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Darling–Erdős Theorems for Martingales

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This paper considers Darlin–Erdős theorems for sums of martingale differences. Our main theorem provides an optimal result for the case of bounded martingale difference sequences. A number of other results are presented, which deal with the unbounded case and which specialize to the case of independent summands. Previous related work on this problem has been based on deep strong approximation theorems. One of the novel features of our approach is that our methods rely on the more easily accessible Skorokhod-type embeddings.

KEY WORDS: Martingale difference sequences; Darling–Erdős theorems; law of the iterated logarithm.

1. INTRODUCTION

Let $\{X_n, \mathscr{F}_n, n \geqslant 0\}$ be a martingale difference sequence with finite second moments where $X_0 = 0$ and $\mathscr{F}_0 = \{\varnothing, \Omega\}$ is the trivial σ-field such that

$$s_n^2 := \sum_{j=1}^{n} E(X_j^2 \mid \mathscr{F}_{j-1}) \to \infty \qquad \text{as} \quad n \to \infty \text{ a.s.} \tag{1.1}$$

Set $S_n = X_1 + \cdots + X_n$ for $n \geqslant 1$ and $S_0 = 0$. Define the partial sum process on $[0, \infty)$ based on these random variables to be

$$S(t) = S_n \qquad \text{where} \quad s_n^2 \leqslant t < s_{n+1}^2, \qquad n \geqslant 0 \tag{1.2}$$

where $s_0^2 := 0$. We introduce the notation for $T \geqslant 0$,

$$LT = \log(T \vee e), \qquad a(T) = (2LLT)^{1/2}$$

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and

$$b(T) = 2LLT + 2^{-1}LLLT - 2^{-1}L(4\pi)$$

Darling and Erdős[2] proved that if $X_1, X_2, \ldots,$ are independent random variables with mean zero, common finite positive variance, and uniformly bounded finite absolute third moments, then

$$a(T) \sup_{1 \leqslant t \leqslant T} \frac{S(t)}{\sqrt{t}} - b(T) \xrightarrow{\mathscr{D}} E \quad \text{as} \quad T \to \infty \tag{1.3}$$

and

$$a(T) \sup_{1 \leqslant t \leqslant T} \frac{|S(t)|}{\sqrt{t}} - b(T) \xrightarrow{\mathscr{D}} E \vee E' \quad \text{as} \quad T \to \infty \tag{1.4}$$

where E and E' are independent random variables with common extreme value distribution function $\exp[-\exp(-t)]$, $-\infty < t < \infty$.

Oodaira[9] and Shorack[12] noticed that since (1.3) and (1.4) hold when S is replaced by a standard Wiener process W on $[0, \infty)$, one can obtain the same results for any right continuous process S defined on $[0, \infty)$ for which a suitable probability space can be constructed on which one has for some $0 < \delta < 1/2$

$$|S(t) - W(t)|/\sqrt{t} = O(t^{-\delta}) \quad \text{as} \quad t \to \infty \tag{1.5}$$

A slight refinement of their methods shows that to obtain the Darling–Erdős theorem, that is (1.3) and (1.4), from the corresponding theorem for W one only needs that on a suitable probability space

$$|S(t) - W(t)|/\sqrt{t} = o((LLt)^{-1/2}) \quad \text{as} \quad t \to \infty \tag{1.6}$$

This approach, however, does not always lead to optimal results. Einmahl[3] has established that when S is the partial sum process formed from a sequence of i.i.d. random variables, one has (1.6) if and only if $E(X_1^2 LL|X_1|) < \infty$. On the other hand, Einmahl[4] has proved by means of a truncation technique of Feller[5] that (1.3) and (1.4) hold in the i.i.d. case if and only if

$$LLtE\{X_1^2 1(|X_1| \geqslant t)\} \to 0 \quad \text{as} \quad t \to \infty$$

The main purpose of the present paper is to derive an optimal condition under which the Darling–Erdős theorem holds when $X_1, X_2, \ldots,$ form a bounded martingale difference sequence. Philipp and Stout[10] have obtained (1.6) for martingale difference sequence satisfying

$$|X_n| \leqslant \varepsilon_n s_n/(LLs_n)^{5/2} \text{ a.s.} \tag{1.7}$$

where ε_n is a sequence of positive constants with $\varepsilon_n \searrow 0$. Moreover, under the more restrictive assumption that $X_1, X_2,...,$ are independent random variables, it follows from a strong approximation result announced by Sakhanenko[11] that (1.6) is true when in (1.7) one replaces 5/2 by 3/2. His proof is based on a refinement of the delicate approximation method of Komlós, Major, and Tusnády.[8] Thus we see by the above remarks that the Darling–Erdős theorem is true for martingale difference sequences satisfying (1.7) and for independent sequences of random variables under Sakhanenko's weaker bounding condition. We will show more generally that the Darling–Erdős theorem holds for martingale difference sequences when

$$|X_n| \leqslant \varepsilon_n s_n/(LLs_n)^{3/2} \text{ a.s.} \tag{1.8}$$

where $\varepsilon_n \in \mathscr{F}_{n-1}$ and $\varepsilon_n \searrow 0$ a.s. To obtain our result we shall use a method of proof that avoids the need of a direct approximation of type (1.6) for the process S. Furthermore, we shall demonstrate by example that our condition is close to being optimal in the sense that if the exponent 3/2 in (1.8) is replaced by any $1 \leqslant p < 3/2$ then the Darling–Erdős theorem can fail. By Kolmogorov's law of the iterated logarithm or by the Stout[14] extension of it to martingale difference sequences this says that even in the bounded situation there are cases when the law of the iterated logarithm holds but not the Darling–Erdős theorem.

Our main theorem for bounded martingale difference sequence is proved in Section 2. Our example is detailed in Section 3. In Section 4 we provide some sufficient conditions for the Darling–Erdős theorem to be valid in the unbounded case. An appendix contains the proof of a technical result required in Section 2.

2. THE MAIN RESULT

Theorem 1. Whenever

$$s_n^2 \to \infty \text{ a.s.} \qquad \text{as} \quad n \to \infty \tag{2.1}$$

and there exists a positive sequence $\varepsilon_n \in \mathscr{F}_{n-1}$ with $\varepsilon_n \searrow 0$ a.s. such that

$$|X_n| \leqslant \varepsilon_n s_n/(LLs_n)^{3/2} \text{ a.s.} \tag{2.2}$$

then as $T \to \infty$

$$a(T) \sup_{1 \leqslant t \leqslant T} S(t)/t^{1/2} - b(T) \xrightarrow{\mathscr{D}} E \tag{2.3}$$

and

$$a(T) \sup_{1 \leqslant t \leqslant T} |S(t)|/t^{1/2} - b(T) \xrightarrow{\mathscr{D}} E \vee E' \tag{2.4}$$

