MARIA LUISA DALLA CHIARA
ROBERTO GIUNTINI

The Logics of Orthoalgebras

Abstract. The notion of unsharp orthoalgebra is introduced and it is proved that the category of unsharp orthoalgebras is isomorphic to the category of D-posets. A completeness theorem for some partial logics based on unsharp orthoalgebras, orthoalgebras and orthomodular posets is proved.

Key words and phrases: orthoalgebra, unsharp orthoalgebra, D-poset, partial quantum logic

1. Introduction

Orthodox quantum logic as well as many other standard logics can be described as *total*, both from the syntactical and the semantic point of view. Namely, the language is generally closed under the basic logical constants, whereas sentences receive, in the semantics, a well determined interpretation. In the framework of the logico-algebraic approach to quantum mechanics, many authors have noticed that the structure of the *quantum events* cannot be adequately represented as closed under conjunction and disjunction. This is a natural consequence of the non existence of *joint distributions of strongly incompatible observables*. Suppose that α and β describe two strongly incompatible events (for instance: "the value for the spin in the x- direction is up"; "the value for the spin in the y- direction is down"). It is quite natural to regard the conjunction of α and β as meaningless, since it represents an experimentally non conceivable event. However, from a strictly logical point of view, partial logics (where the basic connectives are semantically described as partial operations) turn out to be somewhat problematic.

Among the partial structures that have been analysed as good candidates in order to represent the quantum events one should mention at least the following:

1) *partial Boolean algebras* and *transitive partial Boolean algebras* (introduced by S. Kochen, E. Specker ([11]) and J. Czelakowski ([3])).

2) *orthomodular posets* (which P. Suppes has called *quantum mechanical algebras*).

Received May 11, 1994;

© 1995 *Kluwer Academic Publishers. Printed in the Netherlands.*

3) *orthoalgebras* (investigated by D. Foulis, C. Randall ([7]), G. Hardegree, P. Lock ([10]) and D. Foulis, R. Greechie, G. Rüttimann ([6])).

4) *unsharp orthoalgebras* and *D-posets*, which turn out to be equivalent structures ([8], [5], [12]).

Transitive partial Boolean algebras are orthomodular posets which are orthoalgebras, which in turn are unsharp orthoalgebras; but not the other way around.

In this paper we will study three forms of partial quantum logic, which are based respectively on the class of all unsharp orthoalgebras, of all orthoalgebras and of all orthomodular posets.

2. Sharp and Unsharp Orthoalgebras, D-posets and Orthomodular Posets

Sharp and unsharp orthoalgebras, D-posets and orthomodular posets are examples of partial algebraic structures, where the basic operations are not always defined. When an operation $\circ$ is defined for two elements a, b we will write $\exists(a \circ b)$. Let us first investigate the notion of unsharp orthoalgebra.

DEFINITION 2.1. An *unsharp orthoalgebra* is a partial algebraic structure $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0}\rangle$, where $\mathbf{1}$ and $\mathbf{0}$ are two distinct elements of A and $\oplus$ is a partial binary operation on A which satisfies the following conditions:

(U1) *Weak commutativity*
$\exists(a \oplus b) \implies \exists(b \oplus a)$ and $a \oplus b = b \oplus a$.

(U2) *Weak associativity*
$[\exists(b \oplus c)$ and $\exists(a \oplus (b \oplus c))] \implies [\exists(a \oplus b)$ and $\exists((a \oplus b) \oplus c))$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c]$.

(U3) *Strong excluded middle*
For any a, there exists a unique x s.t. $a \oplus x = \mathbf{1}$.

(U4) *Weak consistency*
$\exists(a \oplus \mathbf{1}) \implies a = \mathbf{0}$.

An orthogonality relation, a partial order relation and a generalized complement can be defined in any unsharp orthoalgebra.

DEFINITION 2.2. Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0}\rangle$ be an unsharp orthoalgebra and let $a, b \in A$.

(i) a is *orthogonal* to b ($a \perp b$) iff $a \oplus b$ is defined in A.

(ii) *a precedes b* ($a \sqsubseteq b$) iff $\exists c \in A$ s.t. $a \perp b$ and $b = a \oplus c$.

(iii) The *generalized complement* of a is the unique element a' s.t. $a \oplus a' = \mathbf{1}$ (the definition is justified by the strong excluded middle condition (U3)).

The notion of unsharp orthoalgebra morphism (UO morphism) is defined in the standard way.

DEFINITION 2.3. Let $\mathcal{A}_1 = \langle A_1, \oplus_1, \mathbf{1}_1, \mathbf{0}_1 \rangle$ and $\mathcal{A}_2 = \langle A_2, \oplus_2, \mathbf{1}_2, \mathbf{0}_2 \rangle$ be two unsharp orthoalgebras. An *UO morphism* is a map $h : A_1 \to A_2$ s.t.

(i) $h(\mathbf{1}_1) = \mathbf{1}_2$.

(ii) $\exists(a \oplus_1 b) \implies [\exists(h(a) \oplus_2 h(b))$ and $h(a \oplus_1 b) = h(a) \oplus_2 h(b)]$.

It is easy to see that the class of all unsharp orthoalgebras with the UO morphisms determines a category. This category will be denoted by $\mathcal{UO}$.

LEMMA 2.1. *Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ be an unsharp orthoalgebra and let $a, b \in A$. The following properties hold:*

(i) $a \perp b \Longrightarrow b \perp a$.

(ii) $a'' = a$.

(iii) $\mathbf{1}' = \mathbf{0}$ *and* $\mathbf{0}' = \mathbf{1}$.

(iv) $a \perp \mathbf{0}$ *and* $a \oplus \mathbf{0} = a$.

(v) $a \perp b,\ a \oplus b = \mathbf{0} \Longrightarrow a = b = \mathbf{0}$

PROOF. (i) follows from U1).
(ii) By U3), $a \oplus a' = \mathbf{1}$ and $a' \oplus a'' = \mathbf{1}$. By U1), we have $a' \oplus a = \mathbf{1}$; hence, by U3), $a = a''$.
(iii) By U3), $\mathbf{1} \oplus \mathbf{1}' = \mathbf{1}$. Thus, by U1) and U4), $\mathbf{1}' = \mathbf{0}$; therefore, by (ii), $\mathbf{0}' = \mathbf{1}'' = \mathbf{1}$.
(iv) $\mathbf{1} = \mathbf{1} \oplus \mathbf{1}' = (a \oplus a') \oplus \mathbf{1}' = (a \oplus a') \oplus \mathbf{0} = (a' \oplus a) \oplus \mathbf{0} = a' \oplus (a \oplus \mathbf{0})$. Hence, by U3): $a \oplus \mathbf{0} = a'' = a$.
(v) Suppose $a \perp b$ and $a \oplus b = \mathbf{0}$. Now, $a = a \oplus \mathbf{0} = a \oplus (a \oplus b)$. Thus, $\mathbf{1} = [a \oplus (a \oplus b)] \oplus a' = a \oplus [(a \oplus b) \oplus a'] = a \oplus (b \oplus \mathbf{1})$. Thus, $b \oplus \mathbf{1}$ is defined; by U4), we obtain $b = \mathbf{0}$. Similarly, one can prove that $a = \mathbf{0}$. ■

