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Cut–Free Sequent Calculi for Some Tense Logics

Abstract. We introduce certain enhanced systems of sequent calculi for tense logics, and prove their completeness with respect to Kripke–type semantics.

1. Introduction

In this paper, we consider propositional tense logics which have four modal (tense) connectives $\Box^P$, $\Box^F$, $\Diamond^P$ and $\Diamond^F$ representing "at all past times", "at all future times", "at some past times" and "at some future times", respectively. First we define some tense logics by considering conditions "transitive", "reflexive", "connected" and "total" on Kripke–type models. Then we present extended Gentzen–type sequent calculi for these logics. We prove the completeness theorem for our sequent calculi without cut, and as its corollary, we show the cut–elimination theorem for them.

The novelty of our systems is that the sequents we deal with are not the usual ones but they are "nested"[1]. So at first sight it may look a little complicated, but in this way we can get cut–free systems. (To the author's knowledge, there has been no other cut–free sequent calculus for such tense logics.)

In Section 2, we define our formulas, Kripke–type models, and eight tense logics. In Section 3, we present eight sequent calculi for the eight tense logics, and prove the soundness theorem for them. In Section 4, we introduce eight other sequent calculi for the eight tense logics, and prove the completeness theorem for the cut–free parts of them. Moreover we prove that those calculi are equivalent to the sequent calculi in the previous section. The cut–elimination theorem is shown as their corollary. The difference between the sequent calculi in the two sections is that the inference rules in Section 3 operate always on the root of a nested sequent, and inference rules in Section 4 operate on all the nodes in a nested sequent. Due to this difference, the former is suitable for proving the soundness theorem, and the latter for the completeness theorem. In Section 5, we give a remark on the relation

[1] This is a refinement of the method invented by Sato [3] for modal logic S5. This technique is independently discovered by Bull [1] for dynamic logic.

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between our systems and semantic tableaux, and also on the inference rule "contraction".

2. Tense logics

Our language consists of the following symbols:

- propositional variables: $v_0, v_1, \ldots$;
- logical connectives: $\wedge, \vee, \neg$;
- modal connectives: $\Box^P, \Box^F, \Diamond^P, \Diamond^F$;
- auxiliary symbols: $(,)$, ${}^P\{$, ${}^F\{$, $\}$, comma.

We define *formulas* of our systems as follows:

1. If v is a propositional variable, then v and $\neg v$ are formulas;
2. If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$, $\Box^P A$, $\Box^F A$, $\Diamond^P A$, and $\Diamond^F A$ are formulas.

Note that negation symbol $\neg$ is allowed only in front of propositional variables. We will use letters $u, v, \ldots$ for propositional variables and $A, B, \ldots$ for formulas.

A *model* is a triple $\langle T, R, V \rangle$ such that:

1. T is a non–empty set (of moments of time);
2. R is a binary relation on T (the "earlier–later relation");
3. V is a function assigning to each propositional variable v a subset $V(v)$ of T.

For a formula A, a model $\mathcal{M} = \langle T, R, V \rangle$ and a time t in T, the relation "A is true at t in M", denoted

$$\mathcal{M} \models_t A,$$

is defined inductively as follows:

- $\mathcal{M} \models_t v \Leftrightarrow t \in V(v)$;
- $\mathcal{M} \models_t \neg v \Leftrightarrow t \notin V(v)$;
- $\mathcal{M} \models_t (A \wedge B) \Leftrightarrow \mathcal{M} \models_t A$ and $\mathcal{M} \models_t B$;
- $\mathcal{M} \models_t (A \vee B) \Leftrightarrow \mathcal{M} \models_t A$ or $\mathcal{M} \models_t B$;

- $\mathcal{M} \models_t \Box^P A \Leftrightarrow (sRt \Rightarrow \mathcal{M} \models_s A)$ for all $s \in T$;
- $\mathcal{M} \models_t \Box^F A \Leftrightarrow (tRs \Rightarrow \mathcal{M} \models_s A)$ for all $s \in T$;
- $\mathcal{M} \models_t \Diamond^P A \Leftrightarrow (sRt$ and $\mathcal{M} \models_s A)$ for some $s \in T$;
- $\mathcal{M} \models_t \Diamond^F A \Leftrightarrow (tRs$ and $\mathcal{M} \models_s A)$ for some $s \in T$.

If $\mathcal{M} \models_t A$ holds for every t in T, then we write $\mathcal{M} \models A$.

We consider the following four conditions Tr, Re, Co and To on a model $\langle T, R, V\rangle$:

Tr : $((sRt$ and $tRu) \Rightarrow sRu)$ for all s, t, u in T. (Transitive)
Re : (tRt) for all t in T. (Reflexive)
Co : $(sRt$ or tRs or $s = t)$ for all s, t in T. (Connected)
To : $(sRt$ or $tRs)$ for all s, t in T. (Total)

Then we define eight tense logics K_t, K_tTr, K_tRe, K_tCo, K_tTo, K_tTrRe, K_tTrCo, and K_tTrTo, as the sets of formulas:

$K_t = \{A|\mathcal{M} \models A$ for every model $\mathcal{M}\}$;
$K_tTr = \{A|\mathcal{M} \models A$ for every model $\mathcal{M}$ which satisfies the condition $Tr\}$;

⋮

$K_tTrTo = \{A|\mathcal{M} \models A$ for every model $\mathcal{M}$ which satisfies the conditions Tr and $To\}$.

Note that the condition To is equivalent to (Re and Co). Hence, by those four conditions, we can characterize only the eight logics above. (For example, $K_tReCoTo = K_tTo$.)