Proof. As in Philipp and Stout,[10] we can assume without loss of generality that there exists a standard Wiener process W and a sequence of nonnegative random variables $\tau_n \in \mathscr{F}_n$, $n \geqslant 1$, such that

$$S_n = W(T_n), \qquad n \geqslant 1 \tag{2.5}$$

where $T_n = \tau_1 + \cdots + \tau_n$, $n \geqslant 1$, and $T_0 = 0$,

$$E(\tau_n \mid \mathscr{F}_{n-1}) = E(X_n^2 \mid \mathscr{F}_{n-1}), \qquad n \geqslant 1 \tag{2.6}$$

$$E(\tau_n^r \mid \mathscr{F}_{n-1}) \leqslant 2\left(\frac{8}{\pi^2}\right)^{r-1} r!\, E(X_n^{2r} \mid \mathscr{F}_{n-1}), \qquad r \geqslant 2, \quad n \geqslant 1 \tag{2.7}$$

and moreover for all $T_n \leqslant t < T_{n+1}$, $n \geqslant 0$,

$$|W(t) - W(T_n)| \leqslant \varepsilon_{n+1} s_{n+1}/(LLs_{n+1})^{3/2} \tag{2.8}$$

For (2.5), (2.6), and (2.7) see Hall and Heyde[6] page 269 and for (2.8) refer to Skorokhod[13] page 163, noticing that the proof given there is also valid for martingale differences.

By a refinement of the proof of Lemma 2.3 of Philipp and Stout,[10] see Lemma a of the Appendix, under (2.1) and (2.2) one has

$$T_n - s_n^2 = o(s_n^2/LLs_n) \text{ a.s.} \tag{2.9}$$

Observe that by (2.2) we also have

$$s_{n+1}^2 - s_n^2 = o(s_{n+1}^2/LLs_{n+1}) \text{ a.s.} \tag{2.10}$$

Set

$$\tilde{S}(t) = S_n \qquad \text{whenever} \quad T_n \leqslant t < T_{n+1}, \qquad n \geqslant 0$$

Lemma 1. As $t \to \infty$ we have

$$\tilde{S}(t) = W(t) + o(t^{1/2}/(LLt)^{1/2}) \text{ a.s.} \tag{2.11}$$

Proof. First note that by (2.9) and (2.10) we have

$$T_{n+1} - T_n = o(s_{n+1}^2/LLs_{n+1}) \text{ a.s.} \tag{2.12}$$

and

$$T_n/s_n^2 \to 1 \text{ a.s.} \qquad \text{as} \quad n \to \infty \tag{2.13}$$

Now for $T_n \leqslant t < T_{n+1}$, we have by (2.5) and (2.8)

$$|\tilde{S}(t) - W(t)| \leqslant \varepsilon_{n+1} s_{n+1} (LLs_{n+1})^{-3/2} \tag{2.14}$$

The lemma is now an easy consequence of (2.12), (2.13), and (2.14). □

Lemma 2.

$$\limsup_{t \to \infty} |\tilde{S}(t)|/(2tLLt)^{1/2} = 1 \text{ a.s.} \tag{2.15}$$

and

$$\limsup_{t \to \infty} |S(t)|/(2tLLt)^{1/2} = 1 \text{ a.s.} \tag{2.16}$$

Proof. Assertion (2.15) is immediate from (2.11). From (2.15) it is straightforward to show that

$$\limsup_{n \to \infty} |S_n|/(2T_n LLT_n)^{1/2} = 1 \text{ a.s.}$$

This combined with (2.13) yields

$$\limsup_{n \to \infty} |S_n|/(2s_n^2 LLs_n)^{1/2} = 1 \text{ a.s.}$$

from which (2.16) easily follows. □

Let

$$U(s) = e^{-s} W(e^{2s}), \qquad s \geqslant 0 \tag{2.17}$$

be the Ornstein–Uhlenbeck process. It is known, cf. Darling and Erdős[2] and Shorack[12] that

$$a(T) \sup_{0 \leqslant s \leqslant 2^{-1} \log T} U(s) - b(T) \xrightarrow{\mathscr{D}} E \tag{2.18}$$

Lemma 3. For all $0 < a < b < \infty$ and $T \geqslant 1/a$

$$\sup_{aT \leqslant t \leqslant bT} \frac{W(t)}{\sqrt{t}} \overset{\mathscr{D}}{=} \sup_{0 < s \leqslant 2^{-1} \log(b/a)} U(s) \tag{2.19}$$

and as $T \to \infty$

$$\sup_{aT \leqslant t \leqslant bT} \frac{\tilde{S}(t)}{\sqrt{t}} \xrightarrow{\mathscr{D}} \sup_{0 < s \leqslant 2^{-1} \log(b/a)} U(s) \tag{2.20}$$

Proof. Assertion (2.19) follows from the transformation (2.17) and stationarity of U and from (2.11) we get (2.20). □

For any $T \geqslant 1$ let

$$h(T) = \exp[(\log T)^{1/2}]$$

also for any $T \geqslant 1$ and $\lambda \geqslant 1$ define the random sets of indices

$$A_\lambda(T) = \{k: \lambda^{-1}h(T) \leqslant T_k \leqslant \lambda T\}$$

and

$$B_\lambda(T) = \{k: \lambda^{-1}h(T) \leqslant s_k^2 \leqslant \lambda T\}$$

Lemma 4. For all $\lambda > 0$, as $T \to \infty$

$$a(T) \left| \sup_{\lambda^{-1}h(T) \leqslant t \leqslant \lambda T} \frac{\tilde{S}(t)}{\sqrt{t}} - \max_{k \in A_\lambda(T)} \frac{S_k}{\sqrt{T_k}} \right| = o_p(1) \tag{2.21}$$

and

$$a(T) \left| \sup_{\lambda^{-1}h(T) \leqslant t \leqslant \lambda T} \frac{S(t)}{\sqrt{t}} - \max_{k \in B_\lambda(T)} \frac{S_k}{S_k} \right| = o_p(1) \tag{2.22}$$

Proof. First consider (2.21). Notice that the left side of (2.21) is less than or equal to

$$a(T) \max_{k \in A_\lambda(T)} \frac{|S_k| \{\sqrt{T_{k+1}} - \sqrt{T_k}\}}{\sqrt{T_{k+1}}\sqrt{T_k}} + a(T) \max_{k \in A_\lambda(T)} \frac{|S_k| \{\sqrt{T_k} - \sqrt{T_{k-1}}\}}{\sqrt{T_k}\sqrt{T_{k-1}}}$$

Assertion (2.21) now follows easily from (2.12), (2.13), and (2.15). The second part of the lemma is proved in the same way. □

A simple argument based on (2.18) and (2.11) combined with the convergence of types theorem shows that for all $\lambda > 1$, as $T \to \infty$

$$a(T) \sup_{\lambda^{-1}h(T) \leqslant t \leqslant \lambda T} \frac{W(t)}{\sqrt{t}} - b(T) \xrightarrow{\mathscr{D}} E \tag{2.23}$$

and

$$a(T) \sup_{\lambda^{-1}h(T) \leqslant t \leqslant \lambda T} \frac{\tilde{S}(t)}{\sqrt{t}} - b(T) \xrightarrow{\mathscr{D}} E \tag{2.24}$$

