Existence of Steady Vortex Rings in an Ideal Fluid

A. AMBROSETTI & M. STRUWE

Communicated by H. BREZIS

§ 0. Introduction

Consider an ideal fluid occupying all of $\boldsymbol{R}^3$ with axisymmetric velocity field $\boldsymbol{q}$. A vortex ring $\mathcal{R}$ is a toroidal region in $\boldsymbol{R}^3$ such that $\operatorname{curl} \boldsymbol{q} = 0$ in $\boldsymbol{R}^3 \setminus \mathcal{R}$ while $\operatorname{curl} \boldsymbol{q} \neq 0$ in $\mathcal{R}$.

In cylindrical coordinates, in terms of the Stokes stream function Ψ the problem can be reduced to a free boundary problem on the half plane $\Pi = \{(r, z) : r > 0\}$ of the form (*cf.* § 1):

$$-L\Psi = 0 \quad \text{on } \Pi \setminus A, \tag{0.1}$$

$$-L\Psi = \lambda r^2 f(\Psi) \quad \text{on } A, \tag{0.2}$$

$$\Psi(0, z) = -k \leqq 0, \tag{0.3}$$

$$\Psi_{|\partial A} = 0, \tag{0.4}$$

$$\Psi_r/r \to -W, \quad \Psi_z/r \to 0 \quad \text{as } r^2 + z^2 \to \infty. \tag{0.5}$$

Above, L stands for a second order elliptic differential operator. A is the (*a priori* unknown) cross section of the vortex ring. f is called the "vorticity function" with coupling strength parameter $\lambda > 0$. k is the flux constant measuring the flow rate between the z-axis and ∂A. The constant $W > 0$ is the "propagation speed", namely the limit of the velocity field q at infinity. Subscripts denote partial derivatives.

When $k = 0$ and f is a positive constant, an explicit solution of (0.1–0.5) was found by HILL [12]. It corresponds to a spherical vortex, *Hill's vortex*.

Papers [6, 14] deal with the existence of vortex rings bifurcating from Hill's vortex and [4, 5] study uniqueness questions.

Global existence of vortex rings was first established in [10] to which we also refer for a description of the physical significance of the problem. However, in [10] a nonlinear *eigenvalue* problem is solved and the coupling constant λ arises as a Lagrange parameter which is left undetermined. For physical applications, however, existence results for *fixed* λ, say $\lambda = 1$, are desirable.

Motivated by [10], problem (0.1–0.5) for fixed $\lambda = 1$ has been studied in [13], and, independently, in [1] assuming that the vorticity f is *superlinear*. In both [1] and [13] it is assumed that $f(0) = 0$, even if from the physical point of view a strictly positive vorticity f is more appropriate. Lastly, the case of a *superlinear* f with $f(0) > 0$ *small* is investigated in [8]. Let us point out that when the free boundary problem (0.1–0.5) is translated, as usual, into a semilinear elliptic problem on $\boldsymbol{R}^2$ by extending $f(s) = 0$ for $s < 0$, then if $f(0)$ is strictly positive the corresponding nonlinearity will be *discontinuous* at 0.

Besides [6] and [14], where $f \equiv$ constant, we do not know any existence results for vortex rings for given strength parameter λ and bounded, positive vorticity function f.

The purpose of this paper is to study such a case. More precisely, in our Theorem 4.1 we establish the existence of a solution Ψ of (0.1–0.5), corresponding to a bounded, symmetric vortex core A, under the assumptions that k, λ, W are prescribed and the vorticity function f is *bounded* and positive, and so gives rise to a *discontinuous* nonlinearity, as in [10].

Our approach would apply to superlinear f as well; also for this case in the present generality the existence of vortex rings would be new, extending the results of [1], [8], [13]. However, to limit the paper to a reasonable length, we discuss in detail only the case of bounded vorticity, which seems to be the most interesting one.

Problem (0.1–0.5) is first approximated by a semilinear Dirichlet boundary value problem on a ball B_R centered in 0, passing then to the limit as $R \to \infty$. The approximate problem is accessible by variational methods and possesses, for R large, two *nontrivial*, cylindrically symmetric solutions: v_R, the absolute minimum of the associated energy; and u_R, corresponding to a "Mountain Pass" critical point [2].

It is worth noting that, strikingly, in the limit the energetically unstable solutions u_R survive, while the stable ones, v_R, diverge. To perform the limit procedure we use the variational characterization of the "Mountain Pass" solution u_R and derive, by arguments somewhat related to those of [16], a uniform bound for $|\nabla u_R|$ in L^2 for a sequence $R_m \to \infty$. When f is superlinear, this bound could be obtained by a more direct argument from the equation itself (*cf.* [1]) but the latter approach does not seem to work in the case of a *bounded* f. In contrast, the approach we use here could be employed to solve more general semilinear elliptic variational problems in $\boldsymbol{R}^n$ under suitable symmetries.

