Journal of Dynamics and Differential Equations, Vol. 5, No. 2, 1993

Wavefronts for a Cooperative Tridiagonal System of Differential Equations

D. Hankerson[1] and B. Zinner[1]

Received June 22, 1992

Consider the infinite system of nonlinear differential equations $\dot{u}_n = f(u_{n-1}, u_n, u_{n+1})$, $n \in \mathbb{Z}$, where $f \in C^1$, $D_1 f > 0$, $D_3 f > 0$, and $f(0, 0, 0) = 0 = f(1, 1, 1)$. Existence of wavefronts—i.e., solutions of the form $u_n(t) = U(n + ct)$, where $c \in \mathbb{R}$, $U(-\infty) = 0$, $U(+\infty) = 1$, and U is strictly increasing—is shown for functions f which satisfy the condition: there exists a, $0 < a < 1$, such that $f(x, x, x) < 0$ for $0 < x < a$ and $f(x, x, x) > 0$ for $a < x < 1$.

KEY WORDS: Traveling waves; Nagumo equation; cooperative systems; comparision principles.

1. INTRODUCTION

Consider the infinite system of coupled nonlinear differential equations

$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \qquad n \in \mathbb{Z} \tag{1}$$

where d is a positive number and f is a Lipschitz continuous function for which $f(0) = f(a) = f(1) = 0$, $f(x) < 0$ for $0 < x < a$ and $f(x) > 0$ for $a < x < 1$; e.g., $f(x) = x(x - \frac{1}{4})(1 - x)$. Note that system (1) is the discrete analogue to the well-known Nagumo equation (McKean, 1970),

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u)$$

The discrete Nagumo equation, system (1), was proposed by Bell (1981) as a model for conduction in myelinated nerve axons and was studied by a number of authors (Bell, 1981; Bell and Cosner, 1984; Chi

[1] Department of Discrete and Statistical Sciences, Auburn University, Auburn, Alabama 36849-5307.

1040-7294/93/0400-0359$07.00/0

et al., 1986; Keener, 1987; Zinner, 1991, 1992). In particular, the existence of a traveling wave with positive speed was shown under the additional assumptions that $\int_0^1 f(s)\, ds > 0$ and d is sufficiently large (Zinner, 1992).

It is the aim of this paper to show the existence of wavefronts for a more general discrete Nagumo equation. More precisely we will consider the tridiagonal system

$$\dot{u}_n = f(u_{n-1}, u_n, u_{n+1}), \qquad n \in \mathbb{Z} \tag{2}$$

where f is of class C^1 and satisfies the following conditions:

(C1) $D_1 f(x, y, z) > 0$ and $D_3 f(x, y, z) > 0$ whenever $0 \leqslant x \leqslant y \leqslant z \leqslant 1$.

(C2) There exists $a \in (0, 1)$ such that $f(0, 0, 0) = f(a, a, a) = f(1, 1, 1) = 0$, $f(x, x, x) < 0$ for $x \in (0, a)$, and $f(x, x, x) > 0$ for $x \in (a, 1)$.

(C3) $\sum_{i=1}^{3} D_i f(a, a, a) > 0$.

Condition (C1) says that (2) is cooperative, condition (C2) captures the important features of the Nagumo dynamics, and condition (C3) is a technical assumption needed in the proof. The main result is the following theorem.

Theorem 1. *Suppose that the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ satisfies the conditions (C1), (C2), and (C3). Then there exist $c \in \mathbb{R}$ and a strictly increasing function U with $U(-\infty) = 0$ and $U(\infty) = 1$ such that system (2) has a solution $u(t) = \{u_n(t)\}_{n=-\infty}^{\infty}$ satisfying $u_n(t) = U(n + ct)$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$.*

Note that the existence of waves with speed c of system (2) is equivalent to the existence of a solution of the functional differential equation

$$cU'(z) = f(U(z-1), U(z), U(z+1)), \qquad z \in \mathbb{R} \tag{3}$$

with $U(-\infty) = 0$ and $U(\infty) = 1$. The existence theory for this type of functional differential equation is not well developed yet, and to our knowledge results are known only for linear functions or small perturbations of linear functions f (Rustichini, 1989a, b).

The main idea for the proof of Theorem 1 can be traced back to the classical paper by Kolmogoroff *et al.* (1937). They showed that the solution $u(x, t)$ of a certain initial value problem of Fisher's equation has the property that $u(x_0, t_0) = u(y, t)$ and $t > t_0$ imply $u_x(x_0, t_0) \geqslant u_x(y, t)$ (Kolmogoroff *et al.*, 1937, Thm. 11). We will prove the analogue result for system (2). The approach taken here is also closely related to the idea of the lap number or integer valued Liapunov function which have been used by many authors (see, e.g., Smith, 1990, and references therein).

2. COMPARISON OF SOLUTIONS

Discussions concerning existence, uniqueness, and continuous dependence are given by Walter (1970). For our purposes, it suffices to consider the following case. Let $a < b \in \mathbb{Z} \cup \{\pm\infty\}$, let $\Omega = \{n \in \mathbb{Z} \mid a < n < b\}$, and let B be the space of bounded sequences $(x_n)_{n \in \Omega} \subseteq \mathbb{R}$ with the supremum norm. The partial order on B is component-wise; that is, $u \leqslant v$ if and only if $u_n \leqslant v_n$, $n \in \Omega$.

Assume that J is an interval, $0 \in J \subseteq [0, \infty)$, and $f = f(t, u)$ is a continuous function from $J \times B$ to B which satisfies a Lipschitz condition with respect to u. In addition, assume that f is *quasi-monotone increasing in* u; that is, $f_n(t, u) \leqslant f_n(t, v)$ whenever $u \leqslant v$ and $u_n = v_n$. Then we have local existence, uniqueness, and convergence of successive approximations for the initial value problem $\dot{u} = f(t, u)$, $u(t_0) = u^0$ (Walter, 1970, Sec. 12).

Further, the following monotonicity result holds: If u and v are functions from J to B satisfying $u(0) \leqslant v(0)$ and $\dot{u} - f(t, u) \leqslant \dot{v} - f(t, v)$ in J, then $u \leqslant v$ in J (Walter, 1970, Th. 12.XIV). As a consequence, we obtain a comparison result.

