MELVIN FITTING

Tableaus for Many-Valued Modal Logic*

Abstract. We continue a series of papers on a family of many-valued modal logics, a family whose Kripke semantics involves many-valued accessibility relations. Earlier papers in the series presented a motivation in terms of a multiple-expert semantics. They also proved completeness of sequent calculus formulations for the logics, formulations using a cut rule in an essential way. In this paper a novel cut-free tableau formulation is presented, and its completeness is proved.

1. Introduction

If we have a many-valued logic L whose truth values constitute a complete lattice, a natural many-valued version of a Kripke model can be easily constructed. The notion of a frame is as usual, but now truth values at possible worlds are members of L and not just *true* and *false*. Propositional connectives are dealt with in the obvious way. And for the modal connectives, one sets the truth value of $\Box X$ at a world to be the inf of the truth values of X at all accessible worlds. This kind of generalization of Kripke semantics has been explored by several people [16, 19, 15, 9, 11, 10, 12]. The key thing to note is that, although a many-valued truth value space has been introduced, the underlying notion of a Kripke frame remains classical.

In [5] and especially [6] I introduced a somewhat more complicated generalization in which the notion of frame itself was modified: the classical accessibility relation is replaced by a many-valued relation. For this to work it is not enough that the many-valued logic have the structure of a complete lattice — now we need a complete *Heyting algebra*. But the resulting family of modal logics has a natural interpretation: it can be thought of as representing the opinions of a set of experts who are not necessarily independent of each other. (In order to make this paper relatively self-contained, I sketch this motivation below.) Nonetheless, I feel this family of many-valued modal logics is worth exploring mathematically for its own sake, and not just because of a connection with multiple experts. For instance, just as the classical modal logics have non-monotonic versions, there is a non-monotonic

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version of this many-valued modal logic as well [7]. Also, the details of the completeness argument are of particular interest. I think we may be able to better understand even the classical modal logics by looking at them in a broader context of 'similar' logics.

As in [5, 6] I will confine things here to *finite* many-valued logics. In the papers just cited I gave Gentzen calculi for the many-valued modal logics, but a cut rule played an essential role. In this paper I present a tableau formulation, though it is equivalent to a Gentzen-style system, of course. The key point is that there is no cut rule. The general formulation is somewhat unusual for tableau treatments of many-valued logic, but my real interest is in the form the modal rules take. I begin with a brief presentation of the background motivation in terms of multiple experts, then I turn to a formal presentation of the tableau system.

2. Many Experts, Many Values

We give the informal background for the many-valued modal logic we are considering, leading up to a formal presentation of the semantics in the next section. The material here is developed more fully and rigorously in [6].

Suppose we have several experts and we are interested in the opinions of each, not just about how things are, but about how they might be. That is, we want to hear from each expert answers to questions like: "if the world were thus-and-so, what do you think would be the case?" But this is not enough, since "thus-and-so" may be extremely unlikely. So we also want to hear from each expert an answer to "do you think the world being thus-and-so is a serious possibility?" Now this can be represented rather easily using Kripke models. We have a set of possible worlds; each expert has his or her own opinion on the truth of atoms at each possible world; and each expert has his or her own accessibility relation. The details are straightforward.

Now we complicate the picture. Suppose some experts dominate others: anything a dominant expert declares to be the case will also be asserted to be so by any dominated expert. To keep the discussion manageable, say we have just two experts, e_1 and e_2, and e_1 dominates e_2. If e_1 says A is true at world w, e_2 will also say A is true at w. On the other hand, if e_1 does not say A is true at w, e_2 is free to say anything: A is true at w, or A is false at w. Thus there is a lack of symmetry built in: assertion (truth) outranks non-assertion (falsehood). What was just said about truth of formulas at worlds applies as well to accessibility relations: if e_1 says world v is accessible from world w, e_2 will also say this.

The picture just described has an intuitionistic flavor to it. For example, suppose e_2 believes A is true at world w but B is not. Since expert e_1 dominates e_2, e_1 can not believe that $A \supset B$ is true at w because otherwise e_2 would also have to believe it, but e_2 does not. Thus the calculation of truth values is no longer 'local,' an expert will accept $A \supset B$ provided that expert, *and every expert that one dominates*, accepts B provided he or she accepts A. This is essentially the treatment of implication in Kripke intuitionistic models.

Non-locality carries over to modal notions as well. For e_1 to accept $\Box A$ at world w it is not enough for e_1 to accept A at all worlds e_1 believes are alternatives to w; we also need that the dominated expert, e_2, should accept A at all the worlds e_2 believes are alternatives to w. The conditions on $\Diamond$ are simpler, but we won't go into the details here — things are similar to the characterization of $\forall$ and $\exists$ in Kripke intuitionistic models, including the impossibility of inter-defining them in general.

A formal version of the semantics sketched above can be found in [6]. In effect, the multiple-expert modal model has features that are modal and features that are intuitionistic. It is, in fact, a version of the semantics of [17, 13, 1].

A natural way of simplifying the structure outlined above is to move to a many-valued picture, treating *sets* of experts as truth values. Instead of saying both e_1 and e_2 accept A at world w, we could say the truth value of A at w is $\{e_1, e_2\}$. Of course, not every set of experts gives us a truth value; since e_1 dominates e_2, the set $\{e_1\}$ can not be the value of any formula, since no formula can be true for e_1 alone. What we want as truth values are *sets of experts closed under dominance*. All we require of the dominance relation is that it be reflexive and transitive. It is a standard result that the collection of sets closed under such a relation is a *Heyting* algebra, [14]. (We give a definition in the next section.) Consequently, what we want to characterize is the notion of a many-valued modal logic where the space of truth values is a Heyting algebra.

We can take the truth conditions worked out for multiple-expert modal models and simply translate them into conditions appropriate for a many-valued modal model. Some of them are straightforward. For instance, if S_1 is the set of experts who think A is true at w, and S_2 is the set of experts who think B is true at w, clearly $S_1 \cap S_2$ will be the set of experts who think $A \wedge B$ is true at w. Thus $\wedge$ in the formal language corresponds to $\cap$ in the family of sets of experts, and this in fact is the meet operation of the corresponding Heyting algebra.

All the multiple-expert truth conditions convert into quite natural many-valued conditions. This gives rise to a notion of *many-valued modal model*;

details can be found in [6]. We do not repeat them here. In this paper we start the formal work at this point.

3. Syntax and Semantics

In this section we give the formal syntax and semantics for our many-valued modal logic. First, however, we sketch the basic ideas of Heyting algebras, since these will be needed in what follows. The primary source for Heyting algebras is [14].

Suppose we have a lattice (we denote meet and join by $\wedge$ and $\vee$). The *pseudo-complement* of a relative to b is the greatest member of the lattice, c, such that $a \wedge c \leq b$. A pseudo-complement for two members need not exist in general. If the pseudo-complement of a relative to b does exist, it is denoted $a \Rightarrow b$. Pseudo-complements meet the condition (and in fact are determined by it):

$$x \leq (a \Rightarrow b) \text{ iff } (x \wedge a) \leq b.$$

Definition 3.1. A *Heyting algebra* is a lattice $\mathcal{T}$ with a bottom element (which we denote *false*), in which all relative pseudo-complements exist.