LEMMA 2.2. *Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ be an unsharp orthoalgebra and let $a, b \in A$ s.t. $a \perp b$. Then,*

$$a \perp (a \oplus b)' \quad \textit{and} \quad b' = a \oplus (a \oplus b)'$$

PROOF. $\mathbf{1} = (a \oplus b) \oplus (a \oplus b)' = a \oplus [b \oplus (a \oplus b)'] = [b \oplus (a \oplus b)'] \oplus a = b \oplus [(a \oplus b)' \oplus a] = b \oplus [a \oplus (a \oplus b)']$. Thus, by U3), $b' = a \oplus (a \oplus b)'$. ■

LEMMA 2.3. *Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ be an unsharp orthoalgebra and let $a, b \in A$. Then,*

$$a \perp b \quad \textit{iff} \quad a \sqsubseteq b'$$

PROOF. Suppose $a \perp b$. Then, by Lemma 2.2, $b' = a \oplus (a \oplus b)'$. Thus, by Definition 2.2(i) $a \sqsubseteq b'$. Viceversa, suppose $a \sqsubseteq b'$. Then, $\exists c \in A$ s.t. $a \perp c$ and $b' = a \oplus c$. Thus, $\mathbf{1} = (a \oplus c) \oplus b = (c \oplus a) \oplus b = c \oplus (a \oplus b)$. Therefore, $a \oplus b$ is defined so that $a \perp b$. ■

Lemma 2.3 shows that the orthogonality relation has here the usual meaning.

LEMMA 2.4. *(Orthomodularity)*
Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ be an unsharp orthoalgebra and let $a, b \in A$. Then,

$$a \sqsubseteq b \Longrightarrow b = a \oplus (a \oplus b')'$$

PROOF. Suppose $a \sqsubseteq b$. Then, $a \sqsubseteq b''$. By Lemma 2.3, $a \perp b'$. Thus, by Lemma 2.2, $b = b'' = a \oplus (a \oplus b')'$. ■

LEMMA 2.5. *(Cancellation law)*
Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ be an unsharp orthoalgebra and let $a, b, \in$ s.t. $a \perp c$ and $b \perp c$. Then,

(i) $a \oplus c = b \oplus c \Longrightarrow a = b$.

(ii) $a \oplus c \sqsubseteq b \oplus c \Longrightarrow a \sqsubseteq b$.

PROOF. (i) Suppose $a \oplus c = b \oplus c$. Then, $\mathbf{1} = (a \oplus b) \oplus (a \oplus b)' = (b \oplus c) \oplus (a \oplus c)'$. Thus, $a \oplus [c \oplus (a \oplus c)'] = b \oplus [c \oplus (a \oplus c)'] = \mathbf{1}$. By U3), we get $a = b$.
(ii) Suppose $a \oplus c \sqsubseteq b \oplus c$. Then, $\exists d \in A$ s.t. $d \perp (a \oplus c)$ and $b \oplus c = (a \oplus c) \oplus d = a \oplus (d \oplus c) = (a \oplus d) \oplus c$. By (i), $b = a \oplus d$, so that $a \sqsubseteq b$. ■

THEOREM 2.1. *Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ be an unsharp orthoalgebra. Then, $\langle A, \sqsubseteq, ', \mathbf{1}, \mathbf{0} \rangle$ is an involutive bounded poset.*

PROOF. Let us first prove that $\sqsubseteq$ is a partial order. The reflexivity of $\sqsubseteq$ follows from Lemma 2.1(iv).
Let $a \sqsubseteq b$ and $b \sqsubseteq a$. We have to prove that $a = b$. By hypothesis: $\exists c, d \in A$ s.t. $a \perp c$, $b \perp d$ and $b = a \oplus c, a = b \oplus d$. Then, $(a \oplus \mathbf{0}) = a = b \oplus d =$

$(a \oplus c) \oplus d = a \oplus (c \oplus d)$. By the cancellation law, $c \oplus d = \mathbf{0}$. Thus, by Lemma 2.1(v), $c = d = \mathbf{0}$. Then, $a = b \oplus d = b \oplus \mathbf{0} = b$.
Let $a \sqsubseteq b$ and $b \sqsubseteq c$. We have to prove that $a \sqsubseteq c$. By hypothesis: $\exists d, e \in A$ s.t. $a \perp d$, $b \perp e$ and $b = a \oplus d$, $c = b \oplus e$. Then, $c = (a \oplus d) \oplus e = a \oplus (d \oplus e)$. Thus $a \sqsubseteq c$.
That $\mathbf{0}$ is the minimum of the poset follows from Lemma 2.1(iv). That $\mathbf{1}$ is the maximum of the poset follows from U3).
Finally, let us prove that $'$ is an involutive antiautomorphism.
(i) By Lemma 2.1(ii): $\forall a \in A$: $a'' = a$.
(ii) Suppose $a \sqsubseteq b$. We have to prove that $b' \sqsubseteq a'$. By Lemma 2.3, $a \perp b'$ and therefore, $b' \perp a$. Again, by Lemma 2.3, $b' \sqsubseteq a'$. ∎

The notion of unsharp orthoalgebra turns out to be equivalent to the notion of *weak orthoalgebra*, that has been first investigated in [8].

DEFINITION 2.4. A *weak orthoalgebra* is a structure $\mathcal{A} = \langle A, \perp, \oplus, ', \mathbf{1}, \mathbf{0}\rangle$, where $\perp$ is a binary relation on A, $\oplus$ is a partial operation whose domain is $\perp$ and $\mathbf{1}, \mathbf{0}$ are two distinct elements of A. The following conditions hold:

(W1) $a \perp b \implies b \perp a$ and $a \oplus b = b \oplus a$.

(W2) $a \perp \mathbf{0}$ and $a \oplus \mathbf{0} = a$.

(W3) $a \perp a'$ and $a \oplus a' = \mathbf{1}$.

(W4) $a \perp b \implies a \perp (a \oplus b)'$ and $b' = a \oplus (a \oplus b)'$.

(W5) $a \perp (a' \oplus b) \implies b = \mathbf{0}$.

(W6) $[a \perp b$ and $c \perp (a \oplus b)] \implies [b \perp c,\ a \perp (b \oplus c)$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c]$.

Similarly to the case of unsharp orthoalgebras, one can define the notion of *WO morphism*. The category determined by the class of all weak orthoalgebras and the WO morphisms will be denoted by $\mathcal{WO}$. We want to show that $\mathcal{UO}$ are $\mathcal{WO}$ are isomorphic. This result is a direct consequence of Lemma 2.7.

LEMMA 2.6. *Let $\mathcal{A} = \langle A, \perp, \oplus, ', \mathbf{1}, \mathbf{0}\rangle$ be a weak orthoalgebra. The following conditions are satisfied:*

(i) $\mathbf{0}' = \mathbf{1}$ *and* $\mathbf{1}' = \mathbf{0}$.

(ii) $\forall a \in A$: $a'' = a$.