For any $\lambda > 1$ set

$$C_\lambda(T) = A_\lambda(T) \cap B_1(T)$$

Lemma 5. For all $\lambda > 1$,

$$P(C_\lambda(T) = B_1(T)) \to 1 \qquad \text{as} \quad T \to \infty \tag{2.25}$$

Proof. The proof follows from (2.13). □

Lemma 6. For all $\lambda > 1$

$$a(T) \left| \max_{k \in C_\lambda(T)} \frac{S_k}{\sqrt{T_k}} - \max_{k \in C_\lambda(T)} \frac{S_k}{s_k} \right| = o_p(1) \tag{2.26}$$

Proof. The left side of (2.26) is less than or equal to

$$a(T) \max_{k \in C_\lambda(T)} \frac{|S_k|}{\sqrt{T_k}\, s_k} |\sqrt{T_k} - s_k|$$

which is equal to $o_p(1)$ by (2.15), (2.13), and (2.9). □

Lemma 7. For all $\lambda > 1$, as $T \to \infty$

$$a(T) \max_{k \in A_\lambda(T) - C_\lambda(T)} \frac{S_k}{\sqrt{T_k}} - b(T) \xrightarrow{P} -\infty \tag{2.27}$$

Proof. Set for $T \geqslant 1$ and $\lambda > 1$

$$D_\lambda(T) = \{k: \lambda^{-1} h(T) \leqslant T_k \leqslant \lambda h(T)\}$$

and

$$E_\lambda(T) = \{k: \lambda^{-1} T \leqslant T_k \leqslant \lambda T\}$$

Notice that by (2.13)

$$P(A_\lambda(T) - C_\lambda(T) \subset D_\lambda(T) \cup E_\lambda(T)) \to 1 \qquad \text{as} \quad T \to \infty \tag{2.28}$$

Also for large T

$$a(T) \max_{k \in D_\lambda(T)} \frac{S_k}{\sqrt{T_k}} - b(T) \leqslant a(T) \sup_{1 \leqslant t \leqslant \lambda h(T)} \frac{\tilde{S}(t)}{\sqrt{t}} - b(T) \tag{2.29}$$

and

$$a(T) \max_{k \in E_\lambda(T)} \frac{S_k}{\sqrt{T_k}} - b(T) \leqslant a(T) \sup_{\lambda^{-1} T \leqslant t \leqslant \lambda T} \frac{\tilde{S}(t)}{\sqrt{t}} - b(T) \tag{2.30}$$

Now from (2.15) it is easy to argue that the right side of expression (2.29) converges in probability to $-\infty$ as $T \to \infty$ and from (2.20) that the same is true for the right side of expression (2.30). Therefore from (2.28) we have (2.27). □

We are now prepared to finish the proof of (2.3). From (2.22) we conclude

$$a(T)\left\{\sup_{h(T) \leqslant t \leqslant T} \frac{S(t)}{\sqrt{t}} - \max_{k \in B_1(T)} \frac{S_k}{S_k}\right\} = o_p(1) \tag{2.31}$$

and for any $\lambda > 1$ we obtain from (2.21) and (2.24) that

$$a(T) \max_{k \in A_\lambda(T)} \frac{S_k}{\sqrt{T_k}} - b(T) \xrightarrow{\mathscr{D}} E \tag{2.32}$$

which when combined with (2.25) and (2.27) yields

$$a(T) \max_{k \in B_1(T)} \frac{S_k}{\sqrt{T_k}} - b(T) \xrightarrow{\mathscr{D}} E \tag{2.33}$$

Hence from (2.31), (2.26), and (2.25) in combination with (2.33) we get

$$a(T) \sup_{h(T) \leqslant t \leqslant T} \frac{S(t)}{\sqrt{t}} - b(T) \xrightarrow{\mathscr{D}} E \tag{2.34}$$

From (2.16) one easily obtains as $T \to \infty$

$$a(T) \sup_{1 \leqslant t \leqslant h(T)} \frac{S(t)}{\sqrt{t}} - b(T) \xrightarrow{P} -\infty \tag{2.35}$$

Assertion (2.3) is now an obvious consequence of (2.34) and (2.35).

The proof of assertion (2.4) is essentially the same, so the details are omitted. This completes the proof of Theorem 1. □

Remark 1. Theorem 1 remains true if one were to replace the assumption that $\varepsilon_n \searrow 0$ a.s. by $\varepsilon_n \to 0$ a.s. This, however, makes the proof much more technical, thus obscuring the main ideas. For the proof of this refinement, Lemma a is required with arbitrary $0 < \varepsilon < 1$ combined with a truncation argument.

3. EXAMPLE

Here we construct the example mentioned in the Introduction, which shows that our Theorem 1 provides a sharp result.

Proposition. For each $1 \leqslant p < 3/2$ there exists a sequence $X_1, X_2,...,$ of independent mean zero random variables such that

$$s_n^2 = \sum_{i=1}^{n} EX_i^2 \to \infty \qquad \text{as} \quad n \to \infty \tag{3.1}$$

$$|X_n| \leqslant s_n/(LLs_n)^p, \qquad n \geqslant 1, \text{ a.s.} \tag{3.2}$$

and at the same time the Darling–Erdős theorem does not hold.

Proof. We require the following exponential inequality.

Lemma 8. Let $X_1, X_2,...,$ be a sequence of independent mean zero random variables satisfying (3.1) and (3.2). Then we have for all $0 \leqslant x \leqslant 3(LLs_n)^{1/2}$, $n \geqslant 1$,

$$P(S_n \geqslant xs_n) \leqslant A \exp\left(-\frac{x^2}{2} + \frac{x^3}{6s_n^3} \sum_{i=1}^{n} EX_i^3\right) \tag{3.3}$$

where A is a universal constant.

Proof. For any $t \geqslant 0$, we have

$$P(S_n \geqslant xs_n) \leqslant \exp\left[\sum_{i=1}^{n} L_i(t) - txs_n\right] \tag{3.4}$$

where

$$L_i(t) = \log R_i(t) \quad \text{and} \quad R_i(t) = E\exp(tX_i), \qquad 1 \leqslant i \leqslant n$$

A Taylor expansion yields

$$\sum_{i=1}^{n} L_i(t) = \frac{t^2}{2} \sum_{i=1}^{n} L_i^{(2)}(0) + \frac{t^3}{6} \sum_{i=1}^{n} L_i^{(3)}(0) + \frac{t^4}{24} \sum_{i=1}^{n} L_i^{(4)}(\bar{t}) \tag{3.5}$$

where $0 < \bar{t} < t$. Notice that for each $1 \leqslant i \leqslant n$, $L_i(0) = L_i^{(1)}(0) = 0$. Furthermore, it is easily checked that

$$L_i^{(2)}(0) = EX_i^2 \quad \text{and} \quad L_i^{(3)}(0) = EX_i^3, \qquad 1 \leqslant i \leqslant n$$

After some calculation and then applying Hölder's inequality, it is not difficult to see that

$$|L_i^{(4)}(\bar{t})| \leqslant \bar{A}E\{X_i^4 \exp(t|X_i|)\}, \qquad 1 \leqslant i \leqslant n \tag{3.6}$$

where $\bar{A}$ is a universal constant.