The rest of the paper is divided into 4 sections. In § 1 the problem is described in more detail; in § 2 the existence of solutions of the approximating problems is derived; § 3 contains the *a priori* estimates which enable us to pass to the limit; in § 4 we state the main results.

A preliminary announcement of this paper has appeared in Applied Mathematics Letters.

§ 1. Setting of the problem

As stated in § 0, by axisymmetry the vortex problem can be formulated in the half space $\Pi = \{(r, z) : r > 0\}$. As is shown for example in [10], if q is the velocity field, there is a stream function Ψ such that

$$q = \left(\frac{1}{r}\frac{-\partial\Psi}{\partial z},\ 0,\ \frac{1}{r}\frac{\partial\Psi}{\partial r}\right).$$

Let L denote the operator

$$L = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2};$$

then the vorticity of the flow, curl q, has cylindrical components $(0, \omega = -r^{-1} L\Psi, 0)$. Finally, the laws of hydrodynamics demand that ω/r is constant on stream-surfaces $\Psi =$ constant. Thus the problem of finding a vortex ring with cross section $A \subset \Pi$, flux constant $k \geqq 0$ and propagation speed $W > 0$, amounts to determining a function $\Psi \in C^1(\Pi) \cap C^2(\Pi \setminus \partial A)$ satisfying (0.1–0.5), for some function f and constants λ, k and W.

Without loss of generality we may take $\lambda = 1$, $W = 2$. We also set $\psi(r, z) = \Psi(r, z) + r^2 + k$, the reduced flow potential, and introduce the functions $h, g: \boldsymbol{R} \to \boldsymbol{R}$

$$h(s) = 0 \quad \text{if } s \leqq 0, \qquad h(s) = 1 \quad \text{if } s > 0;$$

$$g(s) = h(s) f(s).$$

In this notation (0.1–0.5) become:

$$-L\Psi = r^2 g(\psi - r^2 - k) \quad \text{in } \Pi,$$
$$\psi(0, z) = 0, \tag{1.1}$$
$$|\nabla\psi|/r \to 0 \quad \text{as } r^2 + z^2 \to \infty.$$

A solution of (1.1) is a $\psi \in C^2(\Pi \setminus \partial A) \cap C^1(\Pi)$ which solves the first equation in (1.1) almost everywhere. By the maximum principle any solution ψ of (1.1) is positive; the set $A = \{\Psi > 0\} = \{(r, z) : \psi(r, z) > r^2 + k\}$ corresponds to the vortex core.

Following Ni [13], we introduce as new unknown the function u, related to ψ by

$$\psi(r, z) = r^2 u(r, z).$$

Then, formally, we have $L\psi = r^2 \Delta u$, where Δ denotes the Laplacian in cylindrical coordinates (r, z) in $\boldsymbol{R}^5$, with

$$r = \sqrt{x_1^2 + \ldots + x_4^2}, \quad z = x_5.$$

Hence if $u(r, z)$ solves

(P) $\qquad -\Delta u = g(r^2 u - r^2 - k) \quad \text{in } \boldsymbol{R}^5, \quad u \to 0 \quad \text{as } |x| \to \infty,$

then $\psi(r, z) = r^2 u(r, z)$ is the desired solution of (1.1).

Observe that if the vortex core $\{(r, z) : u(r, z) > 1 + k/r^2\}$ is bounded, then the decay condition "$u \to 0$ as $|x| \to \infty$" implies (0.5).

Let $B(R) = \{x \in \mathbf{R}^5 : |x| < R\}$ denote the ball in $\mathbf{R}^5$ centered at $x = 0$ with radius R. It is natural to approximate of (P) by the following boundary value problem:

$$(\mathrm{P})_R \qquad -\Delta u = g(r^2 u - r^2 - k) \quad \text{in } B(R), \quad u = 0 \quad \text{on } \partial B(R),$$

This problem has a physical interest in itself. It will be studied in the following section.

§ 2. The approximate problem

Problem $(\mathrm{P})_R$ will be solved by variational methods. We will use standard notations for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $H^{m,p}(\Omega)$, for any domain $\Omega \subset \mathbf{R}^5$. The norm in $L^2(B(R))$ will be denoted by $|u|_{2,R}$. $H(R)$ will denote the space of cylindrically symmetric u in $H_0^{1,2}(B(R))$ and will be equipped with scalar product and norm, respectively

$$((u, v))_R = \int_{B(R)} \nabla u \cdot \nabla v,$$

$$\|u\|_R^2 = ((u, u))_R.$$

In the sequel we will suppose

(*f*) *f is bounded, continuous, positive and nondecreasing on* $]0, \infty[$.

