N, H)\}$ with H continuous. (ii) However, the result is not necessarily true if we use only that H is continuous on a complete, separable metric space: weak compactness of $\mathscr{P}_\pi$ was essential for our proof.

To derive Theorem 4.1, we now apply Corollary A.1 with the above wide sense SAEF $\{(F_N, F)\}$ playing the role of $\{(H_N, H)\}$ there. However, since F is not continuous, one must check to see what is the corresponding result and apply the condition $s(\mathring{\Delta}) = s(\bar{\Delta})$. We remark, finally, that using the above lemma in conjunction with Theorems A.1 and A.2, we obtain an independent proof of Theorem 2.1 in Section 2.

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