Now let $0 \leqslant x \leqslant 3(LLs_n)^{1/2}$ be fixed and set $t_x = x/s_n$. Then it is immediate from (3.2) that

$$E\{X_i^4 \exp(t_x |X_i|)\} \leqslant e^3 (EX_i^2) s_n^2/(LLs_n)^2, \qquad 1 \leqslant i \leqslant n \tag{3.7}$$

Using (3.6) and (3.7) we get for an appropriate universal constant $\bar{\bar{A}}$

$$\frac{t_x^4}{24} \sum_{i=1}^{n} |L_i^{(4)}(\bar{t}_x)| \leqslant \bar{\bar{A}} \tag{3.8}$$

Assertion (3.3) now follows from (3.4), (3.5), and (3.8) with $t = t_x$. □

To construct our example let Y be a random variable such that

$$|Y| \leqslant 2 \text{ a.s.} \tag{3.9}$$

and

$$EY = 0, \qquad EY^2 = 1, \quad \text{and} \quad EY^3 = -a < 0 \tag{3.10}$$

Let $Y_1, Y_2, \ldots$, be independent copies of Y. Choose $1 \leqslant p < 3/2$ and set

$$b_j = \exp[\lambda j/(Lj)^{2p}]/(L_j)^p, \qquad j \geqslant 1$$

with $\lambda > 0$ to be determined later. Define

$$X_j = b_j Y_j, \qquad j \geqslant 1 \tag{3.11}$$

Then we have

$$s_n^2 = \sum_{j=1}^{n} b_j^2 = \sum_{j=1}^{n} \exp[2\lambda j/(Lj)^{2p}]/(Lj)^{2p}$$

From an integral approximation of this sum we obtain

$$s_n^2 \sim (2\lambda)^{-1} \exp[2\lambda n/(Ln)^{2p}] \qquad \text{as} \quad n \to \infty \tag{3.12}$$

from which we get

$$LLs_n \sim Ln \qquad \text{as} \quad n \to \infty \tag{3.13}$$

By choosing $\lambda > 0$ small enough we have immediately

$$|X_n| \leqslant 2b_n \leqslant s_n/(LLs_n)^p \text{ a.s.} \tag{3.14}$$

Moreover, it follows from the definition of the sequence $X_1, X_2, \ldots$, that

$$-\sum_{j=1}^{n} EX_j^3 = a \sum_{j=1}^{n} b_j^3 \geqslant a(Ln)^{-p} \sum_{j=1}^{n} \exp[3\lambda j/(Lj)^{2p}]/(Lj)^{2p}$$
$$\sim a(Ln)^{-p}(3\lambda)^{-1} \exp[3\lambda n/(Ln)^{2p}]$$

which for some positive constant c_1 is

$$\sim c_1 s_n^3/(LLs_n)^p \qquad \text{as} \quad n \to \infty \tag{3.15}$$

We will now show that

$$P(S_n \geqslant (2s_n^2 LLs_n)^{1/2}, \text{i.o.}) = 0 \tag{3.16}$$

Notice that (3.16) implies that for the partial sum process formed by $X_1, X_2, \ldots,$

$$a(T) \sup_{1 \leqslant t \leqslant T} S(t)/\sqrt{t} - b(T) \to -\infty \text{ a.s.}$$

from which it is obvious that the Darling–Erdős theorem does not hold for this process.

We shall now establish (3.16). Let n_k be a subsequence of $\mathbb{N}$ such that for large enough k

$$\exp(k/Lk) \leqslant s_{n_k}^2 \leqslant \exp[(k+1)/Lk] \tag{3.17}$$

The existence of such a subsequence can be inferred from (3.1) and (3.2). To prove (3.16) it suffices to show that

$$\sum_{k=1}^{\infty} P(\max_{n_k < n \leqslant n_{k+1}} S_n \geqslant (2s_{n_k}^2 LLs_{n_k})^{1/2}) < \infty \tag{3.18}$$

Using the following Lévy type inequality for $1 \leqslant m \leqslant n$ and $x \geqslant 0$

$$p(\max_{m \leqslant j \leqslant n} S_j \geqslant x) \leqslant 2P\left(S_n \geqslant x - \left(2 \sum_{j=m}^{n} EX_j^2\right)^{1/2}\right)$$

and the definition of the subsequence n_k we get for all large k

$$\begin{aligned} &P(\max_{n_k < n \leqslant n_{k+1}} S_n \geqslant (2s_{n_k}^2 LLs_{n_k})^{1/2}) \\ &\quad \leqslant 2P(S_{n_{k+1}} \geqslant (2s_{n_k}^2 LLs_{n_k})^{1/2} - [2(s_{n_{k+1}}^2 - s_{n_k}^2)]^{1/2}) \\ &\quad \leqslant 2P(S_{n_{k+1}} \geqslant \sqrt{2} s_{n_{k+1}}[(LLs_{n_k})^{1/2} - c_2(LLs_{n_k})^{-1/2}]) \end{aligned}$$

where $c_2 > 0$ is a constant [use (3.17)]. Taking into account (3.13) and (3.15) we infer from Lemma 8 that for all large enough k, this last expression is

$$\leqslant c_3 \exp[-LLs_{n_k} - c_4(LLs_{n_k})^{3/2-p}]$$

where $c_3, c_4 > 0$ are constants. By our construction of the sequence n_k we see immediately that (3.18) holds. This completes the proof of the proposition. □

4. RESULTS FOR UNBOUNDED SEQUENCES

In this section we present some sufficient conditions for the Darling–Erdős theorem to hold for unbounded martingale difference sequence. Our first theorem considers independent sequences.

Theorem 2. Let $X_1, X_2, \ldots,$ be a sequence of independent mean zero random variables with finite variances. Set $\sigma_n^2 = EX_n^2$ and $s_n^2 = \sum_{i=1}^n \sigma_i^2$, $n \geqslant 1$. Assume that

$$s_n^2 \to \infty \qquad \text{as} \quad n \to \infty \tag{4.1}$$

Suppose further that for some $\delta > 0$ and all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P(|X_n| > \delta s_n (LLs_n)^{1/2}) < \infty \tag{4.2}$$

$$\sum_{n=1}^{\infty} s_n^{-2}(LLs_n)^{-1} E\{X_n^2 1[\varepsilon s_n/(LLs_n)^{3/2} < |X_n| \leqslant \delta s_n (LLs_n)^{1/2}]\} < \infty \tag{4.3}$$

and as $n \to \infty$

$$\frac{LLs_n}{s_n^2} \sum_{i=1}^{n} E\{X_i^2 1[|X_i| > \varepsilon s_i/(LLs_i)^{3/2}]\} \to 0 \tag{4.4}$$

Then the conclusions (2.3) and (2.4) of Theorem 1 hold.