There are several easy facts concerning pseudo-complements which we will need. Since $a \Rightarrow b \leq a \Rightarrow b$, it follows from the equivalence above, taking x to be $a \Rightarrow b$, that $a \wedge (a \Rightarrow b) \leq b$. Also, since $x \wedge \mathit{false} = \mathit{false} \leq \mathit{false}$, it follows that $x \leq (\mathit{false} \Rightarrow \mathit{false})$ for any x. Consequently if we set $\mathit{true} = (\mathit{false} \Rightarrow \mathit{false})$ we have a top element for the Heyting algebra. Since $\mathit{true} \wedge a = a$, we have $\mathit{true} \leq (a \Rightarrow b)$ iff $a \leq b$. Finally, $a \leq (b \Rightarrow c)$ iff $b \leq (a \Rightarrow c)$, since the first is equivalent to $(a \wedge b) \leq c$, the second is equivalent to $(b \wedge a) \leq c$, and the meet operation is commutative.

It is shown in [14] that Heyting algebras are distributive. On the other hand, a *finite* distributive lattice must be a Heyting algebra. We have been tacitly assuming we had a finite number of experts — consequently we have finite Heyting algebras to deal with, and this translates into the probably more familiar notion of a finite, distributive lattice.

Notation Convention For the rest of this paper, $\mathcal{T} = \langle \mathcal{T}, \leq \rangle$ is a finite, distributive lattice; equivalently, a finite Heyting algebra. We write *false* for the smallest element of it; and *true* for the largest element.

Now we begin the business of introducing a many-valued modal logic, $L(\mathcal{T})$, based on the Heyting algebra $\mathcal{T}$. We begin with syntax. Classical logic is intended to be two-valued, and counterparts of the two truth values are available in the language, $A \wedge \neg A$ for *false* and $A \vee \neg A$ for *true*. When more than two truth values are allowed it may not be possible to find counterparts of all the truth values in the language. We need them, so we build them in. From now on we assume the language of $L(\mathcal{T})$ has propositional constants, corresponding to the members of $\mathcal{T}$. To keep the notation simple, we will just assume the members of $\mathcal{T}$ themselves are constant symbols of $L(\mathcal{T})$.

DEFINITION 3.2. The language of $L(\mathcal{T})$ is specified as follows.

1. *Atomic formulas* are the members of $\mathcal{T}$, called *propositional constants,* and a countable list of *propositional variables*, $A_1, A_2, \ldots$.
2. *Formulas* are built up from atomic formulas in the usual way, allowing the connectives $\wedge$, $\vee$, $\supset$, $\Box$, and $\Diamond$.

Note that there is no negation in the language of $L(\mathcal{T})$. A standard way of introducing one in such a context is to set $\neg X = (X \supset \mathit{false})$. If $\mathcal{T}$ happens to be not just a Heyting algebra but a Boolean algebra, this yields the expected negation. We find it simpler to omit direct treatment of negation here. We also use $\wedge$ and $\vee$ to denote meet and join in $\mathcal{T}$; there should be no confusion between their algebraic roles and their roles syntactically in $L(\mathcal{T})$.

Now we introduce the intended semantics for $L(\mathcal{T})$, beginning with the non-modal part.

DEFINITION 3.3. A *valuation* is a mapping v from propositional variables to $\mathcal{T}$.

We will refer to members of $\mathcal{T}$ as *truth values*, or *$\mathcal{T}$-truth values* if it is necessary to be more specific.

DEFINITION 3.4. Valuations are extended to all non-modal formulas as follows. Let v be a valuation.

1. If t is a propositional constant, $v(t) = t$.
2. $v(A \wedge B) = v(A) \wedge v(B)$.
3. $v(A \vee B) = v(A) \vee v(B)$.
4. $v(A \supset B) = v(A) \Rightarrow v(B)$.

Now we extend these notions to the full language, introducing a suitably generalized version of a Kripke model. This is taken from [5, 6].

DEFINITION 3.5. A $\mathcal{T}$-*modal model* is a structure $\langle \mathcal{G}, \mathcal{R}, w \rangle$ where $\mathcal{G}$ is a non-empty set (of possible worlds), $\mathcal{R}$ is a mapping from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{T}$, and w maps worlds to valuations.

The map $\mathcal{R}$ should be thought of as a many-valued accessibility relation. If $\mathcal{T}$ is the Boolean algebra $\{false, true\}$, $\mathcal{R}$ corresponds to a classical relation in the obvious way. To keep notation simple, we will write $w(\Gamma, X)$ instead of $w(\Gamma)(X)$. Now we extend w to arbitrary formulas. We use $\bigvee$ and $\bigwedge$ for arbitrary $\vee$ and $\wedge$; meaningfulness of the operations is immediate, since $\mathcal{T}$ is assumed finite.

DEFINITION 3.6. Let $\langle \mathcal{G}, \mathcal{R}, w \rangle$ be a $\mathcal{T}$-modal model. The map w is extended as follows. For any $\Gamma \in \mathcal{G}$:

1. The action of w, at each world, with respect to $\wedge$, $\vee$, and $\supset$ is as in Definition 3.4.
2. $w(\Gamma, \Box A) = \bigwedge\{\mathcal{R}(\Gamma, \Delta) \Rightarrow w(\Delta, A) \mid \Delta \in \mathcal{G}\}$.
3. $w(\Gamma, \Diamond A) = \bigvee\{\mathcal{R}(\Gamma, \Delta) \wedge w(\Delta, A) \mid \Delta \in \mathcal{G}\}$.

Some examples of the calculation of truth values for non-atomic formulas are given in [6]. We do not repeat them here.

DEFINITION 3.7. We say a formula X is *valid* in the $\mathcal{T}$-modal model $\langle \mathcal{G}, \mathcal{R}, w \rangle$ provided, for each $\Gamma \in \mathcal{G}$, $w(\Gamma, X) = true$.

We return to the non-modal setting for a moment, for some explanation. Generally, when working with many-valued logics one has some family of *designated* truth values, say $\{d_1, \ldots, d_k\}$, in mind, and a formula is considered valid if it always takes on some designated truth value. Suppose we do this, but we also assume the set of designated truth values is closed under meet (a common assumption). Set $d = (d_1 \wedge \ldots \wedge d_k)$. Then saying X has a designated truth value under every valuation really amounts to saying that, if v is any valuation, $d \leq v(X)$. Since we have propositional constants in our language, this is equivalent to saying that $v(d) \leq v(X)$, and this in turn is equivalent to $true \leq (v(d) \Rightarrow v(X))$, which finally is equivalent to $v(d \supset X) = true$.

What all this amounts to is simple. The notion of validity as we gave it above is general enough to capture the 'designated value' version of validity. Of course we are using the Heyting algebra structure to carry out this reduction — for many-valued logics with less structure such a thing is not generally possible. It also suggests a special role for implications involving propositional constants, and that indeed is the case, as will be seen shortly.

4. Non-Modal Tableau Rules

In presenting the tableau rules for the logic $L(\mathcal{T})$ we postpone the modal rules for now, and concentrate on the underlying non-modal many-valued logic. The rules we give are rather straightforward, and are designed to serve as a framework to which we can add modal rules. We assume the reader is familiar with tableau systems for classical logic — if not, [18, 4] will serve as references. We use *signed formulas*, following the main development of [18].

To begin with, all formulas appearing in our tableaus will be implications of a special kind: $a \supset A$ or $A \supset a$, where a is a propositional constant. We call these *bounding implications*. Informally, think of $a \supset A$ as asserting that, under some many-valued valuation, the value of A is at least a, that is, $\geq a$; likewise $A \supset a$ informally asserts that the value of A is $\leq a$. The tableau completeness proof will show to what extent the rules capture this intention. In addition we use signs, T and F, familiar from classical logic. If X is a formula, $T\,X$ and $F\,X$ are signed formulas. Think of $T\,X$ as asserting X, and $F\,X$ as denying X.