3. Sequent calculi

Sequents are defined inductively as follows:

1. A formula is a sequent;
2. If Γ is a sequent then ${}^P\{\Gamma\}$ and ${}^F\{\Gamma\}$ are sequents;
3. If $n \geqslant 0$ and each Γ_i $(1 \leqslant i \leqslant n)$ is a sequent, then the sequence $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is a sequent.

In the following, we will use letters $\Gamma, \Delta, \ldots$ for sequents.

For a sequent Γ, we define the "meaning formula" Γ^* inductively as follows:

1. If $\Gamma = A$, then $\Gamma^* = A$;
2. $({}^P\{\Delta\})^* = \Box^P \Delta^*$, and $({}^F\{\Delta\})^* = \Box^F \Delta^*$;
3. If $n > 0$, then $(\Gamma_1, \Gamma_2, \ldots, \Gamma_n)^* = \Gamma_1^* \vee \Gamma_2^* \vee \cdots \vee \Gamma_n^*$. If $n = 0$, then $(\,)^* = (v_0 \wedge \neg v_0)$.

Suppose that there is exactly one occurrence of the propositional variable v_0 in a sequent Γ, and v_0 is not a subformula of any other formulas. Then by $\Gamma[\Delta]$, we will mean the sequent obtained from Γ by replacing v_0 by the sequent Δ. Also, by $\Gamma[\Delta_0; \Delta_1]$, we will mean the sequent obtained from Γ by replacing v_i by Δ_i ($i = 0, 1$) provided that both v_0 and v_1 satisfy the above condition stated for v_0.

To denote a sequence of formulas, for example, $A_1, A_2, \ldots, A_n$ ($n \geqslant 0$), we will write $\overline{A}$; and then $\overline{\Diamond^P A}$ will mean the sequence $\Diamond^P A_1, \Diamond^P A_2, \ldots, \Diamond^P A_n$.

Now we define sequent calculi for our tense logics.

First we define the basic system SK_t.

The axioms in SK_t: $\qquad v, \neg v$

The inference rules in SK_t: "exchange" — "cut" as follows:

$$\frac{\Gamma, \Delta, \Pi, \Sigma}{\Gamma, \Pi, \Delta, \Sigma}\text{ exchange} \qquad \frac{\Gamma}{\Gamma, \Delta}\text{ weakening} \qquad \frac{\Gamma, A, A}{\Gamma, A}\text{ contraction}$$

$$\frac{\Gamma, {}^F\{\Delta\}}{{}^P\{\Gamma\}, \Delta}\text{ turn} \qquad \frac{\Gamma, {}^P\{\Delta\}}{{}^F\{\Gamma\}, \Delta}\text{ turn}$$

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, (A \wedge B)}\wedge \qquad \frac{\Gamma, A, B}{\Gamma, (A \vee B)}\vee$$

$$\frac{\Gamma, {}^P\{A\}}{\Gamma, \Box^P A}\Box^P \qquad \frac{\Gamma, {}^F\{A\}}{\Gamma, \Box^F A}\Box^F$$

$$\frac{\Gamma, {}^P\{\Delta, A\}}{\Gamma\, {}^P\{\Delta\}, \Diamond^P A}\Diamond^P \qquad \frac{\Gamma, {}^F\{\Delta, A\}}{\Gamma\, {}^F\{\Delta\}, \Diamond^F A}\Diamond^F$$

$$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}\text{ cut}$$

where $\neg A$ is inductively defined as: $\neg\neg v = v$, $\neg(A \wedge B) = (\neg A \vee \neg B)$, $\neg(A \vee B) = (\neg A \wedge \neg B)$, $\neg\Box^P A = \Diamond^P \neg A$, $\neg\Box^F A = \Diamond^F \neg A$, $\neg\Diamond^P A = \Box^P \neg A$,

and $\neg\Diamond^F A = \Box^F \neg A$.
Now we consider the following inference rules:

$$\frac{\Gamma,\ {}^P\{\Delta, \Diamond^P A\}}{\Gamma,\ {}^P\{\Delta\}, \Diamond^P A}\Diamond^P Tr \qquad \frac{\Gamma,\ {}^F\{\Delta, \Diamond^F A\}}{\Gamma,\ {}^F\{\Delta\}, \Diamond^F A}\Diamond^F Tr$$

$$\frac{\Gamma, A}{\Gamma, \Diamond^P A}\Diamond^P Re \qquad \frac{\Gamma, A}{\Gamma, \Diamond^F A}\Diamond^F Re$$

$$\frac{\Gamma[\overline{A}], \overline{B} \qquad \Gamma[\overline{C}], \overline{D} \qquad \Gamma[\overline{E}], \overline{F}}{\Gamma[\overline{\Diamond^F B}, \overline{\Diamond^P D}, \overline{F}], \overline{\Diamond^P A}, \overline{\Diamond^F C}, \overline{E}}Co$$

$$\frac{\Gamma[\overline{A}, \overline{\Diamond^P A}], \overline{B}, \overline{\Diamond^F B} \qquad \Gamma[\overline{C}, \overline{\Diamond^F C}], \overline{D}, \overline{\Diamond^P D} \qquad \Gamma[\overline{E}], \overline{F}}{\Gamma[\overline{\Diamond^F B}, \overline{\Diamond^P D}, \overline{F}], \overline{\Diamond^P A}, \overline{\Diamond^F C}, \overline{E}}TrCo$$

$$\frac{\Gamma[\overline{A}], \overline{B} \qquad \Gamma[\overline{C}], \overline{D}}{\Gamma[\overline{\Diamond^F B}, \overline{\Diamond^P D}], \overline{\Diamond^P A}, \overline{\Diamond^F C}}To$$

$$\frac{\Gamma[\overline{A}, \overline{\Diamond^P A}], \overline{B}, \overline{\Diamond^F B} \qquad \Gamma[\overline{C}, \overline{\Diamond^F C}], \overline{D}, \overline{\Diamond^P D}}{\Gamma[\overline{\Diamond^F B}, \overline{\Diamond^P D}], \overline{\Diamond^P A}, \overline{\Diamond^F C}}TrTo$$

Then sequent calculi SK_tTr, SK_tRe, SK_tCo, SK_tTo, SK_tTrRe, SK_tTrCo, and SK_tTrTo are defined as the systems obtained from SK_t by adding the inference rules according to Table 1.