Proof. We first note that by Theorem 3, page 345 of Chow and Teicher,[1] any sequence $X_1, X_2, \ldots,$ as above satisfies the law of the iterated logarithm. Thus, by arguing as in the proof of Theorem 1, it suffices to show that as $T \to \infty$

$$a(T) \max_{k \in B_1(T)} S_k/s_k - b(T) \xrightarrow{\mathscr{D}} E \tag{4.5}$$

and

$$a(T) \max_{k \in B_1(T)} |S_k|/s_k - b(T) \xrightarrow{\mathscr{D}} E \vee E' \tag{4.6}$$

We will only show (4.5); the proof of (4.6) is similar. It is clear that (4.5) would follow if we were to establish

$$a(s_n^2) \max_{k \in K_n} S_k/s_k - b(s_n^2) \xrightarrow{\mathscr{D}} E \qquad \text{as} \quad n \to \infty \tag{4.7}$$

where $K_n = B_1(s_n^2) = \{k: s_k^2 \geqslant h(s_n^2),\ 1 \leqslant k \leqslant n\}$ with h defined as in the proof of Theorem 1.

From the above assumptions it is easily seen that one can find two sequences of real numbers $\delta_n \uparrow \infty$ and $\varepsilon_n \uparrow 0$ such that (4.3) and (4.4) remain valid when δ and ε are replaced by δ_n and ε_n, respectively, so that

$$\sum_{n=1}^{\infty} s_n^{-2}(LLs_n)^{-1} E\{X_n^2 1[\varepsilon_n s_n/(LLs_n)^{3/2} < |X_n| \leqslant \delta_n s_n (LLs_n)^{1/2}]\} < \infty \tag{4.8}$$

and as $n \to \infty$

$$\frac{LLs_n}{s_n^2} \sum_{i=1}^{n} E\{X_i^2 1[|X_i| > \varepsilon_i s_i^2/(LLs_i)^{3/2}]\} \to 0 \tag{4.9}$$

We set for $n \geqslant 1$,

$$X_n' = X_n 1[|X_n| \leqslant \varepsilon_n s_n/(LLs_n)^{3/2}], \qquad \bar{X}_n' = X_n' - EX_n'$$

$$X_n''' = X_n 1[|X_n| > \delta_n s_n (LLs_n)^{1/2}], \qquad \bar{X}_n''' = X_n''' - EX_n'''$$

$$X_n'' = X_n - X_n' - X_n''', \qquad \bar{X}_n'' = X_n'' - EX_n''$$

Denote the corresponding sums by S_n', $\bar{S}_n'$, S_n''', $\bar{S}_n'''$, S_n'', and $\bar{S}_n''$.

We now apply a truncation argument of Feller[5] in a similar way as was done by Einmahl.[4]

Lemma 9. Under the above assumptions we have

$$P(\{|\bar{S}_n''| > s_n/LLs_n\} \cap \{|\bar{S}_n'| > s_n(LLs_n)^{1/2}\}, \text{ i.o.}) = 0$$

Proof. The proof is very similar to that of Lemma 1 of Einmahl,[4] so it will be enough to only indicate the main arguments.

First note that (4.8) and Kronecker's lemma immediately imply that

$$\bar{S}_n'' = o(s_n(LLs_n)^{1/2}) \text{ a.s.} \tag{4.10}$$

Next, without loss of generality we may assume that the sequence $\varepsilon_n \searrow 0$ has been chosen so that $\varepsilon_n \geqslant (LLs_n)^{-1/2}$, which in combination with (4.8) gives

$$\sum_{n=1}^{\infty} \frac{E|X_n''|}{s_n(LLs_n)^3} \leqslant \sum_{n=1}^{\infty} \frac{E(X_n'')^2}{s_n^2 LLs_n} < \infty \tag{4.11}$$

This when combined with Kronecker's lemma yields

$$ES_n'' = o(s_n(LLs_n)^3) \qquad \text{as} \quad n \to \infty \tag{4.12}$$

Furthermore, using the same argument as in Lemma 3 of Einmahl,[4] one gets from (4.11)

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2 \beta_n}{s_n^3 (LLs_n)^3} < \infty$$

where

$$\beta_n = \sum_{j=1}^{n} E|X_j''|, \qquad n \geqslant 1$$

Define two subsequence m_k and n_k of $\mathbb{N}$ as follows:

$$m_k = \min\{n \colon s_n^2 \geqslant 2^{k-2}/(Lk)^8\}$$

$$n_k = \min\{n \colon s_n^2 \geqslant 2^k\}, \qquad k \geqslant 1$$

Note that because of (4.1) both subsequences are well defined. We define the sets

$$F_k = \bigcup_{n=n_{k-1}}^{n_k - 1} \{|\bar{S}_n''| > s_n/LLs_n\} \cap \{|\bar{S}_n'| > s_n(LLs_n)^{1/2}\}$$

$$G_k = \bigcup_{n=m_k}^{n_k - 1} \{X_n'' \neq 0\}$$

$$H_k = \bigcup_{n=n_{k-1}}^{n_k - 1} \{|\bar{S}_n'| > s_n(LLs_n)^{1/2}\}$$

for $k \geqslant k_0$, where k_0 is a positive integer such that $n_k > n_{k-1}$ for all $k \geqslant k_0$. Such a k_0 exists since by (4.4) $s_{n+1}^2/s_n^2 \to 1$ as $n \to \infty$.

Notice that for $\omega \notin G_k$

$$|S_n''(\omega)| = |S_{m_k}''(\omega)|, \qquad n_{k-1} \leqslant n < n_k \tag{4.13}$$

which by (4.10), (4.12), and the definition of m_k implies that for any $\omega \notin G_k \cup N$, where N is a null set, with $k \geqslant k_1(\omega)$ one has

$$|S_n''(\omega)| < \tfrac{1}{2} s_n/LLs_n, \qquad n_{k-1} \leqslant n < n_k \tag{4.14}$$

Using (4.14) it is easy to see that to complete the proof of Lemma 8 it is sufficient to show

$$\gamma_1 := \sum_{k \in \mathbb{N}_1} P(G_k \cap H_k) < \infty \tag{4.15}$$

and

$$\gamma_2 := \sum_{k \in \mathbb{N}_2} P(H_k) < \infty \tag{4.16}$$

where $\mathbb{N}_1 := \{k: |ES''_n| \leqslant \frac{1}{2} s_n/LLs_n \text{ for all } n_{k-1} \leqslant n < n_k\}$ and $\mathbb{N}_2 = \mathbb{N} - \mathbb{N}_1$, since

$$\sum_{k=1}^{\infty} P(F_k) \leqslant \gamma_1 + \gamma_2$$

This can be accomplished by a straightforward modification of the proof of the corresponding relations (15) and (16) of Einmahl.[4] □

Lemma 10. Under the above assumptions, we have

$$P(\{|\bar{S}''_n| > s_n/LLs_n\} \cap \{|S_n| > \tfrac{5}{4} s_n (LLs_n)^{1/2}\}, \text{i.o.}) = 0$$