Tableau systems are refutation systems. To establish something, we begin by denying it, and derive some sort of syntactical contradiction. In our case, if we want to show X is valid under all many-valued interpretations (in $\mathcal{T}$), we start a tableau with $F\,(\mathit{true} \supset X)$, thus informally asserting there could be an interpretation in which X is not (at least) *true*. Then a tree is constructed, using the *Branch Extension Rules* given below. Think of the tree as the disjunction of its branches, and a branch as the conjunction of the signed formulas on it. A branch is *closed* if it contains an 'obvious' contradiction, again specified below. If each branch is closed, the tableau is closed. A closed tableau beginning with $F\,(\mathit{true} \supset X)$ constitutes a tableau proof of X. Somewhat more generally, a tableau proof of Z is a closed tableau starting with $F\,Z$, and we are specially interested in formulas Z of the form $\mathit{true} \supset X$.

Notice that we are allowing bounding implications of both forms, $a \supset A$ and $A \supset a$, in tableaus, both lower and upper bounds. Now we have the formal presentation of the rules. We begin with those for closing branches — termination rules, so to speak.

Branch Closure Conditions A tableau branch is *closed* if it contains:

$T\,(a \supset b)$ where $a \not\leq b$
$F\,(a \supset b)$ where $a \leq b$ and $a \neq \mathit{false}, b \neq \mathit{true}$

$F\,(\mathit{false} \supset X)$
$F\,(X \supset \mathit{true})$
$\left.\begin{array}{l} T\,(b \supset X) \\ F\,(a \supset X) \end{array}\right\}$ where $a \leq b$

REMARK 4.1. The rule covering $F\,(a \supset b)$ has the restrictions it does simply because these cases are covered by the two rules immediately following it.

There is no negation symbol in our language. Even so, there are analogs of negation rules. Classically, X is equivalent to $\mathit{true} \supset X$ and $\neg X$ is equivalent to $X \supset \mathit{false}$, so the usual classical rule, to infer $T\,X$ from $F\,\neg X$, is equivalent to a rule saying: infer $T\,(\mathit{true} \supset X)$ from $F\,(X \supset \mathit{false})$. What we need are more rules like this, suitable for $\mathcal{T}$, allowing us to switch signs by reversing implications. There are four such rules.

Reversal Rules In these rules, X is restricted to be any formula other than a propositional constant.

$F \geq$

$$\frac{F\,(a \supset X)}{T\,(X \supset t_1) \mid \ldots \mid T\,(X \supset t_n)}$$

Where $t_1, \ldots, t_n$ are all maximal members of $\mathcal{T}$ not above a, and $a \neq \mathit{false}$.

$T \geq$

$$\frac{T\,(a \supset X)}{F\,(X \supset t_i)}$$

Where t_i is any maximal member of $\mathcal{T}$ not above a, and $a \neq \mathit{false}$.

$F \leq$

$$\frac{F\,(X \supset a)}{T\,(u_1 \supset X) \mid \ldots \mid T\,(u_k \supset X)}$$

Where $u_1, \ldots, u_k$ are all minimal members of $\mathcal{T}$ not below a, and $a \neq \mathit{true}$.

$T \leq$

$$\frac{T\,(X \supset a)}{F\,(u_i \supset X)}$$

Where u_i is any minimal member of $\mathcal{T}$ not below a, and $a \neq \mathit{true}$.

The intuition behind these rules is straightforward; consider $F \geq$ as representative. Suppose we have $F\,(a \supset X)$ on a tableau branch and so, under some valuation v, $a \not\leq v(X)$. Let S be $\{z \in \mathcal{T} \mid a \not\leq z\}$; then $v(X) \in S$, so S is not empty. Since $\mathcal{T}$ is finite, $v(X)$ is below some maximal member of S. If we designate the maximal members of S by $t_1, \ldots, t_n$ we have $v(X) \leq t_1$ or $\ldots$or $v(X) \leq t_n$, and so the tableau branches to the possible continuations $T\,(X \supset t_1), \ldots, T\,(X \supset t_n)$.

We just used an argument involving maximal members of $\mathcal{T}$ meeting a certain condition. We will use such arguments frequently, and analogous ones concerning minimal ones. So once and for all we state the general principle involved. As noted above, it follows from the finiteness of $\mathcal{T}$.

General Principle If $x \not\leq y$ then: (1) there is some $w \leq x$ such that w is a minimal member of $\mathcal{T}$ not below y; and (2) there is some $z \geq y$ such that z is a maximal member of $\mathcal{T}$ not above x.

Rule $F \geq$ does not make sense if $a = \mathit{false}$ since there are no members of $\mathcal{T}$ that are not above *false*. But this case is covered by the Branch Closure Condition allowing closure of a branch containing $F(\mathit{false} \supset X)$. Rule $T \geq$ has a similar restriction but for a different reason. A signed formula of the form $T(\mathit{false} \supset X)$ gives no useful information, since everything in $\mathcal{T}$ is above *false* and so cannot be expected to enter meaningfully into a tableau construction. The formal justification of the restriction comes when we show the system is complete in the presence of the restriction. Similar comments apply to Rules $F \leq$ and *true*. Finally we have rules for the various propositional connectives, in which there are similar restrictions, for similar reasons.

Conjunction Rules

$T\wedge$ $$\frac{T(t \supset (A \wedge B))}{\begin{array}{c} T(t \supset A) \\ T(t \supset B) \end{array}}$$ Where $t \neq \mathit{false}$.

$F\wedge$ $$\frac{F(t \supset (A \wedge B))}{F(t \supset A) \mid F(t \supset B)}$$ Where $t \neq \mathit{false}$.

Disjunction Rules

$T\vee$ $$\frac{T((A \vee B) \supset t)}{\begin{array}{c} T(A \supset t) \\ T(B \supset t) \end{array}}$$ Where $t \neq \mathit{true}$.

$F\vee$ $$\frac{F((A \vee B) \supset t)}{F(A \supset t) \mid F(B \supset t)}$$ Where $t \neq \mathit{true}$.

Finally we have the rules for implication. These are somewhat more complicated, but the motivating idea is clear. Suppose we have $F(t \supset (A \supset B))$ on a tableau branch, so under some valuation v, $t \not\leq (v(A) \Rightarrow v(B))$. Since $\mathcal{T}$ is a Heyting algebra this is equivalent to $(t \wedge v(A)) \not\leq v(B)$. Let $t_i = t \wedge v(A)$. It follows that: $t_i \leq v(A)$, $t_i \not\leq v(B)$; and $t_i \leq t$. Thus we should be able to extend the tableau branch by adding $T(t_i \supset A)$ and $F(t_i \supset B)$, for some $t_i \leq t$. Of course we can rule out the case of $t = \mathit{false}$, since then a Branch Closure Condition applies, and similarly we do not need to consider the possibility that $t_i = \mathit{false}$. This should be enough to motivate the following rules.

Implication Rules

$F \supset$

$$\frac{F(t \supset (A \supset B))}{\begin{array}{c|c|c} T(t_1 \supset A) & \ldots & T(t_n \supset A) \\ F(t_1 \supset B) & & F(t_n \supset B) \end{array}}$$

Where $t \neq$ *false* and $t_1, \ldots, t_n$ are all the members of $\mathcal{T}$ below t except *false*.

$T \supset$

$$\frac{T(t \supset (A \supset B))}{F(t_i \supset A) \mid T(t_i \supset B)}$$

Where $t \neq$ *false* and t_i is any member of $\mathcal{T}$ below t except *false*.