PROOF. (i) By W1), W2) and W3), $\mathbf{1} = \mathbf{0} \oplus \mathbf{0}' = \mathbf{0}' \oplus \mathbf{0} = \mathbf{0}'$. By W3) and W2), $\mathbf{0}' \oplus \mathbf{0}'' = \mathbf{1}$ and $\mathbf{0} \perp (\mathbf{0}' \oplus \mathbf{0}'')$. Thus, by W5), $\mathbf{0}'' = \mathbf{0}$; hence, $\mathbf{0}' = \mathbf{1}$.
(ii) By W3), $a \oplus a' = \mathbf{1}$. By W4), $a'' = a \oplus (a \oplus a')' = a \oplus \mathbf{1}' = a \oplus \mathbf{0} = a$. ■

LEMMA 2.7.

(i) *Let* $\mathcal{A} = \langle A, \perp, \oplus, ', \mathbf{1}, \mathbf{0} \rangle$ *be a weak orthoalgebra. Then,* $\langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ *is an unsharp orthoalgebra.*

(ii) *Let* $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ *be an unsharp orthoalgebra. Let* $\perp$ *and* $'$ *be defined according to Definition 2.2 (i)-iii)). Then, the structure* $\mathcal{A} = \langle A, \perp, \oplus, ', \mathbf{1}, \mathbf{0} \rangle$ *is a weak orthoalgebra.*

PROOF. (i) U1) is W1) and U2) is W6) .
U3) Let $a \in A$. By W3), $a \oplus a' = \mathbf{1}$. Suppose $\exists b \in A$ s.t. $a \oplus b = \mathbf{1}$. By W4), $a \perp (a \oplus b)'$ and $b' = a \oplus (a \oplus b)' = a \oplus \mathbf{1}' = a \oplus \mathbf{0} = a$. Thus, $b = b'' = a'$.
U4) Suppose $a \perp \mathbf{1} = a \oplus a'$. Then, by W5), $a = \mathbf{0}$.
(ii) W1) is U1); W2) is Lemma 2.1(iv); W3) is U3). W4) is Lemma 2.2.
W5) Suppose $a \perp (a' \oplus b)$. Then, $a \oplus (a' \oplus b) = (a \oplus a') \oplus = \mathbf{1}$. By U3), we obtain $b = \mathbf{0}$.
W6) is U2). ■

Recently, Kôpka and Chovanec [12] have proposed a new (partial) algebraic structure, called *D-poset*, as a "natural generalization of quantum logics, real vector lattices and orthoalgebras". We will show that the category of D-posets is isomorphic to the category $\mathcal{UO}$ of unsharp orthoalgebras.

DEFINITION 2.5. A *difference poset* (shortly, D-*poset*), is a structure $\langle A, \sqsubseteq, \ominus, \mathbf{1} \rangle$, where: $\sqsubseteq$ is a partial order with maximum ($\mathbf{1}$), and $\ominus$ (the difference) is a partial operation s.t. $\forall a, b \in A$, $b \ominus a$ is defined iff $a \sqsubseteq b$. Further, the following conditions hold $\forall a, b, c \in A$:

(D1) $\exists(b \ominus a) \Longrightarrow (b \ominus a) \sqsubseteq b$.

(D2) $\exists(b \ominus a) \Longrightarrow [\exists(b \ominus (b \ominus a))$ and $b \ominus (b \ominus a) = a]$.

(D3) $a \sqsubseteq b \sqsubseteq c \Longrightarrow c \ominus b \sqsubseteq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

DEFINITION 2.6. Let $\mathcal{A}_1 = \langle A_1, \sqsubseteq_1, \ominus_1, \mathbf{1}_1 \rangle$ and $\mathcal{A}_2 = \langle A_2, \sqsubseteq_2, \ominus_2, \mathbf{1}_2 \rangle$ be two D-posets. A *DP morphism* is a map $h : A_1 \to A_2$ s.t.

(i) $h(\mathbf{1}_1) = \mathbf{1}_2$.

(ii) $a \sqsubseteq_1 b \implies [\exists(h(b) \ominus_2 h(a))$ and $h(b \ominus_1 b) = h(b) \ominus_2 h(b)]$.

The class of all D-posets with the D morphisms form a category. The following statements have been proved in [12]:

LEMMA 2.8. $\mathcal{A} = \langle A, \sqsubseteq, \ominus, \mathbf{1} \rangle$. *Then, the following properties hold:*

(i) $\mathbf{1} \ominus \mathbf{1}$ *(denoted by* $\mathbf{0}$*) is the minimum of* $\mathcal{A}$.

(ii) $a \ominus \mathbf{0} = a$.

(iii) $a \ominus a = \mathbf{0}$.

(iv) $a \sqsubseteq b \implies [b \ominus a = \mathbf{0} \iff b = a]$.

(v) $a \sqsubseteq b \implies [b \ominus a = b \implies a = \mathbf{0}]$.

(vi) $a \sqsubseteq b \sqsubseteq c \implies b \ominus a \sqsubseteq c \ominus a$ *and* $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.

(vii) $b \sqsubseteq c,\ a \sqsubseteq c \ominus b \implies b \sqsubseteq c \ominus a$ *and* $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.

(viii) $a \sqsubseteq b \sqsubseteq c \implies a \sqsubseteq c \ominus (b \ominus a)$ *and* $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.

REMARK 2.1. A D-poset can be equivalently defined (see [13]) as a bounded poset $\langle A, \sqsubseteq, \mathbf{1}, \mathbf{0} \rangle$ with a partial binary operation $\ominus$ on A s.t. $b \ominus a$ is defined iff $a \sqsubseteq b$, and the following conditions hold $\forall a, b, c \in A$:

(i) $a \ominus \mathbf{0} = a$.

(ii) $a \sqsubseteq b \sqsubseteq c \implies c \ominus b \sqsubseteq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

A generalized complement $'$ can be defined in any D-poset $\mathcal{A}$. It is sufficient to state $\forall a \in A$:

$$a' := \mathbf{1} \ominus a.$$

One can easily check that $\langle A, \sqsubseteq, ', \mathbf{1}, \mathbf{0} \rangle$ is an involutive bounded poset. Namely, $a'' = a$ follows from D2). Suppose $a \sqsubseteq b$. There holds: $b \sqsubseteq \mathbf{1}$. Hence, by D3) and D2): $b = \mathbf{1} \ominus (\mathbf{1} \ominus b) = \mathbf{1} \ominus b' \sqsubseteq \mathbf{1} \ominus a = a'$.

