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Interface Motion in Models with Stochastic Dynamics

Herbert Spohn[1]

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We derive the phenomenological dynamics of interfaces from stochastic "microscopic" models. The main emphasis is on models with a nonconserved order parameter. A slowly varying interface has then a local normal velocity proportional to the local mean curvature. We study bulk models and effective interface models and obtain Green–Kubo-like expressions for the mobility. Also discussed are interface motion in the case of a conserved order parameter, pure surface diffusion, and interface fluctuations. For the two-dimensional Ising model at zero temperature, motion by mean curvature is established rigorously.

KEY WORDS: Ginzburg–Landau models A and B; spin-flip models; lattice gases; interfacial dynamics; motion by mean curvature; Green–Kubo formula for the interfacial mobility.

1. INTRODUCTION

Commonly, distinct thermodynamic phases are spatially separated through a sharp transition region where the order parameter changes rapidly from one phase to the other. For many reasons it is of interest to understand the dynamical behavior of this interface. In the simplest case the bulk dynamics does not conserve the order parameter. Then, according to the standard phenomenological theory,[1-3] one represents the interface as a smooth surface and postulates that the local normal velocity is proportional to the local mean curvature. Our aim is to relate this phenomenological theory to microscopic models.

As a general program, our task is comparable to the derivation of fluid dynamics either from the Boltzmann equation or, more fundamentally, from the Hamiltonian dynamics of interacting microscopic particles. In

[1] Theoretische Physik, Ludwig-Maximilians-Universität, D-8000 Munich 2, Germany.

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both cases there are slowly varying "modes." For fluids they arise from the conservation laws: transport over macroscopic distances is constrained by conservation and therefore takes a long time. For interfaces these are the slow Goldstone modes due to the broken translational symmetry, which force the intrinsic interface motion to be much slower than the bulk relaxation. For a fluid the hydrodynamic fields vary on a macroscopic scale. Locally the fluid is in equilibrium. Similarly a small piece of the interface should be flat on the average with the local statistics of the order parameter given by the equilibrium distribution. Thus the common issue is a separation of space-time scales which allows for a macroscopic (contracted) description.

The derivation of hydrodynamics is a very traditional subject, well exposed in many textbooks on kinetic theory and nonequilibrium statistical mechanics. The common belief (which does not secure against surprises) is that the physical picture is clear and the derivation well understood on a formal level. To push such a program further toward mathematical rigor leads to interesting, deep dynamical questions to many of which we do not have even the glimpse of an answer. Still one expects (and hopes) that the traditional physics wisdom will be corroborated eventually. For interface motion such a coherent picture is missing or, at least, has not been stated with the standard of precision which we are used to in equilibrium statistical mechanics. Our primary goal is to supply the missing pieces.

At this point it is necessary to spell out the microscopic models and the corresponding phenomenological theory in more detail. A prototypical model is the ferromagnetic Ising model at sufficiently low temperatures and at phase coexistence (external field $h=0$). We assume stochastic spin-flip dynamics satisfying the condition of detailed balance. In particular, the order parameter is *not* conserved. We further assume to have the + phase coexisting with the − phase separated by an interface which varies slowly on the scale of the lattice. Therefore, on a scale large compared to the lattice constant, we can represent the interface as a smooth surface Σ embedded in $\mathbb{R}^3$. The driving force for the interface motion is supplied by a lowering of the surface free energy. Let us introduce the surface tension σ. In general it depends on the local orientation of the surface element, i.e., on the local normal $\hat{n}$. The surface free energy is then defined by

$$F=\int_{\Sigma} df\, \sigma(\hat{n}) \tag{1.1}$$

Phenomenologically, it is postulated that the interface velocity along the local normal is given by

$$v_n=-\mu\frac{\delta F}{\delta \Sigma} \tag{1.2}$$

Here μ is called the mobility of the interface. In general, it depends also on the local orientation. $\delta F/\delta \Sigma$ is the change in free energy due to a small variation of the surface. If σ and μ are isotropic, then v_n is proportional to the local mean curvature. Therefore we refer to (1.2) as the *mean curvature equation* in general.

The core of our paper deals with the derivation of Eq. (1.2) from the underlying stochastic model. For this undertaking, at least, the quantities appearing in the mean curvature equation should have a precise microscopic meaning. The surface tension has a well-established statistical mechanics definition,[4,5] which will be briefly recalled at the end of Section 2. On the other hand, the mobility has been discussed so far only in rudimentary form. Thus our first task is to find a microscopic expression for the mobility. This will be done in Section 3, where through a linear response argument we obtain a formula rather analogous to the Green–Kubo formula for transport coefficients in fluids. Having identified the quantities on the right side of Eq. (1.2), we turn in Section 5 to the problem of establishing that the motion of the interface is governed by mean curvature in the limit of slow variation.

On the microscopic scale the interface is noisy and there is no sharp distinction between bulk phases and the transition region. A typical interface width is expected to be of the order of the bulk correlation length. For many applications such an intrinsic structure can be ignored. This results then in effective interface models, for which the bulk is structureless by definition and the interface is a single-valued function with respect to a given reference plane. The reasoning leading to the phenomenological theory still applies and a slowly varying interface should also be governed by Eq. (1.2). Effective interface models will be discussed in Sections 6 and 7, where we also point out a direct link to the hydrodynamic limit of lattice gases in the case of two bulk dimensions.

Once motion by mean curvature is established, one can study small Gaussian fluctuations around the deterministic behavior. Our discussion in Section 9 is standard and included only for the sake of completeness.

There has been considerable mathematical activity with the goal to establish the existence of the large-scale limit for stochastic models.[6,7] So far, only bulk dynamics with a conservation law has been studied. It is therefore a natural problem to investigate whether our reasoning can be pushed to mathematical rigor. Our results are rather modest. We prove motion by mean curvature for the two-dimensional Ising model at zero temperature (Appendix A). In Appendix B we present partial results on the Ginzburg–Landau effective interface model. Basically, we lack a uniqueness theorem for the corresponding equilibrium Gibbs measures. We hope that the appendices serve as an invitation to further investigations.

For the everyday phase coexistence of liquid and vapor the conservation laws are of crucial importance for the interface dynamics. There is no difficulty to incorporate such a conservation law into the stochastic model. For example, in our prototypical example of the Ising model one simply has to substitute the spin-flip dynamics by spin-exchange dynamics. Then the order parameter is dynamically conserved. Clearly, the interface motion then changes drastically and Eq. (1.2) is no longer applicable. In Section 10 we derive the interface dynamics in the case of a conserved order parameter. In fact, beyond bulk diffusion no further transport coefficients appear in the equations of motion. There is a somewhat different route of how a conservation law may show up. Effective interface models are used to describe a crystal in equilibrium with its gas phase. If ad- and desorption processes are very slow, then relaxation is governed by pure surface diffusion and therefore the height relative to the reference plane is conserved. The corresponding equations of motion turn out to be a conventional modification of Eq. (1.2). However, the analog of the mobility acquires a very different physical interpretation.

In (1.2) it is implicitly assumed that the surface tension depends smoothly on $\hat{n}$. This assumption breaks down below the roughening temperature of a high-symmetry plane, where the surface tension has a cusp. In Section 8 we explain how Eq. (1.2) extends to such a singular situation. Among other novel properties we find that facets may be formed spontaneously.

2. MICROSCOPIC MODELS, SURFACE TENSION

Models with a nonconserved order parameter are standardized.[8–10] One distinguishes Ginzburg–Landau models where the field variables take continuous values and spin-flip models with values ± 1 only. Their common feature is that at zero temperature they have two configurations of minimal energy related by flip symmetry. Neighboring variables are sufficiently strongly coupled so that at small temperatures two stable phases persist.

Let us first describe the Ginzburg–Landau model on the simple hypercubic lattice $\mathbb{Z}^d$, $d \geqslant 2$. At each lattice site $x \in \mathbb{Z}^d$ there is a continuous field variable $\phi(x)$. A field configuration ϕ in volume Λ has the energy

$$H(\phi) = \sum_{\langle x, y \rangle, x, y \in \Lambda} \tfrac{1}{2}\kappa[\phi(x) - \phi(y)]^2 + \sum_{x \in \Lambda} V(\phi(x)) \tag{2.1}$$

Here $\langle \cdot, \cdot \rangle$ denotes a pair of nearest neighbors, $\kappa > 0$, and the potential V has the usual reflection-symmetric double-well shape. In particular, $V(\phi) =$

$V(-\phi)$, $V(\phi) \geqslant c_1 \phi^2 + c_2$, $c_1 > 0$, $V'(m_+) = V'(m_-) = V'(0) = 0$, $m_+ = -m_- = m^* > 0$, $V''(m_+) = V''(m_-) > 0$, $V''(0) < 0$.

The order parameter field ϕ is governed by the stochastic differential equations

$$\frac{d}{dt}\phi_t(x) = \kappa \Delta \phi_t(x) - V'(\phi_t(x)) + (2/\beta)^{1/2} \dot{w}_t(x) \tag{2.2}$$

$x \in \Lambda$. Here Δ is the lattice Laplacian. $\dot{w}_t(x)$ is normalized white noise, independent at distinct lattice sites, $\langle \dot{w}_s(x)\, \dot{w}_t(y) \rangle = \delta_{xy}\delta(s-t)$. β stands for the inverse temperature. Clearly, the drift in (2.2) is the gradient of H. Therefore, in finite volume Λ, the dynamics (2.2) has as unique invariant measure the Gibbs distribution

$$Z^{-1} e^{-\beta H(\phi)} \prod_{x \in \Lambda} d\phi(x) \tag{2.3}$$

and the dynamics is stochastically reversible with respect to this measure. If β is sufficiently large, then in infinite volume the reflection symmetry is broken. There are two distinct, translation-invariant phases, $\langle \cdot \rangle_+$ and $\langle \cdot \rangle_-$, with $\langle \phi(x) \rangle_+ = -\langle \phi(x) \rangle_- > 0$, both stationary under the dynamics (2.2).

In the literature[9] one often considers the formal continuum limit of (2.2), yielding

$$\frac{\partial}{\partial t}\phi(x, t) = \kappa \Delta \phi(x, t) - V'(\phi(x, t)) + (2/\beta)^{1/2} \dot{w}(x, t) \tag{2.4}$$

with $\dot{w}$ normalized space-time white noise. For $d \geqslant 2$, Eq. (2.4) is not well defined and some short-distance cutoff, as in (2.2), is needed. In the zero-noise limit, $\beta \to \infty$, Eq. (2.4) goes over to the Allen–Cahn equation[11]

$$\frac{\partial}{\partial t}\phi(x, t) = \kappa \Delta \phi(x, t) - V'(\phi(x, t)) \tag{2.5}$$

In the case of spin-flip dynamics we choose specifically the ferromagnetic Ising model with nearest neighbor couplings. At each site of the lattice there is a spin variable $\sigma(x)$ taking values ± 1. A spin configuration is denoted by σ. It has the energy

$$H(\sigma) = -J \sum_{\langle x, y \rangle, x, y \in \Lambda} \sigma(x)\, \sigma(y) \tag{2.6}$$

with the coupling constant $J > 0$. (We could allow for more general couplings, as long as they are translation invariant, ferromagnetic, symmetric under spin flips, and of finite range.)

In order to define the dynamics, we have to specify the flip rates. We allow only single spin flips. The spin at site x reverses its sign with rate $c_x(\sigma)$, depending on σ only through the spin configuration close to x. The specific functional form of c_x is not important. We only require that $c_x(\sigma) > 0$, $c_x(\sigma)$ is translation invariant, and $c_x(\sigma)$ satisfies the condition of detailed balance in the form

$$c_x(\sigma) = c_x(\sigma^x) \exp\{-\beta[H(\sigma^x) - H(\sigma)]\} \tag{2.7}$$

Here σ^x stands for the configuration σ with the sign of $\sigma(x)$ reversed and $H(\sigma^x) - H(\sigma)$ is the energy difference in the flip from $\sigma(x)$ to $-\sigma(x)$. The generator of the flip process is defined by

$$Lf(\sigma) = \sum_{x \in \Lambda} c_x(\sigma)[f(\sigma^x) - f(\sigma)] \tag{2.8}$$

A probability distribution on spin configurations $\rho_t(\sigma)$ evolves according to the master equation

$$\frac{d}{dt}\rho_t(\sigma) = \sum_{x \in \Lambda} [c_x(\sigma^x)\,\rho_t(\sigma^x) - c_x(\sigma)\,\rho_t(\sigma)] \equiv L^*\rho_t(\sigma) \tag{2.9}$$

In finite volume the master equation has the unique stationary solution

$$Z^{-1} e^{-\beta H(\sigma)} \tag{2.10}$$

As for the Ginzburg–Landau model, if β is sufficiently large, then for infinite volume the spin-flip symmetry is broken. There are two ferromagnetic phases, $\langle \cdot \rangle_+$ and $\langle \cdot \rangle_-$, with $\langle \sigma(x) \rangle_+ = -\langle \sigma(x) \rangle_- > 0$, both stationary for (2.9).