This completes the system of non-modal tableau rules. We conclude the section with a sketch of a proof of $(A \supset (B \supset A))$, or rather, of *true* $\supset (A \supset (B \supset A))$. The tableau begins with F *true* $\supset (A \supset (B \supset A))$. Since every member of $\mathcal{T}$ is below *true*, an application of $F \supset$ yields many branches, each containing signed formulas of the form:

$$\begin{array}{l} T(u \supset A) \\ F(u \supset (B \supset A)) \end{array}$$

where $u \neq$ *false*. We continue with a typical such branch in Figure 1 In it we have used Rule $F \supset$, with $t_1, \ldots, t_k$ being all the members of $\mathcal{T}$ below u, except for *false*. Since $t_i \leq u$ for each i, each branch closes using one of the Branch Closure Conditions.

$$\begin{array}{ccc} & \begin{array}{l} T(u \supset A) \\ F(u \supset (B \supset A)) \end{array} & \\ \swarrow & & \searrow \\ \begin{array}{l} T(t_1 \supset B) \\ F(t_1 \supset A) \end{array} & \ldots & \begin{array}{l} T(t_k \supset B) \\ F(t_k \supset A) \end{array} \end{array}$$

Figure 1: A Proof of $(A \supset (B \supset A))$

5. A Three-Valued Example

A concrete example can often be an aid to understanding. The simplest example after the classical two-valued case is three-valued; we present this system here, and continue with it once we get to the modal rules. We take for truth values $\mathcal{T}(3) = \{$*false*, *half*, *true*$\}$, with the ordering *false* $<$ *half* $<$ *true*. We observed in Section 2. that the kind of many-valued logics we were considering can be identified with a logic of multiple experts. This applies to $\mathcal{T}(3)$ in the following way. Suppose there are two experts, A and B, with A dominating B. Then there are three sets that can serve as truth values: $\emptyset$, which corresponds to *false*; $\{B\}$, which corresponds to *half*; and $\{A, B\}$, which corresponds to *true*. If we had used the same set of experts but assumed neither dominated the other, a four-valued logic would have arisen — we leave it to the reader to formulate rules for it.

The Branch Closure Conditions from Section 4. specialize to the following. A branch of a $\mathcal{T}(3)$ tableau is closed if it contains

$T\,(\mathit{half} \supset \mathit{false})$
$T\,(\mathit{true} \supset \mathit{half})$
$T\,(\mathit{true} \supset \mathit{false})$
$F\,(\mathit{half} \supset \mathit{half})$
$F\,(\mathit{false} \supset X)$
$F\,(X \supset \mathit{true})$
$T\,(\mathit{half} \supset X)$ and $F\,(\mathit{half} \supset X)$
$T\,(\mathit{true} \supset X)$ and $F\,(\mathit{true} \supset X)$
$T\,(\mathit{true} \supset X)$ and $F\,(\mathit{half} \supset X)$

There are eight Reversal Rules.

$$\frac{F\,\mathit{half} \supset X}{T\,X \supset \mathit{false}} \qquad \frac{T\,X \supset \mathit{false}}{F\,\mathit{half} \supset X} \qquad \frac{F\,\mathit{true} \supset X}{T\,X \supset \mathit{half}} \qquad \frac{T\,X \supset \mathit{half}}{F\,\mathit{true} \supset X}$$

$$\frac{T\,\mathit{half} \supset X}{F\,X \supset \mathit{false}} \qquad \frac{F\,X \supset \mathit{false}}{T\,\mathit{half} \supset X} \qquad \frac{T\,\mathit{true} \supset X}{F\,X \supset \mathit{half}} \qquad \frac{F\,X \supset \mathit{half}}{T\,\mathit{true} \supset X}$$

The Conjunction and Disjunction Rules are straightforward, and we omit them here. Finally, there are five Implication Rules for $\mathcal{T}(3)$.

$$\frac{F\,(\mathit{half} \supset (A \supset B))}{\begin{array}{c} T\,(\mathit{half} \supset A) \\ F\,(\mathit{half} \supset B) \end{array}} \qquad\qquad \frac{T\,(\mathit{half} \supset (A \supset B))}{F\,(\mathit{half} \supset A) \;\big|\; T\,(\mathit{half} \supset B)}$$

$$\frac{F\,(\mathit{true} \supset (A \supset B))}{\begin{array}{c|c} T\,(\mathit{half} \supset A) & T\,(\mathit{true} \supset A) \\ F\,(\mathit{half} \supset B) & F\,(\mathit{true} \supset B) \end{array}}$$

$$\frac{T\,(\mathit{true} \supset (A \supset B))}{F\,(\mathit{true} \supset A) \mid T\,(\mathit{true} \supset B)} \qquad \frac{T\,(\mathit{true} \supset (A \supset B))}{F\,(\mathit{half} \supset A) \mid T\,(\mathit{half} \supset B)}$$

Figure 2 shows an example of a tableau proof for $\mathcal{T}(3)$. We leave it to the reader to provide justifications for the various steps.

$$\begin{array}{l}
F\,\mathit{half} \supset \{[(\mathit{half} \supset A) \wedge (\mathit{true} \supset (A \supset B))] \supset B\} \\
T\,\mathit{half} \supset [(\mathit{half} \supset A) \wedge (\mathit{true} \supset (A \supset B))] \\
F\,\mathit{half} \supset B \\
T\,\mathit{half} \supset (\mathit{half} \supset A) \\
T\,\mathit{half} \supset (\mathit{true} \supset (A \supset B))
\end{array}$$

$$F\,\mathit{half} \supset \mathit{half} \qquad T\,\mathit{half} \supset A$$

$$F\,\mathit{half} \supset \mathit{true} \qquad T\,\mathit{half} \supset (A \supset B)$$

$$F\,\mathit{half} \supset A \qquad T\,\mathit{half} \supset B$$

Figure 2: A Non-Modal Proof in $\mathcal{T}(3)$

6. Non-Modal Soundness and Completeness

All entries in tableaus are signed bounding implications. We want to show that there is a closed tableau for $F\,(a \supset A)$ if and only if every valuation v assigns A a value that is $\geq a$. (And similarly for $A \supset a$, and for a sign of T instead of F.) We begin with the 'only if,' or soundness half. All tableau soundness proofs are essentially the same. One defines what it means for a tableau to be satisfiable, proves the rules preserve satisfiability, but a closed tableau is not satisfiable. There are no surprises here.

DEFINITION 6.1. A non-modal tableau is *satisfiable* if at least one branch is satisfiable. A branch is satisfiable if the set of signed formulas on it is satisfiable. A set of signed formulas is satisfiable if some valuation v satisfies each member. A signed formula is satisfiable under the valuation v if it is $T\,(X \supset Y)$ and $v(X) \leq v(Y)$; or if it is $F\,(X \supset Y)$ and $v(X) \not\leq v(Y)$. (Recall that for propositional constants, $v(c) = c$.)

Each of the tableau rules from Section 4. preserves satisfiability. We leave the verification of this to you (there was a sketch of the argument for the rule $F \supset$ immediately before that rule was given in Section 4.).

It is also easy to verify that a closed tableau is not satisfiable. Now we proceed in the customary way. If we have a closed tableau for $F(a \supset A)$ it must be the case that every valuation assigns A a value that is $\geq a$. For if not, $F(a \supset A)$ would be a satisfiable formula; the tableau construction would thus begin with a satisfiable tableau; every subsequent tableau would be satisfiable; and tableau construction would terminate with a closed tableau that was satisfiable.

Now we turn to the completeness half. We begin with some terminology and notation.

Definition 6.2. Let S be a set of signed bounding implications. We say S is *consistent* if no tableau beginning with a finite subset of S closes. Also S is *maximally consistent* if it is consistent and has no proper consistent extensions.

Next, for each maximal consistent set we define *two* mappings to $\mathcal{T}$. If a cut rule were part of the tableau formulation, these two mappings would easily be seen to coincide, and each would be a valuation. As it is, we do not know this, and must work with somewhat weaker properties.