THEOREM 2.2. *Let* $\langle A, \sqsubseteq, \ominus, \mathbf{1} \rangle$ *be a D-poset. For all* $a, b \in A$ *s.t* $a \sqsubseteq b'$ *let* $a \oplus b := (b' \ominus a)'$ *and* $\mathbf{0} := \mathbf{1}'$. *Then,* $\langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ *is an unsharp orthoalgebra.*

PROOF. For the proof of U1)-U3), see [13].
U4) Suppose $\exists(a \oplus \mathbf{1})$. Then, $a \sqsubseteq \mathbf{1}'$. Hence: $a = \mathbf{0}$. ■

THEOREM 2.3. *Let* $\langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ *be an unsharp orthoalgebra. Suppose* $a \sqsubseteq b$. *By Definition 2.2(ii),* $\exists c \in A$ *s.t.* $a \oplus c$ *is defined and* $b = a \oplus c$. *We define* $b \ominus a$ *as follows:* $b \ominus a = c$. *Then, the structure* $\langle A, \sqsubseteq, \ominus, \mathbf{1} \rangle$ *is a D-poset.*

PROOF. By Theorem 2.1, $\langle A, \sqsubseteq, \mathbf{1}\rangle$ is a poset with maximum element ($\mathbf{1}$). The operation $\ominus$ is well defined. Let us suppose $a \sqsubseteq b$. Then, $\exists c \in A$ s.t. $a \oplus c$ is defined and $b = a \oplus c$. Suppose $\exists d \in A$ s.t. $d \oplus a$ is defined and $b = a \oplus d$. Then, by the cancellation law, $c = d$.
(D1) Suppose $b \ominus a$ is defined. Then, $a \sqsubseteq b$ so that $b = a \oplus c$, for a certain c. Thus, $b \ominus a = c \sqsubseteq a \oplus c = b$.
(D2) Suppose $b \ominus a$ is defined. Then $b = a \oplus c$, for a certain c. By (D1): $(b \ominus a) \sqsubseteq b$. Hence, for a certain d: $d \oplus (b \ominus a) = b$. Whence, $d \oplus (b \ominus a) = d \oplus c = b = a \oplus c$. Consequently, by cancellation, $a = d$. Thus, from $b = d \oplus (b \ominus a)$, we obtain $d = b \ominus (b \ominus a)$ and $a = b \ominus (b \ominus a)$.
(D3) Suppose $a \sqsubseteq b$ and $b \sqsubseteq c$. Then, $\exists d, e \in A$ s.t. $b = a \oplus d$ and $c = b \oplus e$. Thus, $c = (a \oplus d) \oplus e = a \oplus (d \oplus e)$. Further, $b \ominus a = d$, $c \ominus b = e$ and $c \ominus a = d \oplus e$. Hence, $c \ominus b = e \sqsubseteq d \oplus e = c \ominus a$. Then, $\exists f \in A$ s.t. $c \ominus a = f \oplus (c \ominus b) = f \oplus e$. Therefore, $(c \ominus a) \ominus (c \ominus b) = f$. Now, $c \ominus a = d \oplus e$. Thus, by the cancellation law, $f = d = b \ominus a$. ∎

COROLLARY 2.1. *$\mathcal{UO}$ and $\mathcal{DP}$ are isomorphic.*

Unsharp orthoalgebras and D-posets represent "fuzzy" generalizations of the notion of *orthoalgebras* (or *sharp orthoalgebras*), first introduced by Foulis and Randall ([7]) and furtherly investigated in [6] and [10].

DEFINITION 2.7. An *orthoalgebra* is a partial algebraic structure $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0}\rangle$, where $\mathbf{1}$ and $\mathbf{0}$ are two distinct elements of A and $\oplus$ is a partial binary operation on A which satisfies the following conditions:

(U1) *Weak commutativity*
$\exists(a \oplus b) \Longrightarrow \exists(b \oplus a)$ and $a \oplus b = b \oplus a$.

(U2) *Weak associativity*
$[\exists(b \oplus c)$ and $\exists(a \oplus (b \oplus c))] \Longrightarrow [\exists(a \oplus b)$ and $\exists((a \oplus b) \oplus c))$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c]$.

(U3) *Strong excluded middle*
For any a, there is a unique x s.t. $a \oplus x = \mathbf{1}$.

(U4) *Consistency*:
$\exists(a \oplus a) \Longrightarrow a = \mathbf{0}$.

We have seen that any unsharp orthoalgebra gives rise to an involutive bounded poset (Theorem 2.1). Orthoalgebras, instead, always determine an orthoposet. Let $\mathcal{A}$ be an orthoalgebra. The structure $\langle A, \sqsubseteq, ', \mathbf{1}, \mathbf{0}\rangle$ is an *orthoposet*. Namely, for any $a \in A$: the *infimum* of a and a' ($a \sqcap a'$) exists and is equal to $\mathbf{0}$. At the same time, the *supremum* of a and a' ($a \sqcup a'$) exists and is equal to $\mathbf{1}$.

The following theorems have been proved in [6].

THEOREM 2.4. *Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0}\rangle$ be an orthoalgebra. If $a, b \in A$ and $a \perp b$, then $a \oplus b$ is a minimal upper bound for a and b in A, i.e., $\forall c \in A$: $a, b \sqsubseteq c \sqsubseteq a \oplus b \Longrightarrow c = a \oplus b$.*

As a Corollary of Theorem 2.1, we obtain that if $a \perp b$ and $a \sqcup b$ exists, then $a \sqcup b = a \oplus b$. However, generally, $a \oplus b$ is not the supremum of a and b.

DEFINITION 2.8. An *orthomodular poset* is an orthoposet $\langle A, \sqsubseteq, ', \mathbf{1}, \mathbf{0}\rangle$ where $\forall a, b \in A$ s.t. $a \sqsubseteq b'$, $a \sqcup b$ exists in A. Further, the *orthomodular property* holds:

$$a \sqsubseteq b \implies b = a \sqcup (a \sqcup b')' .$$

Any orthomodular poset $\langle A, \sqsubseteq, ', \mathbf{1}, \mathbf{0}\rangle$ determines an orthoalgebra $\langle A, \oplus, \mathbf{1}, \mathbf{0}\rangle$, where: $a \oplus b$ is defined iff $a \sqsubseteq b'$. Further, when defined, $a \oplus b = a \sqcup b$.

DEFINITION 2.9. An orthoalgebra $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0}\rangle$ is called *orthocoherent* iff $\forall a, b, c \in A$:

$$a \perp b, a \perp c, b \perp c \implies a \perp (b \oplus c).$$

THEOREM 2.5. *Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0}\rangle$ be an orthoalgebra. The following conditions are equivalent:*

(i) $\langle A, \sqsubseteq, ', \mathbf{1}, \mathbf{0}\rangle$ *is an orthomodular poset.*

(ii) $\mathcal{A}$ *is orthocoherent.*

(iii) $\forall a, b \in A$: $a \perp b \implies a \sqcup b$ *exists.*

3. Some concrete and physical examples

In this section we will present some concrete and physical examples of unsharp orthoalgebras, orthoalgebras and orthocoherent orthoalgebras.

EXAMPLE 3.1. (*Fuzzy sets*)
Let X be a non-empty set and let $[0,1]^X$ be the class of all *fuzzy sets*, i.e., the class of all functionals

$$f : X \to [0,1]$$

A partial operation $\oplus$ can be defined on $[0,1]^X$. For any $f, g \in [0,1]^X$:

$$(1) \qquad \exists (f \oplus g) \iff \forall x \in X : f(x) + g(x) \leq 1$$

where $+$ is the usual sum on the reals.