Next we recall the definition of the surface tension, $\sigma(\hat{n})$. The basic idea is to compute the excess free energy due to the presence of an interface. We choose a box Λ such that $|x_j| \leqslant l$, $j = 1, \ldots, d-1$, and $|x_d| \leqslant M$, $M \gg l$. We force an interface with average normal $\hat{n}$ by imposing $+-$ boundary conditions. This means to fix $\sigma(x) = 1$ for $x \in \partial\Lambda \cap \{x \mid \hat{n} \cdot x \leqslant 0\}$ and $\sigma(x) = -1$ for $x \in \partial\Lambda \cap \{x \mid \hat{n} \cdot x > 0\}$ with $\partial\Lambda$ the set of sites bordering Λ. Let $Z_{+-}(l, M)$ be the corresponding partition function and let $Z_{++}(l, M)$ be the partition function for the all $+$ boundary condition. The excess free energy is proportional to the interface area. Therefore

$$\sigma(\hat{n}) = |\hat{n}_d| \lim_{l \to \infty} (2l+1)^{-(d-1)} \times \lim_{M \to \infty} \left\{ -\frac{1}{\beta} [\log Z_{+-}(l, M) - \log Z_{++}(l, M)] \right\} \tag{2.11}$$

The prefactor $|\hat{n}_d|$ is included because $\sigma(\hat{n})$ is defined as the free energy excess per unit *Euclidean area.* Clearly the full $\sigma(\hat{n})$ has to be pieced together from the various sectors where (2.11) makes sense geometrically. $\sigma(\hat{n})$ is strictly positive and vanishes as $\beta \downarrow \beta_c$. For $d=2$, $\sigma(\hat{n})$ is known explicitly.[5,12]

In the case of the Ginzburg–Landau model we adopt the same route. The field is now fixed at $\phi(x)=m_+$ for $x \in \partial\Lambda \cap \{x \mid \hat{n} \cdot x \leqslant 0\}$ and at $\phi(x)=m_-$ for $x \in \partial\Lambda \cap \{x \mid \hat{n} \cdot x > 0\}$. Here $Z_{+-}(l, M)$ is the corresponding partition function and $Z_{++}(l, M)$ the one with all + boundary conditions. The surface tension $\sigma(\hat{n})$ is again defined through (2.11).

3. MOBILITY

The mobility plays the same role for interfaces as transport coefficients do for fluids. Thus we would expect also some sort of Green–Kubo formula. To derive it, we employ a linear response argument, which we carry out for the case of the stochastic Ising model. We study the response of the interface to a uniform small external magnetic field h, $h>0$. Let $\langle \sigma(x) \rangle_+ = m_+ = -m_- = m^*$ be the spontaneous magnetization. Then, on the macroscopic scale, a small deformation of the surface yields the extra free energy $-h(m_+ - m_-)\,\delta\Sigma$. Therefore the local normal velocity becomes [compare with (1.2)]

$$v_n = -\mu \left[\frac{\delta F}{\delta \Sigma} - h(m_+ - m_-) \right] \tag{3.1}$$

(We adopt here the convention that the normal points into the − phase.) Thus the magnetic field induces the systematic velocity

$$v = \mu h (m_+ - m_-) \tag{3.2}$$

If we could obtain a microscopic expression for the systematic interface velocity, Eq. (3.2) would identify the mobility μ.

Let us enforce an equilibrium interface with average normal $\hat{n}$ by imposing appropriate + − boundary conditions as explained at the end of the previous section. Note that the boundary conditions also enter into the flip rates. We impose now a small external field by modifying $c_x(\sigma)$ to the new flip rates $c_x^{(h)}(\sigma)$. As before they must satisfy the condition of detailed balance

$$c_x^{(h)}(\sigma) = c_x^{(h)}(\sigma^x) \exp\{-\beta[H(\sigma^x) - H(\sigma) - 2h\sigma(x)]\} \tag{3.3}$$

We study the change in magnetization,

$$\frac{d}{dt}<\sum_x \sigma(x)>(t) = -2\sum_x \langle c_x^{(h)}(\sigma)\,\sigma(x)\rangle(t) \tag{3.4}$$

where $\langle\cdot\rangle(t)$ denotes expectation with respect to the spin distribution at time t. It contains two contributions: (1) a change $v(m_+ - m_-)\,|\hat{n}_d|^{-1}$ $(2l+1)^{d-1}$ due to the displacement of the interface with average velocity v, and (2) a bulk change. We hope to subtract out this second contribution by the change in magnetization when starting initially in an equilibrium state with all + boundary conditions. If we do this, we obtain

$$\begin{aligned} v \cong\ & |\hat{n}_d|\,(m_+ - m_-)^{-1}\,(2l+1)^{-(d-1)} \\ & \times \sum_x (-2)[\langle c_x^{(h)}(\sigma)\,\sigma(x)\rangle_{+-}(t) - \langle c_x^{(h)}(\sigma)\,\sigma(x)\rangle_{++}(t)] \end{aligned} \tag{3.5}$$

Here the index $+-$ $(++)$ indicates the initial state and the boundary conditions for the dynamics. For the interface velocity v to be approximately time independent, t has to be large compared to the bulk relaxation time. On the other hand, t has to be small compared to the time when the pinning at the boundary becomes important and to the time when nucleation of the + phase in the bulk − phase sets in, which would then completely destroy the interface.

As always in linear response arguments of this kind, one hopes to single out the plateau value by first letting $h \to 0$ and then $t \to \infty$. We differentiate (3.4) at $h=0$, denoted by a prime. This yields

$$\begin{aligned} & \left[-2\sum_{x\in\Lambda}\langle c_x^{(h)}(\sigma)\,\sigma(x)\rangle_{+-}(t)\right]' \\ & = -2\beta\sum_{x\in\Lambda}\left[\langle e^{Lt}c_x'(\sigma)\,\sigma(x)\rangle_{+-} + \int_0^t ds\,\langle e^{L(t-s)}L' e^{Ls}c_x(\sigma)\,\sigma(x)\rangle_{+-}\right] \\ & = 2\beta\left[\sum_{x\in\Lambda}\langle c_x(\sigma)\rangle_{+-} - 2\int_0^t ds \sum_{x,y\in\Lambda}\langle c_x(\sigma)\,\sigma(x)\,e^{Ls}c_y(\sigma)\,\sigma(y)\rangle_{+-}\right] \end{aligned} \tag{3.6}$$

and correspondingly for $++$. We used the condition (3.3) of detailed balance and that the initial state $\langle\cdot\rangle_{+-}$ is stationary. Properly speaking, we should add indices $+-$ both at the generator L and the rates $c_x(\sigma)$ because they depend on frozen boundary spins in $\partial\Lambda$.

Since the interface is pinned at the boundary, we first take the limit $l, M \to \infty$ so as to get rid of spatial boundary effects. Afterward we take the

limit $t \to \infty$ in order to single out the stationary response. By comparing with (3.2), we arrive at the following Green–Kubo-like formula for the mobility:

$$\mu(\hat{n}) = 2\beta(m_+ - m_-)^{-2} |\hat{n}_d| \left\{ \lim_{l \to \infty} (2l+1)^{-(d-1)} \right.$$
$$\times \lim_{M \to \infty} \sum_{x \in \Lambda} (\langle c_x(\sigma) \rangle_{+-} - \langle c_x(\sigma) \rangle_{++}) - 2 \int_0^\infty dt \lim_{l \to \infty} (2l+1)^{-(d-1)}$$
$$\times \lim_{M \to \infty} \sum_{x, y \in \Lambda} (\langle c_x(\sigma)\, \sigma(x)\, e^{Lt} c_y(\sigma)\, \sigma(y) \rangle_{+-}$$
$$\left. - \langle c_x(\sigma)\, \sigma(x)\, e^{Lt} c_y(\sigma)\, \sigma(y) \rangle_{++}) \right\} \tag{3.7}$$

Note that only *equilibrium* time correlations enter.

The mobility depends on the orientation of the interface, which is hidden in the definition of the $\langle \cdot \rangle_{+-}$ state. The first sum is the average flip rate compared in a state with and without interface. It is rather plausible that this difference decays rapidly with the distance away from the interface and that therefore the first sum is of order $(2l+1)^{(d-1)}$. The second term in (3.7) reflects that the response to the external field is not instantaneous.

The Green–Kubo integrand is fairly complicated. We will discuss its zero-temperature form in the following section, expecting that this reflects correctly the whole low-temperature regime, $\beta \gg \beta_c$.

The linear response argument for the Ginzburg–Landau model runs in complete parallel. The term corresponding to $c_x(\sigma)$ in (3.6) equals one and therefore the first difference in (3.7) cancels. The term corresponding to $2c_x(\sigma)\, \sigma(x)$ in (3.6) equals $\kappa \Delta\phi(x) - V'(\phi(x))$. Therefore the mobility in the Ginzburg–Landau model is given by

$$\mu(\hat{n}) = -\beta(m_+ - m_-)^{-2} \int_0^\infty dt \lim_{l \to \infty} (2l+1)^{-(d-1)}$$
$$\times \lim_{M \to \infty} \sum_{x, y \in \Lambda} \{ \langle [\kappa \Delta\phi_0(x) - V'(\phi_0(x))][\kappa \Delta\phi_t(y) - V'(\phi_t(y))] \rangle_{+-}$$
$$- \langle [\kappa \Delta\phi_0(x) - V'(\phi_0(x))][\kappa \Delta\phi_t(y) - V'(\phi_t(y))] \rangle_{++} \} \tag{3.8}$$

with $m_+ = \langle \phi(x) \rangle_+ = -m_-$.

According to the fluctuation-dissipation theorem, the Green–Kubo formula must also yield the fluctuations in the total magnetization. To be

specific, let us consider the stochastic Ising model. The total magnetization in volume Λ at time t equals then

$$M_\Lambda(\sigma_t) = \sum_{x \in \Lambda} \sigma_t(x) \tag{3.9}$$

Now

$$M_\Lambda(\sigma_t) - M_\Lambda(\sigma) = -2 \int_0^t ds \sum_{x \in \Lambda} c_x(\sigma_s)\, \sigma_s(x) + \mathcal{M}(t) \tag{3.10}$$

with $\mathcal{M}(t)$ a martingale. Since σ_t is a Markov process, we have

$$\langle \mathcal{M}(t)^2 \rangle = 2t \langle M_\Lambda(\sigma)(-L)\, M_\Lambda(\sigma) \rangle \tag{3.11}$$

assuming that $\langle \cdot \rangle$ is stationary. In (3.10) we take the integral to the left side, square, and average. Since the cross term vanishes by reversibility, we have

$$\begin{aligned} &\langle [M_\Lambda(\sigma_t) - M_\Lambda(\sigma)]^2 \rangle \\ &\quad = 4t \sum_{x \in \Lambda} \langle c_x(\sigma) \rangle - 4 \int_0^t ds \int_0^t ds' \sum_{x, y \in \Lambda} \langle c_x(\sigma_s)\, \sigma_s(x)\, c_y(\sigma_{s'})\, \sigma_{s'}(y) \rangle \end{aligned} \tag{3.12}$$

Now if $\langle \cdot \rangle = \langle \cdot \rangle_{++}$, then the first summand proportional to t is canceled by the leading contribution from the time integral. For large t we obtain then a constant proportional to the volume $|\Lambda|$. On the other hand, if $\langle \cdot \rangle = \langle \cdot \rangle_{+-}$, then for large t there is a constant contribution proportional to $|\Lambda|$, which may be canceled by subtracting $\langle \cdot \rangle_{++}$, and in addition a contribution which grows linearly in time and is proportional to the interface area. It is this "diffusion coefficient" which is singled out by Eq. (3.7),

$$\begin{aligned} \mu(\hat{n}) = \beta(m_+ - m_-)^{-2} \Big\{ &\lim_{t \to \infty} \frac{1}{2t} \lim_{l \to \infty} (2l+1)^{-(d-1)} \\ &\times \lim_{M \to \infty} (\langle [M_\Lambda(\sigma_t) - M_\Lambda(\sigma)]^2 \rangle_{+-} - \langle [M_\Lambda(\sigma_t) - M_\Lambda(\sigma)]^2 \rangle_{++}) \Big\} \end{aligned} \tag{3.13}$$

Just as for transport coefficients in fluids, the mobility can be obtained either from a linear response to a bulk magnetic field or from the fluctuations in the bulk magnetization.

4. ZERO-TEMPERATURE (NOISE) LIMIT

It is instructive to check our linear response argument first for the Allen–Cahn equation as one of the few microscopic models with a nonconserved order parameter for which the mobility has been computed before. Clearly, in this equation there is no mechanism for nucleation. Thus we can let $t \to \infty$ first, thereby obtaining a constant interface velocity, and then $h \to 0$.

The Allen–Cahn equation with an external field h, $h > 0$, reads

$$\frac{\partial}{\partial t}\phi = \kappa \Delta\phi - V'(\phi) + h \tag{4.1}$$

Since only the direction orthogonal to the interface is relevant, we may as well take $d = 1$ in (4.1). We are looking for a traveling front (kink) solution of the form $w_h(x - v(h)t)$ which satisfies

$$\kappa w_h'' + v(h)\, w_h' - V'(w_h) + h = 0 \tag{4.2}$$

We choose h sufficiently small, such that there are still only two stable fixed points, $m_+(h)$ and $m_-(h)$, satisfying $V'(m_\pm(h)) = h$. The kink solution is from one stable to the other stable fixed point and satisfies the boundary conditions $w_h(\mp\infty) = m_\pm(h)$. Equation (4.2) then has a unique solution provided we fix the origin, e.g., by requiring $w_h(0) = 0$. For $h = 0$, we have $v(h) = 0$. We expand as $v(h) = v_1 h + \mathcal{O}(h^2)$ and $w_h = w + h\psi + \mathcal{O}(h^2)$, where w satisfies (4.2) for $h = 0$,

$$\kappa w'' - V'(w) = 0, \qquad w(\mp\infty) = m_\pm, \qquad w(0) = 0 \tag{4.3}$$

For later purposes it is also convenient to introduce the operator

$$H_w \psi = \left[-\kappa \frac{d^2}{dx^2} + V''(w(x)) \right] \psi \tag{4.4}$$

obtained from the linearization of (4.1) at w. H_w is a one-dimensional Schrödinger operator. Its ground state is w' with energy zero. The continuum edge starts at $V''(m_+)$. Since the potential in Eq. (4.4) decays exponentially, there are only a finite number of bound states in between. With this notation the linearization of (4.2) at $h = 0$ reads

$$H_w \psi - v_1 w' - 1 = 0 \tag{4.5}$$

Multiplying on the left by w' results in

$$v(h) = \left[\int dx\, w'(x)^2 \right]^{-1} (m_+ - m_-)h + \mathcal{O}(h^2) \tag{4.6}$$

Therefore, according to the Allen–Cahn equation, the mobility is

$$\mu_{\mathrm{AC}} = \left[\int dx\, w'(x)^2 \right]^{-1} \tag{4.7}$$

Since the surface tension σ_{AC} equals

$$\sigma_{\mathrm{AC}} = \kappa \int dx\, w'(x)^2 \tag{4.8}$$

we are in agreement with ref. 11. Note that, according to Eq. (1.2), this means that for the Allen–Cahn equation

$$v_n = \kappa K \tag{4.9}$$

independently of the particular shape of the potential V, where K denotes the local mean curvature.

