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DRAG OF POROUS CYLINDERS IN A VISCOUS FLUID AT LOW REYNOLDS NUMBERS

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The problem of flow past a permeable cylinder at low Reynolds numbers is of interest for the solution of a number of problems in chemical technology in, for example, the design of porous electrodes and porous catalysts and in the calculation of nonstationary filtration of aerosols by fibrous filters. In the present paper, we solve the problem of transverse flow of a viscous fluid past a continuous cylinder in a porous shell and, in particular, in the case of a porous cylinder under conditions of constrained flow (system of cylinders) and an isolated cylinder at arbitrary permeability. The analogous problem of Stokes flow past permeable spheres has been solved in a number of papers [1-3].

1. Formulation of the Problem

We consider stationary flow of a viscous incompressible fluid past a system of cylinders with porous shells placed at right angles to the flow at small Reynolds numbers. To describe such a system, we use the cell hydrodynamic model of [4]: we consider plane motion of fluid past a cylinder only in a region bounded by a circle of radius $\rho_2^* = a\alpha^{-1/2}$, where a is the cylinder radius and α is the fraction of the volume (area) occupied by the solid cylinders (Fig. 1). The cylinder is surrounded by a porous permeable cylindrical shell of radius $\rho_1^* = a\rho_1$. The asterisk denotes the dimensional quantities.

In the region outside the porous cylinder (region I), the motion of the fluid is described by the equations of hydrodynamics in the Stokes approximation, and in the region of the porous shell (region II) by the generalized Darcy equation [2, 5]:

$$\operatorname{grad} p_1 = \Delta \mathbf{v}_1, \qquad \operatorname{div} \mathbf{v}_1 = 0 \tag{1.1}$$

$$\operatorname{grad} p_2 = \Delta \mathbf{v}_2 - s^2 \mathbf{v}_2, \qquad \operatorname{div} \mathbf{v}_2 = 0 \tag{1.2}$$

$$v_{2\rho} = 0, \qquad v_{2\theta} = 0 \quad \text{for } \rho = 1 \tag{1.3}$$

$$v_{1\rho} = v_{2\rho}, \quad v_{1\theta} = v_{2\theta}, \quad \tau_{1\rho\theta} = \tau_{2\rho\theta}, \quad p_1 = p_2 \quad \text{for } \rho = \rho_1 \tag{1.4}$$

Here, $\mathbf{v} = \mathbf{v}^*/v_0$ is the dimensionless flow velocity, v_0 is the flow velocity at infinity, $p = p^* a/\mu v_0$ is the dimensionless pressure, μ is the dynamic viscosity, θ and $\rho = \rho^*/a$ are dimensionless polar coordinates, $s^2 = a^2/\varkappa$, $\varkappa$ is the permeability of the porous shell, and $\tau_{\rho\theta}$ are the tangential components of the viscous stress tensor. The subscripts 1 and 2 are appended to the variables in regions I and II, respectively.

2. Solution in Region I

Outside the porous cylinder, where Eqs. (1.1) hold, the flow function Ψ_1 satisfies the following equation and boundary conditions for $\rho = \alpha^{-1/2}$:

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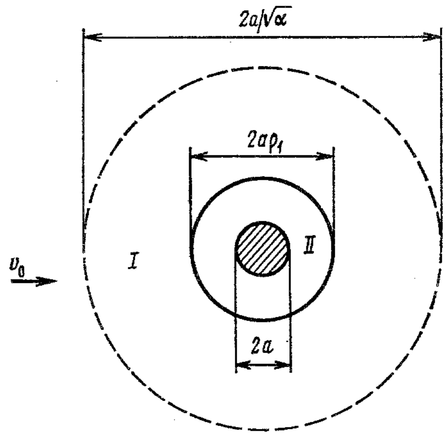

Fig. 1

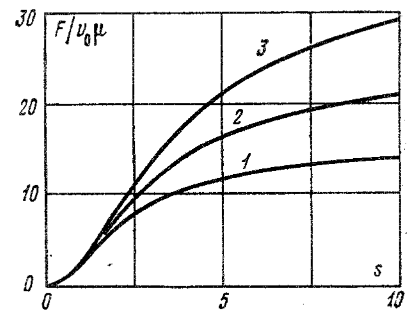

Fig. 2

$$\Delta\Delta\Psi_1=0, \quad \frac{\partial^2\Psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\Psi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial\theta^2}=0, \quad \frac{1}{\rho}\frac{\partial\Psi}{\partial\theta}=\cos\theta \tag{2.1}$$