Let

$$G(r, u) = \int_0^u g(r^2 v - r^2 - k)\, dv$$

and define $J_R, E_R : H(R) \to \mathbf{R}$ by setting

$$J_R(u) = \int_{B(R)} G(r, u),$$

$$E_R(u) = \tfrac{1}{2} \|u\|_R^2 - J_R(u).$$

Note that E_R is well defined on $H(R)$ and is the difference of a quadratic and a Lipschitz continuous and convex term. Therefore, although E_R is not Fréchet differentiable in $H(R)$, it possesses a set-valued super-gradient $dE_R(u) = u - dJ_R(u) \subset H(R)$ at any point $u \in H(R)$, where dJ_R is the sub-gradient of J_R, represented by g, the maximal monotone extension of the map $u \to g(r^2 u - r^2 - k)$ obtained by filling up the jump of g at 0. One has

$$v \in dE_R(u) \Leftrightarrow E_R(u + w) - E_R(u) - ((v, w))_R \leqq o(\|w\|_R), \quad \forall\, w \in H(R).$$

Moreover, the map $u \to dE_R(u)$ is weakly upper semi-continuous, see [9, Prop. 6, p. 105], and compact.

A critical point of E_R is a $u \in H(R)$ such that $0 \in dE_R(u)$.

Lemma 2.1. *$u \in H(R)$ is a critical point of E_R if and only if u is a positive solution of* $(\mathrm{P})_R$ *almost everywhere.*

Proof. If $0 \in dE_R(u)$ then the results of Section 2.2 of [4] imply readily

$$-\Delta u \in g(r^2 u - r^2 - k), \quad u \in H(R) \cap H^{2,2}(B(R)).$$

Let $\Gamma = \{(r, z): r^2 u = r^2 + k\}$. By a theorem of STAMPACCHIA $-\Delta u = -\Delta(k/r^2) = 0$ a.e. on Γ. Since we defined $g(0) = 0$, $-\Delta u = g(r^2 u - r^2 - k)$ a.e. in $B(R)$, and u is a solution a.e. of $(\mathrm{P})_R$. By the maximum principle $u > 0$. The converse is obvious. □

Remark 2.2. Actually, for the critical points obtained below one has meas $(\Gamma) = 0$, and therefore the value $g(0)$ could be defined in an arbitrary way. □

Note that $(\mathrm{P})_R$ always has the *trivial* solution $u \equiv 0$. In order to prove the existence of solutions $u \not\equiv 0$ we next derive some lemmas which will enable us to employ variational principles. Some of the arguments are rather standard and will be sketched only.

Lemma 2.3. *Suppose (f) holds. Then*

(i) *for any* $R > 0$, E_R *is bounded from below, weakly lower semicontinuous and coercive on* $H(R)$;

(ii) *for any* $R > 0$ *the function* $u \equiv 0$ *is a (strict) relative minimizer of* E_R *and for any* $\varrho_0 > 0$ *there exists* $0 < \varrho < \varrho_0, \alpha > 0$, *such that* $E_R(u) \geqq \alpha$, $\forall u: \|u\|_R = \varrho$;

(iii) $\exists\, R_0 > 0$ *and* $u_1 \in H(R_0)$ *such that* $E_{R_0}(u_1) < 0$. *Moreover, setting* $u_1 = 0$ *outside* $B(R_0)$, *then* $u_1 \in H(R)$ *and* $E_R(u_1) < 0$, $\forall\, R \geqq R_0$.

Proof. (i) is trivial because g is bounded.

(ii) From the fact that $g(r, u)$ is monotone in u and vanishes for $r^2 u < r^2 + k$, by the Sobolev inequality we have

$$\int_{B(R)} G(r, u) \leqq \int_{B(R)} g(r^2 u - r^2 - k)\, u \leqq C \int_{\{x: u(x) \geqq 1\}} u \leqq C \int_{B(R)} |u|^{\frac{10}{3}} \leqq c\, \|u\|_R^{\frac{10}{3}}.$$

Hence (ii) follows.

(iii) Let $\phi \in H(1)$ satisfy $J_1(\phi) > 0$. Scaling $\phi_R(x) = \phi(x/R) \in H(R)$, we have

$$\|\phi_R\|_R^2 = R^3\, \|\phi\|_1^2. \tag{2.0}$$

Moreover, by the monotonicity of g

$$\begin{aligned} J_R(\phi_R) &= \int_{B(R)} \left\{ \int_0^{\phi_R(x)} g(r^2(s-1) - k)\, ds \right\} dx \\ &\geqq R^5 \int_{B(1)} \left\{ \int_0^{\phi\left(\frac{x}{R}\right)} g\left(\left(\frac{r}{R}\right)^2 (s-1) - k\right) ds \right\} d\left(\frac{x}{R}\right) \qquad (2.1) \\ &= R^5\, J_1(\phi), \end{aligned}$$

for all $R \geqq 1$. Hence

$$E_R(\phi_R) \to -\infty \quad (R \to \infty), \tag{2.2}$$

and (iii) follows. □

The next lemma is concerned with the Palais-Smale condition which for non-smooth functionals like E_R can be replaced by the following:

Lemma 2.4. *Let f satisfy (f) and suppose that $u_m \in H(R)$ is a sequence such that*

$$|E_R(u_m)| \leqq c, \quad \inf\{\|v\|_R : v \in dE_R(u_m)\} \to 0. \tag{2.3}$$

Then, up to a sub-sequence, $u_m \to u$ in $H(R)$ and $0 \in dE_R(u)$.