We have to check whether the linear response argument of Section 3 agrees with (4.7). We will do this in two different ways. First, we simply repeat our argument for the Allen–Cahn equation. Second, we expand (3.8) in $1/\beta$. We will find that all three results agree with each other. The second approach has the advantage of yielding some information on the Green–Kubo integrand.

We linearize Eq. (4.1) as $w + h\psi_t$. Then

$$\frac{\partial}{\partial t}\psi_t = -H_w \psi_t + 1, \qquad \psi_0 = 0 \tag{4.10}$$

On the other hand, linearizing around $w = m_+$ yields

$$\frac{\partial}{\partial t}\psi_t^+ = -H_+ \psi_t^+ + 1, \qquad \psi_0^+ = 0 \tag{4.11}$$

with

$$H_+ = -\kappa \frac{d^2}{dx^2} + V''(m_+) \tag{4.12}$$

Therefore the subtracted change in magnetization to linear order in h equals

$$\frac{d}{dt} \int dx\, [\psi_t(x) - \psi_t^+(x)] = \int dx\, [e^{-H_w t} 1(x) - e^{-V''(m_+)t}] \tag{4.13}$$

Note that the subtraction is needed, because otherwise the x integration would diverge. To understand the large-t behavior in (4.13), we use the Feynman–Kac formula as

$$\int dx\, \mathsf{E}_x\left(\exp\left[-\int_0^t ds\, V''(w(w(s)))\right]-\exp\left[-\int_0^t ds\, V''(m_+)\right]\right) \tag{4.14}$$

Here E_x denotes average with respect to the Brownian motion $\omega(\cdot)$ starting at x. We split the x integration into $|x|\leqslant t$ and $|x|>t$. For starting points x with $|x|>t$ the Brownian motion has an exponentially small probability to reach the neighborhood of the origin. But away from the origin the integrand is exponentially small as a function of x. Thus the integral for $|x|>t$ is bounded by const $\cdot \exp[-V''(m_+)t]$. For $|x|\leqslant t$, we consider each term in (4.13) separately. From the eigenfunction expansion of H_w we get the ground-state contribution $[\int dx\, w'(x)]^2/\int dx\, w'(x)^2$. All other terms are bounded by $2t\exp(-E_1 t)$, where $E_1>0$ is the energy of the first excited state (or the continuum edge if there is none). Thus, taking the limit $t\to\infty$ in (4.13) yields

$$(m_+-m_-)^2\Big/\int dx\, w'(x)^2 \tag{4.15}$$

which, according to (3.2) and (3.5), results in the mobility μ_{AC}. Note that in (4.13) the limit $t\to\infty$ is approached exponentially fast.

We now turn directly to the large-β behavior of the Green–Kubo formula (3.8). At this stage we are not specifically interested in lattice effects. Therefore we take the formal continuum limit, which at the perturbative level causes no harm. (To recover the lattice, we only have to substitute everywhere the lattice Laplacian.) We choose the interface normal to be $\hat{n}=(1,0,\ldots,0)$ and split accordingly $x=(x_1, x_\perp)$, $\Delta=\Delta_1+\Delta_\perp$. Also, let us denote the Green–Kubo integrand by

$$\begin{aligned} C_t(x,y)=-\beta\{&\langle[\kappa\,\Delta\phi_0(x)-V'(\phi_0(x))][\kappa\,\Delta\phi_t(y)-V'(\phi_t(y))]\rangle_{+-}\\ &-\langle[\kappa\,\Delta\phi_0(x)-V'(\phi_0(x))][\kappa\,\Delta\phi_t(y)-V'(\phi_t(y))]\rangle_{++}\} \end{aligned} \tag{4.16}$$

In the limit $\beta\to\infty$, $\langle\cdot\rangle_{+-}$ concentrates at w. Since $\kappa\,\Delta w-V'(w)=0$, C_t vanishes. To next order we expand the solution of the Ginzburg–Landau equation (2.4) as $\phi(x,t)=w(x)+\beta^{-1/2}\psi_t(x)$. Then

$$\frac{\partial}{\partial t}\psi_t=-(H_w-\kappa\,\Delta_\perp)\,\psi_t+\sqrt{2}\,\dot{w}(t) \tag{4.17}$$

In this approximation $\langle\cdot\rangle_{+-}$ becomes a Gaussian measure with covariance

$$\langle\psi(x)\,\psi(y)\rangle_{+-} = \langle x|\,(H_w - \kappa\,\Delta_\perp)^{-1}\,|y\rangle \tag{4.18}$$

Also, expanding in (4.16),

$$\kappa\,\Delta\phi(x) - V'(\phi(x)) \cong -\beta^{-1/2}(H_w - \kappa\,\Delta_\perp)\,\psi(x) \tag{4.19}$$

Thus, to leading order in β^{-1} the Green–Kubo integrand can be evaluated in the Gaussian approximation. Inserting (4.17) and (4.19) in (4.16) yields

$$C_t(x, y) \cong \frac{d}{dt}\left[(\langle x_1|\,e^{-H_w t}\,|y_1\rangle - \langle x_1|\,e^{-H_+ t}\,|y_1\rangle)\langle x_\perp|\,e^{\kappa\,\Delta_\perp t}\,|y_\perp\rangle\right] \tag{4.20}$$

We note the following decay properties: If we perform the limits as in (3.8), then

$$\int_0^t ds \lim_{l\to\infty}(2l+1)^{-(d-1)} \lim_{M\to\infty}\int d^d x \int d^d y\, C_s(x, y)$$

$$= \int dx_1 \int dy_1\,(\langle x_1|\,e^{-H_w t}\,|y_1\rangle - \langle x_1|\,e^{-H_+ t}\,|y_1\rangle) \tag{4.21}$$

This quantity came up already in the context of the Allen–Cahn equation. It converges exponentially fast to its limit $(m_+ - m_-)^2/\int dx_1\, w'(x_1)^2$. On the other hand, for $t > 1/E_1$ we can approximate the difference in (4.20) by $w'(x_1)\,w'(y_1)$. Therefore

$$\int_0^t ds\, C_s(x, y) \cong w'(x_1)\,w'(y_1)\langle x_\perp|\,e^{\kappa\,\Delta_\perp t}\,|y_\perp\rangle \tag{4.22}$$

which decays on a length $(\kappa t)^{1/2}$ along the interface.

Next we turn our attention to the two-dimensional Ising model. As $\beta\to\infty$, energy-increasing flips are forbidden. The interface dynamics is determined by flips which conserve the energy. The corresponding rate is denoted by c_0. It sets the overall time scale and, in physical applications, would still depend on the temperature. Note that, in contrast to the Ginzburg–Landau equations, at zero temperature the dynamics is still noisy. Therefore the mobility must be determined through Eq. (3.7).

Let us first consider the surface tension, which can be computed

explicitly.[12] Let θ denote the angle between the interface and the 1 axis. Then, including the first-order correction in β^{-1},

$$\sigma(\theta) = 2J(|\cos\theta| + |\sin\theta|) + \beta^{-1}\{|\sin\theta| \log[|\sin\theta|/(|\cos\theta| + |\sin\theta|)] + |\cos\theta| \log[|\cos\theta|/(|\cos\theta| + |\sin\theta|)]\} \tag{4.23}$$

The surface tension is a difference between energy and $\beta^{-1} \times$ entropy.

To evaluate the Green–Kubo formula (3.7), it is convenient to regard the interface location as a function relative to the diagonal. Even if not so initially, this function will become single-valued exponentially fast. We label the interface slopes as $\eta_j = \pm 1$, $j = 0, \pm 1, \ldots$. With respect to the measure $\langle \cdot \rangle_{+-}$, in the limit $\beta \to \infty$, the η_j are independent with average $\langle \eta_j \rangle = u = (\sin\theta - \cos\theta)/(\sin\theta + \cos\theta)$, $0 \leqslant \theta \leqslant \pi/2$. The average flip rate, $\langle c_x(\sigma) \rangle_{+-}$, translates to $\frac{1}{4}c_0\langle(\eta_{j+1} - \eta_j)^2\rangle = \frac{1}{2}c_0(1-u^2)$. The term $\langle c_x(\sigma)\, \sigma(x)\, e^{Lt} c_y(\sigma)\, \sigma(y) \rangle_{+-}$ transcribes to

$$\tfrac{1}{4}c_0\langle(\eta_{j+1} - \eta_j)_0\, (\eta_{i+1} - \eta_i)_t\rangle \tag{4.24}$$

It cancels out to zero upon spatial summation. By inserting everything in (3.7) we obtain for the mobility

$$\mu(\theta) = \frac{\beta c_0}{4} \frac{|\sin 2\theta|}{|\sin\theta| + |\cos\theta|} \tag{4.25}$$

It is of interest to compute also the normal velocity. In the variational derivative $\delta F/\delta\Sigma$ the energy makes no contribution. Therefore, in our approximation, the normal velocity does not depend on β (except through c_0) and is given by

$$v_n = c_0(|\sin\theta| + |\cos\theta|)^{-2} K \tag{4.26}$$

with K the local curvature.

5. SCALING LIMIT: MOTION BY MEAN CURVATURE

Up to now we have discussed only pieces of the interface which are flat on the average. The link to the mean curvature equation (1.2) is still missing. To establish it, we first have to explain the relevant length and time scales. The bulk equilibrium phases have a finite correlation length ξ which is of the order of the lattice spacing. Deviations from equilibrium relax exponentially fast, defining a relaxation time τ which is of the order of a typical flip time per spin. (Actually it has been argued convincingly that in the pure phases the generator has no gap in spectrum. Still, the

magnetization is expected to decay according to a stretched exponential.) In the traditional, capillary wave picture the interface has an intrinsic width of the order of the correlation length. At best up to this precision we can define meaningfully the interface location. We are interested in a situation where the interface is slowly varying such that

$$K\xi \ll 1 \tag{5.1}$$

with K its typical mean curvature. The interface will then move slowly and we must consider times such that

$$t/\tau \gg 1 \tag{5.2}$$

We will argue that motion by mean curvature becomes exact in these limits *provided* they are linked diffusively. In particular, away from the interface the bulk maintains equilibrium.

Even in thermal equilibrium the interface undergoes ondulations and shape fluctuations on a scale large compared to ξ. For a reference plane with linear dimension l these are of the order $\sqrt{l}$ in two and of the order $\log l$ in three dimensions. For $d>3$ the fluctuations are bounded with the same behavior in $d=2$ below the roughening transition. Since we study deformations of the order l, such ondulations are negligible in the limit considered here. Note that very close to the critical point ξ and τ are large and the scale separation (5.1), (5.2) may become meaningless. The merger of interface fluctuations and critical behavior is a fascinating topic in itself, but is outside the present scope.

Let us choose a smooth (codimension-one) surface Σ_0 in $\mathbb{R}^d$, e.g., the surface of a droplet, and let it evolve according to Eq. (1.2) to the surface Σ_t. As explained in Sections 2 and 3, the mobility and the surface tension must be an input from the microscopic model. We should mention that in general the interface motion will be rather complex. For example, minority droplets have the tendency to shrink and will disappear after a finite time. Before doing so, droplets may split, at least in dimension $d\geqslant 3$. On the other hand, for the spinodal decomposition after a symmetric quench, one obtains a hierarchical network of interlaced $+$ and $-$ phases. Under mean curvature, this network will coarsen as $\sqrt{t}$.[13,14]

We approximate the continuum by an underlying lattice with lattice spacing εa, i.e., by $(\varepsilon a\mathbb{Z})^d$. Here a is a length and the scaling parameter ε will tend to zero. To be specific, let us discuss the Ginzburg–Landau model. The initial measure for the stochastic evolution should reproduce approximately the interface Σ_0. Let Λ_0^+ be the open domain of the $+$ phase and Λ_0^- be that of the $-$ phase at the initial time $t=0$. Σ_0 is their common boundary. Now, one possible choice would be to impose, at

time $t=0$, $\phi_0(x)=m_+$ $[m_-]$ for $x\in\Lambda_0^+\cap(\varepsilon a\mathbf{Z})^d$ $[\Lambda_0^-\cap(\varepsilon a\mathbf{Z})^d]$, where $m_+=-m_-$ is the spontaneous magnetization of the equilibrium phases. Note that when viewed on the scale of the lattice the interface has a mean curvature of order ε, thus is of slow variation. At the initial stage the interface will be essentially frozen and the field $\phi_t(x)$ will approach a local equilibrium distribution. To observe a macroscopic interface motion, we have to go to the diffusive time scale of order $\varepsilon^{-2}t$. At such long time we conjecture the following:

(i) (*Mean curvature equation*) Let Σ_t be the surface at time t as obtained through motion by mean curvature (1.2) from Σ_0 and let Λ_t^+, resp. Λ_t^-, be the corresponding open domains of the $+$, resp. $-$ phases. Then

$$\lim_{\varepsilon\to 0}\langle\phi_{\varepsilon^{-2}t}(r)\rangle=\begin{cases}m_+ & \text{if } r\in\Lambda_+^t\\ m_- & \text{if } r\in\Lambda_-^t\end{cases}\tag{5.3}$$

Note that r is a finite macroscopic distance, i.e., a distance $\varepsilon^{-1}a$ on the scale of the lattice, away from the interface Σ_t.