Lemma 2. *Let $f(t, u)$ be a continuous function from $J \times B$ to B which satisfies a Lipschitz condition with respect to u, where $f = (f_n)_{n \in \Omega}$ has the form $f_n(t, u) = f_n(t, u_{n-1}, u_n, u_{n+1})$ [if $n = a + 1$, then we mean that $f_n(t, u) = f_n(t, u_n, u_{n+1})$; a similar statement holds in the case that $n = b - 1$]. Assume that $f_n(t, x_1, y, z_1) \leqslant f_n(t, x_2, y, z_2)$ if $x_1 \leqslant x_2$ and $z_1 \leqslant z_2$, where the inequality is strict if $x_1 < x_2$ or $z_1 < z_2$ (again, if $n = a + 1$, then the "x" term is not present; a similar statement holds if $n = b - 1$). If $u = u(t)$ and $v = v(t)$ satisfy*

(a) $u(0) \leqslant v(0)$ and $u(0) \neq v(0)$,

(b) $\dot{u} \leqslant f(t, u)$ and $\dot{v} \geqslant f(t, v)$ on J,

then $u_n(t) < v_n(t)$ for all $0 < t \in J$.

Proof. The monotonicity result mentioned above shows that $u \leqslant v$ in J. Suppose, on the contrary, that there exist $t > 0$ and $n \in \Omega$ such that $u_n(t) = v_n(t)$. Consider the case that $a + 1 < n < b - 1$. Since $u_{n-1}(t) \leqslant v_{n-1}(t)$ and $u_{n+1}(t) \leqslant v_{n+1}(t)$, it follows that

$$\dot{u}_n(t) \leqslant f_n(t, u_{n-1}(t), u_n(t), u_{n+1}(t)) \leqslant f_n(t, v_{n-1}(t), v_n(t), v_{n+1}(t)) \leqslant \dot{v}_n(t)$$

Using the monotonicity condition, we see that $\dot{u}_n(t) = \dot{v}_n(t)$. The conditions of f then imply that $u_{n-1}(t) = v_{n-1}(t)$ and $u_{n+1}(t) = v_{n+1}(t)$. The process can be repeated to show that $u = v$, contradicting the uniqueness condition.

The cases $n=a+1$ and $n=b-1$ are similar and will be omitted. This completes the proof. ▮

Corollary 3. *Assume (u_n) is the solution of the initial value problem (2),*

$$u_n(0)=0, \quad \textit{for} \quad n\leqslant 0, \qquad u_n(0)=1, \quad \textit{for} \quad n>0, \quad n\in\mathbb{Z} \tag{4}$$

Then $0<u_n(t)<u_{n+1}(t)<1$ for all $t>0$, $n\in\mathbb{Z}$.

Proof. Standard arguments using Lemma 2 prove the corollary. ▮

We now use Lemma 2 to compare solutions u and v for which condition (a) in Lemma 2 does not hold.

Definition. Let $x=(x_n)_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{R}$. We define the following *types* for the sequence x.

(T1) There exists $n_0\in\mathbb{Z}$ such that $x_n<0$ for $n<n_0$, $x_{n_0}\leqslant 0$, and $x_n>0$ for $n>n_0$.

(T2) There exists $n_0\in\mathbb{Z}$ such that $x_n<0$ for $n<n_0$, and $x_n=0$ for $n\geqslant n_0$.

(T3) $x_n<0$ for all $n\in\mathbb{Z}$.

(T4) There exists $n_0\in\mathbb{Z}$ such that $x_n=0$ for $n\leqslant n_0$, and $x_n>0$ for $n>n_0$.

(T5) $x_n>0$ for all $n\in\mathbb{Z}$.

Lemma 4. *Let $f_n=f_n(t,x,y,z)$ have partial derivatives which satisfy $\partial f_n/\partial x>0$ and $\partial f_n/\partial z>0$. Further, assume that $u=(u_n)$ and $v=(v_n)$ are solutions to the equations*

$$\dot{y}_n=f_n(t,y_{n-1},y_n,y_{n+1}), \qquad n\in\mathbb{Z}$$

Set $x=u-v$. If $x(0)$ is of type T1, then $x(t)$ may change its type with increasing t only according to the following diagram:

$$\begin{tikzcd}
\mathrm{T2} \arrow[dd] & & \mathrm{T4} \arrow[dd] \\
& \mathrm{T1} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
\mathrm{T3} & & \mathrm{T5}
\end{tikzcd}$$

Proof. It follows from Lemma 2 that T2 changes to T3 and T4 changes to T5. In addition, the lemma shows that no change of type is possible from T3 and T5. It remains to be shown that the only possible changes from T1 are to T2, T3, T4, and T5.

Let $T=\inf\{t\geqslant 0\colon x(t)$ is not of type T1$\}$ and suppose that $T<\infty$. We will show that $x(T)$ is of type T2, T3, T4, or T5. The proof uses only the comparison result.

First, we claim that if $x(t_0)$ is of type T1, then there exists $t_1>t_0$ such that $x(t)$ is of type T1 for all $t\in[t_0, t_1)$. To see this, note that, by definition, there exists $n_0\in\mathbb{Z}$ such that $x_n(t_0)<0$ for $n<n_0$, $x_{n_0}(t_0)\leqslant 0$, and $x_n(t_0)>0$ for $n>n_0$. Let $t_1=\inf\{t\geqslant t_0\colon x_{n_0-1}(t)\geqslant 0$ or $x_{n_0+1}(t)\leqslant 0\}$. Note that $t_1>t_0$ and that the following hold:

(i) $u_n(t_0)<v_n(t_0)$ for all $n\leqslant n_0-2$, and $u_{n_0-1}(t)<v_{n_0-1}(t)$ for all $t\in[t_0, t_1)$, and

(ii) $u_n(t_0)>v_n(t_0)$ for all $n\geqslant n_0+2$, and $u_{n_0+1}(t)>v_{n_0+1}(t)$ for all $t\in[t_0, t_1)$.

Lemma 2 applied to (i) and (ii) establishes the claim.

It follows that $T>0$ and $x(T)$ is not of type T1. Necessarily, there exists $n_0\in\mathbb{Z}\cup\{-\infty,\infty\}$ such that $x_n(T)\leqslant 0$ for $n\leqslant n_0$ and $x_n(T)\geqslant 0$ for $n>n_0$.