Proof. On account of Lemma 9 and (4.10), we only have to show that

$$\bar{S}'''_n = o(s_n/(LLs_n)^{1/2}) \text{ a.s.} \tag{4.17}$$

First, by (4.2) and the Borel–Cantelli lemma we have

$$S'''_n = O(1) \text{ a.s.}$$

Next

$$\begin{aligned} |ES'''_n| &\leqslant \sum_{j=1}^{n} E\{|X_j| \, 1[|X_j| > \delta_j s_j (LLs_j)^{1/2}]\} \\ &\leqslant \sum_{j=1}^{n} \delta_j^{-1} \sigma_j^2 / [s_j (LLs_j)^{1/2}] \\ &= o(s_n/(LLs_n)^{1/2}) \qquad \text{as} \quad n \to \infty \end{aligned}$$

This finishes the proof of Lemma 10. □

We are now prepared to complete the proof of Theorem 2. Observing that (4.17) implies

$$\limsup_{n \in \bar{\mathbb{N}}(\cdot)} \frac{|S_n - \bar{S}_n'|}{s_n/(LLs_n)^{1/2}} = 0 \text{ a.s.}$$

where $\bar{\mathbb{N}}(\cdot)$ is the random set of indices

$$\bar{\mathbb{N}}(\cdot) = \{n \in \mathbb{N}: |\bar{S}_n''| \leqslant s_n/LLs_n\}$$

we obtain from Lemmas 9 and 10 exactly as in the proof of Theorem 1 of Einmahl[4] [from (31) to (35)] that (4.7) is equivalent to

$$a(s_n^2) \max_{k \in K_n} \bar{S}_k'/s_k - b(s_n^2) \xrightarrow{\mathscr{D}} E \qquad \text{as} \quad n \to \infty \tag{4.18}$$

Set

$$\bar{s}_n^2 = \sum_{i=1}^{n} E(\bar{X}_i')^2, \qquad n \geqslant 1$$

Noticing that Kolmogorov's law of the iterated logarithm applies to the sequence $\bar{X}_1', \bar{X}_2',...,$ and using (4.4) we get

$$a(s_n^2) |\max_{k \in K_n} \bar{S}_k'/s_k - \max_{k \in K_n} \bar{S}_k'/\bar{s}_k| = o_p(1) \tag{4.19}$$

This last statement combined with the convergence of types theorem shows that (4.18) is equivalent to

$$a(\bar{s}_n^2) \max_{k \in K_n} \bar{S}_k'/\bar{s}_k - b(\bar{s}_n^2) \xrightarrow{\mathscr{D}} E \qquad \text{as} \quad n \to \infty \tag{4.20}$$

which in turn follows from Theorem 1 and its proof. Therefore we have established (4.7). □

Corollary 1. Let $X_1, X_2,...,$ be an i.i.d. sequence of mean zero random variables with variance one. Suppose that

$$LLtE\{X_1^2 1(|X_1| \geqslant t)\} \to 0 \qquad \text{as} \quad t \to \infty$$

Then we have as $n \to \infty$

$$a(n) \max_{1 \leqslant k \leqslant n} S_k/\sqrt{k} - b(n) \xrightarrow{\mathscr{D}} E$$

and

$$a(n) \max_{1 \leqslant k \leqslant n} |S_k|/\sqrt{k} - b(n) \xrightarrow{\mathscr{D}} E \vee E'$$

Proof. We apply Theorem 2. Conditions (4.1) and (4.2) are trivial to verify in this case. For the proof of (4.3) see Lemma 4 of Feller.[5] Finally (4.4) is an easy consequence of the above assumption. □

Corollary 2. Let $X_1, X_2, \ldots,$ be independent mean zero random variables with finite variances. Suppose that $s_n^2 \to \infty$ and for some $\eta > 0$

$$\limsup_{n \to \infty} E\{X_n^2 L|X_n| (LLL|X_n|)^{1+\eta}\}/EX_n^2 < \infty$$

Then the conclusions of Theorem 1 hold.

Proof. From Markov's inequality we obtain

$$\sum_{n=1}^{\infty} P(|X_n| > \delta s_n (LLs_n)^{1/2}) \leqslant C \sum_{n=1}^{\infty} \frac{EX_n^2}{s_n^2 LLs_n Ls_n (LLLs_n)^{1+\eta}} < \infty$$

Hence (4.2) holds. Similarly, one can verify (4.3) and (4.4). Thus the assertion is implied by Theorem 2. □

We now return to the martingale case. Since the Feller truncation method has not yet been extended to the martingale situation, we must apply somewhat different methods to obtain good results in this general setting.

Theorem 3. Whenever

$$s_n^2 \to \infty \qquad \text{as} \quad n \to \infty \ \text{a.s.} \tag{4.21}$$

and there exists a sequence $\varepsilon_n \in \mathscr{F}_{n-1}$ with $\varepsilon_n \to 0$ a.s. such that

$$\sum_{n=1}^{\infty} P(|X_n| > \varepsilon_n s_n/(LLs_n)^{1/2} \mid \mathscr{F}_{n-1}) < \infty \ \text{a.s.} \tag{4.22}$$

$$\sum_{n=1}^{\infty} \frac{(LLs_n)^2}{s_n^4} E(X_n^4 1[|X_n| \leqslant \varepsilon_n s_n/(LLs_n)^{1/2}] \mid \mathscr{F}_{n-1}) < \infty \ \text{a.s.} \tag{4.23}$$

and

$$\frac{LLs_n}{s_n^2} \sum_{i=1}^{n} E(X_i^2 1[|X_i| > \varepsilon_i s_i/(LLs_i)^{1/2}] \mid \mathscr{F}_{i-1}) \to 0 \ \text{a.s.} \tag{4.24}$$

then the conclusions of Theorem 1 hold.

Proof. Since the arguments are very similar to those employed in Theorems 1 and 2, we shall only give a brief sketch of the proof.

Wer first note that the proof can be reduced to a problem for bounded martingale difference sequences. Set for $n \geqslant 1$

$$\bar{X}_n = X_n 1[|X_n| \leqslant \varepsilon_n s_n/(LLs_n)^{1/2}]$$
$$\tilde{X}_n = \bar{X}_n - E(\bar{X}_n \mid \mathscr{F}_{n-1})$$

Then using (4.22) and (4.24) it follows by standard arguments that

$$\sum_{i=1}^{n} \tilde{X}_i - \sum_{i=1}^{n} X_i = o(s_n/(LLs_n)^{1/2}) \text{ a.s.} \qquad \text{as} \quad n \to \infty$$

Thus, it is enough to show that the Darling–Erdős theorem holds for the process $\tilde{S}_1(t)$, $t \geqslant 0$, defined to be

$$\tilde{S}_1(t) = \sum_{i=1}^{k} \tilde{X}_i, \qquad s_k^2 \leqslant t < s_{k+1}^2, \qquad k \geqslant 0$$