(ii) (*Local equilibrium*) If $r\in\Lambda_+^t$ (Λ_-^t), then the joint distribution of $\{\phi_{\varepsilon^{-2}t}(r+\varepsilon ax)\mid x\in\Lambda\subset\mathbf{Z}^d,\ |\Lambda|<\infty\}$ approaches as $\varepsilon\to 0$ the equilibrium distribution corresponding to the $+$ $(-)$ phase. [Properly phrased, r should be replaced by the closest point to r on the lattice $(\varepsilon a\mathbf{Z})^d$.]

Our conjecture may be supported by a formal computation familiar from the theory of fluids.[15] This will not be repeated, simply because it does not shed any additional light on the validity of the mean curvature equation. Basically, one has to assume that certain (nonstationary) space-time correlations decay sufficiently rapidly, which in itself is already rather close to (i) and (ii).

For the Allen–Cahn equation our conjecture has been proved by de Mottoni and Schatzmann[16] and Evans *et al.*[17] The latter work covers a more general situation. Both groups assume initial local equilibrium. In this context it means that the initial profile transverse to the interface is given by the kink solution of the Allen–Cahn equation. In Appendix A we prove motion by curvature for the two-dimensional Ising model at zero temperature in full agreement with the results in Section 4. Our conjecture is further supported by effective interface models, as to be discussed in the following section.

We carefully avoided any statement on shape fluctuations of the interface. We only claim that it becomes sharp on the macroscopic scale. This is true even if the shape fluctuations would be as large as $\varepsilon^{-1+\delta}a$ with some $\delta>0$.

6. EFFECTIVE INTERFACE MODELS I: MOBILITY

As we have seen, the zero-temperature Ising model has the drastic simplification that, at least locally, the interface can be represented as a single-valued function with respect to a reference plane, e.g., the one orthogonal to the (111) direction. As the temperature is increased there is some probability for overhangs. Also, bubbles of the wrong phase occur. Effective interface models are based on the idea of avoiding such complications by fiat. They can be derived from bulk models in the limit of strongly anisotropic couplings. We are not so much interested in this particular limit here, and study effective interface models in their own right. Our point is that the mean curvature equation is so general that it must apply also to effective interface models.

We represent the interface as a single-valued function over a reference plane which is taken as $\mathbf{Z}^{d-1}$, $d \geqslant 2$. At each site $x \in \Lambda \subset \mathbf{Z}^{d-1}$ there is a height variable $\phi(x)$, where either $\phi(x) \in \mathbf{R}$ or $\phi(x) \in \mathbf{Z}$. The region $\{x, x_d \mid x_d < \phi(x)\} \subset \Lambda \times \mathbf{R}$ (resp. $\Lambda \times \mathbf{Z}$) corresponds to the $+$ phase and the region $\{x, x_d \mid x_d \geqslant \phi(x)\}$ corresponds to the $-$ phase.

There is considerable freedom of how to choose the energy function. The crucial physical constraint is that the energy remains unchanged under the global shift $\phi(x)$ to $\phi(x)+b$ for all $x \in \Lambda$ and all $b \in \mathbf{R}$ (resp. $\mathbf{Z}$). In the case of continuous height variables a standard example is

$$H(\phi) = \sum_{\langle x, y \rangle, x, y \in \Lambda} V(\phi(x) - \phi(y)) \tag{6.1}$$

with V an even convex function, $V(\phi) \geqslant c\phi^2$ with $c > 0$. [As will be seen, less stringent conditions may suffice, e.g., for $d = 2$ only $V(\phi) \geqslant c\,|\phi|^{1+\delta}$ for some $\delta > 0$ is needed.] If we regard $\phi(x)$ as the (scalar) displacements of the atoms in a crystal, then (6.1) is the usual anharmonic elastic energy. To model the underlying lattice $\mathbf{Z}^d$, one would have to add to (6.1) the term

$$H_{\text{latt}}(\phi) = \sum_{x \in \Lambda} \cos[2\pi\phi(x)] \tag{6.2}$$

For discrete height variables the usual energy is

$$H(\phi) = 2J \sum_{\langle x, y \rangle, x, y \in \Lambda} |\phi(x) - \phi(y)|^{\alpha} \tag{6.3}$$

$J > 0$. Here $\alpha = 2$ is the discrete Gaussian model and $\alpha = 1$ is the solid-on-solid (SOS) model. In the SOS model *admissible* spin configurations have an energy as in (2.6), up to a global constant.

To define the dynamics, we follow the standard recipe. We regard the energy as potential and add noise. Then

$$\frac{d}{dt}\phi_t(x) = -\sum_{\langle x,y\rangle, x,y\in\Lambda} V'(\phi_t(x)-\phi_t(y)) + (2/\beta)^{1/2}\,\dot{w}_t(x) \tag{6.4}$$

$x\in\Lambda$. As before, the white noises are independent for distinct sites. With free boundary conditions the invariant measure $e^{-\beta H}$ has infinite total weight because of the shift invariance of $\phi(x)$ to $\phi(x)+b$. To have a finite normalization, we must break this symmetry through appropriate boundary conditions.

In case $\phi_t(x)$ takes discrete values, we specify the rate $c_x^+(\nabla\phi)$ for the jump from $\phi(x)$ to $\phi(x)+1$ and the rate $c_x^-(\nabla\phi)$ for the jump $\phi(x)$ to $\phi(x)-1$. Our notation is supposed to indicate that the rates depend on the height configuration only through height differences to ensure that the shift symmetry is preserved. The jump rates satisfy the condition of detailed balance,

$$c_x^+(\nabla\phi) = c_x^-(\nabla\phi^{x+})\exp\{-\beta[H(\phi^{x+})-H(\phi)]\} \tag{6.5}$$

Here ϕ^{x+} (ϕ^{x-}) stands for the configuration ϕ with the height variable at x increased (decreased) by one. The generator for the dynamics is given by

$$Lf(\phi) = \sum_{x\in\Lambda}\{c_x^+(\nabla\phi)[f(\phi^{x+})-f(\phi)] + c_x^-(\nabla\phi)[f(\phi^{x-})-f(\phi)]\} \tag{6.6}$$

We can now follow the path laid out by the bulk models already. Before doing so, let us write the mean curvature equation in our specific choice of coordinates. By assumption, the interface is described by the single-valued function h_t on $\mathbb{R}^{d-1}$. Let $\sigma(\nabla h)$ be the (surface) free energy per unit area in the reference plane of the interface with slope ∇h. Then, in Eq. (1.2),

$$F = \int d^{d-1}x\,\sigma(\nabla h(x)) \tag{6.7}$$

and

$$-\frac{\delta F}{\delta h(x)} = \sum_{\alpha=1}^{d-1}\frac{\partial}{\partial x_\alpha}\sigma_\alpha(\nabla h(x)) \tag{6.8}$$

with $\sigma_\alpha = \partial\sigma/\partial u_\alpha$, $u_\alpha = \partial h/\partial x_\alpha$. Let μ be the mobility relative to the reference plane. According to (1.2), the normal velocity is then given by

$$v_n = -\frac{\mu}{[1+(\nabla h)^2]^{1/2}}\frac{\delta F}{\delta h} \tag{6.9}$$

But $[1+(\nabla h)^2]^{1/2} v_n$ is just the velocity along the h axis. Therefore the mean curvature equation reads

$$\frac{\partial}{\partial t} h_t = \mu(\nabla h_t) \sum_{\alpha,\beta=1}^{d-1} \sigma_{\alpha\beta}(\nabla h_t) \frac{\partial^2}{\partial x_\alpha \partial x_\beta} h_t \tag{6.10}$$

with $\sigma_{\alpha\beta} = \partial^2\sigma/\partial u_\alpha \partial u_\beta$. We emphasize that in (6.10) the mobility μ and the free energy σ are defined relative to the reference plane. Equation (6.10) has the structure of a nonlinear diffusion equation, only the "diffusion matrix" $\sigma_{\alpha\beta}$ depends on ∇h_t rather than h_t itself.

The (surface) free energy can be defined either canonically or through a suitable slope chemical potential. In the first case we consider the box $\Lambda = [-l, l]^{d-1}$. We pick an average slope u and impose the boundary conditions $\phi(x) = u \cdot x$ for $x \in \partial\Lambda$, the sites bordering Λ. Let $H_{\Lambda,u}$ be the corresponding energy. Then (in the case of continuous height variables)

$$\sigma(u) = -\beta^{-1} \lim_{l\to\infty} (2l+1)^{-(d-1)} \log \int \prod_{x\in\Lambda} d\phi(x) \exp[-\beta H_{\Lambda,u}(\phi)] \tag{6.11}$$

Note that we can transform (6.11) to zero boundary conditions by the shift $\tilde{\phi}(x) = \phi(x) - u \cdot x$. Then $\tilde{\phi}(x) = 0$ for $x \in \partial\Lambda$ and in terms of $\tilde{\phi}$ the bond $(x, x+e_\alpha)$ has the interaction energy $V([\tilde{\phi}(x+e_\alpha) - \tilde{\phi}(x)] + u_\alpha)$, $\alpha = 1,..., d-1$. $\sigma(u)$ is convex (we are not aware of a proof). This can be seen from (6.10). If $\sigma_{\alpha\beta}$ had a negtive eigenvalue, then the interface motion would be unstable.

In the grand-canonical prescription we add a slope chemical potential to the energy as

$$\sum_{-l\le x_i<l, i=1,...,d-1} \sum_{\alpha=1}^{d-1} \lambda_\alpha \nabla_\alpha \phi(x) \tag{6.12}$$

Here $\nabla_\alpha \phi(x) = \phi(x+e_\alpha) - \phi(x)$ with e_α the αth unit vector. This can be read as modifying the interaction potential for the bond $(x, x+e_\alpha)$ to $V(\nabla_\alpha\phi(x)) - \lambda_\alpha \nabla_\alpha \phi(x)$. (6.12) sums to a pure boundary term of the form

$$\sum_{\alpha=1}^{d-1} \sum_{\hat{x}^\alpha} (\phi(x_1,..., l,..., x_{d-1}) - \phi(x_1,..., -l,..., x_{d-1})) \tag{6.13}$$

where $\hat{x}^\alpha = (x_1,..., x_{d-1})$ with deleted αth coordinate. Thus the slope chemical potential corresponds to adding a linear boundary term to H. We define the "pressure"

$$p(\lambda) = -\beta^{-1} \lim_{l\to\infty} (2l+1)^{-(d-1)} \log \int \prod_{x\in\Lambda} d\phi(x)\, \delta(\phi(0)) \times \exp\left[-\beta H(\phi) + \beta \sum_{-l\le x_i<l, i=1,...,d-1} \sum_{\alpha=1}^{d-1} \lambda_\alpha \nabla\phi_\alpha(x)\right] \tag{6.14}$$

To break the translation symmetry, we pinned the height $\phi(0)$ at zero. Clearly, with $|\Lambda|$ the number of sites in Λ,

$$\exp[-\beta p(\lambda)\,|\Lambda|]$$

$$= \int d^{d-1}u\,|\Lambda|^{d-1} \exp[\beta(\lambda\cdot u)\,|\Lambda|] \int \prod_{x\in\Lambda} d\phi(x)\,\delta(\phi(0)) \exp(-\beta H)$$

$$\prod_{\alpha=1}^{d-1} \delta\left(\sum_x \nabla_\alpha \phi(x) - u_\alpha\,|\Lambda|\right)$$

$$\cong \int d^{d-1}u\,|\Lambda|^{d-1} \exp\{-\beta[\sigma(u)-\lambda\cdot u]\,|\Lambda|\} \tag{6.15}$$

Therefore $p(\lambda)$ is the Legendre transform of the free energy,

$$p(\lambda) = \inf_u (\sigma(u) - \lambda\cdot u) \tag{6.16}$$

and hence concave.

Mobility is the next item on the list. The most direct approach is through the fluctuations in the magnetization, which is taken to be m_+ below and m_- above the interface. Then, up to a global constant, the magnetization is simply

$$M_\Lambda(\phi_t) = (m_+ - m_-) \sum_{x\in\Lambda} \phi_t(x) \tag{6.17}$$

for the height configuration ϕ_t. We repeat our argument from the end of Section 3. Expectations with respect to $\langle\cdot\rangle_{+-}$ are now replaced by expectations with respect to the Gibbs measure $\langle\cdot\rangle(u)$, where the average slope of the interface is fixed to be u through the boundary conditions $\phi(x) = u\cdot x$, $x\in\partial\Lambda$. We obtain then

$$\mu(u) = -\beta(m_+ - m_-)^{-2} \left\{ \lim_{l\to\infty} (2l+1)^{-(d-1)} \langle M_\Lambda(\phi)(-L)\,M_\Lambda(\phi)\rangle(u) \right.$$

$$\left. - \int_0^\infty dt \lim_{l\to\infty} (2l+1)^{-(d-1)} \langle LM_\Lambda(\phi).e^{Lt}LM_\Lambda(\phi)\rangle(u) \right\} \tag{6.18}$$

Properly speaking, L should also carry the index u, because the dynamics depends on the boundary conditions.

For the Ginzburg–Landau dynamics

$$L \sum_{x\in\Lambda} \phi(x) = -\sum_{x\in\Lambda} \sum_{\langle x,y\rangle} V'(\phi(x)-\phi(y)) \tag{6.19}$$

In the second term of (6.18), LM_Λ sums up to a boundary term and therefore vanishes as $l \to \infty$.

For the first term in (6.18) we use

$$\left\langle \left[\sum_{w \in \Lambda} \phi(w) \right] \left[\sum_{x \in \Lambda} \sum_{\langle x, y \rangle} V'(\phi(x) - \phi(y)) \right] \right\rangle$$

$$= -\beta^{-1} Z^{-1} \int \prod_{y \in \Lambda} d\phi(y) \left[\sum_{w \in \Lambda} \phi(w) \right] \left[\sum_{x \in \Lambda} \frac{\partial}{\partial \phi(x)} \exp[-\beta H_{\Lambda, u}(\phi)] \right]$$

$$= \beta^{-1} |\Lambda| \tag{6.20}$$

Therefore the mobility of the Ginzburg–Landau model equals

$$\mu = 1 \tag{6.21}$$

This result is rather special. Already if we added the cosine potential to (6.4), then both terms in (6.18) would contribute and the mobility would acquire a nontrivial inclination dependence which is no longer computable.