Next, consider the case that $x_n(T)=0$ for some n. We claim that $x_{n-1}(T)\leqslant 0$. Suppose that the claim is false; then $x_{n-1}(T)>0$ and it follows that $\dot{x}_n(T)>0$. Hence, $x_n(T-\varepsilon)<0$ for $\varepsilon>0$ sufficiently small. Thus, there exists $\varepsilon>0$ such that $x_{n-1}(T-\varepsilon)>0$ and $x_n(T-\varepsilon)<0$, in contradiction to $x(T-\varepsilon)$ is of type T1. A similar argument shows that $x_{n+1}(T)\geqslant 0$.

As a consequence, if $x_n(T)=0$ and $x_{n+1}(T)>0$, then $x_m(T)>0$ for all $m\geqslant n+1$. Similarly, if $x_n(T)=0$ and $x_{n-1}(T)<0$, then $x_m(T)<0$ for all $m\leqslant n-1$.

Finally, suppose that $x(T)$ is not of type T2, T3, T4, or T5. Then there exists $n,m\in\mathbb{Z}$, $n<m$, such that

$$x_{n-1}(T)<x_n(T)=0=x_{n+1}(T)=\cdots=x_m(T)<x_{m+1}(T)$$

Hence, $\dot{x}_n(T)<0$ and $\dot{x}_m(T)>0$ and it follows that there exists $\varepsilon>0$ such that $x_n(T-\varepsilon)>0$ and $x_m(T-\varepsilon)<0$, which contradicts the fact that $x(T-\varepsilon)$ is of type T1. This completes the proof. ▮

Since the variational equation of (2) also satisfies condition (C1), one may apply Lemma 4 to the difference between the trivial solution and the derivative of a solution of (2) to obtain the following corollary.

Corollary 5. *Suppose that (u_n) is the solution of the initial value problem (2), (4). If $\dot{u}_{n_0}(t_0) \geqslant 0$ for some $n_0 \in \mathbb{Z}$ and $t_0 > 0$, then $\dot{u}_n(t_0) \geqslant 0$ for all $n < n_0$. Similarly, if $\dot{u}_{n_0}(t_0) \leqslant 0$ for some $n_0 \in \mathbb{Z}$ and $t_0 > 0$, then $\dot{u}_n(t_0) \leqslant 0$ for all $n > n_0$. In both cases, the inequality on $\dot{u}_n(t_0)$ is strict if the inequality on $\dot{u}_{n_0}(t_0)$ is strict.*

Proof. Set $v_n = \dot{u}_n$ and $w_n = 0$. Then (v_n) and (w_n) are solutions of

$$\dot{y}_n = h_n(t, y_{n-1}, y_n, y_{n+1}), \qquad n \in \mathbb{Z}$$

where

$$h_n(t, x_1, x_2, x_3) = \sum_{i=1}^{3} D_i f(u_{n-1}(t), u_n(t), u_{n+1}(t)) x_i$$

Condition (C1) guarantees that the functions h_n satisfy the hypotheses of Lemma 4. From (C1) and (C2), it follows that

$$w_n(0) = 0 = \dot{u}_n(0) = v_n(0) \qquad \text{for} \quad n < 0$$

$$w_0(0) = 0 < f(0, 0, 1) = \dot{u}_0(0) = v_0(0).$$

Set $\Omega = \{n \in \mathbb{Z} \mid n < 0\}$ and let B be bounded real sequences $(x_n)_{n \in \Omega}$. Let $g = g(t, x)$ be defined on $[0, \infty) \times B$ by $g_{-1}(t, x) = h_{-1}(t, x_{-2}, x_{-1}, v_0(t))$, and, for $n < -1$, $g_n(t, x) = h_n(t, x_{n-1}, x_n, x_{n+1})$. Let $T > 0$ be such that $v_0(t) > 0$ on $[0, T]$. By considering the natural projection of $v(t)$ onto B, which is also denoted by $v(t)$, we may write $\dot{v}(t) = g(t, v(t))$ on $[0, T]$. Similarly, $\dot{w}(t) \leqslant g(t, w(t))$ on $[0, T]$. An application of Lemma 2 shows that $0 = w_n(t) < v_n(t)$ for $t > 0$ sufficiently small and $n < 0$.

Similar arguments show that $0 > v_n(t)$ for $t > 0$ sufficiently small and $n > 0$. Hence, $-v(t)$ is of type T1 for all $t > 0$ sufficiently small, and the result follows by Lemma 4. ∎

Note that if $u(t) = (u_n(t))$ is a solution of (2), then so is $w(t) = (w_n(t))$, $w_n(t) = u_n(t + t_0)$ for any constant $t_0 \in \mathbb{R}$. Applying Lemma 4 to the difference of these, we obtain the following corollary.

Corollary 6. *Suppose that (u_n) is the solution of the initial value problem (2), (4), and assume that $u_k(t_1) = u_m(t_2)$ for some k, $m \in \mathbb{Z}$ and $0 \leqslant t_1 \leqslant t_2$. Then for all $n \in \mathbb{N}$, $[u_{m-n}(t_2), u_{m+n}(t_2)] \subset [u_{k-n}(t_1), u_{k+n}(t_1)]$.*

3. AN A PRIORI BOUND

In this section, an a priori bound is established on the number of terms which can lie in a given interval for the solution u of (2), (4). We begin with two supporting lemmas.

Lemma 7. *Suppose that* $(u_{n_k}(t_k))$ *converges and* (t_k) *diverges to infinity. Then* $\lim(u_{m_k}(s_k)) = \lim(u_{n_k}(t_k))$ *and* $\lim(s_k) = \infty$ *imply that* $\lim(\dot{u}_{m_k}(s_k)) = \lim(\dot{u}_{n_k}(t_k))$.

Proof. Assume that Lemma 7 is not true. Then we may assume that $\lim(u_{m_k}(s_k)) = \lim(u_{n_k}(t_k))$, $(\dot{u}_{n_k}(t_k))$, and $(\dot{u}_{m_k}(s_k))$ converge, and $\lim(\dot{u}_{m_k}(s_k)) \neq \lim(\dot{u}_{n_k}(t_k))$.

At least one of the sequences $(\dot{u}_{m_k}(s_k))$ and $(\dot{u}_{n_k}(t_k))$ does not converge to zero. It suffices to consider the case where $\lim(\dot{u}_{m_k}(s_k)) \neq 0$.