Definition 6.3. Let S be a maximally consistent set of bounding implications. For each formula X set:

$$\begin{aligned} bound^S(X) &= \bigwedge\{a \mid T(X \supset a) \in S\} \\ bound_S(X) &= \bigvee\{a \mid T(a \supset X) \in S\} \end{aligned}$$

Essentially we will show that for a maximal consistent set S, any valuation between $bound_S$ and $bound^S$ satisfies S. First we must establish that the very notion of 'between' is meaningful.

Lemma 6.4. *If S is maximally consistent, then for every formula X, $bound_S(X) \leq bound^S(X)$.*

Proof. It is enough to show that for every $a \in \{a \mid T(a \supset X) \in S\}$ and for every $b \in \{b \mid T(X \supset b) \in S\}$, $a \leq b$. So, we assume $T(a \supset X)$ and $T(X \supset b)$ are both in S, $a \not\leq b$, and we derive a contradiction.

Since $a \not\leq b$, and $\mathcal{T}$ is finite, there must be a minimal $u_i \leq a$ such that $u_i \not\leq b$. Since $T(X \supset b) \in S$, by Reversal Rule $T \leq$, $S \cup \{F(u_i \supset X)\}$ is consistent. Then since S is maximally consistent, $F(u_i \supset X) \in S$. But then a tableau starting with members of S can close immediately by one of the Branch Closure Conditions, since $T(a \supset X)$ and $F(u_i \supset X)$ are in S and $u_i \leq a$, so S is inconsistent, a contradiction. ∎

Next, some basic properties of $bound_S$ and $bound^S$. These play a crucial role in the completeness proof both for the non-modal and the modal cases.

PROPOSITION 6.5. *Let S be maximal consistent, and let X be any formula.*

1. *If $T(c \supset X) \in S$ then $c \leq bound_S(X)$.*
2. *If $T(X \supset c) \in S$ then $bound^S(X) \leq c$.*
3. *If $F(c \supset X) \in S$ then $c \not\leq bound^S(X)$.*
4. *If $F(X \supset c) \in S$ then $bound_S(X) \not\leq c$.*

PROOF. Items 1 and 2 are immediate from the definitions of $bound_S$ and $bound^S$. For item 3, suppose $F(c \supset X) \in S$ but $c \leq bound^S(X)$; we derive a contradiction. Using Reversal Rule $F \geq$, for some $t_i \not\geq c$, $S \cup \{T(X \supset t_i)\}$ is consistent, so by maximality of S, $T(X \supset t_i) \in S$. Then by item 2, $bound^S(X) \leq t_i$. But then, $c \leq t_i$, and this is a contradiction. Item 4 has a similar proof. ∎

Note that both the Lemma and the Proposition above use only the Reversal Rules in their proofs. Consequently both hold for the systems with and without modal rules. Now the main item we need to establish completeness of the non-modal system.

PROPOSITION 6.6. *Let S be a maximally consistent set of signed bounding implications, and let v be any valuation such that, for* propositional variables *P,*

$$bound_S(P) \leq v(P) \leq bound^S(P).$$

Then for any *non-modal formula X,*

$$bound_S(X) \leq v(X) \leq bound^S(X).$$

PROOF. The argument is by induction on the degree of X. The atomic case is by definition. Now suppose $X = (A \wedge B)$ and the result is known for each of A and B; we show it for X. (The other two cases are similar and are omitted.)

Let a be an arbitrary member of $\mathcal{T}$ and suppose $T(a \supset (A \wedge B)) \in S$. Using Conjunction Rule $T\wedge$ (and maximality of S) it follows that $T(a \supset A)$

and $T(a \supset B)$ are both in S. By Proposition 6.5., $a \leq bound_S(A)$ and $a \leq bound_S(B)$. Then it follows from the induction hypothesis that $a \leq v(A)$ and $a \leq v(B)$. But then $a \leq v(A) \wedge v(B) = v(A \wedge B)$. Since a was arbitrary, this establishes that $bound_S(A \wedge B) \leq v(A \wedge B)$.

For the other half of the conjunction case, to show $v(A \wedge B) \leq bound^S(A \wedge B)$ it is enough to show that whenever $T((A \wedge B) \supset a) \in S$ it follows that $v(A \wedge B) \leq a$. We do this by contradiction: suppose there is an $a \in \mathcal{T}$ such that $T((A \wedge B) \supset a) \in S$ but $v(A \wedge B) \not\leq a$. From the set of members of $\mathcal{T}$ that are below $v(A \wedge B)$ but are not below a choose a minimal member — call it u_i. Thus $u_i \leq v(A \wedge B)$, and is minimal such that $u_i \not\leq a$. Now by Reversal Rule $T \leq$ (and maximality of S), $F(u_i \supset (A \wedge B)) \in S$. Then by Conjunction Rule $F\wedge$, either $F(u_i \supset A) \in S$ or $F(u_i \supset B) \in S$. By Proposition 6.5., $u_i \not\leq bound^S(A)$ or $u_i \not\leq bound^S(B)$. It follows from the induction hypothesis that $u_i \not\leq v(A)$ or $u_i \not\leq v(B)$, and hence $u_i \not\leq v(A) \wedge v(B) = v(A \wedge B)$, and this is our contradiction. ■

THEOREM 6.7. *Any consistent set of signed bounding formulas is satisfiable, and hence the non-modal tableau rules are complete.*

PROOF. Suppose S_0 is consistent. In the usual way it can be extended to a maximal consistent set S by systematically adding each signed bounding implication that preserves consistency. Pick an arbitrary valuation v such that on propositional variables v is between $bound_S$ and $bound^S$. Now, if $F(c \supset X) \in S_0 \subseteq S$, by Proposition 6.5., $c \not\leq bound^S(X)$. But by Proposition 6.6., $v(X) \leq bound^S(X)$, so $c \not\leq v(X)$. This means v satisfies $F(c \supset X)$. The argument is similar for the other cases of signed bounding implications. Now completeness follows in the usual way. ■

7. Modal Rules

There are several varieties of tableau rules for modal logics based on classical, two-valued logic. Here we are interested in the so-called *destructive* style, see [3], and also [2, 8]. We begin this section with a brief sketch of the rules for two-valued K, then we present the many-valued analog.

Let S be a set of signed formulas of modal logic in the conventional sense, taking both $\Box$ and $\Diamond$ as primitive. We define a set $S^\#$ as follows.

$$S^\# = \{T\,X \mid T\,\Box X \in S\} \cup \{F\,X \mid F\,\Diamond X \in S\}$$

The idea is, in a conventional Kripke model, if S is the set of formulas true at a world, and we move from that world to a generic alternative world, the members of $S^{\#}$ will be true there. Now the classical K-rules are easily given. They are destructive: instead of adding formulas to branches, whole branches are replaced with new ones. The branch replacement rules are these.

Classical K Branch Replacement Rules

$$\frac{S, T\,\Diamond X}{S^{\#}, T\,X} \qquad\qquad \frac{S, F\,\Box X}{S^{\#}, F\,S}$$

These rules are applied as follows: if $S \cup \{T\,\Diamond X\}$ is the set of formulas on a tableau branch, that branch can be *replaced* with a new branch whose formula set is $S^{\#} \cup \{T\,X\}$. Similarly for the other rule. We assume the reader has some familiarity with this style of tableau, and do not elaborate further here.

Now, to present modal rules in this style for a many-valued logic we first need an analog of the # operation. Classically # corresponds to a move from a world to an alternative one. But the classical accessibility relation is two-valued, while now we have a many-valued one. Consequently we need a # operation for each of the truth-values (other than *false*).