(2) $$\exists(f \oplus g) \implies f \oplus g = f + g$$

where: $\forall x \in X$:

$$(f+g)(x) := f(x) + g(x)$$

Let $\underline{1}$ and $\underline{0}$ be the functionals defined as follows:

$$\forall x \in X: \quad \underline{1}(x) = 1, \quad \underline{0}(x) := 0.$$

The structure $\mathcal{F}(X) = \left\langle [0,1]^X, \oplus, \underline{1}, \underline{0} \right\rangle$ is an unsharp ortholgebra. It turns out that $\forall f \in [0,1]^X$:

$$f' = \underline{1} - f$$

where, $\forall x \in X$:

$$f'(x) := 1 - f(x)$$

Further, the partial order relation $\sqsubseteq$, defined according to Definition 2.2(ii), coincides with the usual partial order of fuzzy set theory:

$$f \leq g \iff \forall x \in X: \ f(x) \leq g(x)$$

The structure $\left\langle [0,1]^X, \sqsubseteq, ', \underline{1}, \underline{0} \right\rangle$ is an involutive distributive bounded lattice (called also De Morgan lattice), where $\forall x \in X$:

$$(f \sqcap g)(x) = \min_{x \in X} \{f(x), g(x)\}$$

and

$$(f \sqcup g)(x) = \max_{x \in X} \{f(x), g(x)\}.$$

However, $\mathcal{F}(X)$ is not an orthoalgebra. As a counterexample, consider the functional $\underline{\frac{1}{2}}$, defined as follows: $\forall x \in X$: $\underline{\frac{1}{2}}(x) := \frac{1}{2}$. It turns out that $\exists \left(\underline{\frac{1}{2}} \oplus \underline{\frac{1}{2}} \right)$; but $\underline{\frac{1}{2}} \neq \underline{0}$.

A standard Hilbert-space exemplification of an unsharp orthoalgebra can be constructed with the class $E(\mathcal{H})$ of all *effects* in a Hilbert space $\mathcal{H}$. From an intuitive point of view, effects represent a kind of *maximal* possible notion of *experimental property*, in agreement with the probabilistic rules of quantum mechanics. Mathematically, an effect is a linear bounded operator E s.t. for any statistical operator W, $Tr(WE) \in [0,1]$. In other words, effects are all and only the linear bounded operators for which a Born probability can be conveniently defined. Any projection is an effect, but not viceversa. An important intuitive difference between projections and proper effects is the following: effects can be associated to unsharp properties asserting that "the value for a given observable lies in a certain *fuzzy* Borel set" ([2]). As a consequence, there are effects E (different from the null projection) which are verified with certainty by no state (for any state W: $Tr(WE) \neq 1$). A limit case is represented by the *semitransparent effect* $\frac{1}{2}\mathbb{I}$ (where $\mathbb{I}$ is the identity operator) to which any state W assigns probability-value $\frac{1}{2}$.

EXAMPLE 3.2. (*Effects of a Hilbert space*)
Let $E(\mathcal{H})$ be the class of all effects of a Hilbert space $\mathcal{H}$. Let us introduce on $E(\mathcal{H})$ a partial sum $\oplus$ in the following way. For any $E, F \in E(\mathcal{H})$:

(1) $$\exists(E \oplus G) \iff E + F \in E(\mathcal{H})$$

(in other words, the orthoalgebraic sum $E \oplus F$ is defined iff the usual operator-sum $E + F$ is an effect operator).

(2) $$\exists(E \oplus F) \implies E \oplus F = E + F$$

(if defined, the orthoalgebraic sum coincides with the usual sum).
Finally, $\mathbf{0}$ and $\mathbf{1}$ are identified with the null and the identity operator ($\mathbb{O}$ and $\mathbb{I}$, respectively). It is easy to check that the structure $\mathcal{E}(\mathcal{H}) = \langle E(\mathcal{H}), \oplus, \mathbf{1}, \mathbf{0}\rangle$ is an unsharp orthoalgebra. Differently from the case of fuzzy sets, the involutive bounded poset $\langle E(\mathcal{H}), \sqsubseteq, ', \mathbf{1}, \mathbf{0}\rangle$ is *not* a lattice.

The unsharp approach to quantum mechanics has been generalized by Bugajski ([1]) in order to provide a classical (phase-space) representation of quantum theory, by means of a "delinearization procedure". Mathematically, this generalization is based on *order-unit normed Banach spaces*, which play a fundamental role in many statistical theories.

EXAMPLE 3.3. (*Effects of an order-unit normed Banach space*)
Let $\mathcal{B}$ be an order-unit normed Banach space with e (the unit), $\underline{0}$ (the origin) and the partial order $\preceq$. Let

$$\mathcal{B}_{[\underline{0},e]} := \{a \in \mathcal{B} \mid \underline{0} \preceq a \preceq e\}$$

($\mathcal{B}_{[\underline{0},e]}$ is the mathematical representation of the set of all *effects* of the statistical theory that $\mathcal{B}$ is supposed to describe). Similarly to effects in a Hilbert space, we can introduce a partial operation $\oplus$ on $\mathcal{B}_{[\underline{0},e]}$; for any $f, g \in \mathcal{B}_{[\underline{0},e]}$:

(1) $$\exists(f \oplus g) \iff f + g \preceq \underline{e}$$

where $+$ is the restriction to $\mathcal{B}_{[\underline{0},e]}$ of the corresponding linear operation on $\mathcal{B}$.

$$(2) \qquad \exists(f \oplus g) \implies f \oplus g = f + g$$

One can prove that the structure $\left\langle \mathcal{B}_{[\underline{0},e]}, \oplus, e, \underline{0} \right\rangle$ is a partial unsharp orthoalgebra.

The smallest unsharp orthoalgebra which is not an orthoalgebra is given by the set $A = \left\{0, \frac{1}{2}, 1\right\}$ equipped with the partial operation $\oplus$ defined as follows:

$$0 \oplus 0 = 0, \quad 0 \oplus \frac{1}{2} = \frac{1}{2} \oplus 0 = \frac{1}{2}, \quad 0 \oplus 1 = 1 \oplus 0 = 1 \quad \frac{1}{2} \oplus \frac{1}{2} = 1.$$

The structure $\langle A, \oplus, 1, 0 \rangle$ is an unsharp orthoalgebra that is not an orthoalgebra. For $\frac{1}{2} \oplus \frac{1}{2}$ is defined; at the same time $\frac{1}{2} \neq 0$.

Let us now present an example of a "genuine" orthoalgebra, which is based on a particular subset of $E(\mathcal{H})$.