Proof. Use the fact that E_R is coercive on $H(R)$ and dE_R is weakly upper semi-continuous and compact. □

In the sequel a sequence u_m in $H(R)$ satisfying (2.3) will be referred to as a PS-sequence. For $u \in H(R)$, $u \geqq 0$, we denote by u^* the Steiner symmetrization of u with respect the $z = x_5$ axis, namely $u^* \in H(R)$, $u^*(r, z) = u^*(r, -z)$, u^* is non-increasing in $|z|$ and equi-measurable with u for fixed r:

$$\text{meas}\,\{z : u^*(r, z) > c\} = \text{meas}\,\{z : u(r, z) > c\}, \quad \forall\, c \geqq 0, \quad r \geqq 0.$$

Note that $\|u^*\|_R \leqq \|u\|_R$, $J_R(u^*) = J_R(u)$ for $u \in H(R)$. Hence, in particular,

$$E_R(u^*) \leqq E_R(u), \quad \forall\, u \in H(R). \tag{2.4}$$

We are now in position to state the main result of this section:

Theorem 2.5. *Suppose (f) holds. Then for $R \geqq R_0$ defined in Lemma 2.3(iii) problem* $(\mathrm{P})_R$ *has at least two positive symmetric solutions $u_R = u_R^*$ and $v_R = v_R^*$ satisfying:*

$$J_R(v_R) = \min\{J_R(u) : u \in H_R\} < 0;$$

$$J_R(u_R) = \inf_{p \in \Lambda(R)} \max\{J_R(p(t)) : 0 \leqq t \leqq 1\}, \tag{2.5}$$

where

$$\Lambda(R) = \{p \in C([0, 1]; H(R)) : p(0) = 0, p(1) = u_1\}.$$

Moreover, for u_R the free boundary Γ has zero measure.

Proof. By Lemma 2.3(i) J_R attains the minimum on some $v_R \in H(R)$. By Lemma 2.3(iii) $J_R(v_R) < 0$ for R large and hence $v_R \neq 0$. By (2.4) we may assume that $v_R = v_R^*$.

Lemmas 2.4, 2.3(ii) and (iii) enable us to apply the "Mountain Pass" theorem [2] in the form stated in [9] (suitable for Lipschitz functionals) yielding the existence of a critical point $u_R \neq 0$ satisfying (2.5). Similarly, (2.4) and the arguments of [7, Theorem 3.4 p. 403–405] allow us to find a critical point $u_R = u_R^*$ satisfying (2.5) and such that $\partial u_R/\partial z < 0$ for $z > 0$. In particular it follows that meas $(\Gamma) = 0$.

Both u_R and v_R give rise to positive solutions of $(\mathrm{P})_R$ according to Lemma 2.1 (see also Remark 2.2). □

Remarks 2.6. (i) The preceding theorem is related to the results of [3], where an approach based on a dual variational principle is employed. Actually, the approach of [3] furnishes an alternative proof of Theorem 2.5.
(ii) Let us point out that the symmetry of the solutions does not follow (at least in a direct way) from the result by GIDAS, NI & NIRENBERG [11] because g is discontinuous. In [3] a rather simple proof of the symmetry results needed here can be found. □

§ 3. A priori estimates for u_R

In order to obtain *a priori* bounds on suitable critical points u_R characterized by the min-max principle (2.5) and suitable for passing to the limit $R \to \infty$ we need to take a closer look at the mechanism for constructing u_R.

We set

$$\gamma(R) = \inf_{p \in \Lambda(R)} \sup_{u \in p} E_R(u) > 0$$

where $\Lambda(R)$ has been defined in the preceding section.

Recall that, for $R' < R$, we may regard $H(R') \subset H(R)$ (simply extend $u \in H(R')$ by setting $u = 0$ outside $B(R')$) and, still denoting the extended function by u, we conclude that $E_{R'}(u) = E_R(u)$. It follows that $\Lambda(R') \subset \Lambda(R)$, whence $\gamma(R') \geqq \gamma(R)$. In other words $\gamma(R)$ is non-increasing, hence a.e. differentiable and

$$\int_{R_0}^{\infty} \left| \frac{d}{dR} \gamma(R) \right| dR \leqq \gamma(R_0) - \liminf_{R \to \infty} \gamma(R) \leqq \gamma(R_0) < \infty .$$

As a consequence, there is a sequence $R_m \to \infty$ such that

$$\lim_{m \to \infty} R_m \frac{d}{dR} \gamma(R_m) = 0 . \tag{3.1}$$

Before stating the *a priori* estimates, we need some preliminary results.