We may assume that $(u_{m_k-1}(s_k))$, $(u_{n_k-1}(t_k))$, $(u_{m_k+1}(s_k))$, and $(u_{n_k+1}(t_k))$ converge, otherwise we take suitable subsequences. Since $\lim(\dot{u}_{m_k}(s_k)) \neq \lim(\dot{u}_{n_k}(t_k))$, at least one of the following holds:

(i) $\lim(u_{m_k-1}(s_k)) < \lim(u_{n_k-1}(t_k))$,

(ii) $\lim(u_{m_k-1}(s_k)) > \lim(u_{n_k-1}(t_k))$,

(iii) $\lim(u_{m_k+1}(s_k)) < \lim(u_{n_k+1}(t_k))$,

or

(iv) $\lim(u_{m_k+1}(s_k)) > \lim(u_{n_k+1}(t_k))$.

Suppose (i) holds; the other cases are treated similarly. One may assume that $s_k > t_k + 1$ for all k; otherwise one takes suitable subsequences. Since $\lim(\dot{u}_{m_k}(s_k)) \neq 0$, there exists a sequence (μ_k) such that $\lim(\mu_k) = 0$ and $u_{m_k}(s_k + \mu_k) = u_{n_k}(t_k)$. It follows that $u_{m_k-1}(s_k + \mu_k) \geqslant u_{n_k-1}(t_k)$ for $\mu_k < 1$ by Corollary 6. Hence $\lim(u_{m_k-1}(s_k)) = \lim(u_{m_k-1}(s_k + \mu_k)) \geqslant \lim(u_{n_k-1}(t_k))$, in contradiction to (i). ∎

Lemma 8. *Given* $T > 0$ *and* $\mu > 0$, *there exist* $N \in \mathbb{N}$ *and* $\delta > 0$ *such that* $\dot{u}_{n+j}(t_0) < \delta$ *for* $j = -N, \ldots, N$, *implies* $\dot{u}_n(t_0 + t) < \mu$ *for all* $t \in [0, T]$.

Proof. Let $N \in \mathbb{N}$, $p, q > 0$, and consider the following initial value problem: $\dot{w}_j = qw_{j-1} + qw_j + qw_{j+1}$, $w_{-N}(0) = p$, $w_j(0) = 0$ for $j = -N+1, \ldots, N-1$, $w_N(0) = p$, where $w_{-N-1} \equiv w_{-N}$ and $w_{N+1} \equiv w_N$.

Note that by symmetry $w_j(t) = w_{-j}(t)$ for $t \geqslant 0$, and using comparison methods it can be shown that $w_{-N}(t) \geqslant w_{-N+1}(t) \geqslant \cdots \geqslant w_0(t)$ for $t \geqslant 0$.

Therefore $\dot{w}_{-N} \leqslant 3qw_{-N}$ and $\dot{w}_{-N+k} \leqslant qw_{N+k-1} + 2qw_{-N+k}$ for $k = 1, \ldots, N$. One shows by induction that for $k = 1, \ldots, N$,

$$w_{-N+k}(t) \leqslant p\left(e^{qt} - \sum_{j=0}^{k-1} \frac{(qt)^j}{j!}\right) e^{2qt}$$

and in particular, for $k = N$,

$$w_0(t) \leqslant p\left(e^{qt} - \sum_{j=0}^{N-1} \frac{(qt)^j}{j!}\right) e^{2qt}$$

Let μ, $T>0$. Since $(\sum_{j=0}^{N-1}((qt)^j/j!))$ converges uniformly on $[0, T]$ to e^{qt}, we conclude that for sufficiently large $N\in\mathbb{N}$, $w_0(t)<\mu/2$ for all $t\in[0, T]$.

If follows by continuous dependence on initial data that there exists $\delta>0$ such that the solution $(w_n(t))$ of the initial value problem $\dot{w}_j=qw_{j-1}+qw_j+qw_{j+1}$, $w_{-N}(0)=p$, $w_j(0)=\delta$ for $j=-N+1,..., N-1$, $w_N(0)=p$, where $w_{-N-1}\equiv w_{-N}$ and $w_{N+1}\equiv w_N$, satisfies $w_0(t)<\mu$ for all $t\in[0, T]$.

Let $v_j(t)=\dot{u}_{n+j}(t)$, $a_j(t)=D_1f(u_{n+j-1}, u_{n+j}, u_{n+j+1})$, $b_j(t)=D_2f(u_{n+j-1}, u_{n+j}, u_{n+j+1})$, and $c_j(t)=D_3f(u_{n+j-1}, u_{n+j}, u_{n+j+1})$. Then $\dot{v}_j=a_jv_{j-1}+b_jv_j+c_jv_{j+1}$.

Let $p=\sup\{v_j(t)\colon j\in\mathbb{Z}, t\geqslant 0\}$, $q=\sup\{a_j(t), b_j(t), c_j(t)\colon j\in\mathbb{Z}, t\geqslant 0\}$, and assume that $\dot{u}_{n+j}(t_0)<\delta$ for $j=-N,..., N$. Then $v_{n+j}(t_0)\leqslant\delta=w_j(0)$, and by the comparison principle $v_{n+j}(t_0+t)\leqslant w_j(t)$ for $j=-N,..., N$, $t\geqslant 0$. In particular, $\dot{u}_n(t_0+t)<\mu$ for all $t\in[0, T]$. ∎

Lemma 9. *Suppose that $(u_n(t))$ is the solution of the initial value problem*

$$\dot{y}_n=f(y_{n-1}, y_n, y_{n+1}),\ y_n(0)=0 \quad \textit{for } n\leqslant 0,\ y_n(0)=1 \quad \textit{for } n>0,\ n\in\mathbb{Z}$$

where f satisfies (C1), (C2), and (C3). Then the number of u_n's of $(u_n(t))$ which are in $J=[\varepsilon, 1-\varepsilon]$, $\varepsilon>0$ is a priori bounded where the bound depends on ε but not on t.

The proof is presented in a series of claims. Claim 1 points out a symmetry in the problem and so reduces the number of cases to consider in the proofs of Claim 4 and Claim 5.

Claim 2 states that the number of u_n's in $[\varepsilon, 1-\varepsilon]$, $\varepsilon>0$, is finite and therefore well defined. The proof consists in showing that the time it takes for two consecutive u_n's to enter J is bounded away from zero.

Claim 3 states that if there is a large number of u_n's in an interval contained in $[0, 1]$ at some time, then there is also a large number of u_n's in this interval at any future time. So in order to prove Lemma 9 it suffices to show that the number of u_n's is a priori bounded for all sufficiently large t. Claim 3 follows easily from Corollary 6.