EXAMPLE 3.4. (*Special effects of a Hilbert space*)
Let $E(\mathcal{H})$ be the class of all effects of a Hilbert space $\mathcal{H}$. A *special effect* (*unsharp property* in Bush terminology) is an effect E s.t. for at least two density operators W_1, W_2: $Tr(W_1 E) < \frac{1}{2}$ and $Tr(W_2 E) > \frac{1}{2}$. Let $E_{se}(\mathcal{H})$ be the class of all special effects with $\mathbb{O}$ and $\mathbb{I}$. As proved in [8], $E_{se}(\mathcal{H})$ is not closed under the partial sum $\oplus$ defined on the class of all effects. Let us introduce a new partial sum $\oplus'$ on $E_{se}(\mathcal{H})$ in the following way; for any $E, F \in E_{se}(\mathcal{H})$:

$$(1) \qquad \exists(E \oplus' F) \iff (E \oplus F) \in E_{se}(\mathcal{H})$$

$$(2) \qquad \exists(E \oplus' F) \implies E \oplus' F = E \oplus F$$

It turns out that the strcture $\mathcal{E}(\mathcal{H})_{se} = \langle E_{se}(\mathcal{H}), \oplus', \mathbf{1}, \mathbf{0} \rangle$ is an orthoalgebra that is not orthocoherent ([8]). As a consequence, the orthoposet $\langle E_{se}(\mathcal{H}), \sqsubseteq, ', \mathbf{1}, \mathbf{0} \rangle$ is not orthomodular.

Finally, we will introduce two examples of orthocoherent orthoalgebras which represent a kind of "sharp counterpart" of the corresponding unsharp examples.

EXAMPLE 3.5. (*Crisp sets*)
Let $[0,1]^X$ be the class of all fuzzy sets of X (see Example 3.1) and let $P(X)$ be the class of all *characteristic functionals* (*crisp sets*) of X. In other words: $f \in P(X)$ iff $f \in [0,1]^X$ and $\forall x \in X$: $f(x) = 0$ or $f(x) = 1$. Clearly, $P(X) \subset [0,1]^X$. Let $\oplus_{P(X)}$ be the restriction to $P(X)$ of the partial sum defined on $[0,1]^X$. Then, the structure $\left\langle P(X), \oplus_{P(X)}, \underline{1}, \underline{0} \right\rangle$ is an orthocoherent orthoalgebra. Further, the orthoposet $\langle P(X), \sqsubseteq, ', \mathbf{1}, \mathbf{0} \rangle$ is a complete Boolean algebra.

EXAMPLE 3.6. (*Projections in a Hilbert space*)
Let $E(\mathcal{H})$ be the class of all effects of a Hilbert space $\mathcal{H}$ (see Example 3.2) and let $P(\mathcal{H})$ be the class of all *projections* in $\mathcal{H}$. Clearly, $P(\mathcal{H}) \subset E(\mathcal{H})$. Let $\oplus_{P(\mathcal{H})}$ be the restriction to $P(\mathcal{H})$ of the partial sum $\oplus$ defined on $E(\mathcal{H})$. Then, the structure $\mathcal{P}(\mathcal{H}) = \left\langle P(\mathcal{H}), \oplus_{P(\mathcal{H})}, \mathbf{1}, \mathbf{0} \right\rangle$ is an orthocoherent orthoalgebra. Further, the orthoposet $\langle P(\mathcal{H}), \sqsubseteq, ', \mathbf{1}, \mathbf{0} \rangle$ is a complete orthomodular lattice.

As shown by R. Wright, the smallest orthoalgebra that is not orthocoherent is given by the set

$$A = \{\mathbf{0}, \mathbf{1}, a, b, c, d, e, f, a', b', c', d', e', f'\}$$

where, apart the obvious cases, the partial sum $\oplus$ is defined as folows:

$$a \oplus b = d \oplus e = c'$$
$$b \oplus c = e \oplus f = a'$$
$$c \oplus d = f \oplus a = e'$$
$$c \oplus e = d', \ a \oplus c = b', \ e \oplus a = f'$$

Let us check that the orthoalgebra $\langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ is not orthocoherent. By Theorem 2.5, it is sufficient to find two elements a, b s.t. $a \perp b$ but $a \oplus b$ is not the supremum of a and b. Let us consider the elements $a, c \in A$. By construction, $a \oplus c = b'$. Now, $a \sqsubseteq a \oplus f = e'$ and $c \sqsubseteq c \oplus d = e'$. But $b' \not\sqsubseteq e'$; hence, $a \oplus c$ is not the supremum of a and c.

4. Partial quantum logics

How to give a semantic characterization for different forms of quantum logic, corresponding respectively to the class of all unsharp orthoalgebras, of all orthoalgebras and of all orthomodular posets? We will call these logics: *unsharp partial quantum logic* (UPaQL), *weak partial quantum logic* (WPaQL) and *strong partial quantum logic* (SPaQL).

Let us first consider the case of UPaQL, that represents the "logic of unsharp orthoalgebras". We will work in the framework of an algebraic semantics based on the following intuitive idea: interpreting a formal language essentially means transforming any *sentence* of the language into a *proposition* (or alternatively into a *generalized truth value*) represented by an element of a certain algebraic structure. The interpretation must preserve the logical form of the sentences.

Let us first introduce the language of UPaQL, whose alphabet will contain:

a) a denumerably infinite list of atomic sentences;

b) two primitive connectives: the *negation* $\neg$ and the *exclusive disjunction* $\circledvee$ (aut).

The set of sentences is defined in the usual way. Let $\alpha\,\beta, \gamma, \ldots$ represent metalinguistic variables ranging over sentences. A *conjunction* is metalinguistically defined, via De Morgan law:

$$\alpha \circledwedge \beta := \neg(\neg\alpha \circledvee \neg\beta).$$

The intuitive idea underlying our semantics for UPaQL is the following: disjunctions and conjunctions are considered "legitimate" from a mere linguistic point of view. However, semantically, a disjunction $\alpha \circledvee \beta$ will have the intended meaning only in the "well behaved cases" (where the values of α and β are orthogonal in the corresponding unsharp orthoalgebra). Otherwise, $\alpha \circledvee \beta$ will have any meaning whatsoever (generally not connected with the meanings of α and β). A similar semantic "trick" is used in some classical treatments of the description operator ι ("the unique individual satisfying a given property"; for instance, "the present king of Italy").

As is customary in the algebraic semantics, we have to define the following basic semantic notions: *model*, *truth* and *consequence in a given model*, *logical consequence* and *logical truth.*

DEFINITION 4.1. A *model* of UPaQL is a pair $\mathfrak{M} = \langle \mathcal{A}, v \rangle$, where

(a) $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ is an unsharp orthoalgebra (see Definition 2.7).

(b) v (the interpretation function) satisfies the following conditions:

$v(\alpha) \in A$, for any atomic sentence α.

$v(\neg\beta) = v(\beta)'$, where $'$ is the generalized complement defined in $\mathcal{A}$.

$$v(\beta \circledvee \gamma) = \begin{cases} v(\beta) \oplus v(\gamma), & \text{if } v(\beta) \oplus v(\gamma) \text{ is defined in } \mathcal{A} \\ \text{any element}, & \text{otherwise} \end{cases}$$

DEFINITION 4.2. *Truth* and *consequence* in a model $\mathfrak{M}$

(a) A sentence α is *true* in $\mathfrak{M}$ ($\models_{\mathfrak{M}} \alpha$) iff $v(\alpha) = \mathbf{1}$.

(b) β is a consequence of α in $\mathfrak{M}$ ($\alpha \models_{\mathfrak{M}} \beta$) iff $v(\alpha) \sqsubseteq v(\beta)$ (where $\sqsubseteq$ is the partial order of $\mathcal{A}$).