Setting

$$\tilde{s}_k^2 = \sum_{i=1}^{k} E(\tilde{X}_i^2 \mid \mathscr{F}_{i-1}), \qquad k \geqslant 1, \quad \tilde{s}_0^2 = 0$$

it is straightforward to deduce from condition (4.24) that

$$\tilde{S}_1(t) - \tilde{S}(t) = o((t/LLt)^{1/2}) \text{ a.s.} \qquad \text{as} \quad t \to \infty \tag{4.25}$$

where $\tilde{S}(t)$, $t \geqslant 0$ is the "natural" partial sum process defined by

$$\tilde{S}(t) = \sum_{i=1}^{k} \tilde{X}_i, \qquad \tilde{s}_k^2 \leqslant t < \tilde{s}_{k+1}^2, \qquad k \geqslant 0$$

On account of (4.25) to complete the proof, we need only establish the Darling–Erdős theorem for the process $\tilde{S}(t)$, $t \geqslant 0$. As in Section 2, let τ_n be stopping times such that for an appropriate Wiener process W

$$\sum_{i=1}^{n} \tilde{X}_i = W(T_n), \qquad n \geqslant 1$$

where

$$T_n = \sum_{i=1}^{n} \tau_i, \qquad n \geqslant 1$$

Combining (2.7) with (4.23) yields

$$\sum_{i=1}^{\infty} \frac{(LLs_i)^2}{s_i^4} E([\tau_i - E(\tau_i \mid \mathscr{F}_{i-1})]^2 \mid \mathscr{F}_{i-1}) < \infty \text{ a.s.}$$

which by Theorem 3.3.9, page 156 of Stout[15] gives

$$T_n - \bar{s}_n^2 = o(s_n^2/LLs_n) = o(\bar{s}_n^2/LL\bar{s}_n) \text{ a.s.} \tag{4.26}$$

The bounds on the martingale difference sequence $\tilde{X}_1, \tilde{X}_2,...,$ combined with (4.26) now allows us to repeat the proof of Theorem 1, nearly verbatim, thus establishing the Darling–Erdős theorem for the process $\tilde{S}(t)$, $t \geq 0$. □

Remark 2. Specializing to the case when $X_1, X_2,...,$ are i..i.d. mean zero random variables with variance one, it can be readily verified that the conditions of Theorem 3 hold if and only if $E(X_1^2 LL|X_1|) < \infty$. So in this situation Theorem 3 is not as sharp as Theorem 2; however, it still gives the best result that can be obtained by the direct strong approximation approach described in the Introduction.

Remark 3. Theorem 3 should be compared with Theorem 3.1 of Jain *et al.*[7] When $1 < \alpha < 2$ in their theorem the conditions there imply (4.22), (4.23), and (4.24). In this case they obtain the rate of approximation

$$|S(t) - W(t)|/\sqrt{t} = o((LLt)^{(1-\alpha)/2}) \qquad \text{as} \quad t \to \infty$$

Therefore it is not possible to infer our Theorem 3 from their result.

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APPENDIX

Lemma a. Let $\{X_n, \mathscr{F}_n, n \geq 0\}$ with $X_0 = 0$ and $\mathscr{F}_0 = \{\varnothing, \Omega)$ be a martingale difference sequence with finite second moments. Assume that for some $0 \leq \varepsilon < 1$ there exist sequences $\varepsilon_n \in \mathscr{F}_{n-1}$ with $\varepsilon_n \searrow \varepsilon$ a.s. and $V_n^2 \in \mathscr{F}_{n-1}$ with $V_n^2 \nearrow \infty$ a.s. such that for $n \geq 1$

$$s_n^2 - s_{n-1}^2 \leq V_n^2 - V_{n-1}^2 \text{ a.s.} \tag{5.1}$$

where $V_0^2 = s_0^2 = 0$, and for some $\rho \geqslant 1/2$

$$|X_n| \leqslant \varepsilon_n V_n/(LLV_n)^\rho \text{ a.s.} \tag{5.2}$$

Then for any sequence of nonnegative random variables $\tau_n \in \mathscr{F}_n$ satisfying for $n \geqslant 1$

$$E(\tau_n \mid \mathscr{F}_{n-1}) = E(X_n^2 \mid \mathscr{F}_{n-1}) \tag{5.3}$$

and for $r \geqslant 2$

$$E(\tau_n^r \mid \mathscr{F}_{n-1}) \leqslant 2\left(\frac{8}{\pi^2}\right)^{r-1} r!\, E(X_n^{2r} \mid \mathscr{F}_{n-1}) \tag{5.4}$$

we have

$$\limsup_{n \to \infty} \frac{|T_n - s_n^2|}{V_n^2/(LLV_n)^{\rho - 1/2}} \leqslant 32\varepsilon \text{ a.s.} \tag{5.5}$$

where $T_n = \sum_{j=1}^n \tau_j$, $n \geqslant 1$, and $T_0 = 0$.

Proof. Define the stopping times t_k, $k \geqslant 1$, by the following recursion:

$$t_1 = \max\{m: \varepsilon_m \geqslant 1) \vee \max\{m: V_m^2 \leqslant 1\}$$

where $\max \varnothing = 0$, and

$$t_{k+1} = \max\{m: V_m^2 \leqslant 2V_{t_k+1}^2\}, \qquad k \geqslant 1$$

Further, for each fixed integer $k \geqslant 1$ construct the sequence of stopping times

$$\alpha_j^{(k)} = (t_k + j) \wedge t_{k+1}, \qquad j \geqslant 0$$

Also for $k \geqslant 1$ set

$$Z_0^{(k)} = 0, \qquad Z_j^{(k)} = \sum_{i=1}^{j} [\tau_{\alpha_i^{(k)}} - E(\tau_{\alpha_i^{(k)}} \mid \mathscr{F}_{\alpha_{i-1}^{(k)}})], \qquad j \geqslant 1$$

and

$$Y_j^{(k)} = Z_j^{(k)} - Z_{j-1}^{(k)}, \qquad j \geqslant 1$$

Obviously we have for $k \geqslant 1$

$$E(Y_j^{(k)} \mid \mathscr{F}_{\alpha_{j-1}^{(k)}}) = 0, \qquad j \geqslant 1$$

Notice that since $Y_j^{(k)}=0$ when $\alpha_j^{(k)}=\alpha_{j-1}^{(k)}$, we have for all $r\geqslant 2$, $k\geqslant 1$, and $j\geqslant 1$

$$E(|Y_j^{(k)}|^r\,|\,\mathscr{F}_{\alpha_{j-1}^{(k)}})=\sum_{i=0}^{t_{k+1}-1} 1(\alpha_{j-1}^{(k)}=i)\,E(|\tau_{i+1}-E(\tau_{i+1}\,|\,\mathscr{F}_i)|^r\,|\,\mathscr{F}_i) \quad (5.6)$$

It easily follows from (5.2) and the definition of the stopping time t_k that

$$|X_n|\leqslant c_k \qquad \text{for} \quad t_k<n\leqslant t_{k+1} \quad (5.7)$$

where

$$c_k=2^{1/2}\varepsilon_{t_k+1}V_{t_k+1}/(LLV_{t_k+1})^\rho\in\mathscr{F}_{\alpha_0^{(k)}}$$

For any $k\geqslant 1$ and $\theta\in\mathscr{F}_{\alpha_0^{(k)}}$ set $M_0^{(k)}=1$ and for $j\geqslant 1$, let

$$M_j^{(k)}(\theta)=\exp(\theta Z_j^{(k)})\exp\left\{-\frac{128\theta^2}{\pi^2}c_k^2(V^2_{\alpha_j^{(k)}}-V^2_{\alpha_0^{(k)}})\,1(\alpha_{j-1}^{(k)}<t_{k+1})\right\}$$

From now on for notational convenience we shall suppress the superscript (k).