We use a divide and conquer strategy. The interval $J=[\varepsilon, 1-\varepsilon]$ is divided into the three subintervals $J_1=[\varepsilon, a-\varepsilon]$, $J_2=[a-\varepsilon, a+\varepsilon]$, and $J_3=[a+\varepsilon, 1-\varepsilon]$.

Claim 4 states that the number of u_n's in J_1 and J_3 must be a priori bounded. The proof proceeds by showing that if the number of u_n's were not a priori bounded in J_3, then there would exist an interval in $[0, 1]$ which more u_n's would leave than enter.

The proof that the number of u_n's is also a priori bounded in J_2 is accomplished as follows. Claims 5–8 show that one may assume that the derivatives of the u_n's which are leaving J_2 are positive and bounded away from zero. Claim 9 shows that one may also assume that the derivatives of the u_n's which are entering J_2 are positive and bounded away from zero. Using (C3) it is shown in the conclusion of the proof that all derivatives in J_2 are positive, from which it then follows easily that the number of u_n's must be a priori bounded in J_2.

Claim 1. Let $g(x, y, z) = -f(1-z,\ 1-y,\ 1-x)$. Then $(u_n(t))$ is the solution of the initial value problem

$$\dot{y}_n = f(y_{n-1}, y_n, y_{n+1}),\ y_n(0) = 0 \quad \text{for } n \leqslant 0,\ y_n(0) = 1 \quad \text{for } n > 0,\ n \in \mathbb{Z}$$

if and only if $(w_n(t))$, $w_n(t) = 1 - u_{-n+1}(t)$ is the solution of the same initial value problem, except that f is replaced by g. Furthermore, g satisfies conditions (C1), (C2), and (C3). In particular, $D_1 g(b, b, b) = D_3 f(a, a, a)$, $D_3 g(b, b, b) = D_1 f(a, a, a)$, and $g(b, b, b) = 0$, where $b = 1 - a$.

The proof of Claim 1 is a straightforward calculation.

Claim 2. The number of u_n's in $[\varepsilon, 1-\varepsilon]$, $\varepsilon > 0$ is finite for each $t > 0$.

Proof. Let $t_k = \inf\{t \geqslant 0\colon u_{-k}(t) = \varepsilon\}$, $x = u_{-k-2}(t_k)$, $y = u_{-k-1}(t_k)$, and $z = u_{-k}(t_k)$. Then by (C1) and Corollary 3,

$$f(y, y, z) \geqslant f(x, y, z) = \dot{u}_{-k-1}(t_k) \geqslant 0$$

Since $f(y, y, y) < 0$, by (C2), there exists $\delta_1 > 0$ such that $z - y \geqslant \delta_1$, where δ_1 is independent of k. Hence

$$\delta_1 \leqslant u_{-k-1}(t_{k+1}) - u_{-k-1}(t_k) \leqslant |\dot{u}_{-k-1}(s)|\,(t_{k+1} - t_k)$$

for some $s \in (t_k, t_{k+1})$, by the mean value theorem. This implies that there exists $\delta_2 > 0$ such that $t_{k+1} - t_k \geqslant \delta_2$ for all k, because $|\dot{u}_{-k-1}(s)|$ is a priori bounded.

One infers that only finitely many of the u_n's with negative index can enter $[\varepsilon, 1-\varepsilon]$ in a finite time. The proof for the u_n's with positive index is similar. ∎

Claim 3. Let $0 < b_1 < b_2 < 1$ and suppose that at some time t_1 the number of u_n's in $[b_1, b_2]$ is K. Then the number of u_n's in $[b_1, b_2]$ at any time $t > t_1$ is at least $K - 2$.

Proof. Assume the contrary of Claim 3; i.e., assume that at some time t_1 the number of u_n's in $[b_1, b_2]$ is K, and at some time $t_2 > t_1$ the number is less than $K-2$. Then at least three of the u_n's which were in $[b_1, b_2]$ at time t_1 must have left $[b_1, b_2]$ during the time interval (t_1, t_2) and at least two of those three must have left through either b_1 or b_2. More precisely, there exist $m, k \in \mathbb{Z}$, t_3, $t_4 \in (t_1, t_2)$, $t_3 < t_4$, such that the number of u_n's in $[b_1, b_2]$ at time t_3 is larger than at time t_4 and either (i) $u_k(t_3) = b_1 = u_m(t_4)$, or (ii) $u_k(t_3) = b_2 = u_m(t_4)$. In either case we obtain a contradiction from Corollary 6. ∎

Claim 4. The number of u_n's in J_1 and J_3 is a priori bounded.

Proof. The proof is by contradiction. We consider one of the contrary cases; specifically, we suppose that the number of u_n's in J_3 is not a priori bounded. The first step is to show that the number of u_n's in J_1 is a priori bounded.

Since $f(x, x, x) > 0$ for all $x \in J_3$, we may choose $v > 0$ such that $f(x, y, z) > 0$ whenever $y \in J_3$, $x \leqslant y \leqslant z$, and $z - x < v$. Choose $N \in \mathbb{N}$ sufficiently large so that whenever N points are chosen from J_3, there are at least three points in some interval of length v. Choose t_0 such that at least $N+2$ terms of $(u_n(t_0))$ are in J_3. By Claim 3, there are at least N terms of $(u_n(t))$ in J_3 for all $t \geqslant t_0$.

For $t \geqslant t_0$, we must have $\dot{u}_n(t) > 0$ whenever $u_n(t) \in J_1 \cup J_2$. To see this, note that if $u_n(t) \in J_1 \cup J_2$, then there exists $p > n$ such that $u_p(t) \in J_3$ and $u_{p+1}(t) - u_{p-1}(t) < v$. It follows that $\dot{u}_p(t) > 0$, and hence $\dot{u}_n(t) > 0$ by Corollary 5. Since $f(x, x, x) < 0$ for $x \in J_1$, it follows that the number of u_n's in J_1 is a priori bounded.

At this stage, we see that (i) the number of u_n's in J_1 is a priori bounded, (ii) $\dot{u}_n(t) > 0$ whenever $u_n(t) \in J_1 \cup J_2$ for $t \geqslant t_0$, and (iii) f is bounded. It follows that the rate at which the u_n's enter $J_1 \cup J_2$ is bounded, say by $K-1$. (More precisely, the number of u_n's which enter $J_1 \cup J_2$ is less than K during one unit of time.)