Lemma 3.1. *For* $R_0 < R' = sR < R$ *and* $u \in H(R)$ *we let* $u_s(x) = u(x/s) \in H(R')$. *Then, if* $s < 1$ *and sufficiently close to* 1,

$$\gamma(sR) = \inf_{p \in \Lambda(R)} \sup_{u \in p} E_{sR}(u_s) .$$

Proof. Let us consider the maps

$$u \to \tilde{u} = u(\cdot/s),$$

$$v \to \hat{v} = v(s\cdot) .$$

which yield an isomorphism between $H(R)$ and $H(sR)$ and induce mappings $\Lambda(R) \to \Lambda(sR)$ and $\Lambda(sR) \to \Lambda(R)$ as follows: for $p \in \Lambda(R)$ with $p(1) = u_1$

let $\tilde{p} \in \Lambda(sR)$ be the path

$$\tilde{p}(t) = \left(p\left(\frac{t}{s}\right)\right)^{\sim} \quad \text{for } 0 \leqq t \leqq s; \quad \tilde{p}(t) = u_1(\cdot/t) \quad \text{for } s \leqq t \leqq 1.$$

Conversely, for $p \in \Lambda(sR)$ let $\hat{p} \in \Lambda(R)$ be the path

$$\hat{p}(t) = \left(p\left(\frac{t}{s}\right)\right)^{\wedge} \quad \text{for } 0 \leqq t \leqq s; \quad \hat{p}(t) = u_1(t\cdot) \quad \text{for } s \leqq t \leqq 1.$$

It is easy to verify that for all s sufficiently close to 1 and all $s \leqq t \leqq 1$ there results $E_{sR}(u_1(\cdot/t)), E_R(u_1(t\cdot)) < 0$. Moreover, given a path $p \in \Lambda(sR)$, let $q = \tilde{\hat{p}} \in \Lambda(sR)$ be the path obtained composing the above maps $\hat{p}$ and $\tilde{p}$. Note that

$$\sup_{u \in q} E_{sR}(u) = \sup_{u \in p} E_{sR}(u) \geqq \gamma(sR) > 0.$$

Hence if we let $\tilde{\Lambda} = \{\tilde{p} : p \in \Lambda(R)\}$ and define

$$\gamma = \inf_{\tilde{p} \in \tilde{\Lambda}} \sup_{u \in \tilde{p}} E_{sR}(u) = \inf_{p \in \Lambda(R)} \sup_{u \in p} E_{sR}(\tilde{u}),$$

it follows that $\tilde{\gamma} = \gamma(sR)$. □

Proposition 3.2. *Suppose $R \to \gamma(R)$ is differentiable at R, $R > R_0$. Then there is a (positive) solution u_R of $(\mathrm{P})_R$ satisfying*

$$\|u_R\|_R^2 \leqq C \cdot (\gamma(R) + 2R\,|\gamma'(R)| + 5),$$

with a constant C independent of R.

Proof. *Step* 1. We set $u_s(x) = u(x/s)$ for $0 < s < 1$ close to 1. By the preceding lemma, for any $\varepsilon \in\,]0, 1]$ there exists $p \in \Lambda = \Lambda(R)$ such that

$$\sup_{u \in p} E_{sR}(u_s) \leqq \gamma(sR) + \varepsilon(1 - s^5). \tag{3.2}$$

Moreover, let $u \in p$ satisfy

$$E_R(u) \geqq \gamma(R) - \varepsilon(1 - s^5). \tag{3.2'}$$

From (3.2–2′) it follows:

$$E_{sR}(u_s) - E_R(u) \leqq \gamma(sR) - \gamma(R) + 2\varepsilon(1 - s^5). \tag{3.3}$$

First we estimate the left-hand side of (3.3). By (2.1) $J_{sR}(u_s) \leqq s^5 J_R(u)$ and

$$\frac{s^5}{1 - s^5}(J_R(u) - J_{sR}(u_s)) \geqq J_{sR}(u_s).$$

On the other hand, by (2.0) one has

$$\|u_s\|_{sR}^2 = s^3 \|u_s\|_R^2$$

whence

$$\frac{s^5}{1-s^5}(\|u\|_R^2 - \|u_s\|_{sR}^2) = \frac{s^2 - s^5}{1-s^5}\|u_s\|_{sR}^2.$$