	$\Diamond^P Tr$ $\Diamond^F Tr$	$\Diamond^P Re$ $\Diamond^F Re$	Co	$TrCo$	To	$TrTo$
SK_t						
SK_tTr	○					
SK_tRe		○				
SK_tCo			○			
SK_tTo					○	
SK_tTrRe	○	○				
SK_tTrCo	○			○		
SK_tTrTo	○					○

Table 1

For example, $SK_tTrCo = SK_t + \Diamond^P Tr + \Diamond^F Tr + TrCo$.

EXAMPLE OF PROOF IN SK_tTrCo:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v,\neg v}{v,\neg v,\Diamond^P\neg v,{}^F\{\}}\text{w.}}{{}^P\{v,\neg v,\Diamond^P\neg v\}}\text{turn}}{{}^P\{v,\neg v,\Diamond^P\neg v\},{}^P\{\}}\text{w.}}{{}^F\{{}^P\{v,\neg v,\Diamond^P\neg v\}\}}\text{turn}\qquad \dfrac{\dfrac{\dfrac{\dfrac{v,\neg v}{v,\neg v,\Diamond^F\neg v,{}^F\{\}}\text{w.}}{{}^P\{v,\neg v,\Diamond^F\neg v\}}\text{turn}}{{}^P\{v,\neg v,\Diamond^F\neg v\},{}^P\{\}}\text{w.}}{{}^F\{{}^P\{v,\neg v,\Diamond^F\neg v\}\}}\text{turn}\qquad \dfrac{\dfrac{\dfrac{\dfrac{v,\neg v}{v,\neg v,{}^F\{\}}\text{w.}}{{}^P\{v,\neg v\}}\text{turn}}{{}^P\{v,\neg v\},{}^P\{\}}\text{w.}}{{}^F\{{}^P\{v,\neg v\}\}}\text{turn}}{{}^F\{{}^P\{v\}\},\Diamond^P\neg v,\Diamond^F\neg v,\neg v}\text{TrCo}}{\Diamond^P\neg v,\Diamond^F\neg v,\neg v,{}^F\{{}^P\{v\}\}}\text{e.}}{{}^P\{\Diamond^P\neg v,\Diamond^F\neg v,\neg v\},{}^P\{v\}}\text{turn}}{{}^P\{\Diamond^P\neg v,\Diamond^F\neg v,\neg v\},\Box^P v}\Box^P$$

$$\vdots$$

$$\dfrac{\Diamond^P\neg v,\Diamond^F\neg v,\neg v,{}^F\{\Box^P v\}}{\Diamond^P\neg v,\Diamond^F\neg v,\neg v,\Box^F\Box^P v}\Box^F$$

$$\vdots$$

$$\Diamond^P\neg v\lor\Diamond^F\neg v\lor\neg v\lor\Box^F\Box^P v$$

THEOREM 3.1. (SOUNDNESS THEOREM) *Let $\mathcal{L}$ be arbitrary one of the eight logics. If Γ is provable in $S\mathcal{L}$, then $\Gamma^* \in \mathcal{L}$.*

PROOF. We prove the following: *Let $\mathcal{R}$ be an inference rule in $S\mathcal{L}$ where $\Pi_1,\ldots,\Pi_n$ are its upper–sequents* (n=1,2 *or* 3), *Σ is its lower–sequent, and $\Sigma^* \notin \mathcal{L}$ holds. Then $\Pi_i^* \notin \mathcal{L}$ holds for some i.*

We distinguish cases according to $\mathcal{R}$, and show only the following cases.

Case 1 $\mathcal{R}$ is

$$\dfrac{\Gamma,{}^F\{\Delta\}}{{}^P\{\Gamma\},\Delta}\text{turn}$$

If $({}^P\{\Gamma\},\Delta)^* \notin \mathcal{L}$, then there is a model $\mathcal{M} = \langle T,R,V\rangle$ such that $\mathcal{M} \not\models_t \Box^P\Gamma^* \lor \Delta^*$ (i.e. $\mathcal{M} \not\models_t \Box^P\Gamma^*$ and $\mathcal{M} \not\models_t \Delta^*$) for some $t \in T$. This leads $\mathcal{M} \not\models_s \Gamma^*$ and $\mathcal{M} \not\models_s \Box^F\Delta^*$ for some $s \in T$ such that sRt. Therefore we have $(\Gamma,{}^F\{\Delta\})^* \notin \mathcal{L}$.

Case 2 $\mathcal{R}$ is $TrCo$, and of the form

$$\dfrac{\Phi_1,\Lambda,B,\Diamond^F B \qquad \Phi_2,\Lambda,D,\Diamond^P D \qquad \Phi_3,\Lambda,F}{{}^P\{{}^F\{\Diamond^F B,\Diamond^P D,F,\Gamma\},\Delta\},\Lambda,\Diamond^P A,\Diamond^F C,E}TrCo,$$

where:

$\Phi_1 = {}^P\{{}^F\{A, \diamond^P A, \Gamma\}\Delta\}$;

$\Phi_2 = {}^P\{{}^F\{C, \diamond^F C, \Gamma\}\Delta\}$;

$\Phi_3 = {}^P\{{}^F\{E, \Gamma\}\Delta\}$.