For the models with discrete height variables

$$L \sum_{x \in \Lambda} \phi(x) = \sum_{x \in \Lambda} [c_x^+(\nabla\phi) - c_x^-(\nabla\phi)] \tag{6.22}$$

Therefore, using detailed balance (6.5),

$$\mu = \beta^{-1} \left\{ \lim_{l \to \infty} (2l+1)^{-(d-1)} \left\langle \sum_{x \in \Lambda} c_x^+(\nabla\phi) \right\rangle (u) \right.$$

$$- \int_0^\infty dt \lim_{l \to \infty} (2l+1)^{-(d-1)} \sum_{x \in \Lambda} \sum_{y \in \Lambda} \langle [c_x^+(\nabla\phi) - c_x^-(\nabla\phi)]$$

$$\left. \times e^{Lt} [c_y^+(\nabla\phi) - c_y^-(\nabla\phi)] \rangle(u) \right\} \tag{6.23}$$

7. EFFECTIVE INTERFACE MODELS II: SCALING LIMIT

Locally an interface configuration is rough and fluctuates. To compare it with the smooth solution of the mean curvature equation (6.10), we average the height variables over some small macroscopic region. Mathematically it is more convenient to sum them over an arbitrary test function f. In the scaling limit distances both in the reference plane and along the ϕ direction have to be rescaled. We want to make sure that the

initial ϕ field represents a smooth interface with slow variation on the scale of the lattice. Therefore we require, at time $t=0$,

$$\lim_{\varepsilon \to 0} \varepsilon^{(d-1)} \sum_x f(\varepsilon x)\, \varepsilon \phi_0(x) = \int d^{d-1} r\, f(r)\, h_0(r) \tag{7.1}$$

with probability one. Note that the first prefactor, $\varepsilon^{(d-1)}$, comes from normalizing the sum and the second prefactor, ε, from rescaling the ϕ direction. We conjecture as follows:

(i) (*Mean curvature equation*) For $t>0$ we have

$$\lim_{\varepsilon \to 0} \varepsilon^{(d-1)} \sum_x f(\varepsilon x)\, \varepsilon \phi_{\varepsilon^{-2}t}(x) = \int d^{d-1} r\, f(r)\, h_t(r) \tag{7.2}$$

with probability one and h_t satisfies Eq. (6.10) with initial condition h_0.

(ii) (*Local equilibrium*) Locally the interface fluctuates and the mean location is changing. Therefore the local statistics is better studied in terms of height differences. We pick a macroscopic point r_0 which on the scale of the lattice becomes $[\varepsilon^{-1} r_0]$ ($[\cdot]$ denoting the integer part). We consider the differences

$$\phi_{\varepsilon^{-2}t}([\varepsilon^{-1} r_0] + y) - \phi_{\varepsilon^{-2}t}([\varepsilon^{-1} r_0] + x)$$

As $\varepsilon \to 0$ their joint distribution should be given by the equilibrium measure with slope $\nabla h_t(r_0)$.

On a formal level, the validity of (7.2) is easily demonstrated. We have

$$\begin{aligned} &\frac{d}{dt} \varepsilon^{(d-1)} \sum_x f(\varepsilon x)\, \varepsilon \phi_{\varepsilon^{-2}t}(x) \\ &\quad = -\varepsilon^{(d-1)} \sum_x \sum_{\alpha=1}^{d-1} \varepsilon^{-1} (f(\varepsilon x + \varepsilon e_\alpha) - f(\varepsilon x))\, V'(\nabla_\alpha \phi_{\varepsilon^{-2}t}(x)) \\ &\qquad + \varepsilon^{(d-1)} \sum_x f(\varepsilon x)\, \varepsilon^{-1} \dot{w}_{\varepsilon^{-2}t}(x) \end{aligned} \tag{7.3}$$

The square of the noise is

$$\varepsilon^{2(d-1)} \sum_x \sum_y f(\varepsilon x)\, f(\varepsilon y)\, \varepsilon^{-2} \delta_{xy}\, \delta(\varepsilon^{-2}(s-t)) = \delta(s-t)\, \varepsilon^{2(d-1)} \sum_x f(\varepsilon x)^2 \tag{7.4}$$

which vanishes as $\varepsilon \to 0$. If at the long time $\varepsilon^{-2}t$ the interface is locally in equilibrium, then

$$-\varepsilon^{(d-1)} \sum_x \partial_\alpha f(\varepsilon x)\, V'(\nabla_\alpha \phi_{\varepsilon^{-2}t}(x))$$

$$\cong -\int d^{d-1}r\, \partial_\alpha f(r) \langle V'(\nabla_\alpha \phi(0)) \rangle (\nabla h_t(r)) \tag{7.5}$$

$\partial_\alpha f = \partial f / \partial r_\alpha$. Now, for the equilibrium measure, $\langle \cdot \rangle(u)$, with boundary conditions $\phi(x) = u \cdot x$, $x \in \partial\Lambda$, we have

$$\sum_{x \in \Lambda} \langle V'(\nabla_\alpha \phi(x)) \rangle(u) = \sum_{x \in \Lambda} \langle V'(\nabla_\alpha \phi(x) + u_\alpha) \rangle(0)$$

$$= |\Lambda|\, \sigma_\alpha(u) \tag{7.6}$$

for large volume $|\Lambda|$. By inserting (7.6) in (7.5), and (7.4) and (7.5) in (7.3), we obtain Eq. (6.10). Thus effective interface models give further support to the mean curvature equation (1.2).

Several parts of our argument can be improved. As the net result, we lack the proof of some plausible properties of the equilibrium states. One reason for our difficulty is certainly the fact that the equilibrium measures are massless and have slowly decaying correlations. Since this discussion is somewhat technical, it is relegated to Appendix B.

There is a further general property of effective interface models: In the case of two dimensions the surface slope is governed by an exchange dynamics. The scaling limit for the interface is equivalent to the hydrodynamic limit for the slope. For notational simplicity we stick to Ginzburg–Landau models, commenting on discrete height variables at the end.

Let us define the slope variables

$$\eta(x) = \phi(x+1) - \phi(x) \tag{7.7}$$

The η's are really bond variables, but we may ignore this feature for a one-dimensional reference lattice. The height differences are governed by the stochastic differential equation

$$\frac{d}{dt}\eta_t(x) = V'(\eta_t(x+1)) - 2V'(\eta_t(x)) + V'(\eta_t(x-1))$$

$$+ \dot{w}_t(x+1) - \dot{w}_t(x) \tag{7.8}$$

These are just the discretized evolution equations for a conserved field (model B). We note that with respect to the equilibrium measure the $\eta(x)$'s

are independent with distribution $Z^{-1}\exp[-\beta V(\eta(x))]\,d\eta(x)$. The boundary constraint $\phi(\pm l)=\pm ul$ means fixing the total slope as

$$\sum_{x=-l}^{l-1}\eta(x)=2lu \tag{7.9}$$

which in terms of the slope gas is the canonical constraint. The scaling limit (7.2) translates to the hydrodynamic limit for (7.8), which has been studied extensively[7] with proofs available.[18] The mean curvature equation corresponds then to a nonlinear diffusion equation for the slope $(\partial/\partial x)\,h_t=u_t$,

$$\frac{\partial}{\partial t}u_t=\frac{\partial}{\partial x}\left(\mu(u_t)\,\sigma_{11}(u_t)\,\frac{\partial}{\partial x}u_t\right) \tag{7.10}$$

$(\sigma_{11})^{-1}$ is the compressibility of the slope gas and μ is its conductivity, i.e., the response in the slope current to a small external driving field. $\mu\sigma_{11}$ is the bulk diffusion coefficient. In general, both σ_{11} and μ depend on the local slope u_t. The conductivity is given by a standard Green–Kubo formula as the space-time integral over the slope current correlations.[7] It agrees with Eq. (6.18) specialized to $d=2$.

We can also turn our construction around by first defining a lattice gas with spin exchange dynamics. If $\zeta_t(x)=\pm 1$ denotes the spin (=slope) variables, then

$$h_t(x)=\sum_{y=-l}^{x}\zeta_t(y) \tag{7.11}$$

with some freedom of how to define the left boundary. $h_t(x)$ can then interpreted as the height of an effective interface model. In our example the height differences are ± 1. They would take integer values in the standard SOS model.

8. ROUGHENING TRANSITION

Some of the models discussed previously undergo a roughening transition at a temperature T_R. For example, in the three-dimensional Ising model with temperature $T<T_R$ a (001) interface is smooth in the sense that typical fluctuations away from the average interface location are bounded, whereas for $T_R<T<T_c$ the interface is rough and fluctuations diverge logarithmically.[19,20] Below the roughening temperature the surface tension has cusps, implying a faceted equilibrium shape. One may wonder whether the mean curvature equation extends to such a somewhat singular situation. We claim that it does so and want to explain how.

To illustrate the situation we consider the $(2+1)$-dimensional sine-Gordon model. To our advantage, the stiffness and the mobility for this model have been computed by Nozières and Gallet[21] by means of a dynamic renormalization group. We will heavily rely on their results. The sine-Gordon model is an effective interface model of Ginzburg–Landau type with energy

$$H = \kappa \sum_{\langle x, y\rangle, x, y \in \Lambda} [\phi(x) - \phi(y)]^2 - \lambda \sum_{x \in \Lambda} \cos[2\pi\phi(x)] \tag{8.1}$$

with $\phi(x) \in \mathbb{R}$ and Λ an $l \times l$ box. λ fixes the strength of the pinning potential, $\lambda > 0$. Following the computation leading to Eq. (6.18), we obtain the mobility

$$\mu(u) = 1 - \beta \int_0^\infty dt \sum_{x \in \mathbb{Z}^2} (2\pi\lambda)^2 \langle \sin[2\pi\phi_0(0)] \sin[2\pi\phi_t(x)] \rangle(u) \tag{8.2}$$

where we have formally taken the limit $l \to \infty$. If $u = 0$, then for β sufficiently large the interface is smooth.[22,23] Therefore in (8.2) we may expand around $\phi_t(x) = 0$. The equilibrium measure is then Gaussian with energy

$$\tilde{H} = \kappa \sum_{\langle x, y\rangle, x, y \in \Lambda} [\phi(x) - \phi(y)]^2 + \lambda(2\pi)^2 \tfrac{1}{2} \sum_{x \in \Lambda} \phi(x)^2 \tag{8.3}$$

The corresponding equations of motion are linear and yield an exponential decay to equilibrium. Inserting in (8.2) and expanding the sine, we finally get

$$\mu(0) = 0 \tag{8.4}$$

in the limit $\beta \to \infty$. In their computation Nozières and Gallet obtain $\mu(0) = 0$ for all $T < T_R$.

The physical mechanism for a vanishing mobility is nucleation: Because of the small excess density in the gas $(=-)$ phase, droplets are formed on the smooth surface $\{\phi(x) = 0\}$. They disappear rapidly unless they have reached a critical size, in which case they expand, cover the whole surface, and form the layer $\{\phi(x) = 1\}$. Such an activated process leads to a growth velocity $v \sim \exp(-c/h)$. Therefore the mobility vanishes. As the interface is slightly tilted, it necessarily contains thermally roughened steps. At kinks, nucleation occurs without cost in energy. A step moves then with a velocity proportional to h. Therefore the interface has a growth velocity proportional to the step density, which implies for the mobility

$$\mu(u) = a\,|u| \tag{8.5}$$

for small $|u|$ and $T < T_R$. Here a is a coefficient depending on T and the model parameters. At larger $|u|$ the mobility levels off. For $T \geqslant T_R$, $a = 0$ and $\mu(u) \to \mu(0) > 0$ smoothly as $u \to 0$. In particular, $\mu(0)$ jumps to a nonzero value at T_R. We do not know how such detailed properties could be extracted from the Green–Kubo formula (8.2).

For the surface tension Nozières and Gallet obtain

$$\sigma(u) = \sigma_0 + b\,|u| + \tfrac{1}{3} c\,|u|^3 \tag{8.6}$$

for $T < T_R$ with σ_0, b, c positive coefficients. In order to emphasize the qualitative behavior, we ignored in our discussion the anisotropy due to the lattice structure of the substrate Λ. Strictly speaking, σ and μ depend on u and not just on $|u|$.

For $T > T_R$, σ and μ depend smoothly on u and the mean curvature equation is well posed. Let us insert then in Eq. (6.10) our findings (8.5), (8.6) valid for $T < T_R$. Since σ has a cusp at $u = 0$, the stiffness matrix $\sigma_{\alpha\beta}$ has a δ-singularity at $u = 0$. However, when multiplied by μ, the singularity gets canceled, because μ is continuous and $\mu(0) = 0$. Therefore

$$\mu\sigma(u) = a\,|u|^{-2}(b + cu^2)\begin{pmatrix} u_2^2 & -u_1u_2 \\ -u_1u_2 & u_1^2 \end{pmatrix} + 2ac\begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix} \tag{8.7}$$

The matrix $\mu\sigma(u)$ has the eigenvector (u_1, u_2) with eigenvalue $2acu^2$ vanishing as $|u| \to 0$. The orthogonal eigenvector has the eigenvalue $a(b + cu^2)$. Equation (6.10) is still well posed. Since $\mu\sigma(u)$ has a zero eigenvalue for $u = 0$, the level set $\{x \mid \nabla h_t(x) = 0\}$ has to be treated as a free boundary. This describes how facets dissolve under mean curvature motion. If h_t depends only on x_1, then the dynamic equation for $u_{1t} = \partial h_t / \partial x_1$ becomes

$$\frac{\partial}{\partial t} u_{1t} = \frac{2}{3} ac \frac{\partial^2}{\partial x_1^2} (u_{1t})^3 \tag{8.8}$$

which is the well-studied porous medium equation.[24] As one of its main features, an initial bump of compact support maintains compact support with a diameter increasing as $t^{1/6}$. For further details we refer to ref. 25.