As the conditions on the cell boundary, we have chosen Kuwabara's conditions [6]. The general solution of Eq. (2.1) can be written, using the boundary conditions, in the form

$$\Psi_1(\rho,\theta)=\Phi_1(\rho)\sin\theta=\left\{\frac{A_1}{\rho}+B_1\rho+C_1\rho\ln\rho+D_1\rho^3\right\}\sin\theta$$

$$B_1=1-A_1\alpha+\frac{C_1}{2}\ln\alpha+\frac{C_1}{4}, \quad D_1=-\frac{C_1\alpha}{4}, \quad p_1=-2C_1\left(\alpha\rho+\frac{1}{\rho}\right)\cos\theta \tag{2.2}$$

3. Solution in Region II

In the region of the porous shell, the flow function Ψ_2 satisfies the equation

$$\Delta\Delta\Psi_2-s^2\Delta\Psi_2=0 \tag{3.1}$$

Solving Eq. (3.1) by separating the variables ($\Psi_2(\rho,\theta)=\Phi_2(\rho)\sin\theta$) and using the boundary conditions (1.3), we obtain

$$\Phi_2(\rho)=-\frac{A_2I_1(s\rho)}{s^2}-\frac{B_2K_1(s\rho)}{s^2}+\frac{C_2\rho}{2}+\frac{D_2}{\rho}, \quad B_2=-\frac{A_2I_0(s)}{K_0(s)}+\frac{C_2s}{K_0(s)}$$

$$D_2=\frac{A_2}{s^2K_0(s)}\{I_1(s)K_0(s)-I_0(s)K_1(s)\}+C_2\left\{\frac{K_1(s)}{sK_0(s)}-\frac{1}{2}\right\}, \quad p_2=-s^2\left\{\frac{C_2\rho}{2}-\frac{D_2}{\rho}\right\}\cos\theta \tag{3.2}$$

Here, $I_0(s)$, $I_1(s)$, $K_0(s)$, and $K_1(s)$ are modified Bessel functions of imaginary argument. The coefficients A_1, C_1, A_2, and C_2 can be found from the conditions (1.4) of matching of the solutions on the boundary of regions I and II. The equations for determining these coefficients are not given here because of their length, but their values can be found for each particular value of the parameters s and α. The force acting on unit length of the cylinder with the porous shell is determined by the value of the coefficient C_1:

$$F=4\pi\mu v_0C_1 \tag{3.3}$$

4. Porous Cylinder

We now consider the simpler case when the cylinders are completely porous ($a = 0$) and have packing density $\beta=\rho_1^2/\rho_2^2$. As characteristic length, we now choose ρ_1, and divide all linear dimensions by this radius. The flow function in region II has the form

$$\Psi_2(\rho,\theta)=\left\{\frac{C_2\rho}{2}-\frac{A_2I_1(s\rho)}{s^2}\right\}\sin\theta \tag{4.1}$$

The coefficients B_2 and D_2 are zero, since the function $\Phi_2(\rho)$ must be bounded at the origin. The function $\Phi_1(\rho)$ is determined, as before, by (2.2). From the conditions (1.4) of matching of the solutions on the surface of the porous cylinder, all the unknown coefficients can be determined:

$$A_1 = \frac{1}{2}\left\{1 - \frac{\beta}{2} - \frac{2(1-\beta)}{s^2}\frac{I_2(s)}{I_1(s)}\right\} C_1, \quad A_2 = -\frac{2(1-\beta)}{I_1(s)} C_1, \quad C_2 = \frac{4(1+\beta)}{s^2} C_1$$
$$C_1 = \left\{-\frac{1}{2}\ln\beta - \frac{3}{4} + \beta - \frac{\beta^2}{4} + \frac{(1-\beta)^2 I_2(s)}{sI_1(s)} + \frac{4}{s^2}\right\}^{-1} \tag{4.2}$$

The parameter s is inversely proportional to the permeability of the porous cylinder, and a model of the structure of the porous cylinder must be specified if it is to be particularized.

The force acting on unit length of the porous cylinder is

$$\frac{F}{\mu v_0} = 4\pi \left\{-\frac{1}{2}\ln\beta - \frac{3}{4} + \beta - \frac{\beta^2}{4} + \frac{(1-\beta)^2 I_2(s)}{sI_1(s)} + \frac{4}{s^2}\right\}^{-1} \tag{4.3}$$

In the limit $s \to \infty$, this goes over into Kuwabara's formula for an impermeable cylinder [6]. In Fig. 2, we have plotted $F/\mu v_0$ as a function of s for β = 0.05, 0.1, and 0.15 (curves 1, 2, and 3, respectively).