If $\Sigma^* \notin \mathcal{L}$, ($\Sigma$ is this lower–sequent), then there is a model $\mathcal{M} = \langle T, R, V\rangle$ such that

(1) $\mathcal{M}$ satisfies the conditions Tr and Co;

(2) $\mathcal{M} \not\models_t \Sigma^*$ for some $t \in T$.

By (2), we have

- $\mathcal{M} \not\models_t \Box^P(\Box^F(\diamond^F B \vee \diamond^P D \vee F \vee \Gamma^*) \vee \Delta^*), \mathcal{M} \not\models_t \Lambda^*, \mathcal{M} \not\models_t \diamond^P A$, $\mathcal{M} \not\models_t \diamond^F C, \mathcal{M} \not\models_t E$;
- $\mathcal{M} \not\models_{t'} \Box^F(\diamond^F B \vee \diamond^P D \vee F \vee \Gamma^*), \mathcal{M} \not\models_{t'} \Delta^*$;
- $\mathcal{M} \not\models_{t''} \diamond^F B, \mathcal{M} \not\models_{t''} \diamond^P D, \mathcal{M} \not\models_{t''} F, \mathcal{M} \not\models_{t''} \Gamma^*$;

for some t', $t'' \in T$ such that $t'Rt$ and $t'Rt''$. By the condition Co, one of (tRt''), $(t''Rt)$ and $(t = t'')$ holds. If (tRt'') holds, then we have $\mathcal{M} \not\models_t \Box^P(\Box^F(C \vee \diamond^F C \vee \Gamma^*) \vee \Delta^*) \vee \Lambda^* \vee D \vee \diamond^P D$, and hence $\Pi_2^* \notin \mathcal{L}$. The condition Tr is used for showing $\mathcal{M} \not\models_{t''} \diamond^F C$ and $\mathcal{M} \not\models_t \diamond^P D$.
Similarly we have $\Pi_1^* \notin \mathcal{L}$ if $t''Rt$ and $\Pi_3^* \notin \mathcal{L}$ if $t = t''$. ■

4. Completeness and cut–elimination

To prove the completeness theorem for our systems, we introduce eight other sequent calculi $S2K_t$, $S2K_tTr$, $S2K_tRe$, $S2K_tCo$, $S2K_tTo$, $S2K_tTrRe$, $S2K_tTrCo$, and $S2K_tTrTo$.

First we define $S2K_t$.

The axioms in $S2K_t$:

$$\Gamma[v, \neg v]$$

The inference rules in $S2K_t$: "exchange" — "cut" as follows:

$$\frac{\Gamma[\Pi, \Sigma]}{\Gamma[\Sigma, \Pi]}\text{ exchange} \qquad \frac{\Gamma[A, A]}{\Gamma[A]}\text{ contraction}$$

$$\frac{\Gamma[A] \qquad \Gamma[B]}{\Gamma[A \wedge B]}\wedge \qquad \frac{\Gamma[A, B]}{\Gamma[A \vee B]}\vee$$

$$\frac{\Gamma[{}^{P}\{A\}]}{\Gamma[\Box^{P}A]}\ \Box^{P} \qquad \frac{\Gamma[{}^{F}\{A\}]}{\Gamma[\Box^{F}A]}\ \Box^{F}$$

$$\frac{\Gamma[{}^{P}\{\Delta, A\}]}{\Gamma[{}^{P}\{\Delta\}, \Diamond^{P}A]}\ \Diamond^{P}\text{ exit} \qquad \frac{\Gamma[{}^{F}\{\Delta, A\}]}{\Gamma[{}^{F}\{\Delta\}, \Diamond^{F}A]}\ \Diamond^{F}\text{ exit}$$

$$\frac{\Gamma[{}^{F}\{\Delta\}, A]}{\Gamma[{}^{F}\{\Delta, \Diamond^{P}A\}]}\ \Diamond^{P}\text{ enter} \qquad \frac{\Gamma[{}^{P}\{\Delta\}, A]}{\Gamma[{}^{P}\{\Delta, \Diamond^{F}A\}]}\ \Diamond^{F}\text{ enter}$$

$$\frac{\Gamma[A] \qquad \Gamma[\neg A]}{\Gamma[\,]}\ \text{cut}$$

Now we consider the following inference rules.

$$\frac{\Gamma[{}^{P}\{\Delta, \Diamond^{P}A\}]}{\Gamma[{}^{P}\{\Delta\}, \Diamond^{P}A]}\ \Diamond^{P}\text{ exit } Tr \qquad \frac{\Gamma[{}^{F}\{\Delta, \Diamond^{F}A\}]}{\Gamma[{}^{F}\{\Delta\}, \Diamond^{F}A]}\ \Diamond^{F}\text{ exit } Tr$$

$$\frac{\Gamma[{}^{F}\{\Delta\}, \Diamond^{P}A]}{\Gamma[{}^{F}\{\Delta, \Diamond^{P}A\}]}\ \Diamond^{P}\text{ enter } Tr \qquad \frac{\Gamma[{}^{P}\{\Delta\}, \Diamond^{F}A]}{\Gamma[{}^{P}\{\Delta, \Diamond^{F}A\}]}\ \Diamond^{F}\text{ enter } Tr$$

$$\frac{\Gamma[A]}{\Gamma[\Diamond^{P}A]}\ \Diamond^{P}Re \qquad \frac{\Gamma[A]}{\Gamma[\Diamond^{F}A]}\ \Diamond^{F}Re$$

$$\frac{\Gamma[\overline{A};\overline{B}] \qquad \Gamma[\overline{C};\overline{D}] \qquad \Gamma[\overline{E};\overline{F}]}{\Gamma[\overline{\Diamond^{F}B}, \overline{\Diamond^{P}D}, \overline{F}; \overline{\Diamond^{P}A}, \overline{\Diamond^{F}C}, \overline{E}]}\ Co$$