Lemma b. For any $k\geqslant 1$ and $\theta\in\mathscr{F}_{\alpha_0}$ such that

$$\frac{16c_k^2|\theta|}{\pi^2}\leqslant\frac{1}{2}\ \text{a.s.} \quad (5.8)$$

$\{M_j(\theta),\ \mathscr{F}_{\alpha_j},\ j\geqslant 0\}$ is a nonnegative supermartingale.

Proof. Notice that for $r\geqslant 2$, $k\geqslant 1$, and $j\geqslant 1$, by (5.6), the c_r-inequality, and (5.3) we have

$$\begin{aligned}E(|\theta Y_j|^r\,|\,\mathscr{F}_{\alpha_{j-1}})&\leqslant|\theta 2|^r\sum_{i=0}^{t_{k+1}-1}1(\alpha_{j-1}=i)\,E(\tau_{i+1}^r\,|\,\mathscr{F}_i)\\&\leqslant|\theta 2|^r\left(\frac{8}{\pi^2}\right)^{r-1}2r!\,E(X_{\alpha_j}^{2r}\,|\,\mathscr{F}_{\alpha_{j-1}})\,1(\alpha_{j-1}<t_{k+1})\ \text{a.s.}\end{aligned}$$

which by application of (5.1) and (5.7) is a.s.

$$\leqslant\frac{\pi^2}{4c_k^2}\left\{|\theta|\,\frac{16c_k^2}{\pi^2}\right\}^r r!\,E(X_{\alpha_j}^2\,|\,\mathscr{F}_{\alpha_{j-1}})\,1(\alpha_{j-1}<t_{k+1})$$

Hence by (5.4) and $1+x\leqslant e^x$, $-\infty<x<\infty$,

$$\begin{aligned}E(\exp(\theta Y_j)\,|\,F_{\alpha_{j-1}})&\leqslant 1+\frac{\pi^2}{4c_k^2}\sum_{r=2}^{\infty}\left\{\frac{|\theta|\,16c_k^2}{\pi^2}\right\}^r E(X_{\alpha_j}^{(2}\,|\,\mathscr{F}_{\alpha_{j-1}})\,1(\alpha_{j-1}<t_{k+1})\\&\leqslant\exp\left\{\frac{128}{\pi^2}\theta^2c_k^2E(X_{\alpha_j}^2\,|\,\mathscr{F}_{\alpha_{j-1}})\,1(\alpha_{j-1}<t_{k+1})\right\}\end{aligned}$$

Using this last inequality it is now easily checked that

$$E(M_j(\theta)\,|\,\mathscr{F}_{\alpha_{j-1}}) \leqslant M_{j-1}(\theta), \qquad j \geqslant 1 \qquad \square$$

For later use we record that under the assumptions of Lemma c for all $\lambda > 0$

$$P(\sup_{j \geqslant 0} M_j(\theta) > \lambda) < \lambda^{-1} \tag{5.9}$$

cf. Corollary 5.4.1, page 299 of Stout.[15]

Set for $k \geqslant 1$

$$\theta_k = \frac{\pi^2 (LLV_{t_k+1})^{1/2+\rho}}{64\, V_{t_k+1}^2 \varepsilon_{t_k+1}}$$

and

$$a_k = \varepsilon_{t_k+1} V_{t_k+1}^2 / (LLV_{t_k+1})^{\rho - 1/2}$$

Since $\varepsilon_n \leqslant 1$ for $n \geqslant t_1$, we have

$$\frac{16\theta_k c_k^2}{\pi^2} \leqslant \frac{1}{2}$$

We also have

$$\theta_k a_k = \frac{\pi^2}{64} LLV_{t_k+1}$$

and by the construction of the stopping times

$$\frac{128}{\pi^2} \theta_k^2 c_k^2 V_{t_k+1}^2 \leqslant \frac{\pi^2}{8} LLV_{t_k+1}$$

Lemma c.

$$P(\max_{t_k < n \leqslant t_{k+1}} |T_n - s_n^2 - (T_{t_k} - s_{t_k}^2)| > 16 a_k, \text{ i.o.}) = 0 \tag{5.10}$$

Proof. Now for $k \geqslant 1$

$$\begin{aligned} &P(\sup_{j \geqslant 0} \exp(\pm \theta_k Z_j) \geqslant \exp(16\theta_k a_k)) \\ &\quad \leqslant P\left(\sup_{j \geqslant 0} M_j(\pm\theta_k) \geqslant \exp\left(16\theta_k a_k - \frac{128}{\pi^2} \theta_k^2 c_k^2 V_{t_k+1}^2\right)\right) \end{aligned}$$

which by the above calculations and $V_{t_k+1} \geqslant 2^{(k-1)/2}$ is

$$\leqslant P\left(\sup_{j\geqslant 0} M_j(\pm\theta_k) \geqslant \exp\left\{\frac{\pi^2}{8}\,[LL2^{(k-1)/2}]\right\}\right)$$

Inequality (5.9) combined with the Borel–Cantelli lemma lemma shows that the probability in (5.10) is equal to zero. □

Recalling the definition of the stopping times t_k, we easily obtain from Lemma c that for all $\delta > 0$ and almost every $\omega \in \Omega$

$$\begin{aligned}|T_j(\omega) - s_j^2(\omega)| &\leqslant c(\omega, \delta) + \sum_{k=1}^{m} 16(\varepsilon+\delta)\, V^2_{t_k(\omega)+1}(\omega)/[LLV_{t_k(\omega)+1}(\omega)]^{\rho-1/2} \\ &\leqslant c(\omega, \delta) + 32(\varepsilon+\delta)\, V^2_{t_m(\omega)+1}(\omega)/[LLV_{t_m(\omega)+1}(\omega)]^{\rho-1/2} \\ &\leqslant c(\omega, \delta) + 32(\varepsilon+\delta)\, V_j^2(\omega)/[LLV_j(\omega)]^{\rho-1/2}\end{aligned}$$

for $t_m(\omega) < j \leqslant t_{m+1}(\omega)$. Thus, we have shown that for all $\delta > 0$

$$\limsup_{n\to\infty} \frac{|T_n - s_n^2|}{V_n^2/(LLV_n)^{\rho-1/2}} \leqslant 32(\varepsilon+\delta) \text{ a.s.}$$

which obviously implies (5.5). □

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