9. INTERFACE FLUCTUATIONS

The scaling limit (7.2) is a law of large numbers in the sense that, on the chosen scale, typical configurations are well approximated by the

solution of the mean curvature equation. Therefore we expect to have a Gaussian fluctuation theory for small-size deviations from the typical profile. For fluids such considerations would lead to fluctuating hydrodynamics.[7]

Let us discuss first effective interface models. For them, by definition, the interface is sharp and the bulk phases have no fluctuations. Thus we can follow the standard procedure. We fix an average inclination u and want to study the static equilibrium fluctuations in the interface gradients. For this purpose we define the fluctuation field

$$\nabla\xi^{\varepsilon}(f) = \varepsilon^{-(d-1)/2} \sum_x f(\varepsilon x)[\nabla\phi(x) - u] \tag{9.1}$$

with f a smooth averaging function, as before. In the limit $\varepsilon \to 0$, $\nabla\xi^{\varepsilon}(f)$ should become jointly Gaussian with mean zero and covariance

$$\langle \nabla_\alpha \xi(f) \nabla_\beta \xi(g) \rangle = \int d^{d-1}k\, \hat{f}(k)^*\, \hat{g}(k)\, k_\alpha k_\beta \left[\sum_{\gamma,\delta=1}^{d-1} \sigma_{\gamma\delta}(u)\, k_\gamma k_\delta \right]^{-1} \tag{9.2}$$

(For particular cases such a scaling limit is proved in refs. 26 and 27.) We recognize under the integral the small-k limit of the full structure function

$$S_{\alpha\beta}(k) = \sum_x e^{ikx} \langle [\nabla_\alpha \phi(x) - u_\alpha][\nabla_\beta \phi(0) - u_\beta] \rangle(u) \tag{9.3}$$

with $k \in \mathrm{BZ}$, the first Brioullin zone. $\sigma_{\alpha\beta}(u)$ is the stiffness matrix at average inclination u [compare with (6.10), (6.11)].

For the time-dependent fluctuations we linearize the mean curvature equation (6.10) around the constant profile $h_t(x) = u \cdot x$. We add noise in such a way that the Gaussian measure (9.2) is stationary. Then the fluctuations are governed by the Langevin equation

$$\frac{\partial}{\partial t} \xi_t = \mu(u) \sum_{\alpha,\beta=1}^{d-1} \sigma_{\alpha\beta}(u)\, \partial_\alpha \partial_\beta \xi_t + [2\mu(u)]^{1/2}\, \dot{w}(\cdot,\, t) \tag{9.4}$$

$\partial_\alpha = \partial/\partial x_\alpha$. The stochastic differential equation (9.4) defines the Gaussian process ξ_t. We expect that

$$\lim_{\varepsilon \to 0} \varepsilon^{-(d-3)/2} \sum_x f(\varepsilon x)[\phi_{\varepsilon^{-2}t}(x) - u \cdot x] = \int dx\, f(x)\, \xi_t(x) \tag{9.5}$$

in the sense of convergence of joint distributions.

We may also consider fluctuations around a nonstationary interface h_t, i.e., the fluctuation field

$$\varepsilon^{-(d-3)/2} \sum_x f(\varepsilon x)[\phi_{\varepsilon^{-2}t}(x) - h_t(\varepsilon x)] \tag{9.6}$$

Their $\varepsilon \to 0$ limit is again governed by a linear Langevin equation. Invoking the extended local equilibrium hypothesis, the noise is assumed to be locally as in (9.4). Therefore

$$\frac{\partial}{\partial t}\xi_t = L_t \xi_t + [2\mu(\nabla h_t)]^{1/2}\, \dot{w}(\cdot, t) \tag{9.7}$$

with L_t the linearization operator for the mean curvature equation,

$$L_t \xi = \sum_{\alpha,\beta=1}^{d-1} \{\mu(\nabla h_t)\, \partial_\alpha [\sigma_{\alpha\beta}(\nabla h_t)\, \partial_\beta \xi] + \mu_\beta(\nabla h_t)[\partial_\alpha \sigma_\alpha(\nabla h_t)]\, \partial_\beta \xi\} \tag{9.8}$$

with $\mu_\beta = \partial\mu/\partial u_\beta$.

For bulk models the quantity of physical interest is the order parameter correlation in a state with interface. Rather than entering into generalities, let us discuss one example, namely an initially flat and uncorrelated interface. We fix an inclination u_0 and impose the + state in the half-space $\{x \mid u_0 \cdot x < 0\}$ and the − state in $\{x \mid u_0 \cdot x \geqslant 0\}$. We would like to compute then the time-dependent correlations

$$\langle \sigma(x)\, \sigma(y)\rangle_t - \langle \sigma(x)\rangle_t \langle \sigma(y)\rangle_t \tag{9.9}$$

say for the stochastic Ising model. For distances larger than the correlation length ξ and times larger than the relaxation time τ we expect that the interface correlations are well described by a fluctuation theory *identical* in form to the one developed for effective models. We only have to identify the appropriate parameters in the Langevin equation (9.4).

With respect to the reference plane $\{x \mid u_0 \cdot x = 0\}$, the free energy equals

$$f(u)[1 + (u - u_0)^2]^{1/2} \tag{9.10}$$

Therefore the stiffness matrix at u_0 becomes

$$\sigma_{\alpha\beta}(u_0) = f_{\alpha\beta}(u_0) + f(u_0)\, \delta_{\alpha\beta} \tag{9.11}$$

$f_{\alpha\beta} = \partial^2 f/\partial u_\alpha \partial u_\beta$. The mobility at u_0 is just $\mu(u_0)$ of the bulk model.

We regard $\sigma(x)$ as defined on $\mathbb{R}^d$ taking values ± 1 in each elementary cube. We set $x = (x_{||}, x_\perp)$ with $x_{||}$ the coordinate in the reference plane. For $|x - y| > \xi$ and $t > \tau$ we approximate

$$\begin{aligned}
&\langle \sigma(x)\, \sigma(y)\rangle_t - \langle \sigma(x)\rangle_t \langle \sigma(y)\rangle_t \\
&\quad \cong (m_+ - m_-)^2 \,[\langle \theta(x_\perp - \xi_t(x_{||}))\, \theta(y_\perp - \xi_t(y_{||}))\rangle \\
&\qquad - \langle \theta(x_\perp - \xi_t(x_{||}))\rangle \langle \theta(y_\perp - \xi_t(y_{||}))\rangle]
\end{aligned} \tag{9.12}$$

with $\theta(z) = -1/2$ for $z > 0$ and $\theta(z) = 1/2$ for $z < 0$. Here $\xi_t(x_{||})$ is the solution of (9.4) with stiffness matrix (9.11), mobility $\mu(u_0)$, and initial condition $\xi_0(x_{||}) = 0$. Therefore $\xi_t(x_{||})$ is Gaussian with mean zero and covariance

$$C_t(x_{||}, y_{||}) = \int_{\{k \mid k \cdot \xi \leqslant 1\}} d^{d-1}k \left[\sum_{\alpha,\beta=1}^{d-1} \sigma_{\alpha\beta}(u_0)\, k_\alpha k_\beta \right]^{-1} \times \left\{ 1 - \exp\left[-2\mu(u_0) \left(\sum_{\alpha,\beta=1}^{d-1} \sigma_{\alpha\beta}(u_0)\, k_\alpha k_\beta \right) t \right] \right\} \times \exp[ik \cdot (x_{||} - y_{||})] \tag{9.13}$$

where we introduced by hand an ultraviolet cutoff. Inserting in (9.12) yields

$$\langle \sigma(x)\, \sigma(y) \rangle_t - \langle \sigma(x) \rangle_t \langle \sigma(y) \rangle_t \cong (m_+ - m_-)^2 \int d^2q \exp\{ -\tfrac{1}{2}[C_t(x_{||}, x_{||})\, q_1^2 + C_t(y_{||}, y_{||})\, q_2^2] \} \times \{ 1 - \exp[-C_t(x_{||}, y_{||})\, q_1 q_2] \} (q_1 q_2)^{-1} \exp[i(q_1 x_\perp + q_2 y_\perp)] \tag{9.14}$$

If $C_t(x_{||}, y_{||}) \leqslant 1$, we may expand the exponential. Then

$$\langle \sigma(x)\, \sigma(y) \rangle_t - \langle \sigma(x) \rangle_t \langle \sigma(y) \rangle_t \cong (m_+ - m_-)^2\, C_t(x_{||}, y_{||}) [2\pi D(t)]^{-1} \exp[-(x_\perp^2 + y_\perp^2)/2D(t)] \tag{9.15}$$

with

$$D(t) = C_t(0, 0) \cong \begin{cases} \sqrt{t} & \text{if } d \simeq 2 \\ \log t, & \text{if } d \simeq 3 \\ 1 & \text{if } d \geqslant 4 \end{cases} \tag{9.16}$$

for large t. For $d > 3$ the limit $t \to \infty$ yields the large-distance part of the static correlations. Since the interface is smooth, i.e., has bounded fluctuations, the correlations orthogonal to the interface decay rapidly. Along the interface one has slowly decreasing correlations, as predicted by capillary wave theory.

10. BULK DYNAMICS WITH CONSERVED ORDER PARAMETER

If the bulk dynamics conserves the order parameter, then the interface motion is strongly constrained by the conservation law. The mean curvature equation has to be replaced by a more complicated, in fact, nonlocal

evolution equation. We will argue that despite additional mathematical complexities the conceptual framework is simple—at least up to a certain degree. In particular, beyond the bulk diffusion coefficient, there is no additional transport coefficient appearing in the macroscopic equation of motion.

For the sake of concreteness we consider a lattice gas with nearest neighbor attractive forces between particles. The occupation variables are $\eta(x) = 0, 1$; 0 standing for empty and 1 for occupied. The energy reads

$$H = -J \sum_{\langle x, y \rangle} \eta(x)\, \eta(y) \tag{10.1}$$

$J > 0$. The equilibrium states are those of the Ising model. Below the critical temperature there is a high-density fluid phase, density ρ_+, and a low-density gas phase, density ρ_-. The dynamics of the lattice gas is specified through the exchange rates $c(x, y, \eta)$ between nearest neighbor lattice sites. We assume detailed balance by requiring

$$c(x, y, \eta) = \begin{cases} g(\Delta_{xy} H(\eta)) & \text{for } |x-y| = 1 \\ 0 & \text{for } |x-y| > 1 \end{cases} \tag{10.2}$$

with $g(\lambda) = e^{-\beta\lambda} g(-\lambda) > 0$. Here $\Delta_{xy} H$ is the difference in energy after and before the exchange between lattice sites x, y.

On the diffusive time scale the bulk density is governed by the nonlinear diffusion equation

$$\frac{\partial}{\partial t} \rho_t = \nabla \cdot [D(\rho_t) \nabla \rho_t] \tag{10.3}$$

D is the bulk diffusion matrix. It is related to the bulk conductivity σ by the Einstein relation

$$D = \chi^{-1} \sigma \tag{10.4}$$

with χ the static compressibility. In the same fashion as the mobility, the conductivity is defined through the linear response in the average current to an external driving force, i.e., to a bias in the exchange rates. Explicitly, σ is given by the Green–Kubo formula

$$\sigma_{\alpha\beta} = \tfrac{1}{2} \delta_{\alpha\beta} \langle c(0, e_\alpha) \rangle(\rho) - \int_0^\infty dt \sum_x \langle j_\alpha(x)\, e^{Lt} j_\beta(0) \rangle(\rho) \tag{10.5}$$

$$j_\alpha(x) = c(x, x + e_\alpha, \eta)[\eta(x) - \eta(x + e_\alpha)] \tag{10.6}$$

Here e^{Lt} is the formal solution of the master equation. Its matrix elements $(e^{Lt})_{\eta\eta'}$ are the transition probability from η to η' in time t. The expectation $\langle\cdot\rangle(\rho)$ is in the translation-invariant equilibrium state with density ρ, $0<\rho\leqslant\rho_-$, resp. $\rho_+\leqslant\rho<1$. Under our particular assumptions D is diagonal, i.e., $D_{\alpha\beta}=\delta_{\alpha\beta}D$, but in general there is no reason for such a simplification.

We note that for $T>T_c$, $D(\rho)>0$ in the whole interval $0\leqslant\rho\leqslant 1$ and the long-time dynamics is properly described by Eq. (10.3). For $T<T_c$ the situation is more complex. First of all by (10.4), (10.5), D is defined only outside the interval of phase coexistence and $D(\rho_-)>0$, $D(\rho_+)>0$. We formally extend the definition of D to $D(\rho)=0$ for $\rho_-<\rho<\rho_+$. The nonlinear diffusion equation must be read then as a free boundary value problem. (For the Ginzburg–Landau model with conserved dynamics our claims are actually mathematical theorems.[28]) Let us assume that our macroscopic volume Λ consists of a domain Λ_t^- where $\rho_t\leqslant\rho_-$ and a domain Λ_t^+ where $\rho_t\geqslant\rho_+$ separated by a sharp interface Σ_t. By conservation of mass its normal velocity is given by

$$(\rho_+-\rho_-)\,v_n=-\hat{n}\cdot[D(\rho_+)\nabla\rho_t^+-D(\rho_-)\nabla\rho_t^-]\,|_{\Sigma_t} \tag{10.7}$$

where $+$, resp. $-$, refers to the limits being taken from the domain Λ_t^+, resp. Λ_t^-. The nonlinear diffusion equation (10.3) has to be solved with the boundary conditions $\rho_t^+=\rho_+$, $\rho_t^-=\rho_-$ at Σ_t and, say, no-flux boundary conditions at $\partial\Lambda$.

We observe that (10.3), (10.7) have many stationary solutions. In fact, for any particular choice of the interface Σ we just have to set $\rho_t=\rho_+$ in Λ_t^+, $\rho_t=\rho_-$ in Λ_t^-. The set of steady states gives us an indication of the mechanism for the long-time behavior: in each phase the lattice gas tries to reach its equilibrium density and the resulting flux imbalance pushes the interfaces. Once the equilibrium densities are reached, there is then no further motion on the diffusive time scale, although the system has not yet reached global equilibrium.