As a consequence, for $0 < 1 - s$ small,

$$s^5 \frac{E_{sR}(u_s) - E_R(u)}{1-s^5} \geqq J_{sR}(u_s) - \tfrac{13}{10}\|u_s\|_{sR}^2.$$

This inequality and (3.3) imply that for s close to 1

$$-\tfrac{3}{10}\|u_s\|_{sR}^2 + J_{sR}(u_s) \leqq R\,|\gamma'(R)| + 3\varepsilon. \tag{3.4}$$

From (3.4) we deduce

$$\begin{aligned} E_{sR}(u_s) &= \tfrac{1}{2}\|u_s\|_{sR}^2 - J_{sR}(u_s) \\ &\geqq \tfrac{1}{5}\|u_s\|_{sR}^2 - R\,|\gamma'(R)| - 3\varepsilon, \end{aligned} \tag{3.5}$$

whence:

$$\begin{aligned} s^3\|u\|_R^2 = \|u_s\|_{sR}^2 &\leqq 5(E_{sR}(u_s) + R\,|\gamma'(R)| + 3\varepsilon) \\ &< 5(\gamma(sR) + R\,|\gamma'(R)| + 4\varepsilon) \leqq 5(\gamma(R) + 2R\,|\gamma'(R)| + 5\varepsilon). \end{aligned} \tag{3.6}$$

Step 2. We claim *there is a PS-sequence* $u_m \in H(R)$ *such that*

(i) $E_R(u_m) \to \gamma(R)$;

and

(ii) $\limsup_{m\to\infty} \|u_m\|_R^2 \leqq c^* =: 5[\gamma(R) + 2R\,|\gamma'(R)| + 5] + 1$.

To see this, for $\delta > 0$ set

$$U_\delta = \{u \in H(R) : \|u\|_R^2 \leqq c^* + \delta,\ |E_R(u) - \gamma(R)| \leqq \delta\}$$

and suppose, by contradiction, that for some $\varepsilon^* > 0$ and any $u \in U_{\varepsilon^*}$

$$\inf\{\|v\|_R : v \in dE_R(u)\} > \varepsilon^*.$$

By [9, Lemma 3.4 and Theorem 3.1], corresponding to $c = \gamma(R)$, $\varepsilon_0 = \min\{\varepsilon^*, \gamma(R)\}$, $N = H(R) \setminus U_{\varepsilon^*}$, we can find $\varepsilon \in]0, \varepsilon_0[$ and a homeomorphism $\Phi : H(R) \to H(R)$ such that

$$\Phi(u) = u \quad \text{if } |E_R(u) - \gamma(R)| \geqq \gamma(R); \tag{3.7}$$

$$\begin{gathered} E_R(\Phi(u)) \leqq E_R(u) \quad \text{for all } u; \\ E_R(\Phi(u)) \leqq \gamma(R) - \varepsilon \quad \text{if } u \in U_{\varepsilon^*},\quad E_R(u) < \gamma(R) + \varepsilon. \end{gathered} \tag{3.8}$$

For $s < 1$ close to 1 choose $p \in \Lambda(R)$ such that

$$\sup\{E_{sR}(u_s) : u \in p\} \leqq \gamma(sR) + (1 - s^5).$$

Then by Step 1 any $u \in p$ where $E_R(u) \geqq \gamma(R) - (1 - s^5)$ satisfies $\|u\|_R^2 \leqq c^*$. In particular, if s is sufficiently close to 1, by using (2.0) and (2.1) we can arrange that for all such u

$$E_R(u) \leqq E_{sR}(u_s) + \varepsilon/2 \leqq \gamma(sR) + (1 - s^5) + \varepsilon/2 \leqq \gamma(R) + \varepsilon,$$

and $u \in U_{\varepsilon^*}$.

Applying Φ to p, by (3.7) we obtain a comparison path $p' = \Phi(p) \in \Lambda(R)$ which satisfies

$$\sup \{E_R(u) : u \in p'\} < \gamma(R). \tag{3.9}$$

In fact, if $E_R(u) \geqq \gamma(R) - (1 - s^5)$ (otherwise there is nothing to prove), by the preceding remarks and (3.8) it follows that $E_R(\Phi(u)) \leqq \gamma(R) - \varepsilon$ for any $u \in p$. Clearly (3.9) contradicts the definition of $\gamma(R)$ and the proof of Step 2 is complete.

The conclusion of Proposition 3.2 now follows immediately from Lemma 2.4. □

Combining Proposition 3.2 and (3.1) with the arguments of [7] we obtain

Corollary 3.3. *There exist a constant* c, *a sequence* $R_m \to \infty$, *and a sequence of symmetric solutions* $u_m = u_{R_m}$ *of* $(P)_{R_m}$ *with*

$$\|u_m\|_{R_m} < c. \tag{3.10}$$

§ 4. Existence of vortex rings

In this final section we prove the existence of a solution of problem (P), or, equivalently, of problem (0.1–0.5), by a limiting procedure.