DEFINITION 7.1. Let S be a set of signed bounding implications, and c be a propositional constant other than *false*.

$$\begin{array}{rcl} S^{\#}(c) & = & \{T\,((a \wedge c) \supset X) \mid T\,(a \supset \Box X) \in S \text{ and } a \wedge c \neq \mathit{false}\} \\ & & \cup \\ & & \{T\,(X \supset (c \Rightarrow a)) \mid T\,(\Diamond X \supset a) \in S \text{ and } c \Rightarrow a \neq \mathit{true}\} \end{array}$$

Caution: the expressions $(a \wedge c)$ and $(c \Rightarrow a)$ in the two parts of the definition above are *not syntactic*. They are intended to be the propositional constants resulting from evaluating these expressions in $\mathcal{T}$. Similar considerations apply to the tableau rules below. In the first clause above, if we allowed $a \wedge c = \mathit{false}$ the formula added to $S^{\#}$ would be $T\,\mathit{false} \supset X$ which is harmless to allow, but of no use in closing a tableau branch. We rule it out in the interests of efficiency. Similarly for the restriction in the second clause. Now, we have the following branch replacement rules. The idea is, if the set of formulas above the double line is the set on some tableau branch, that branch may be *replaced* by the branches below the double line.

Many-Valued Modal Branch Replacement Rules

$F\Box$

$$\frac{\begin{array}{c} S \\ F(a \supset \Box X) \end{array}}{\begin{array}{c|c|c} S^{\#}(t_1) & \cdots & S^{\#}(t_n) \\ F((a \wedge t_1) \supset X) & & F((a \wedge t_n) \supset X) \end{array}}$$

$F\Diamond$

$$\frac{\begin{array}{c} S \\ F(\Diamond X \supset a) \end{array}}{\begin{array}{c|c|c} S^{\#}(t_1) & \cdots & S^{\#}(t_n) \\ F(X \supset (t_1 \Rightarrow a)) & & F(X \supset (t_n \Rightarrow a)) \end{array}}$$

Restrictions In the Modal Rule $F\Box$, $a \wedge t_i \neq$ *false*. In the $F\Diamond$ Rule, $t_i \Rightarrow a \neq$ *true*, or equivalently, $t_i \not\leq a$.

The reasons for the restrictions are similar to those above. In the $F\Box$ rule, branch closure is immediate if a branch contains $F((a \wedge t_i) \supset X)$ and $a \wedge t_i =$ *false*, so this case can be omitted. Similarly for the other rule. Note that if $t_i =$ *false* both restrictions arise, since $a \wedge$ *false* $=$ *false*, and *false* $\leq a$. Consequently we never consider $t_i =$ *false*, which is compatible with the omission of the *false* case in defining the $\#$ operation.

An intuitive justification for the first rule, $F\Box$, is as follows (the other rule is treated similarly). Suppose $S \cup \{F(a \supset \Box X)\}$ is a set of signed bounding implications, and its members are satisfied at the world Γ of a many-valued modal model $\langle \mathcal{G}, \mathcal{R}, w \rangle$ — so in particular, $a \not\leq w(\Gamma, \Box X)$. Now, $w(\Gamma, \Box X) = \bigwedge\{\mathcal{R}(\Gamma, \Delta) \Rightarrow w(\Delta, X) \mid \Delta \in \mathcal{G}\}$, so for some world Δ_0, $a \not\leq \mathcal{R}(\Gamma, \Delta_0) \Rightarrow w(\Delta_0, X)$. Say $\mathcal{R}(\Gamma, \Delta_0) = t_i$; then $a \not\leq (t_i \Rightarrow w(\Delta_0, X))$, so $a \wedge t_i \not\leq w(\Delta_0, X)$. This means the signed formula $F(a \wedge t_i) \supset X$ is satisfied at Δ_0. Continuing the rule justification, assume $T(\Diamond Y \supset b)$ is one of the members of S, and so is satisfied at Γ. Then $w(\Gamma, \Diamond Y) \leq b$, so $\bigvee\{\mathcal{R}(\Gamma, \Delta) \wedge w(\Delta, Y) \mid \Delta \in \mathcal{G}\} \leq b$, and it follows that $\mathcal{R}(\Gamma, \Delta_0) \wedge w(\Delta_0, Y) \leq b$, or $t_i \wedge w(\Delta_0, Y) \leq b$. It follows from this that $w(\Delta_0, Y) \leq (t_i \Rightarrow b)$, and so $T(Y \supset (t_i \Rightarrow b))$ is satisfied at Δ_0. A similar argument applies to each of the members of S.

What we have shown is that if the members of $S \cup \{F(a \supset \Box X)\}$ are satisfied at a world of a many-valued model, then for some $t_i \in \mathcal{T}$, the members of $S^{\#}(t_i) \cup \{F((a \wedge t_i) \supset X)\}$ are also satisfied at some world. In other words, if we have a satisfiable tableau, and we apply one of the modal rules, we get a satisfiable tableau back.

8. The Three-Valued Example Continued

In Section 5. we gave non-modal rules for a three-valued logic, $\mathcal{T}(3)$. We continue that example, give modal rules, and present a tableau example. We begin with the # operation, which has two cases since *false* is omitted.

$$\begin{array}{|lcl|}
\hline
S^{\#}(\mathit{half}) & = & \{T\,(\mathit{half} \supset X) \mid T\,(\mathit{half} \supset \Box X) \in S\} \cup \\
 & & \{T\,(\mathit{half} \supset X) \mid T\,(\mathit{true} \supset \Box X) \in S\} \cup \\
 & & \{T\,(X \supset \mathit{false}) \mid T\,(\Diamond X \supset \mathit{false}) \in S\} \\
\hline
S^{\#}(\mathit{true}) & = & \{T\,(\mathit{half} \supset X) \mid T\,(\mathit{half} \supset \Box X) \in S\} \cup \\
 & & \{T\,(\mathit{true} \supset X) \mid T\,(\mathit{true} \supset \Box X) \in S\} \cup \\
 & & \{T\,(X \supset \mathit{false}) \mid T\,(\Diamond X \supset \mathit{false}) \in S\} \cup \\
 & & \{T\,(X \supset \mathit{half}) \mid T\,(\Diamond X \supset \mathit{half}) \in S\} \\
\hline
\end{array}$$

There are four Modal Branch Replacement Rules for $\mathcal{T}(3)$, as follows.

$$\frac{\begin{array}{c} S \\ F\,(\mathit{half} \supset \Box X) \end{array}}{\begin{array}{c|c} S^{\#}(\mathit{half}) & S^{\#}(\mathit{true}) \\ F\,(\mathit{half} \supset X) & F\,(\mathit{half} \supset X) \end{array}}$$

$$\frac{\begin{array}{c} S \\ F\,(\mathit{true} \supset \Box X) \end{array}}{\begin{array}{c|c} S^{\#}(\mathit{half}) & S^{\#}(\mathit{true}) \\ F\,(\mathit{half} \supset X) & F\,(\mathit{true} \supset X) \end{array}}$$

$$\frac{\begin{array}{c} S \\ F\,(\Diamond X \supset \mathit{false}) \end{array}}{\begin{array}{c|c} S^{\#}(\mathit{half}) & S^{\#}(\mathit{true}) \\ F\,(X \supset \mathit{false}) & F\,(X \supset \mathit{false}) \end{array}}$$

$$\frac{\begin{array}{c} S \\ F\,(\Diamond X \supset \mathit{half}) \end{array}}{\begin{array}{c} S^{\#}(\mathit{true}) \\ F\,(X \supset \mathit{half}) \end{array}}$$

Finally we present a tableau proof for $\mathcal{T}(3)$ that uses the modal rules. It is a proof of:

$$\mathit{half} \supset \{[(\mathit{half} \supset \Diamond X) \wedge (\mathit{true} \supset \Box Y)] \supset \Diamond(X \wedge Y)\}.$$