We may assume that v has been chosen so that $u_n(t + 1/K) \geqslant u_{n+1}(t)$ whenever $u_n(t) \in J_3$ and $u_{n+1}(t) - u_{n-1}(t) < v$. Choose n_0 such that $u_{n_0}(t_0) \in J_3$ and $u_{n_0+1}(t_0) - u_{n_0-1}(t_0) < v$. Note that there exists $t_0 < t_1 \leqslant t_0 + 1/K$ such that $u_{n_0-1}(t_1) = u_{n_0}(t_0)$. By Corollary 6, $u_{n_0}(t_1) - u_{n_0-2}(t_1) < v$. Inductively, we obtain a sequence (t_m) satisfying $u_{n_0-m}(t_m) = u_{n_0-(m-1)}(t_{m-1})$, $t_{m-1} < t_m \leqslant t_{m-1} + 1/K$, and $u_{n_0-m+1}(t_m) - u_{n_0-m-1}(t_m) < v$.

Hence, the rate at which the u_n's leave $[a+\varepsilon, u_{n_0}(t_0)]$ through the right-hand end point is at least K. However, we have shown above that the rate at which the u_n's enter $J_1 \cup J_2$ is bounded by $K-1$, and we have reached a contradiction. ∎

Claim 5. Let $m_\delta(t) = \sup\{\dot{u}_n(t)\colon\ u_n(t) \geqslant a+\delta\}$. Given $\varepsilon > 0$, there exist $\delta > 0$ and $K \in \mathbb{N}$ such that $m_\delta(t) \leqslant \delta$ implies that the number of terms of $(u_n(t))$ in $[\varepsilon, 1-\varepsilon]$ is less than K.

Proof. Suppose, on the contrary, that there exist $\varepsilon > 0$ and a sequence (t_k) such that the number of terms of $(u_n(t_k))$ in $[\varepsilon, 1-\varepsilon]$ is larger than k and $\dot{u}_n(t_k) < 1/k$ whenever $u_n(t_k) > a + 1/k$. A routine argument using Claim 4 shows that there exists a sequence $(x_n)_{n=-\infty}^{0}$ such that $x_0 > a$, $x_{n-1} \leqslant x_n$, $\lim x_n = a$, and $f(x_{n-1}, x_n, x_{n+1}) \leqslant 0$.

Consider first the possibility that there exists n_0 such that $x_{n_0} = a$. Without loss of generality, assume $x_{n_0-1} = x_{n_0} < x_{n_0+1}$. There exist sequences $\{s_j\}$ and $\{r_j\}$ such that $u_{r_j}(s_j) \to x_{n_0}$, $u_{r_j-1}(s_j) \to x_{n_0-1}$, and $u_{r_j+1}(s_j) \to x_{n_0+1}$. Since

$$f(x_{n_0-1}, x_{n_0}, x_{n_0+1}) = f(a, a, x_{n_0+1}) > 0$$

it follows that $\dot{u}_{r_j-1}(s_j) \to 0$, $\dot{u}_{r_j}(s_j) \to \alpha > 0$, and $\ddot{u}_{r_j-1}(s_j) \to \beta > 0$. Consequently, there exists $\delta > 0$ and $j > 0$ such that $\dot{u}_{r_j}(s_j - \delta) > 0$ and $\dot{u}_{r_j-1}(s_j - \delta) < 0$, which contradicts Lemma 5. Hence, we conclude that $x_n > a$ for all $n \in \{0, -1, -2, \ldots\}$.

Let $L = D_1 f(a, a, a)$, $M = D_2 f(a, a, a)$, and $R = D_3 f(a, a, a)$. Then $L + M + R > 0$ by (C3), and we may assume by Claim 1 that $\hat{L} \leqslant R$. Consider

$$\begin{aligned} 0 &\geqslant f(x_{n-1}, x_n, x_{n+1}) \\ &= R(x_{n+1} - x_n) - L(x_n - x_{n-1}) + (L+M+R)(x_n - a) + o(x_{n+1} - a) \end{aligned}$$

Then

$$\frac{R}{L}(x_{n+1} - x_n) + \frac{L+M+R}{L}(x_n - a) + o(x_{n+1} - a) \leqslant x_n - x_{n-1}$$

Setting $q = (L+M+R)/L$, we see that

$$(x_{n+1} - x_n) + q(x_n - a) + o(x_{n+1} - a) \leqslant x_n - x_{n-1} \tag{5}$$

and hence

$$x_{n+1} - a + o(x_{n+1} - a) \leqslant 2(x_n - a)$$

It follows that

$$x_n - a \geqslant \tfrac{1}{4}(x_{n+1} - a)$$

for x_{n+1} sufficiently close to a. Substituting this estimate in (5), we obtain

$$x_{n+1} - x_n \leqslant x_n - x_{n-1}$$

This leads to

$$0 \leqslant x_{n+1} - x_n \leqslant \lim_{k \to -\infty} (x_k - x_{k-1}) = 0$$

for $-n$ sufficiently large; i.e., $x_n = a$ for $-n$ sufficiently large, which is a contradiction. This completes the proof. ∎

Claim 6. For every $\mu_1 > 0$ there exists μ_0, $0 < \mu_0 < \mu_1$, such that (i) $\dot{u}_n < \mu_0$ implies $\dot{u}_{n+1} < \mu_1$, and (ii) $|\dot{u}_n| < \mu_0$ implies $\dot{u}_{n-1} < \mu_1$.

Proof. Assume the contrary of (i). Then there exist (t_k), (n_k), and $\varepsilon_0 > 0$ such that $\dot{u}_{n_k}(t_k) < 1/k$ and $\dot{u}_{n_k+1}(t_k) \geqslant \varepsilon_0$. Let $b_1 = \inf D_3 f(x, y, z)$ and $b_2 = \inf D_2 f(x, y, z)$, where the infima are taken over all x, y, z which satisfy $0 \leqslant x \leqslant y \leqslant z \leqslant 1$. Since $\dot{u}_{n_k}(t_k) \geqslant 0$ by Corollary 5, it follows with (C1) that

$$\ddot{u}_{n_k} = \sum_{i=1}^{3} D_i f(u_{n_k-1}, u_{n_k}, u_{n_k+1}) \dot{u}_{n_k+i-2} \geqslant b_1 \varepsilon_0 - b_2/k$$

Hence $\dot{u}_{n_k}(t_k - s) < 0$ for sufficiently large k and sufficiently small $s > 0$, which contradicts Corollary 5. The proof of (ii) is similar. ∎

Claim 7. Given $N \in \mathbb{N}$ and $\mu_N > 0$, there exists $\mu_0, \mu_1, ..., \mu_{N-1}$, $0 < \mu_0 < \mu_1 < \cdots < \mu_N$, such that $\dot{u}_n < \mu_0$ implies $\dot{u}_{n+j} < \mu_j$ for $j = 1, 2, ..., N$.