Let $u_m \in H(R_m)$ be the sequence found in Corollary 3.3 and set

$$A_m = \{(r, z) \in B(R_m) : r^2 u_m(r, z) > r^2 + k\}.$$

Lemma 4.1. *There exists* $R^* > 0$ *such that* $A_m \subset B(R^*)$ *for all integer* m.

Proof. The lemma would follow from Corollary 3.3 and the estimates of [10, § 5.2] or [13, § 5.3]. Below, taking advantage of the boundedness of f, we report a slightly different, short proof, to make the paper as self-contained as possible.

Extend u_m to all $\mathbf{R}^5$ setting $u_m \equiv 0$ outside $B(R_m)$. Fix r_0; then the following estimate holds (hereafter we use the symbol c to denote possibly different constants, independent both of m and r_0):

$$\begin{aligned}
&\operatorname{meas}\{z : u_m(r_0, z) \geqq \tfrac{1}{2}\} \\
&\quad \leqq c \int_{-\infty}^{\infty} [u_m(r_0, z)]^{\frac{8}{3}}\, dz \\
&\quad \leqq c \int_{r_0}^{\infty} \left| \frac{\partial}{\partial r} \int_{-\infty}^{\infty} [u_m(r, z)]^{\frac{8}{3}}\, dz \right| dr \leqq c \int_{r_0}^{\infty} \int_{-\infty}^{\infty} |\nabla u_m| \cdot u_m^{\frac{5}{3}}\, dr\, dz \\
&\quad \leqq c \frac{1}{r_0^3} \int_{r}^{\infty} \int_{-\infty}^{\infty} |\nabla u_m| \cdot u_m^{\frac{5}{3}} r^3\, dr\, dz \leqq c \frac{1}{r_0^3} \|u_m\|_{R_m} |u_m|_{\frac{10}{3}, R_m}^{\frac{5}{3}} \leqq c \frac{1}{r_0^3} \|u_m\|_{R_m}^{\frac{8}{3}}.
\end{aligned}$$

By use of (3.10)

$$\text{meas}\,\{z : u_m(r_0, z) \geqq \tfrac{1}{2}\} \leqq c r_0^{-3}. \tag{4.1}$$

Moreover,

$$\|\Delta u_m\|_{\infty, R_m} \leqq \sup f < c \tag{4.2}$$

uniformly, and by the L^p-regularity theory the families $\{u_m(. + x_m)\}$ are equibounded in C^1, locally in $\boldsymbol{R}^5$, for any choice of $\{x_m\}$.

First, let $r_m = \max\{r : (r, z) \in A_m$ for some $z\}$; by symmetry, r_m is achieved for $z = 0$. Set $x_m = (r_m, 0)$; then $u_m(x_m) \geqq 1$, and by equicontinuity there exists $z_0 > 0$ (independent on m) such that $u_m(r_m, z_0) \geqq 1/2$. By (4.1) this implies the uniform bound

$$r_m \leqq c z^{-\frac{1}{3}}.$$

Likewise, choose $x_m = (r_m, z_m) \in A_m$, where $z_m = \max\{z : (r, z) \in A_m$ for some $r\}$. As before we conclude the existence of some $r_0 > 0$ such that $u_m(x) \geqq 1/2$ for $x = (r, z_m)$ and $|r - r_m| \leqq r_0$. But then (4.1) implies that

$$z_m \leqq c r_0^{-3}$$

uniformly, and the conclusion follows. □

Next by (4.2) we also conclude that

(a) u_m converges in $C^{1+\alpha}_{\text{loc}}(\boldsymbol{R}^5)$, $0 < \alpha < 1$, to some u, solving (P).

Let us note that $u \not\equiv 0$; otherwise for m large, $u_m < 1$ on $B(R^*)$, whence $r^2 u_m < r^2 + k$. Since u_m is a solution of $(\mathrm{P})_{Rm}$ and $g(r, z) = 0$ for all $r^2 z < r^2 + k$, it would follow that $u_m \equiv 0$, a contradiction. Moreover remark that $u \not\equiv 0$ implies that the vortex core

$$A = \{(r, z) \in \boldsymbol{R}^5 : r^2 u(r, z) > r^2 + k\}$$

is not empty. Finally, by Lemma 4.1, $A \subset B(R^*)$ is bounded. In addition, (a) and Theorem 2.5 imply that

(b) u is symmetric because the u_m were so; moreover, $\partial u/\partial z > 0$ for $z > 0$. Hence ∂A has zero measure.

Finally, also in view of point (a) above, one has:

(c) $\psi = r^2 u$ is a solution of (1.1) in the sense specified in Section 1.