$$\frac{\Gamma[\overline{A}, \overline{\Diamond^{P}A}; \overline{B}, \overline{\Diamond^{F}B}] \qquad \Gamma[\overline{C}, \overline{\Diamond^{F}C}; \overline{D}, \overline{\Diamond^{P}D}] \qquad \Gamma[\overline{E};\overline{F}]}{\Gamma[\overline{\Diamond^{F}B}, \overline{\Diamond^{P}D}, \overline{F}; \overline{\Diamond^{P}A}, \overline{\Diamond^{F}C}, \overline{E}]}\ TrCo$$

$$\frac{\Gamma[\overline{A};\overline{B}] \qquad \Gamma[\overline{C};\overline{D}]}{\Gamma[\overline{\Diamond^{F}B}, \overline{\Diamond^{P}D}; \overline{\Diamond^{P}A}, \overline{\Diamond^{F}C}]}\ To$$

$$\frac{\Gamma[\overline{A}, \overline{\Diamond^{P}A}; \overline{B}, \overline{\Diamond^{F}B}] \qquad \Gamma[\overline{C}, \overline{\Diamond^{F}C}; \overline{D}, \overline{\Diamond^{P}D}]}{\Gamma[\overline{\Diamond^{F}B}, \overline{\Diamond^{P}D}; \overline{\Diamond^{P}A}, \overline{\Diamond^{F}C}]}\ TrTo$$

Then sequent calculi $S2K_tTr$, $S2K_tRe$, $S2K_tCo$, $S2K_tTo$, $S2K_tTrRe$, $S2K_tTrCo$, and $S2K_tTrTo$ are defined as the systems obtained from $S2K_t$ by adding the corresponding inference rules according to Table 1 where $(\Diamond^{P}Tr, \Diamond^{F}Tr)$ corresponds to ($\Diamond^{P}$ exit Tr, $\Diamond^{F}$ exit Tr, $\Diamond^{P}$ enter Tr, $\Diamond^{F}$ enter Tr). For example, $S2K_tTrCo = S2K_t + \Diamond^{P}$ exit $Tr + \Diamond^{F}$exit $Tr +$ $\Diamond^{P}$enter $Tr + \Diamond^{F}$enter $Tr + TrCo$.

LEMMA 4.1. *Let $\mathcal{L}$ be arbitrary one of the eight logics. Γ is provable in $S\mathcal{L}$ if it is provable in $S2\mathcal{L}$. Γ is cut–free provable in $S\mathcal{L}$ if it is cut–free provable in $S2\mathcal{L}$.*

PROOF. For any axioms and inference rules in (cut–free) $S2\mathcal{L}$, we show that the same derivations can be done in (cut–free) $S\mathcal{L}$, as follows:

Case 1 For an axiom

$$^P\{\,^F\{v, \neg v, \Gamma\}, \Delta\}, \Pi$$

in $S2\mathcal{L}$, we have a proof in $S\mathcal{L}$:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{v, \neg v}{v, \neg v, \Gamma, \,^P\{\}}\ \text{weakening}}{^F\{v, \neg v, \Gamma\}}\ \text{turn}}{^F\{v, \neg v, \Gamma\}, \Delta, \,^F\{\}}\ \text{weakening}}{^P\{\,^F\{v, \neg v, \Gamma\}, \Delta\}}\ \text{turn}}{^P\{\,^F\{v, \neg v, \Gamma\}, \Delta\}, \Pi}\ \text{weakening}$$

Case 2 For an inference rule

$$\frac{^P\{\,^F\{\Gamma\}, A, \Delta\}, \Pi}{^P\{\,^F\{\Gamma, \Diamond^P A\}, \Delta\}, \Pi}\ \Diamond^P \text{ enter}$$

in $S2\mathcal{L}$, we have a proof in $S\mathcal{L}$:

$$^P\{\,^F\{\Gamma\}, A, \Delta\}, \Pi$$

$$\vdots \text{ exchange, turn, exchange}$$

$$\frac{^F\{\Gamma\}, A, \Delta, \,^F\{\Pi\}}{^F\{\Gamma\}, \Delta, \,^F\{\Pi\}, A}\ \text{exchange}$$

$$\vdots \text{ exchange, turn, exchange}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma, \,^P\{\Delta, \,^F\{\Pi\}, A\}}{\Gamma, \,^P\{\Delta, \,^F\{\Pi\}\}, \Diamond^P A}\ \Diamond^P}{\Gamma, \Diamond^P A, \,^P\{\Delta, \,^F\{\Pi\}\}}\ \text{exchange}}{^F\{\Gamma, \Diamond^P A\}, \Delta, \,^F\{\Pi\}}\ \text{turn}}{^P\{\,^F\{\Gamma, \Diamond^P A\}, \Delta\}, \Pi}\ \text{turn}$$

The other cases are similar. ■

To show completeness theorem, we need some definitions.

For a sequent Γ, we inductively define a tree $t(\Gamma)$ whose nodes are labeled by finite sets of formulas, and whose edges are labeled by either "P" or "F", in the following way: If Γ is a sequence of $A_1, \ldots, A_m$, ${}^P\{\Pi_1\}, \ldots, {}^P\{\Pi_n\}$, ${}^F\{\Sigma_1\}, \ldots, {}^F\{\Sigma_k\}$ in some order, then $t(\Gamma)$ is the tree in Figure 1.