As an aside we remark that we could choose also the initial density ρ_0 to be inside the interval of phase coexistence, $\rho_-<\rho_0<\rho_+$. The microscopic model is then metastable or even unstable and phase segregates. Equation (10.3) merely predicts $\rho_t=\rho_0$ on the diffusive time scale. This just means that after $\varepsilon^{-2}t$ exchanges per bond the droplets have reached a size proportional to $\varepsilon^{-2/3}t^{1/3}$, which is still small compared to the macroscopic length ε^{-1}. Spinodal decomposition is a topic outside our discussion and we will continue to assume that the *initial* state is locally thermodynamically stable.

Beyond the diffusive time scale the interface motion is governed by the

Gibbs–Thomson effect: for a large equilibrium fluid droplet with radius R the density equals $\rho_+ + c_+/R$ inside and $\rho_- + c_-/R$ outside the droplet with certain positive coefficients c_+, c_-. Our interface is slowly varying on the scale of the lattice. Therefore at pieces of positive curvature (as seen from the + phase) there is a slight excess density on both sides of the interface, whereas at pieces with negative curvature there is a slight depletion. These density variations drive the system eventually to global equilibrium.

As before, let us consider a lattice constant εa, equivalently an interface variation on the scale ε^{-1}. According to Gibbs–Thomson the densities are $\rho_+ + \varepsilon u_t$ in Λ_t^+ and $\rho_- + \varepsilon u_t$ in Λ_t^-. The density gradient is then of order ε and according to Eq. (10.7) the interface velocity is of order ε. Thus, macroscopic interface motion appears on the time scale $\varepsilon^{-3}t$ (exchanges per bond). The nonlinear diffusion equation should still hold and reads on the new scale

$$\varepsilon \frac{\partial}{\partial t} u_t = \nabla \cdot [D(\rho_+ + \varepsilon u_t) \nabla u_t] \tag{10.8}$$

in Λ_t^+ and correspondingly for Λ_t^-. Since $D(\rho_+) > 0$ and $D(\rho_-) > 0$, we obtain

$$\nabla \cdot [D(\rho_{+(-)}] \nabla u_t = 0 \tag{10.9}$$

in Λ_t^+, resp. Λ_t^-. The conservation law (10.7) is already scale invariant and yields

$$(\rho_+ - \rho_-) v_n = -\hat{n} \cdot [D(\rho_+) \nabla u_t^+ - D(\rho_-) \nabla u_t^-] |_{\Sigma_t} \tag{10.10}$$

Of course, such considerations cannot really tell us what happens at the interface. For a more quantitative point of view we turn to the Cahn–Hilliard equation. Since we are away from the critical point, fluctuations are not so important and we should get a reasonable indication of the dynamics of the lattice gas. The Cahn–Hilliard equation reads

$$\frac{\partial}{\partial t} \phi = -L_0 \Delta(\kappa \Delta \phi - V'(\phi)) \tag{10.11}$$

with the Onsager coefficient L_0 fixing the time scale. Equation (10.11) is just like the Allen–Cahn result only modified by $-\Delta$ in front from the conservation law. Pego[29] studies the interface motion under the Cahn–Hilliard equation through a formal asymptotic expansion precisely on the same space-time scale as introduced here. He obtains (10.9) and (10.10) with $D_{\alpha\beta}(\rho_{+(-)}) = \delta_{\alpha\beta} L_0 V''(\rho_{+(-)})$. Clearly, since Eq. (10.11) is

isotropic, D is scalar. For Cahn–Hilliard the conductivity is L_0 and $V''(\rho)=\chi(\rho)^{-1}$, we therefore have agreement with (10.4). At the interface Pego obtains the boundary conditions

$$\begin{aligned}(\rho_+-\rho_-)\,V''(\rho_+)\,u_t^+ &= \kappa\left[\int dx\, w'(x)^2\right]K\Big|_{\Sigma_t}\\ (\rho_+-\rho_-)\,V''(\rho_-)\,u_t^- &= \kappa\left[\int dx\, w'(x)^2\right]K\Big|_{\Sigma_t}\end{aligned} \tag{10.12}$$

They are independent of L_0, implying that only static, equilibrium quantities can be involved. Thus, the correspondence to the lattice gas is unambiguous: $\kappa\int dx\, w'(x)^2$ is the surface tension and $V''(\rho)$ the inverse of the compressibility χ. If we set the chemical potential as

$$\mu_{\mathrm{eq}}+\varepsilon\mu \tag{10.13}$$

then $u=\chi(\rho_{+(-)})\mu$ in Λ_t^+, resp. Λ_t^-. Thus the chemical potential is the quantity that is continuous across the interface.

We conclude that for a stochastic lattice gas on a spatial scale $\varepsilon^{-1}a$ and on a time scale $\varepsilon^{-3}t$ the local chemical potential deviates by order ε from its equilibrium value at coexistence [cf. (10.13)]. This deviation is governed by

$$(\nabla\cdot\sigma_0\nabla\mu_t)=0 \tag{10.14}$$

with boundary condition

$$(\rho_+-\rho_-)\,\mu_t=\sigma(\hat{n})K\,|_{\Sigma_t} \tag{10.15}$$

Here σ_0 is the conductivity matrix as given by the Green–Kubo formula (10.5) evaluated at phase coexistence and $\sigma(\hat{n})$ is the surface tension as defined in (2.11). The normal interface velocity is then determined by

$$(\rho_+-\rho_-)\,v_n=-\hat{n}\cdot\sigma_0(\nabla\mu_t^+-\nabla\mu_t^-)\,|_{\Sigma_t} \tag{10.16}$$

We remark that for a stochastic Ginzburg–Landau model with conserved order parameter, i.e., model B, in the standard lattice discretization we have $(\sigma_0)_{\alpha\beta}=\delta_{\alpha\beta}L_0$ with L_0 the bare global Onsager coefficient. Thus for this model only the static surface tension enters into the macroscopic equation.

A further support of Eqs. (10.14)–(10.16) comes from the observation that they lead to the correct equilibrium interface. We consider a macroscopic domain Λ with a single fluid droplet immersed in the gas phase and

either no-flux or periodic boundary conditions. Then the volumes of Λ_t^+ and of Λ_t^- do not change in the course of time, since

$$\begin{aligned}\frac{d}{dt}|\Lambda_t^+| &= \int_{\Sigma_t} df\, v_n \\ &= -(\rho_+ - \rho_-)^{-1} \int_{\Sigma_t} df\, \hat{n} \cdot \sigma_0 (\nabla \mu_t^+ - \nabla \mu_t^-) \\ &= 0 \end{aligned} \tag{10.17}$$

by (10.14) and the divergence theorem. On the other hand, the surface tension changes as

$$\begin{aligned}\frac{d}{dt} F = \frac{d}{dt} \int_{\Sigma_t} df\, \sigma(\hat{n}) &= \int_{\Sigma_t} df\, \sigma(\hat{n})\, v_n K \\ &= -(\rho_+ - \rho_-)^{-2} \int_{\Sigma_t} df\, \mu_t \hat{n} \cdot \sigma_0 (\nabla \mu_t^+ - \nabla \mu_t^-) \\ &= -(\rho_+ - \rho_-)^{-2} \int_{\Lambda} d^d r\, (\nabla \mu_t \cdot \sigma_0 \nabla \mu_t) \end{aligned} \tag{10.18}$$

again by (10.14) and the divergence theorem. Thus the surface free energy decreases with the constraint of a constant droplet volume. We expect then that μ_t converges to a constant value as $t \to \infty$ and that the stationary droplet shape is given by minimizing the surface energy at constant droplet volume. This is just the equilibrium shape according to the Wulff construction.

11. SURFACE DIFFUSION

A crystalline surface against vacuum relaxes through surface diffusion, i.e., atoms diffuse on the surface and try to equilibrate. The corresponding stochastic model is the obvious modification of the effective interface models in Section 6. As before, at each site of the reference lattice there is a height variable $\phi(x)$, $x \in \Lambda$, taking integer values. The energy of a height configuration is given by (6.3). We prescribe exchange rates $c(x, y, \nabla\phi)$ for nearest neighbor bonds $\langle x, y \rangle$. Here $c(x, y, \nabla\phi)$ is the rate for the jump from $\phi(x)$, $\phi(y)$ to $\phi(x)-1$, $\phi(y)+1$. The rate c depends on ϕ only through the height differences in the neighborhood of the bond $\langle x, y \rangle$. Note that, in general, $c(x, y) \neq c(y, x)$. The exchange rates satisfy the condition of detailed balance with respect to the energy (6.3) in the form

$$c(x, y, \nabla\phi) = c(y, x, \nabla\phi^{xy}) \exp\{-\beta[H(\phi^{xy}) - H(\phi)]\} \tag{11.1}$$

Here ϕ^{xy} denotes the height configuration ϕ with $\phi(x)$, $\phi(y)$ substituted by $\phi(x)-1$, $\phi(y)+1$.

In the phenomenological approach the starting point is the conservation law for the height,

$$\frac{\partial}{\partial t} h + \nabla \cdot j = 0 \tag{11.2}$$

with j the surface current. One then postulates that the surface current is proportional to the gradient of the chemical potential, i.e.,

$$j = -\mu \nabla \frac{\delta F}{\delta h} \tag{11.3}$$

μ is the surface mobility. In general, it depends on the local slope ∇h. Here μ is a $(d-1)\times(d-1)$ matrix. Carrying out the variational derivative in (11.3), we obtain the surface diffusion matrix

$$D_{\alpha\beta} = \sum_{\gamma=1}^{d-1} \mu_{\alpha\gamma}\sigma_{\gamma\beta} \tag{11.4}$$

Since the set of equilibrium states is not altered by the conservation law, the surface free energy is still given by Eq. (6.11). The surface mobility μ is now defined through the linear response in the surface current to a small bias in the exchange rates. Most naturally the bias is imposed by adding a linear external field to the energy in the form

$$H^{(E)}(\phi) = H(\phi) - \sum_x (E \cdot x)\,\phi(x) \tag{11.5}$$

and requiring that the biased rates satisfy detailed balance with respect to $H^{(E)}$. Since the linear response argument follows standard lines (see ref. 7 in the case of lattice gases), we may jump directly to the conclusion. The mobility is given by the following Green–Kubo formula:

$$\begin{aligned}\mu_{\alpha\gamma} = \beta \Big[& \delta_{\alpha\gamma} < c(0, e_\alpha) > (u) \\ & - 2 \int_0^\infty dt \sum_x \langle [c(0, e_\alpha) - c(e_\alpha, 0)] \\ & \times e^{Lt} [c(x, x+e_\alpha) - c(x+e_\alpha, x)] \rangle (u) \Big]\end{aligned} \tag{11.6}$$

$\alpha, \gamma = 1, \ldots, d-1$. As before, β is the inverse temperature and e^{Lt} is the transition probability for the surface dynamics $\langle \cdot \rangle(u)$ is the equilibrium average for a fixed mean slope u.

We expect that the macroscopic dynamics (11.2), (11.3) becomes exact in a scaling limit for which the appropriate time scale is now $\varepsilon^{-4}t$ because of the conservation law. At present, there are no positive results to record. In two dimensions it is of advantage to go over to the slope gas. The equilibrium measure $\langle \cdot \rangle(u)$ becomes then a product measure. The dynamics of the slope gas is unusual, because two neighboring exchanges occur always simultaneously. Such lattice gases have not been studied so far. In fact, we do not have a single example where the time-delayed part of the response vanishes and thus no example for an explicitly computed mobility μ.

At low temperatures the probability of surface excitations is small. We can think of them as a dilute gas of charges with positive sign for excitations above and negative sign for those below the average surface. Under the exchange dynamics they perform almost independent random walks and annihilate each other upon collision. The built-in steps serve as sink and sources. In this picture the mobility is just the conductivity of the gas of excitations. In particular, even for the high-symmetry plane $u=0$ below its roughening transition the mobility does not vanish. This is in marked contrast to the nonconserved (evaporation) dynamics.

As one application of physical interest we mention that the macroscopic equations (11.2), (11.3) describe the spontaneous formation of facets for the high-symmetry plane below T_R. This topic is somewhat off the main track and is more fully explored in ref. 25. Here we just want to explain the mathematical mechanism and to point out the difference from the porous medium equation (8.8) valid for the nonconserved dynamics. Let us consider the physical dimension $d=3$ and let us assume for simplicity that the initial profile depends only on the coordinate along the 1 axis, here denoted by x. It is convenient to write down the dynamics in terms of the slope $u_t = \partial h_t/\partial x$,

$$\frac{\partial}{\partial t} u_t = -\frac{\partial^2}{\partial x^2}\left[\mu(u_t)\frac{\partial^2}{\partial x^2} g(u_t)\right] \tag{11.7}$$

Here $g=\sigma_1$. Therefore g is a strictly increasing function with $g(u)=-g(-u)$. Below T_R the surface tension has a cusp [cf. Eq. (8.6)] and therefore g has a jump discontinuity of magnitude $2b$ at $u=0$. The mobility is even, $\mu(u)=\mu(-u)$, by symmetry, and, as argued above, $\mu(u)>0$. Such types of equations have been studied in the context of nonlinear filtration problems.[30] To have the correct notion of a solution, one defines

$$v = g(u) \tag{11.8}$$

and rewrites (11.7) as

$$\frac{\partial}{\partial t} g^{-1}(v_t) = \frac{\partial^2}{\partial x^2}\left[\mu(g^{-1}(v_t))\frac{\partial^2}{\partial x^2} v_t\right] \tag{11.9}$$

Note that g^{-1} is increasing and vanishes on the interval $[-b, b]$. Thus we have for $|v| \leqslant b$ and every t

$$0 = \frac{\partial^4}{\partial x^4} v_t \tag{11.10}$$

Its solution has to be matched to the solution of Eq. (11.9) for $|v| \geqslant b$ in such a way that the current is continuous. This leads to a free boundary value problem for the set of points $\{x \mid |v_t(x)| = b\}$. If the initial height profile is periodic, then pieces with zero slope expand, which means that facets of the high-symmetry plane form. After a *finite* span of time the facet is completed and the surface healed.