We can conclude by stating:

Theorem 4.2. *Suppose (f) holds and let u_R, v_R be the solutions of* $(\mathrm{P})_R$, *R large, found in Theorem 2.5 and Proposition 3.2, respectively. Then*

(i) *there is a sequence $R_m \to \infty$ and $u \in H^{1,2}(\boldsymbol{R}^5)$ such that $u_{R_m} \to u$ in $H^{1,2}$; $u = u(r, z)$ and $\psi = r^2 u$ is a positive, symmetric solution of* (1.1) *corresponding to a non-empty bounded vortex core;*

(ii) $E_R(v_R) \to -\infty$ *and* $|v_R|_{2,R} \to \infty$.

Proof. (i) This follows from Lemma 4.1 and conclusions (a), (b), (c).

(ii) by (2.2) it follows that $E_R(v_R) \to -\infty$ $(R \to \infty)$.

Finally, let $c > 0$ be a constant such that $G(r, u) < cu^2$. Such a constant exists because g is bounded and $g(r^2u - r^2 - k) = 0$ for all $r^2u < r^2 + k$. Then

$$0 < \tfrac{1}{2} \|v_R\|_R^2 = E_R(v_R) + \int_{B(R)} G(r, v_R) \leqq E_R(v_R) + c\, |v_R|_{2,R}^2 .$$

Hence

$$c\, |v_R|_{2,R}^2 \geqq -E_R(v_R) \to \infty .$$

This completes the proof of the Theorem. □

Remarks 4.3. (i) The arguments concerning the existence of u_R and its convergence to a solution u of (P) work if f is superlinear, as well. However, in such a case, the *a priori* estimates on $|\nabla u_R|_{2,R}$ can be obtained in a more direct way, as, for example, in [1].

(ii) It is clear that the procedure employed above can be used to prove the existence of nontrivial solutions of semilinear elliptic boundary value problems in $\boldsymbol{R}^n$ with bounded nonlinearity, in presence of a suitable symmetry. We leave it to the reader to carry out the details.

(iii) Theorem 4.1 holds if $k \geqq 0$. If $k = 0$, the vortex is spherical. If, in addition, f is identically constant, we would find Hill's spherical vortex, according to the uniqueness result of [4].

Acknowledgments

AMBROSETTI thanks BOB TURNER for several useful discussions.

STRUWE is indebted to the kind hospitality and support of the Scuola Normale Superiore, Pisa, during the writing of this paper.

References

1. AMBROSETTI, A., & G. MANCINI, *On some free boundary problems*. In "Recent contributions to nonlinear partial Differential equations", Ed. H. BERESTYCKI & H. BREZIS, Pitman 1981.
2. AMBROSETTI, A., & P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.
3. AMBROSETTI, A., & R. E. L. TURNER, *Some discontinuous variational problems*, Diff. and Integral Equat. **1–3** (1988), 341–349.
4. AMICK, C. J., & L. E. FRAENKEL, *The uniqueness of Hill's spherical vortex*, Archive Rational Mech. & Anal. **92** (1986), 91–119.
5. AMICK, C. J., & L. E. FRAENKEL, *The uniqueness of a family of steady vortex rings*, Archive Rational Mech. & Anal. (1988), 207–241.
6. AMICK, C. J., & R. E. L. TURNER, *A global branch of steady vortex rings*, J. Reine Angew. Math. (to appear).

7. BONA, J. L., D. K. BOSE & R. E. L. TURNER, *Finite amplitude steady waves in stratified fluids*, J. de Math. Pures Appl. **62** (1983), 389–439.
8. CERAMI, G., *Soluzioni positive di problemi con parte nonlineare discontinua e applicazioni a un problema di frontiera libera*, Boll. U. M. I. **2** (1983), 321–338.
9. CHANG, C. K., *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), 102–129.
10. FRAENKEL, L. E., & M. S. BERGER, *A global theory of steady vortex rings in an ideal fluid*, Acta Math. **132** (1974), 13–51.
11. GIDAS, B., W. M. NI & L. NIRENBERG, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.
12. HILL, M. J. M., *On a spherical vortex*, Phil. Trans. Roy. Soc. London **185** (1894), 213–245.
13. NI, W. M., *On the existence of global vortex rings*, J. d'Analyse Math. **37** (1980), 208–247.
14. NORBURY, J., *A family of steady vortex rings*, J. Fluid Mech. **57** (1973), 417–431.
15. PALAIS, R. S., *Lusternik-Schnirelman theory on Banach manifolds*, Topology **5** (1966), 115–132.
16. STRUWE, M., *The existence of surfaces of constant mean curvature with free boundaries*, Acta Math. **160** (1988), 19–64.

Scuola Normale Superiore
Pisa

&

Mathematik
E. T. H. Zurich

(Received March 20, 1989)