DEFINITION 4.3. *Logical truth* and *logical consequence*

(a) α is a *logical truth* of UPaQL ($\models_{\overline{\text{UPaQL}}} \alpha$) iff α is true in any model $\mathfrak{M}$.

(b) β is a *logical consequence* of α in UPaQL ($\alpha \models_{\overline{\text{UPaQL}}} \beta$) iff β is a consequence of α in any model of UPaQL.

Weak partial quantum logic (WPaQL) and strong partial quantum logic (SPaQL) (formalized in the same language as UPaQL) will be naturally characterized *mutatis mutandis*. It will be sufficient to replace, in the definition of model, the notion of unsharp orthoalgebra with the notion of orthoalgebra and of orthomodular poset (see Definition 2.7 and Definition 2.8). Of course, UPaQL is weaker than WPaQL, which is, in turn, weaker than SPaQL. Partial quantum logics are axiomatizable. We will first present a calculus for UPaQL, which is obtained as a natural transformation of an axiomatization for orthodox quantum logic ([9] and [4]).

Our calculus (that has no axioms) is determined as a set of rules. Any rule has the form:

$$\frac{\alpha_1 \vdash \beta_1, \ldots, \alpha_n \vdash \beta_n}{\alpha \vdash \beta.}$$

(If β_1 is inferred from $\alpha_1, \ldots$, β_n is inferred from α_n, then β can be inferred from α).

The configurations $\alpha_1 \vdash \beta_1, \ldots, \alpha_n \vdash \beta_n$ represent the *premises* of the rule, while $\alpha \vdash \beta$ is the *conclusion*. An *improper rule* is a rule whose set of premises is empty. Instead of $\frac{\emptyset}{\alpha \vdash \beta}$, we will write $\alpha \vdash \beta$.

Rules of UPaQL

(R1) $\alpha \vdash \alpha$ (identity)

(R2) $$\frac{\alpha \vdash \beta \quad \beta \vdash \gamma}{\alpha \vdash \gamma}$$ (transitivity)

(R3) $$\alpha \vdash \neg\neg\alpha$$ (weak double negation)

(R4) $$\neg\neg\alpha \vdash \alpha$$ (strong double negation)

(R5) $$\frac{\alpha \vdash \beta}{\neg\beta \vdash \neg\alpha}$$ (contraposition)

(R6) $$\beta \vdash \alpha \circledvee \neg\alpha$$ (excluded middle)

(R7) $$\frac{\alpha \vdash \neg\beta \quad \alpha \circledvee \neg\alpha \vdash \alpha \circledvee \beta}{\neg\alpha \vdash \beta}$$ (unicity of negation)

(R8) $$\frac{\alpha \vdash \neg\beta \quad \alpha \vdash \alpha_1 \quad \alpha_1 \vdash \alpha \quad \beta \vdash \beta_1 \quad \beta_1 \vdash \beta}{\alpha \circledvee \beta \vdash \alpha_1 \circledvee \beta_1}$$ (weak substitutivity)

(R9) $$\frac{\alpha \vdash \neg\beta}{\alpha \circledvee \beta \vdash \beta \circledvee \alpha}$$ (weak commutativity)

(R10) $$\frac{\beta \vdash \neg\gamma \quad \alpha \vdash \neg(\beta \circledvee \gamma)}{\alpha \vdash \neg\beta}$$

(R11) $$\frac{\beta \vdash \neg\gamma \quad \alpha \vdash \neg(\beta \circledvee \gamma)}{\alpha \circledvee \beta \vdash \neg\gamma}$$

(R12) $$\frac{\beta \vdash \neg\gamma \quad \alpha \vdash \neg(\beta \circledvee \gamma)}{\alpha \circledvee (\beta \circledvee \gamma) \vdash (\alpha \circledvee \beta) \circledvee \gamma}$$

(R13) $$\frac{\beta \vdash \neg\gamma \quad \alpha \vdash \neg(\beta \circledvee \gamma)}{(\alpha \circledvee \beta) \circledvee \gamma) \vdash \alpha \circledvee (\beta \circledvee \gamma)}$$

(R10)-R13) represent a weak associativity condition)

DEFINITION 4.4. A *proof* is a finite sequence of configurations $\alpha \vdash \beta$ where any element of the sequence is either an improper rule or the conclusion of a proper rule whose premises are previous elements of the sequence.

DEFINITION 4.5. β is a *provable* from α ($\alpha \vdash_{\text{UPaQL}} \beta$) iff there is a proof whose last configuration is $\alpha \vdash \beta$.

In order to axiomatize weak partial quantum logic (WPaQL) it is sufficient to add a rule, which requires an *absurdum principle*

$$\text{(R14)} \qquad \frac{\alpha \vdash \neg\alpha}{\alpha \vdash \beta} \qquad \text{(absurdum)}$$

Clearly, R14) corresponds to the axiom 4 (the consistency condition) of our definition of orthoalgebra (see Definition 2.7).

Finally, an axiomatization of strong partial quantum logic (SPaQL) can be obtained, by adding the following rule to R1)-R14).

$$\text{(R15)} \qquad \frac{\alpha \vdash \neg\beta \quad \alpha \vdash \gamma \quad \beta \vdash \gamma}{\alpha \ovee \beta \vdash \gamma}$$

In other words, R15) requires that the disjunction $\ovee$ behaves like a supremum, whenever it has the "right meaning".

Let PaQL represent any instance of our three calculi. We will use the following abbreviations. Instead of $\alpha \vdash_{\text{PaQL}} \beta$ we will write $\alpha \vdash \beta$. When α and β are logically equivalent ($\alpha \vdash \beta$ and $\beta \vdash \alpha$) we will write $\alpha \equiv \beta$.

Let p represent a particular sentential letter of the language: $\mathbf{T}$ will be an abbreviation for $p \ovee \neg p$; whereas $\mathbf{F}$ will be an abbreviation for $\neg(p \ovee \neg p)$.

Some important derivable rules of all calculi are the following

$$\text{(D1)} \qquad \mathbf{F} \vdash \beta, \; \beta \vdash \mathbf{T} \qquad \text{(Duns Scoto)}$$

$$\text{(D2)} \qquad \frac{\alpha \vdash \neg\beta}{\alpha \vdash \alpha \ovee \beta} \qquad \text{(The weak sup rule)}$$

$$\text{(D3)} \qquad \frac{\alpha \vdash \beta}{\beta \equiv \alpha \ovee \neg(\alpha \ovee \neg\beta)} \qquad \text{(Orthomodularity)}$$

$$\text{(D4)} \qquad \frac{\alpha \vdash \neg\gamma \quad \beta \vdash \neg\gamma \quad \alpha \ovee \gamma \equiv \beta \ovee \gamma}{\alpha \equiv \beta} \qquad \text{(Cancellation)}$$

As a consequence, the following syntactical lemma holds:

LEMMA 4.1. *For any* α, β: $\alpha \vdash \beta$ *iff there exists a formula* γ *s.t.*

(i) $\alpha \vdash \neg\gamma$;

(ii) $\beta \equiv \alpha \circledV \gamma$.