5. Isolated Cylinder

We can solve the problem for an isolated porous cylinder similarly. In this case, in region I the flow function near the cylinder takes the form

$$\Psi_1(\rho, \theta) = \{-a_0\rho\ln\rho + a_1\rho + a_2\rho^{-1}\}\sin\theta \tag{5.1}$$

The velocity components far from the cylinder are

$$v_\rho = \cos\theta + c\left\{\frac{4}{\rho\,\mathrm{Re}} - \exp\left(\frac{\mathrm{Re}}{4}\rho\cos\theta\right)\left[K_0\left(\frac{\mathrm{Re}}{4}\rho\right)\cos\theta + K_1\left(\frac{\mathrm{Re}}{4}\rho\right)\right]\right\}$$
$$v_\theta = \left\{-1 + c\exp\left(\frac{\mathrm{Re}}{4}\rho\cos\theta\right)K_0\left(\frac{\mathrm{Re}}{4}\rho\right)\right\}\sin\theta, \quad \mathrm{Re} = \frac{2av_0}{\nu} \tag{5.2}$$

Here, ν is the kinematic viscosity.

From the conditions of matching of the flow fields near and far from the cylinder, the expression (4.1) for the flow function in region II, and from the conditions (1.4), we obtain

$$c = a_0, \quad a_1 = 1 + a_0 + a_0\ln\frac{8}{\gamma\,\mathrm{Re}}, \quad 1 + a_0 + a_0\ln\frac{8}{\gamma\,\mathrm{Re}} + a_2 = \frac{C_2}{2} + \frac{A_2 I_1(s)}{s^2}$$
$$a_0 + 2a_2 = \frac{A_2 I_2(s)}{s}, \quad 4a_2 = -\frac{A_2}{2}\{I_1(s) + I_3(s)\}, \quad 2a_0 = -\frac{s^2 C_2}{2} \tag{5.3}$$

Here, γ = 0.5772 is Euler's constant.

Solving this system for a_0, we obtain the drag

$$\frac{F}{\mu v_0} = 4\pi\left\{\frac{1}{2} + \ln\frac{8}{\gamma\,\mathrm{Re}} + \frac{I_2(s)}{sI_1(s)} + \frac{4}{s^2}\right\}^{-1} \tag{5.4}$$

In the limit $s \to \infty$, this expression goes over into the well-known expression for the resistance of a cylinder to a viscous flow [7]. Note that the result (5.4) differs from the result of [2], in which the expression for F was obtained for a cylinder with small permeability ($s \gg 1$), by numerical factors, which is due to errors in the formulation of the boundary conditions in [2].

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PULSATING FLOW OF A NEWTONIAN FLUID THROUGH AN AXISYMMETRIC TUBE WITH A LOCAL DILATATION

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INTRODUCTION

In recent years, numerical simulations of the flow of blood through large blood vessels as a viscous Newtonian fluid have been made. Among the most frequent pathological alterations of the vessels we can mention local constrictions and dilatations, which are due primarily to atherosclerosic disease of the vessel wall. Such constrictions and dilatations, which are called stenoses and aneurysms, respectively, can have a significant influence on the flow of blood in their neighborhood, cause changes in the pressure gradient, and also facilitate thrombosis.

In [1], the author made a numerical analysis of the pulsating flow of blood, regarded as a Newtonian fluid, through an axisymmetric vessel with a stenosis. The paper [1] also contains a brief bibliography of papers devoted to the solution of problems of the flow of a viscous fluid in the region of a local constriction of the flow region. A calculation of the oscillating flow in a plane channel with a local dilatation was made in [2]. Experimental investigations into the flow of fluid through glass models of aneurysms are reported in [3, 4].

In the present paper, we give the results of numerical solution to the problem of the pulsating flow of blood, treated as a Newtonian fluid, through a rigid axisymmetric dilatation of the vessel (aneurysm) with an arbitrary, continuously differentiable unique dependence of the wall radius R_w on the longitudinal coordinate Z.

1. The motion of blood in large vessels can be well described by the Navier–Stokes equations with the conditions of no penetration and no slip of the fluid on the wall. In the case of an axisymmetric configuration of the vessel wall, and also axial symmetry of the boundary conditions at the entrance and exit to the considered region, the problem can be conveniently solved in cylindrical coordinates in the dimensionless variables of the flow function ψ and vorticity function ω.

As was shown in [1], it is expedient to solve the determining equations numerically in a system of coordinates made dimensionless as follows:

$$\eta=r/r_w(z),\quad \xi=z,\quad \zeta=\theta,\quad \tau=t,\quad r=R/R_0,\quad z=Z/R_0,\quad t=TU_0/R_0,\quad r_w=R_w/R_0$$

where Z and R are the dimensional axial and radial coordinates, T is the time, and R_0 and U_0 are the characteristic values of the length and the velocity.

This makes it possible to reduce a region with arbitrary curvilinear boundaries to a rectangle, which is very advantageous if one is using algorithms for the solution based

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