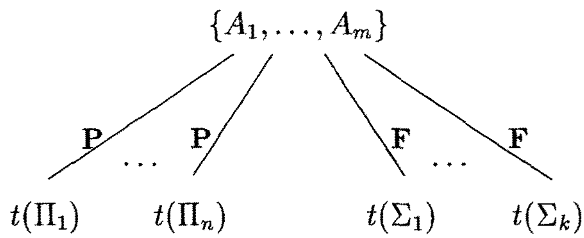

Figure 1

For example,

$$t(A, {}^P\{C\,{}^P\{F\}, {}^F\{\}\}, B, {}^F\{\,{}^P\{G, H\}\}, {}^F\{D, E\})$$

is the tree in Figure 2.

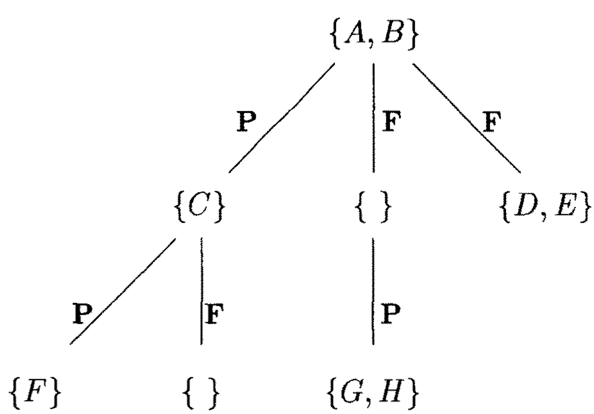

Figure 2

We will use letters $\alpha, \beta, \ldots$ for nodes (or the sets of formulas associated with them). In a tree, if a node α is a child of a node β and the edge between α and β is labeled by P, then we say α is a *P–child* of β. An *F–child* is defined similarly. For example, $\{C\}$ is a P–child and $\{D, E\}$ is an F-child of $\{A, B\}$ in Figure 2. If $t(\Gamma)$ is obtained from $t(\Delta)$ by adding some formulas to the set of formulas at some nodes and/or adding some subtrees to some nodes, then we say $t(\Gamma)$ is an *extension* of $t(\Delta)$. The *depth* of a node α in a tree $t(\Gamma)$, denoted by $d(\alpha)$, is defined as the length of the path from the root of $t(\Gamma)$ to α.

Two binary relations $<$ and $\ll$ between two sets of formulas are defined as follows:

> $\alpha < \beta \Leftrightarrow ((\Diamond^P A \in \beta \Rightarrow A \in \alpha)$ and $(\Diamond^F A \in \alpha \Rightarrow A \in \beta))$ for any formula A.
>
> $\alpha \ll \beta \Leftrightarrow ((\Diamond^P A \in \beta \Rightarrow \Diamond^P A, A \in \alpha)$ and $(\Diamond^F A \in \alpha \Rightarrow \Diamond^F A, A \in \beta))$ for any formula A.

Note that $\ll$ is transitive.

We define conditions $\Diamond^P Tr^\circ, \Diamond^F Tr^\circ, \Diamond^P Re^\circ, \Diamond^F Re^\circ, Co^\circ$, $TrCo^\circ$, To°, and $TrTo^\circ$, on a tree $t(\Gamma)$, as follows:

> $\Diamond^P Tr^\circ$: (α is a P–child of $\beta \Rightarrow \alpha \ll \beta$) for any nodes α, β in $t(\Gamma)$.
>
> $\Diamond^F Tr^\circ$: (α is an F-child of $\beta \Rightarrow \beta \ll \alpha$) for any nodes α, β in $t(\Gamma)$.
>
> $\Diamond^P Re^\circ$: ($\Diamond^P A \in \alpha \Rightarrow A \in \alpha$) for any nodes α in $t(\Gamma)$ and for any formula A.
>
> $\Diamond^F Re^\circ$: ($\Diamond^F A \in \alpha \Rightarrow A \in \alpha$) for any nodes α in $t(\Gamma)$ and for any formula A.
>
> Co° : ($\alpha < \beta$ or $\beta < \alpha$ or $\alpha = \beta\dagger$) for any nodes α, β in $t(\Gamma)$.
>
> $TrCo^\circ$: ($\alpha \ll \beta$ or $\beta \ll \alpha$ or $\alpha = \beta\dagger$) for any nodes α, β in $t(\Gamma)$.
>
> To° : ($\alpha < \beta$ or $\beta < \alpha$ for any nodes α, β in $t(\Gamma)$.
>
> $TrTo^\circ$: ($\alpha \ll \beta$ or $\beta \ll \alpha$ for any nodes α, β in $t(\Gamma)$.

($\dagger$"$\alpha = \beta$" means "α and β are the same sets".)

Let Γ and Δ be sequents and n be a natural number. We say that Δ is an *n–saturation of* Γ *on* K_t if the following conditions are satisfied:

1. $t(\Delta)$ is an extension of $t(\Gamma)$;

2. For any nodes α, β in $t(\Delta)$, and for any formulas A, B,
   (a) $A \wedge B \in \alpha \Rightarrow A \in \alpha$ or $B \in \alpha$;
   (b) $A \vee B \in \alpha \Rightarrow A \in \alpha$ and $B \in \alpha$;
   (c) $\Box^P A \in \alpha$ and $d(\alpha) \leqslant n \Rightarrow$ for some P–child γ of α, $A \in \gamma$;
   (d) $\Box^F A \in \alpha$ and $d(\alpha) \leqslant n \Rightarrow$ for some F–child γ of α, $A \in \gamma$;
   (e) α is a P–child of $\beta \Rightarrow \alpha < \beta$;
   (f) α is an F-child of $\beta \Rightarrow \beta < \alpha$;
3. Each formula in Δ is a subformula of some formula in Γ.