Claim 7 follows from Claim 6 (i) by induction.

Claim 8. One may assume that for all sufficiently small $\varepsilon > 0$,

$$\liminf_{t \to \infty} \{\dot{u}_n(t) \colon u_n(t) \leqslant a + \varepsilon < u_{n+1}(t)\} > 0 \tag{6}$$

Proof. Let $\delta > 0$ and $K \in \mathbb{N}$ be as in Claim 5 and let $\varepsilon \in (0, \delta)$ be so small that $\inf\{\dot{u}_n(t)\colon 1 - \varepsilon \leqslant u_n(t) \leqslant 1,\ t \geqslant 0\} < \delta$. By Claim 4 the number of u_n's in $[a + \varepsilon, 1 - \varepsilon]$ is a priori bounded, say by N, and by Claim 7 there exists $\mu_0 \in (0, \delta)$ such that $\dot{u}_n < \mu_0$ implies $\dot{u}_{n+j} < \delta$ for $j = 1, 2, ..., N$. It follows from Claim 5 that $\dot{u}_n < \mu_0$ and $u_n(t) \leqslant a + \varepsilon < u_{n+1}(t)$ would imply that the number of u_n's in $[a - \varepsilon, a + \varepsilon]$ is bounded above. If (6) does not hold, then Claim 3 can be applied to complete the proof of Lemma 9. Hence, we assume (6) holds.

Claim 8 follows now from Claim 3. ∎

Claim 9. One may assume that there do not exist sequences (n_k) and (t_k) such that $\lim(t_k)=\infty$, $\lim(u_{n_k}(t_k))\in[\varepsilon, a-\varepsilon]$, and $\lim(\dot{u}_{n_k}(t_k))=0$.

Proof. Assume the contrary of the claim; i.e., assume that there exist sequences (n_k) and (t_k) such that $\lim(t_k)=\infty$, $\lim(u_{n_k}(t_k))\in[\varepsilon, a-\varepsilon]$, and $\lim(\dot{u}_{n_k}(t_k))=0$. Recall that by Claim 8 and Corollary 5 we may assume that $u_n(t)\in(0, a+\varepsilon]$ implies $\dot{u}_n(t)>0$ for all sufficiently large t.

Let $b=\lim(u_{n_k}(t_k))$ and for each $n\in\mathbb{N}$ let $s_n=\{\sup s\colon u_{-n}(s)<b\}$. Then by Lemma 7, $\lim(\dot{u}_{-n}(s_n))=0$. Using Claim 6 we conclude that $\lim(\dot{u}_{-n-j}(s_n))=0$ for $j=-N,\dots, N$, $N\in\mathbb{N}$.

If follows from Lemma 8 that the rate at which the u_n's enter the interval $[b, a+\varepsilon]$ converges to zero. This is impossible because by Claim 8 the rate at which the u_n's exit the interval $[b, a+\varepsilon]$ is bounded away from zero. ∎

To conclude the proof of Lemma 9, let $\varepsilon>0$ be such that $a-\varepsilon\leqslant u_{n-1}\leqslant u_n\leqslant u_{n+1}\leqslant a+\varepsilon$ implies $\sum_{i=1}^{3} D_i f(u_{n-1}, u_n, u_{n+1})\geqslant\frac{1}{2}\sum_{i=1}^{3} D_i f(a, a, a)>0$ and Claim 8 holds. Then according to Claim 6, Claim 8, and Claim 9, there exist $\mu_0>0$ and $T>0$ such that for all $t\geqslant T$,

(i) $u_n(t)<a+\varepsilon\leqslant u_{n+1}(t)$ implies $\dot{u}_{n+1}(t)\geqslant\mu_0$, and

(ii) $u_{n-1}(t)\leqslant a-\varepsilon<u_n(t)$ implies $\dot{u}_{n-1}(t)\geqslant\mu_0$.

Let $\mu(t)=\min\{\mu_0, \min\{\dot{u}_n(t)\colon a-\varepsilon<u_n(t)<a+\varepsilon\}\}$. Note that $\mu(t)>0$ for all $t\geqslant T$. If $t\geqslant T$, $a-\varepsilon<u_n(t)<a+\varepsilon$, and $\dot{u}_n(t)=\mu(t)$, then

$$\ddot{u}_n(t)=\sum_{i=1}^{3} D_i f(u_{n-1}, u_n, u_{n+1})\dot{u}_{n-2+i}\geqslant\sum_{i=1}^{3} D_i f(u_{n-1}, u_n, u_{n+1})\,\mu(t)>0$$

Hence $\mu(t)\geqslant\mu(T)>0$ for all $t\geqslant T$.

If the number of u_n's would not be a priori bounded, then by Claim 4 there would exist sequences (n_k) and (t_k) such that $\lim(t_k)=\infty$, $\lim(u_{n_k}(t_k))=a$, and $\lim(\dot{u}_{n_k}(t_k))=0$, in contradiction to $\mu(t)>0$ for all $t\geqslant T$. This completes the proof of Lemma 9.

4. CONCLUSION

In this section, we complete the proof of Theorem 1. For the solution u of (2), (4), one of the following cases must occur.

(A) There exist sequences (n_k) in $\mathbb{Z}$ and (t_k) in $\mathbb{R}$ with $\lim(t_k)=\infty$, $0<\lim(u_{n_k}(t_k))<1$, and $\lim(\dot{u}_{n_k}(t_k))=0$.

(B) There exist sequences (n_k) in $\mathbb{Z}$ and (t_k) in $\mathbb{R}$ with $\lim(t_k)=\infty$, $0<\lim(u_{n_k}(t_k))<1$, and $\lim(\dot{u}_{n_k}(t_k))>0$. Let $b=\lim(u_{n_k}(t_k))$.