The proof is divided among several Figures for convenience. It begins in Figure 3

$T\ half \supset \{[(half \supset \Diamond X) \wedge (true \supset \Box Y)] \supset \Diamond(X \wedge Y)\}$
$T\ half \supset [(half \supset \Diamond X) \wedge (true \supset \Box Y)]$
$F\ half \supset \Diamond(X \wedge Y)$
$T\ half \supset (half \supset \Diamond X)$
$T\ half \supset (true \supset \Box Y)$

$F\ half \supset half$ | $T\ half \supset \Diamond X$

$F\ half \supset true$ | $T\ half \supset \Box Y$
$T\ \Diamond(X \wedge Y) \supset false$
$F\ \Diamond X \supset false$

Figure 3: A Modal Tableau, Part One

All of the steps in Figure 3 are propositional, and we omit most explanations. Note that the left two branches are closed because of their final nodes. Also the last two entries on the rightmost branch result from Reversal Rule applications. Now a modal rule applies to the signed formula $F\ \Diamond X \supset false$ on the rightmost branch, taking for S the remaining formulas on that branch. This replaces the rightmost branch by the pair shown in Figure 4

In Figure 4 the left main branch begins with $S^{\#}(half)$ and the right main branch with $S^{\#}(true)$, which happen to be the same in this case — this gives the first two signed formulas on these branches. In each case the third formula is $F\ X \supset false$, which comes from $F\ \Diamond X \supset false$ in Figure 3 On the left branch (4) is from (2), and (5) is from (3), by Reversal Rules, and (6) and (7) are from (4) by $F\wedge$. The right branch is similar. Finally, each of the four branches is closed.

9. Modal Completeness

We omit proof of the soundness of the modal tableau rules — this is straightforward and may be left to the reader. We proceed directly to their com-

$$
\begin{array}{ll}
(1)\ T\,\mathit{half} \supset Y & (8)\ T\,\mathit{half} \supset Y \\
(2)\ T\,(X \wedge Y) \supset \mathit{false} & (9)\ T\,(X \wedge Y) \supset \mathit{false} \\
(3)\ F\,X \supset \mathit{false} & (10)\ F\,X \supset \mathit{false} \\
(4)\ F\,\mathit{half} \supset (X \wedge Y) & (11)\ F\,\mathit{half} \supset (X \wedge Y) \\
(5)\ T\,\mathit{half} \supset X & (12)\ T\,\mathit{half} \supset X \\
(6)\ F\,\mathit{half} \supset X \quad (7)\ F\mathit{half} \supset Y & (13)\ F\,\mathit{half} \supset X \quad (14)\ F\mathit{half} \supset Y
\end{array}
$$

Figure 4: A Modal Tableau, Part Two

pleteness. As usual, the proof amounts to showing that a consistent tableau is satisfiable. Consistency was characterized in Definition 6.2.; we continue to use that definition (with the understanding that modal rules are allowed now). Similarly for maximal consistency. Likewise satisfiability was characterized in Definition 6.1. We continue to use essentially that definition, with obvious modifications to relativize things to possible worlds. Thus, a set S is satisfiable if there is a $\mathcal{T}$-modal model $\langle \mathcal{G}, \mathcal{R}, w\rangle$ and a world $\Gamma \in \mathcal{G}$ such that each member of S is satisfied at Γ. And so on. With all this understood, we need the following extension of Theorem 6.7.

THEOREM 9.1. *Allowing the modal tableau rules, and using $\mathcal{T}$-modal models, any consistent set of signed bounding formulas is satisfiable, and hence the modal tableau system is complete.*

The proof of this occupies the rest of the section. Not surprisingly, it is along the 'canonical model' line, suitably modified for the space $\mathcal{T}$ of truth values. Let $\mathcal{G}$ be the set of all maximally consistent sets of signed bounding implications. This will be the set of possible worlds of our canonical model. We carry over the notation of Definition 6.3., and use it to define a somewhat unusual many-valued accessibility relation. For $\Gamma, \Delta \in \mathcal{G}$:

$$
\begin{aligned}
\mathcal{R}(\Gamma, \Delta) \quad = \quad & \bigwedge\{\mathit{bound}_{\Gamma}(\Box Y) \Rightarrow \mathit{bound}_{\Delta}(Y) \mid \text{all formulas } Y\} \\
& \wedge \\
& \bigwedge\{\mathit{bound}^{\Delta}(Z) \Rightarrow \mathit{bound}^{\Gamma}(\Diamond Z) \mid \text{all formulas } Z\}
\end{aligned}
$$

Finally, let w be any mapping such that on propositional variables:

$$\mathit{bound}_{\Gamma}(P) \le w(\Gamma, P) \le \mathit{bound}^{\Gamma}(P).$$

This defines a *canonical model* $\langle \mathcal{G}, \mathcal{R}, w \rangle$. The proof is finished once we extend Proposition 6.6. and show that for *any* formula X (even allowing modal operators):

$$bound_{\Gamma}(X) \leq w(\Gamma, X) \leq bound^{\Gamma}(X). \tag{1}$$

The proof of the sequence of inequalities (1) is, of course, by induction on the complexity of X. The propositional connective cases are treated exactly as in the proof of Proposition 6.6., and are not repeated here. The new things are the $\Box$ and $\Diamond$ cases, which we give in detail.

The $\Box$ Case Suppose (1) is known for X; we show it for $\Box X$. Let Γ_0 be a fixed member of $\mathcal{G}$. We begin with the easier half.

By definition of $\mathcal{R}$, for an arbitrary $\Delta \in \mathcal{G}$,

$$\mathcal{R}(\Gamma_0, \Delta) \leq bound_{\Gamma_0}(\Box X) \Rightarrow bound_{\Delta}(X).$$

By the properties of $\Rightarrow$,

$$bound_{\Gamma_0}(\Box X) \leq \mathcal{R}(\Gamma_0, \Delta) \Rightarrow bound_{\Delta}(X).$$

By the induction hypothesis, $bound_{\Delta}(X) \leq w(\Delta, X)$, and it follows that

$$bound_{\Gamma_0}(\Box X) \leq \mathcal{R}(\Gamma_0, \Delta) \Rightarrow w(\Delta, X).$$

Since Δ is arbitrary,

$$bound_{\Gamma_0}(\Box X) \leq \bigwedge\{\mathcal{R}(\Gamma_0, \Delta) \Rightarrow w(\Delta, X) \mid \Delta \in \mathcal{G}\} = w(\Gamma_0, \Box X).$$

Now for the harder half; to show $w(\Gamma_0, \Box X) \leq bound^{\Gamma_0}(\Box X)$ it is enough to show that whenever $T(\Box X \supset c) \in \Gamma_0$ then $w(\Gamma_0, \Box X) \leq c$. To show this, suppose there is some propositional constant c such that $T(\Box X \supset c) \in \Gamma_0$, but $w(\Gamma_0, \Box X) \not\leq c$ — we derive a contradiction.

Since $w(\Gamma_0, \Box X) \not\leq c$, there is some u_i such that $u_i \leq w(\Gamma_0, \Box X)$, and u_i is minimal with $u_i \not\leq c$. Since $T(\Box X \supset c) \in \Gamma_0$, by Reversal Rule $T \leq$, and maximal consistency of Γ_0, $F(u_i \supset \Box X) \in \Gamma_0$. Then by Modal Rule $F\Box$, for some t_j the set $\Gamma_0^{\#}(t_j) \cup \{F((u_i \wedge t_j) \supset X)\}$ is consistent. Extend it to a maximal consistent set Δ_0. Then $\Delta_0 \in \mathcal{G}$ and by Proposition 6.5., $u_i \wedge t_j \not\leq bound^{\Delta_0}(X)$, so by the induction hypothesis, $u_i \wedge t_j \not\leq w(\Delta_0, X)$.