Also, we say that Δ is an *n–saturation of* Γ *on* $\mathcal{L}$ ($\mathcal{L} = K_tTr$, K_tRe, K_tCo, K_tTo, K_tTrRe, K_tTrCo, K_tTrTo) if

1. Δ is an n–saturation of Γ on K_t;
2. $t(\Delta)$ satisfies the corresponding conditions listed in Table 1: i.e., for example, $t(\Delta)$ satisfies $\Diamond^P Tr^\circ$, $\Diamond^F Tr^\circ$ and $TrCo^\circ$ when $\mathcal{L} = K_tTrCo$.

LEMMA 4.2. *Let n be a natural number and $\mathcal{L}$ be arbitrary one of the eight logics. If a sequent Γ is not cut–free provable in $S2\mathcal{L}$, then there is a sequent Δ such that*

- *Δ is an n–saturation of Γ on $\mathcal{L}$;*
- *Δ is not cut–free provable in $S2\mathcal{L}$.*

PROOF. We consider only the case of $\mathcal{L} = K_tTrCo$. The other cases are similar.

Suppose that a sequent Γ is not cut–free provable in $S2K_tTrCo$ and is not an n–saturation of Γ itself. This means that some of the conditions 2a – 2f in the definition of n–saturation on K_t fail for some nodes α, β in $t(\Gamma)$ and some formulas A, B, or some of the conditions $\Diamond^P Tr^\circ$, $\Diamond^F Tr^\circ$ and $TrCo^\circ$ fail for $t(\Gamma)$. Then to get an n–saturation of Γ on K_tTrCo, we extend Γ several times in the following manner:

- If 2a fails, then we consider two sequents Γ_1 and Γ_2 as follows:
  — Γ_1 is obtained from Γ by adding the formula A to the node α.
  — Γ_2 is obtained from Γ by adding the formula B to the node α.

  Because of the inference rule "$\wedge$", either Γ_1 or Γ_2 is not cut–free provable. Then as the required extended sequent, we take one of Γ_i which is not cut–free provable.

- If 2b fails, then we extend Γ by adding the formulas A and B to the node α. This extended sequent is not cut–free provable because of the inference rule "$\vee$".

- If 2c fails, then we extend Γ by adding the new P–child $\{A\}$ to the node α. This extended sequent is not cut–free provable because of the inference rule "$\Box^P$".

- If $\Diamond^P Tr^\circ$ fails, then we define two sets of formulas as:

$$\alpha^F = \{\Diamond^F A | \Diamond^F A \in \alpha\} \cup \{A | \Diamond^F A \in \alpha\}$$

$$\beta^P = \{\Diamond^P A | \Diamond^P A \in \beta\} \cup \{A | \Diamond^P A \in \beta\}$$

  and we extend Γ by adding α^F to the node β and adding β^P to the node α. This extended sequent is not cut–free provable because of the inference rules "$\Diamond^P$ exit", "$\Diamond^F$ enter", "$\Diamond^P$ exit Tr" and "$\Diamond^F$ enter Tr".

- If $TrCo^\circ$ fails, then we define four sets of formulas as:

$$\alpha^P = \{\Diamond^P A | \Diamond^P A \in \alpha\} \cup \{A | \Diamond^P A \in \alpha\}$$

$$\alpha^F = \{\Diamond^F A | \Diamond^F A \in \alpha\} \cup \{A | \Diamond^F A \in \alpha\}$$

$$\beta^P = \{\Diamond^P A | \Diamond^P A \in \beta\} \cup \{A | \Diamond^P A \in \beta\}$$

$$\beta^F = \{\Diamond^F A | \Diamond^F A \in \beta\} \cup \{A | \Diamond^F A \in \beta\}$$

  And we consider three sequents Γ_1, Γ_2 and Γ_3 as follows:

  — Γ_1 is obtained from Γ by adding β^P to the node α and adding α^F to the node β.

  — Γ_2 is obtained from Γ by adding β^F to the node α and adding α^P to the node β.

  — Γ_3 is obtained from Γ by adding β to the node α and adding α to the node β.

  Because of the inference rule "$TrCo$", at least one of Γ_i is not cut–free provable. Then as the required extended sequent, we take one of Γ_i which is not cut–free provable.

- The other cases are similar and omitted.

In this process, we add only the subformulas of Γ, and no new child is added to a node whose depth is greater than n. Hence, after finite iterations of this process, we can get an n–saturation of Γ on K_tTrCo which is not cut–free provable in $S2K_tTrCo$. ■

LEMMA 4.3. *Let $\mathcal{L}$ be arbitrary one of the eight logics. If a sequent Γ is not cut–free provable in $S2\mathcal{L}$, then there is a model $\mathcal{M}$ of $\mathcal{L}$ such that $\mathcal{M} \not\models \Gamma^*$. (We say $\mathcal{M}$ is a* model of $\mathcal{L}$ *if, for example, it satisfies the conditions Tr and Co when $\mathcal{L} = K_tTrCo$.)*

PROOF. By Lemma 4.2, we have an infinite sequence

$$\Gamma_0, \Gamma_1, \Gamma_2, \ldots$$

of sequents such that

- $\Gamma_0 = \Gamma$;
- Γ_n is not cut–free provable in $S2\mathcal{L}$ and is an n–saturation of Γ_{n-1} $(n = 1, 2, \ldots)$.

Let α be a set of formulas. We say α is *maximal* in the sequence $\Gamma_0, \Gamma_1, \ldots$ if the following conditions are satisfied: α is the set associated with a node in $t(\Gamma_k)$ for some k, and the set associated with the corresponding node in $t(\Gamma_{k'})$ remains to be α for all $k' \geqslant k$. The sets associated with the nodes in $t(\Gamma_n)$ are subsets of the set of all subformulas in Γ, which is of course finite. Hence, they cannot be expanded forever. It means that the set associated with a node should eventually be expanded to a maximal one.