(C) There exist sequences (n_k) in $\mathbb{Z}$ and (t_k) in $\mathbb{R}$ with $\lim(t_k) = \infty$, $0 < \lim(u_{n_k}(t_k)) < 1$, and $\lim(\dot{u}_{n_k}(t_k)) < 0$. Let $c = \lim(u_{n_k}(t_k))$.

In the case that (B) holds, it follows from Lemma 7 that there exist positive numbers δ and T such that $u_n(t) = b$ implies $\dot{u}_n(t) > \delta$, and furthermore, for each $n \in \mathbb{N}$ there exists a real number t_n such that $u_{-n}(t_n) = b$.

Similarly, case (C) implies that there exist positive numbers δ and T such that $u_n(t) = c$ implies $\dot{u}_n(t) < -\delta$, and furthermore, for each $n \in \mathbb{N}$ there exists a real number t_n such that $u_n(t_n) = c$.

Assume that (B) and (C) hold. Then either $b < c$ or $c < b$. If $b < c$, then there exists $T > 0$ such that infinitely many u_n's enter $[b, c]$ and none of them leaves during the time $[T, \infty)$, which contradicts Lemma 9. If $c < b$, then there exists $T > 0$ such that infinitely many u_n's must exit $[c, b]$ and no u_n enters during the time $[T, \infty)$, which is also not possible. One infers that (B) and (C) exclude each other.

Suppose that (A) holds. Then there exist sequences (n_k) in $\mathbb{Z}$ and (t_k) in $\mathbb{R}$ such that $\lim(t_k) = \infty$, $0 < \lim(u_{n_k}(t_k)) < 1$, $\lim(\dot{u}_{n_k}(t_k)) = 0$, and $(u_{n_k+j}(t_k))_{k=1}^{\infty}$ converges for all $j \in \mathbb{Z}$. Let $x_j = \lim(u_{n_k+j}(t_k))$ for all $j \in \mathbb{Z}$. Then $\lim_{j \to -\infty}(x_j) = 0$ and $\lim_{j \to \infty}(x_j) = 1$ by Lemma 9, $f(x_{-1}, x_0, x_1) = 0$, and $f(x_{j-1}, x_j, x_{j+1}) \geqslant 0$ for all $j \in \mathbb{Z}$ or $f(x_{j-1}, x_j, x_{j+1}) \leqslant 0$ for all $j \in \mathbb{Z}$, because (B) and (C) exclude each other. Using condition (C1) one can show that $f(x_{j-1}, x_j, x_{j+1}) > 0$ or $f(x_{j-1}, x_j, x_{j+1}) < 0$ for some j would imply (B) and (C). Hence $f(x_{j-1}, x_j, x_{j+1}) = 0$ for all $j \in \mathbb{Z}$; i.e., Theorem 1 holds with $c = 0$.

Suppose now that (A) does not hold but (B) holds. Then there exists a sequence (t_n) in $\mathbb{R}$ such that $u_{-n}(t_n) = b$ for all $n \in \mathbb{N}$. Let X be the Banach space of all bounded sequences in $\mathbb{R}$ equipped with the supremum norm. Then (x^n), given by $x_j^n = u_{-n+j}(t_n)$, converges in X to some $x = (x_j)_{j \in \mathbb{Z}}$ by Corollary 6 and Lemma 9. In particular, it follows from Lemma 9 that $\lim_{j \to -\infty}(x_j) = 0$ and $\lim_{j \to \infty}(x_j) = 1$. Denote by $u(x, t) = (u_n(x, t))_{n \in \mathbb{Z}}$, $x = (x_n)_{n \in \mathbb{Z}}$, the solution of the initial value problem

$$\dot{u}_n = f(u_{n-1}, u_n, u_{n+1}), \qquad u_n(0) = x_n, \qquad n \in \mathbb{Z}$$

and let $\tau_n = t_{n+1} - t_n$. Since (A) does not hold, the sequence (τ_n) is bounded and thus contains a convergent subsequence (τ_{n_k}), with $\lim(\tau_{n_k}) = \tau$. Let $S\colon X \to X$ be the shift operator given by $(Sx)_n = x_{n-1}$. Then $u(x^{n_k}, \tau_{n_k}) = Sx^{n_k+1}$ and hence $u(x, \tau) = Sx$. Note that $\tau > 0$. This proves Theorem 1 with $c = 1/\tau$.

Finally, if (A) does not hold but (C) holds, then one proves similarly that Theorem 1 holds for some $c < 0$.

REFERENCES

Bell, J. (1981). Some threshold results for models of myelinated nerves. *Math. Biosci.* **54**, 181–190.

Bell, J., and Cosner, C. (1984). Threshold behaviour and propagation for nonlinear differential-difference systems motivated by modeling myelinated axons. *Q. Appl. Math.* **42**, 1–14.

Chi, H., Bell, J., and Hassard, B. (1986). Numerical solution of a nonlinear advance-delay-differential equation from nerve conduction theory. *J. Math. Biol.* **24**, 583–601.

Keener, J. P. (1987). Propagation and its failure in coupled systems of discrete excitable cells. *SIAM J. Appl. Math.* **47**, 556–572.

Kolmogoroff, A., Petrovsky, I., and Piscounoff, N. (1937). Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Moscow Univ. Bull. Math.* **1**, 1–25.

McKean, H. K. (1970). Nagumo's equation. *Adv. Math.* **4**, 209–223.

Rustichini, A. (1989a). Functional differential equations of mixed type: The linear autonomous case. *J. Dynam. Diff. Eq.* **1**, 121–143.

Rustichini, A. (1989b). Hopf bifurcation for functional differential equations of mixed type. *J. Dynam. Diff. Eq.* **1**, 145–177.

Smith, H. L. (1990). A discrete Lyapunov function for a class of linear differential equations. *Pacific J. Math.* **144**, 345–360.

Walter, W. (1970). *Differential and Integral Inequalities*, Springer, New York.

Zinner, B. (1991). Stability of traveling wavefronts for the discrete Nagumo equation. *SIAM J. Math. Anal.* **22**, 1016–1020.

Zinner, B. (1992). Existence of traveling wavefront solutions for the discrete Nagumo equation. *J. Diff. Eq.* **96**, 1–27.

Printed in Belgium
Verantwoordelijke uitgever:
Hubert Van Maele
Altenastraat 20 - B-8310 St.-Kruis