Subordinate Result $t_j \leq \mathcal{R}(\Gamma_0, \Delta_0)$.

Proof of Subordinate Result
The argument is in two parts. First we show $t_j \leq \bigwedge\{bound_{\Gamma_0}(\Box Y) \Rightarrow bound_{\Delta_0} \mid \text{all formulas } Y\}$. So, let Y be an arbitrary formula; we show $t_j \leq bound_{\Gamma_0}(\Box Y) \Rightarrow bound_{\Delta_0}(Y)$, or equivalently, that $t_j \wedge bound_{\Gamma_0}(\Box Y) \leq bound_{\Delta_0}(Y)$. Since $\mathcal{T}$ is a distributive lattice,

$$\begin{aligned} t_j \wedge bound_{\Gamma_0}(\Box Y) &= t_j \wedge \bigvee\{a \mid T(a \supset \Box Y) \in \Gamma_0\} \\ &= \bigvee\{(a \wedge t_j) \mid T(a \supset \Box Y) \in \Gamma_0\} \end{aligned}$$

Suppose $T(a \supset \Box Y) \in \Gamma_0$. Then by construction, $T((a \wedge t_j) \supset Y) \in \Delta_0$, so $a \wedge t_j \leq bound_{\Delta_0}(Y)$. It follows that

$$t_j \wedge bound_{\Gamma_0}(\Box Y) \leq bound_{\Gamma_0}(Y).$$

For the second part of the Subordinate Result argument we show $t_j \leq \bigwedge\{bound^{\Delta_0}(Z) \Rightarrow bound^{\Gamma_0}(\Diamond Z) \mid \text{all formulas } Z\}$. Let Z be an arbitrary formula; we show $t_j \leq bound^{\Delta_0}(Z) \Rightarrow bound^{\Gamma_0}(\Diamond Z)$, or equivalently, that $t_j \wedge bound^{\Delta_0}(Z) \leq bound^{\Gamma_0}(\Diamond Z)$. To show this we argue that $t_j \wedge bound^{\Delta_0}(Z)$ is a lower bound for $\{a \mid T(\Diamond Z \supset a) \in \Gamma_0\}$.

Suppose $T(\Diamond Z \supset a) \in \Gamma_0$. Then by construction, $T(Z \supset (t_j \Rightarrow a)) \in \Delta_0$, so $bound^{\Delta_0}(Z) \leq (t_j \Rightarrow a)$. But then, $t_j \wedge bound^{\Delta_0}(Z) \leq t_j \wedge (t_j \Rightarrow a) \leq a$, and this gives us what we need.

This ends the proof of the Subordinate Result.

Now that we have established $t_j \leq \mathcal{R}(\Gamma_0, \Delta_0)$ we return to the main argument. By definition,

$$\begin{aligned} w(\Gamma_0, \Box X) &= \bigwedge\{\mathcal{R}(\Gamma_0, \Delta) \Rightarrow w(\Delta, X) \mid \Delta \in \mathcal{G}\} \\ &\leq \mathcal{R}(\Gamma_0, \Delta_0) \Rightarrow w(\Delta_0, X) \end{aligned}$$

Then

$$w(\Gamma_0, \Box X) \wedge \mathcal{R}(\Gamma_0, \Delta_0) \leq w(\Delta_0, X).$$

But by the Subordinate Result, $t_j \leq \mathcal{R}(\Gamma_0, \Delta_0)$, and by our choice of u_i, $u_i \leq w(\Gamma_0, \Box X)$. But then,

$$u_i \wedge t_j \leq w(\Delta_0, X)$$

contradicting the fact that $u_i \wedge t_j \not\leq w(\Delta_0, X)$, established above.

This contradiction ends the argument that $w(\Gamma_0, \Box X) \leq bound^{\Gamma_0}(\Box X)$ and finishes the $\Box$ case. The argument for the $\Diamond$ case is similar, but since it is short we give it anyway.

The $\Diamond$ Case Suppose (1) is known for X; we show it for $\Diamond X$. Again let Γ_0 be a fixed member of $\mathcal{G}$.

For an arbitrary $\Delta \in \mathcal{G}$,

$$\mathcal{R}(\Gamma_0, \Delta) \leq \mathit{bound}^{\Delta}(X) \Rightarrow \mathit{bound}^{\Gamma_0}(\Diamond X).$$

Then, using the basic properties of $\Rightarrow$ and the part of the induction hypothesis that says $w(\Delta, X) \leq \mathit{bound}^{\Delta}(X)$, we have

$$\mathcal{R}(\Gamma_0, \Delta) \wedge w(\Delta, X) \leq \mathit{bound}^{\Gamma_0}(\Diamond X).$$

Since Δ is arbitrary,

$$w(\Gamma_0, \Diamond X) = \bigvee\{\mathcal{R}(\Gamma_0, \Delta) \wedge w(\Delta, X) \mid \Delta \in \mathcal{G}\} \leq \mathit{bound}^{\Gamma_0}(\Diamond X).$$

For the other half, $\mathit{bound}_{\Gamma_0}(\Diamond X) \leq w(\Gamma_0, \Diamond X)$, it is enough to show that from $T(c \supset \Diamond X) \in \Gamma_0$ and $c \not\leq w(\Gamma_0, \Diamond X)$ we can derive a contradiction. Under these assumptions, there must exist some $u_i \geq w(\Gamma_0, \Diamond X)$ where u_i is maximal not above c. Then by Reversal Rule $T \geq$ and the maximal consistency of Γ_0, $F(\Diamond X \supset u_i) \in \Gamma_0$. Now, using Rule $F\Diamond$, for some $t_j \in \mathcal{T}$, $\Gamma_0^{\#}(t_j) \cup \{F(X \supset (t_j \Rightarrow u_i))\}$ is consistent. Extend it to a maximal consistent set Δ_0; so $\Delta_0 \in \mathcal{G}$.

By the Subordinate Result (which still applies) $t_j \leq \mathcal{R}(\Gamma_0, \Delta_0)$. Now,

$$\begin{array}{rcl} t_j \wedge w(\Delta_0, X) & \leq & \mathcal{R}(\Gamma_0, \Delta_0) \wedge w(\Delta_0, X) \\ & \leq & \bigvee\{\mathcal{R}(\Gamma_0, \Delta) \wedge w(\Delta, X) \mid \Delta \in \mathcal{G}\} \\ & = & w(\Gamma_0, \Diamond X) \\ & \leq & u_i \end{array}$$

so $w(\Delta_0, X) \leq (t_j \Rightarrow u_i)$. By induction hypothesis, $\mathit{bound}_{\Delta_0} \leq w(\Delta_0, X)$, so

$$\mathit{bound}_{\Delta_0}(X) \leq (t_j \Rightarrow u_i).$$

But by construction, $F(X \supset (t_j \Rightarrow u_i)) \in \Delta_0$, so by Proposition 6.5., $\mathit{bound}_{\Delta_0}(X) \not\leq (t_j \Rightarrow u_i)$, and we have the desired contradiction.

Now the proof of Theorem 9.1. is essentially done. If a set S is consistent, extend it to a maximal consistent set Γ. Γ will be a world in a canonical model, and the inequalities (1) directly imply that S is satisfied at Γ. Completeness now follows in the usual way.

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DEPT. MATHEMATICS AND COMPUTER SCIENCE
LEHMAN COLLEGE (CUNY), BRONX, NY 10468
DEPTS. COMPUTER SCIENCE, PHILOSOPHY, MATHEMATICS
GRADUATE CENTER (CUNY), 33 WEST 42ND STREET, NYC, NY 10036

mlflc@cunyvm.cuny.edu