In other words, the logical implication behaves similarly to the partial order relation in the (sharp and unsharp) orthoalgebras.

The following derivable rule holds for WPaQL and for SPaQL:

$$\text{(D5)} \qquad \frac{\alpha \vdash \neg\beta \quad \alpha \vdash \gamma \quad \beta \vdash \gamma \quad \gamma \vdash \alpha \circledV \beta}{\alpha \circledV \beta \vdash \gamma}$$

5. Soundness and Completeness

Our calculi turn out to be adequate with respect to the corresponding semantic characterizations. Soundness proofs are straightforward. Let us sketch the proof of the completeness theorem for our weakest calculus (UPaQL).

THEOREM 5.1. *Completeness*
For any formula α, β:

$$\alpha \models \beta \implies \alpha \vdash \beta$$

PROOF. Following a standard procedure, it is sufficient to construct a canonical model $\mathfrak{M} = \langle \mathcal{A}, v \rangle$ s.t. for any formulas α, β:

$$\alpha \vdash \beta \iff \alpha \models_{\mathfrak{M}} \beta.$$

DEFINITION OF THE CANONICAL MODEL

1) The algebra $\mathcal{M} = \langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ is determined as follows:

   1.1) A is the class of all equivalence classes of logically equivalent formulas: $A := \{[\alpha]_\equiv \mid \alpha \text{ is a formula}\}$. (In the following, we will write $[\alpha]$ instead of $[\alpha]_\equiv$).

   1.2) $[\alpha] \oplus [\beta]$ is defined iff $\alpha \vdash \neg\beta$. If defined, $[\alpha] \oplus [\beta] := [\alpha \circledV \beta]$.

   1.3) $\mathbf{1} := [\mathbf{T}]$; $\mathbf{0} := [\mathbf{F}]$.

2) The interpretation function v is defined as follows: $v(\alpha) = [\alpha]$.

One can easily check that $\mathfrak{M}$ is a "good" model for our logic. The operation $\oplus$ is well defined (by the transitivity, contraposition and weak substitutivity rules). Further, $\mathcal{A}$ is an unsharp orthoalgebra: $\oplus$ is weakly commutative and weakly associative, because of the corresponding rules of our calculus. The strong excluded middle axiom holds by definition of $\oplus$ and in virtue of the following rules: excluded middle, unicity of negation, double negation. Finally, the weak consistency axiom holds by Duns Scoto (D1) and by definition of $\oplus$.

LEMMA OF THE CANONICAL MODEL

For any α, β:

$$[\alpha] \sqsubseteq [\beta] \iff \alpha \vdash \beta.$$

SKETCH OF THE PROOF. By definition of $\sqsubseteq$ (in an unsharp orthoalgebra) one has to prove:

$$\alpha \vdash \beta \iff \text{for a given } \gamma \text{ s.t. } [\alpha] \perp [\gamma] : \; [\alpha] \oplus [\gamma] = [\beta].$$

This equivalence holds by Lemma 4.1 and by definition of $\oplus$.

Finally, let us check that v is a "good" interpretation function. In other words:

1) $v(\neg\beta) = v(\beta)'$

2) $v(\beta \circledvee \gamma) = v(\beta) \oplus v(\gamma)$, if $v(\beta) \oplus v(\gamma)$ is defined.

Proof of 1) By definition of v, we have to show that $[\neg\beta]$ is the unique $[\gamma]$ s.t. $[\beta] \oplus [\gamma] = \mathbf{1} := [\mathbf{T}]$. In other words,

1.1) $[\mathbf{T}] \sqsubseteq [\beta] \oplus [\neg\beta]$.

1.2) $[\mathbf{T}] \sqsubseteq [\beta] \oplus [\gamma] \implies \neg\beta \equiv \gamma$.

This holds by the Lemma of the canonical model, by definition of $\oplus$ and by the following rules: double negation, excluded middle, unicity of negation.
Proof of 2) Suppose $v(\beta) \oplus v(\gamma)$ is defined. Then $\beta \vdash \neg\gamma$. Hence, by definition of $\oplus$ and of v: $v(\beta) \oplus v(\gamma) = [\beta] \oplus [\gamma] = [\beta \circledvee \gamma] = v(\beta \circledvee \gamma)$.

As a consequence, we obtain:

$$\alpha \vdash \beta \iff [\alpha] \sqsubseteq [\beta] \iff v(\alpha) \sqsubseteq v(\beta) \iff \alpha \models_{\mathfrak{M}} \beta$$

■

The completeness argument can be easily transformed, *mutatis mutandis* for the case of weak and strong partial quantum logic.

References

[1] S. BUGAJSKI, 'Nonlinear quantun mechanics is a classical theory', *International Journal of Theoretical Physics* **30**, 1991 , 961–971.

[2] G. CATTANEO and F. LAUDISA, 'Axiomatic quantum mechanics', *Foundations of Physics* **24**, 1994, 631–681.

[3] J. CZELAKOWSKI, 'Logics based on partial Boolean σ-algebras', *Studia Logica* **33**, 1974, 371–396.

[4] M.L. DALLA CHIARA, ' Quantum logic', (ed.)s D. Gabbay, F. Guenthner, in: *Handbook of Philosophical Logic* **III**, Reidel, Dordrecht 1986.

[5] M. L. DALLA CHIARA and R. GIUNTINI, 'Unsharp quantum logics', *Foundations of Physics* **24**, 1994, 1161–1177.

[6] D.J. FOULIS, R.J. GREECHIE and G.T. RÜTTIMANN, 'Filters and supports in orthoalgebras', *International Journal of Theoretical Physics* **31**, 1992 , 787–807.

[7] D.J. FOULIS and C. RANDALL, 'Empirical logics', (ed.) H. Neumann, in: *Interpretations and Foundations of Quantum Mechanics,* Bibliographisches Institut, Mannheim 1981.

[8] R. GIUNTINI and H. GREULING, 'Toward a formal language for unsharp properties', *Foundations of Physics* **20**, 1989, 931–935.

[9] R. GOLDBLATT, 'Semantic analysis of orthologic', *Journal of Philosophical Logic* **2**, 1974, 19–35.

[10] G. HARDEGREE and P. LOCK, 'Connections among quantum logics', *International Journal of Theoretical Physics* **24**, 1984, 43–53.

[11] S. KOCHEN and E. SPECKER, 'The calculus of partial propositional functions', (ed.) Y. Bar-Hillen, in: *Logic, Methodology and the Philosophy of Science,* North-Holland, Amsterdam.

[12] F. KÔPKA and F. CHOVANEC, 'D-posets', *Mathematica Slovaca* **44**, 1994 , 21–34.

[13] M. NAVARA and P. PTÁK, 'Difference posets and orthoalgebras', (submitted).

DIPARTIMENTO DI FILOSOFIA, UNIVERSITÀ DI FIRENZE
VIA BOLOGNESE 52, 50139 FIRENZE, ITALY
E-MAIL: {GIUNTINI, DACHIARA} @ MAILSERVER.IDG.FI.CNR.IT