Now we define a model $\mathcal{M} = \langle T, R, V\rangle$ as follows:

- $T = \{\alpha \mid \alpha \text{ is a maximal set in the sequence } \Gamma_0, \Gamma_1, \ldots\}$
- $\alpha R \beta \Leftrightarrow \begin{cases} \alpha < \beta & \text{when } \mathcal{L} = K_t, K_tRe, K_tCo, K_tTo \\ \alpha \ll \beta & \text{when } \mathcal{L} = K_tTr, K_tTrRe, K_tTrCo, K_tTrTo \end{cases}$
- $V(v) = \{\alpha \mid v \notin \alpha\}$.

Then it is easy to verify that $\mathcal{M}$ is a model of $\mathcal{L}$.

By induction on the length of A, we can show that

$$A \in \alpha \Rightarrow \mathcal{M} \not\models_\alpha A$$

holds for any formula A and any set α in T. This leads $\mathcal{M} \not\models \Gamma^*$. ■

Now, we show the completeness and cut–elimination for our systems.

THEOREM 4.4. (COMPLETENESS THEOREM) *Let $\mathcal{L}$ be arbitrary one of the eight logics. If $\Gamma^* \in \mathcal{L}$, then Γ is cut–free provable in both $S\mathcal{L}$ and $S2\mathcal{L}$.*

PROOF. By Lemmas 4.1 and 4.3. ∎

COROLLARY 4.5. (CUT–ELIMINATION THEOREM) *Let $\mathcal{L}$ be arbitrary one of the eight logics. In $S\mathcal{L}$ and $S2\mathcal{L}$, the inference rule "cut" is redundant.*

PROOF. By Theorem 3.1, Lemma 4.1 and Theorem 4.4. ∎

5. Remarks

(1)

It is known that Gentzen's sequent calculus LK is a dual system of the semantic tableau for classical logic. Likewise, our $S2K_t$ is a dual system of the tableau system for K_t by Rescher and Urquhart [2].

A tableau system for K_tTrCo (the logic of "linear time") is also introduced in [2]. This tableau system and our $S2K_tTrCo$ differ in the ways of axiomatizing "linearity": the former has simpler rules which directly reflect "linearity", and the latter has two separate kinds of inference rules expressing "transitivity" and "connectedness". This may be viewed as an advantage of our formulation, since one cannot express "transitivity" and "connectedness" separately by Rescher and Urquhart's method. On the other hand, "backwards linearity" and "forwards linearity" cannot be separated by our way, whereas they can by Rescher and Urquhart's method.

(2)

It is clear that we can obtain a sequent calculus (modified LK) for propositional classical logic as the subsystem of SK_t such that:

Axioms: the same as SK_t.

Inference rules: exchange, weakening, contraction, cut, $\wedge$, $\vee$.

It can be shown that not only the rule "cut" but "contraction" is redundant in this system. But "contraction" is necessary in our sequent calculi for propositional tense logics. Indeed, the formula

$$\Box^P\Box^F v \vee \Box^P\Box^F\neg v \vee \Diamond^P(\Diamond^F v \wedge \Diamond^F\neg v)$$

is provable in $S2K_t$ as follows:

$$\dfrac{\begin{array}{c}\vdots\, P\\ {}^{P}\{{}^{F}\{v\},\Diamond^F v\},\Phi\end{array} \qquad \dfrac{{}^{P}\{{}^{F}\{v,\neg v\}\},\Phi}{{}^{P}\{{}^{F}\{v\},\Diamond^F\neg v\},\Phi}\;\Diamond^F\text{ exit}}{{}^{P}\{{}^{F}\{v\},\Diamond^F v\wedge\Diamond^F\neg v\},\Phi}\;\wedge$$

$$\vdots$$

$$\dfrac{\Box^P\Box^F v,\Box^P\Box^F\neg v,\Diamond^P(\Diamond^F v\wedge\Diamond^F\neg v),\Diamond^P(\Diamond^F v\wedge\Diamond^F\neg v)}{\Box^P\Box^F v,\Box^P\Box^F\neg v,\Diamond^P(\Diamond^F v\wedge\Diamond^F\neg v)}\;\text{contraction}$$

$$\vdots$$

$$\Box^P\Box^F v\vee\Box^P\Box^F\neg v\vee\Diamond^P(\Diamond^F v\wedge\Diamond^F\neg v)$$

where $\Phi = {}^{P}\{{}^{F}\{\neg v\},\Diamond^F v\wedge\Diamond^F\neg v\}$

P is

$$\dfrac{\dfrac{\dfrac{\chi,{}^{P}\{{}^{F}\{v,\neg v\}\}}{\chi,{}^{P}\{{}^{F}\{\neg v,v\}\}}\;\text{exchange}}{\chi,{}^{P}\{{}^{F}\{\neg v\},\Diamond^F v\}}\;\Diamond^F\text{ exit} \qquad \dfrac{\dfrac{\dfrac{{}^{P}\{{}^{F}\{v\}\},{}^{P}\{{}^{F}\{\neg v\}\},v,\neg v}{{}^{P}\{{}^{F}\{v\}\},v,{}^{P}\{{}^{F}\{\neg v\}\},\neg v}\;\text{exchange}}{\chi,{}^{P}\{{}^{F}\{\neg v\}\},\neg v}\;\Diamond^F\text{ enter}}{\chi,{}^{P}\{{}^{F}\{\neg v\},\Diamond^F\neg v\}}\;\Diamond^F\text{ enter}}{\underbrace{{}^{P}\{{}^{F}\{v\},\Diamond^F v\}}_{\chi},\underbrace{{}^{P}\{{}^{F}\{\neg v\},\Diamond^F v\wedge\Diamond^F\neg v\}}_{\Phi}}\;\wedge$$

and $\chi = {}^{P}\{{}^{F}\{v\},\Diamond^F v\}$

But this is not provable without "contraction".

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