12. CONCLUSIONS

We have introduced a variety of stochastic models with the common feature that in thermal equilibrium they have two spatially coexisting phases. We argued that a slowly varying interface is governed by the appropriate macroscopic equation in a scaling limit. Most importantly we have identified the quantities appearing in the macroscopic equation in terms of well-defined time-dependent equilibrium correlations of the underlying microscopic model.

A novel situation arises when the chemical potential (=external magnetic field) is moved slightly off phase coexistence. Specifically, let us consider the nonconserved case and let us restrict ourselves to times much shorter than the nucleation time of the now metastable phase. Clearly, a planar interface acquires a net velocity depending on the orientation. Thus the macroscopic equation becomes

$$v_n = -\mu \frac{\delta F}{\delta \Sigma} + v(\hat{n}) \tag{12.1}$$

with $v(\hat{n})$ the systematic velocity along $\hat{n}$. Since the mobility and the surface free energy are equilibrium notions, it is slightly inconsistent to keep their form in (12.1) also for the driven interface. Only to linear order in the chemical potential difference are we allowed to ignore such changes and this was used in the identification of the mobility.

The planar interface moving with velocity $v(\hat{n})$ shows one surprising feature, as first noticed in ref. 31 and intensively studied in recent years.[32] In equilibrium the interface has small Gaussian fluctuations around its mean location. As soon as $v(\hat{n}) \neq 0$, fluctuations increase and, more importantly, they are no longer Gaussian. This phenomenon is known as "kinetic roughening," because the interface roughens under systematic motion.

Driving can also be implemented for models with a conserved order parameter. For surface diffusion one could bias particles along the surface, although a physical realization may require some ingenuity. For lattice gases one possibility is to drive through boundary conditions: we impose on the left side of the box a density $\rho < \rho_-$ and on the right side a density $\rho > \rho_+$. There is then a sharp interface parallel to the boundaries of the slab. Its fluctuations are suppressed compared to thermal fluctuations.[33]

APPENDIX A. ZERO-TEMPERATURE, TWO-DIMENSIONAL ISING MODEL

The stochastic dynamics of the zero-temperature, two-dimensional Ising model is defined by the flip rates

$$c_x(\sigma) = \begin{cases} 0 & \text{if } n(x) = 3, 4 \\ 1 & \text{if } n(x) = 2 \\ c_1 & \text{if } n(x) = 1 \\ c_0 & \text{if } n(x) = 0 \end{cases} \tag{A.1}$$

$x \in \mathbb{Z}^2$. Here $n(x)$ is the number of nearest neighbors y of x such that $\sigma(y) = \sigma(x)$. Energy-increasing flips are forbidden. Flips that conserve the energy have rate one, thereby fixing the time scale. The energy-decreasing rates c_0 and c_1 are free parameters in principle. Physically, we expect $c_0 \gg c_1 \gg 1$.

An ambitious goal is to prove that if initially the spins are independent with, say, $\mathsf{E}(\sigma(x)) = 0$, then after the long time $\varepsilon^{-2}t$ one has sharp interfaces which (on the spatial scale ε^{-1}) are governed by mean curvature with mobility (4.25).

A more modest project is to impose initially a domain of + and a domain of − spins such that they are separated by a *single*, non-self-intersecting contour γ (γ is a sequence of connected bonds on the dual lattice). Under the rules (A.1) such a situation is not strictly maintained in time. For example, we could have a + spin with east, west neighbors + and north, south neighbors −. By flipping, γ would split into two. Such complications can either be estimated to have a small probability or be forbidden by a slight modification of (A.1).

In passing we should mention the closely related contour model.[34] Here the configuration space is restricted to single closed, non-self-intersecting contours. Both energy-increasing and energy-decreasing spin flips are allowed (subject to detailed balance), but illegal flips leading out of the configuration space have rate zero. In the stationary measure, a contour of

length $|\gamma|$ has the weight $e^{-\beta|\gamma|}$. For the contour model, motion by mean curvature has not been proved yet.

Returning to zero temperature, we note that a tractable situation is obtained by requiring that γ consists only of bonds e_1, e_2 (i.e., the contour γ consists only of bonds directed to the right and upwards).[35] If we identify e_1 with 0 and e_2 with 1, then the dynamics on the sequences of 0, 1's as induced by the rates (A.1) is precisely the symmetric exclusion process with nearest neighbor exchanges. Proving motion by curvature corresponds to the hydrodynamic limit for symmetric exclusion. This is a well understood subject[7] and results in the mobility (4.25).

We could regard the contour γ also as a single-valued function with respect to the x axis. Let $\phi(x) \in \mathbf{Z}$, $x \in \mathbf{Z}$, be the corresponding height. Then the height differences $\eta(x) = \phi(x+1) - \phi(x)$ are nonnegative and are governed by the zero-range process with rate function $c(n) = 1$ for $n \geqslant 1$, $c(0) = 0$. Again motion by curvature corresponds to the hydrodynamic limit for the zero-range process.

With the present constraint, clearly, we do not cover "critical" points where the tangent vector to the macroscopic interface is parallel to one of the coordinate axes. One may worry whether the mean curvature equation actually applies at such singular points, in particular since the (entropic part of the) surface tension and the mobility vanish and are not differentiable at such points. We are going to use the mapping to zero range to prove that such doubts are unfounded, fortunately.

We consider a cylinder $[1,..., N] \times \mathbf{Z}$ with periodic boundary conditions, $N = [\varepsilon^{-1}]$, $[q]$ denoting the integer part of q. The interface is given by the single-valued function $x \mapsto \phi(x) \in \mathbf{Z}$, $x = 1,..., N$, with spins equal to $+1$ below and equal to -1 above ϕ. We set $c_1 = 2$ and forbid flips that would make ϕ multivalued. The height process is then governed by the rates

$$\begin{aligned} c_x^+(\phi) &= c(\phi(x+1) - \phi(x)) + c(\phi(x-1) - \phi(x)) \\ c_x^-(\phi) &= c(\phi(x) - \phi(x+1)) + c(\phi(x) - \phi(x-1)) \end{aligned} \tag{A.2}$$

where $c(n) = 1$ for $n \geqslant 1$, $c(n) = 0$ for $n \leqslant 0$. The height differences $\eta(x) = \phi(x+1) - \phi(x)$ now take values in $\mathbf{Z}$. We can think of their dynamics as a two-component zero-range process. The component A corresponds to $\eta(x) > 0$, the component B to $\eta(x) < 0$, and "empty" to $\eta(x) = 0$. The A and B particles jump according to zero range. If an A particle jumps to a site with a B particle present (or vice versa), they annihilate each other instantaneously. Thus we have the diffusion-reaction process $A + B \to 0$, well studied in the case of independent particles [i.e., $c(n) = n$, $n > 0$, $c(n) = 0$, $n \leqslant 0$].[36,37]

For our particular coordinate frame, motion by curvature reads

$$\frac{\partial}{\partial t} h_t = \frac{\partial}{\partial x} \sigma' \left(\frac{\partial}{\partial x} h_t \right), \qquad \sigma'(u) = \frac{u}{1+|u|} \tag{A.3}$$

on the circle [0, 1]. Equivalently the slope $u_t = (\partial/\partial x)\, h_t$ is governed by the nonlinear diffusion equation

$$\frac{\partial}{\partial t} u_t = \frac{\partial^2}{\partial x^2} \sigma'(u_t) \tag{A.4}$$

Thus we have to show that the two-component zero-range process is governed by (A.4) on a large scale. This looks like a problem covered by methods which are standard by now. The difficulties with the conventional approach can be understood from the peculiar structure of the set of stationary measures. Clearly for $t \to \infty$ the majority component survives and has then an independent geometric distribution conditioned on the total number of surviving particles. Thus the stationary measures split into a set of stationary A measures and of stationary B measures which are singular with respect to each other. Therefore any simple-minded version of relative entropy for the time t measure of the η_t process is bound to be infinite. Standard methods very heavily rely on entropy and entropy production estimates and it is not clear how they extend to the present situation.

As explained to me by Horng-Tzer Yau, fortunately there is a somewhat different technique available.[38]

Let us introduce the single-site distribution for the stationary measures. We set for $u > 0$

$$p^{(u)}(n) = \begin{cases} u^n (1+u)^{-(n+1)} & \text{if} \quad n \geqslant 0 \\ 0 & \text{if} \quad n < 0 \end{cases} \tag{A.5}$$

for $u < 0$

$$p^{(u)}(n) = \begin{cases} 0 & \text{if} \quad n > 0 \\ |u|^{|n|}\, (1+|u|)^{-(|n|+1)} & \text{if} \quad n \leqslant 0 \end{cases} \tag{A.6}$$

and

$$p^{(0)}(n) = \delta_{0n} \tag{A.7}$$

Here u is the average density, $u = \sum_n p^{(u)}(n) n$.

Theorem 1. Let the scaled height process be given by

$$\phi_t^\varepsilon(x) = \varepsilon \phi_{\varepsilon^{-2} t}(\varepsilon^{-1} x) \tag{A.8}$$

with $x = \varepsilon, 2\varepsilon, \ldots, N\varepsilon$. In the obvious way we regard ϕ_t^ε as a piecewise constant function on the circle $[0, 1]$. Let h_t be the solution of the curvature equation (A.3) with initial datum h_0, assumed to be C^1 on the circle $[0, 1]$. Let the initial measure μ^ε of the ϕ_t processes be a product measure such that

$$\mu^\varepsilon(\eta(x) = n) = p^{(u_0(\varepsilon x))}(n) \tag{A.9}$$

with $u_0 = (d/dx)\, h_0$. Then, for all $t \geqslant 0$,

$$\lim_{\varepsilon \to 0} \mathsf{E}\left(\int_0^1 dx\, [\phi_t^\varepsilon(x) - h_t(x)]^2 \right) = 0 \tag{A.10}$$

Proof. We first discretize (A.3) as

$$\begin{aligned} \frac{\partial}{\partial t} h_t^\varepsilon(x) &= \varepsilon^{-1}(\sigma'(u_t^\varepsilon(x)) - \sigma'(u_t^\varepsilon(x - \varepsilon))) \\ u_t^\varepsilon &= \varepsilon^{-1}[h_t^\varepsilon(x + \varepsilon) - h_t^\varepsilon(x)] \end{aligned} \tag{A.11}$$

$x = \varepsilon, 2\varepsilon, \ldots, N\varepsilon$, with initial condition $h_0^\varepsilon(x) = h_0(x)$. We regard h_t^ε and u_t^ε as piecewise constant functions on the circle $[0, 1]$.

We have

$$\begin{aligned} &\mathsf{E}\left(\varepsilon \sum_{x=1}^{N} [h_t^\varepsilon(\varepsilon x) - \varepsilon \phi_{\varepsilon^{-2}t}(x)]^2 \right) \\ &\quad = \mathsf{E}\left(\varepsilon \sum_{x=1}^{N} [h_0(\varepsilon x) - \varepsilon \phi_0(x)]^2 \right) \\ &\qquad -2 \int_0^t ds\, \varepsilon \sum_{x=0}^{N} \{ u_s^\varepsilon(\varepsilon x)\, \sigma'(u_s^\varepsilon(\varepsilon x)) - \sigma'(u_s^\varepsilon(\varepsilon x))\, \mathsf{E}(\eta_{\varepsilon^{-2}s}(x)) \\ &\qquad - u_s^\varepsilon(\varepsilon x)\, \mathsf{E}(\mathrm{sg}(\eta_{\varepsilon^{-2}s}(x))) + \mathsf{E}(|\eta_{\varepsilon^{-2}s}(x)| - |\mathrm{sg}(\eta_{\varepsilon^{-2}s}(x))|) \} \end{aligned} \tag{A.12}$$

with $\eta_t(x) = \phi_t(x + 1) - \phi_t(x)$ and $\mathrm{sg}(n) = -1$, 0, and 1 for $n < 0$, $n = 0$, and $n > 0$, respectively. Under the time integral we want to average locally exploiting that the η_t process restricted to a bounded interval of lattice sites should be in one of its stationary measures as $\varepsilon \to 0$. Let us denote them by $\langle \cdot \rangle(u)$, where $\langle \eta(x) \rangle(u) = u$. With respect to $\langle \cdot \rangle(u)$, the $\eta(x)$'s are independent and

$$\langle \{\eta(x) = n\} \rangle(u) = p^{(u)}(n) \tag{A.13}$$

We note that

$$\begin{aligned} \langle \mathrm{sg}(\eta(x)) \rangle(u) &= \sigma'(u) \\ \langle |\eta(x)| - |\mathrm{sg}(\eta(x))| \rangle(u) &= u\sigma'(u) \end{aligned} \tag{A.14}$$

We now fix an integer $l>0$ and divide $[1,..., N]$ into blocks B_j of l sites each, $j=1,..., M=[N/l]$. The block $B_{M+1}=[Ml+1,..., N]$ has a length less than l. Let y_j be the left endpoint of B_j. We also define the block averaged density

$$v_t^\varepsilon(j)=\frac{1}{l}\sum_{x\in B_j}\eta_{\varepsilon^{-2}t}(x) \tag{A.15}$$

Note that the $v_t^\varepsilon(j)$'s are random variables. In (A.12) we average locally using that

$$\frac{1}{l}\sum_{x\in B_j} f(\eta_{\varepsilon^{-2}t}(x))\cong \langle f\rangle(v_t^\varepsilon(j)) \tag{A.16}$$

This substitution results in an error W which we will have to estimate. We obtain

$$\begin{aligned}
&\mathsf{E}\left(\varepsilon\sum_{x=1}^{N}[h_t(\varepsilon x)-\varepsilon\phi_{\varepsilon^{-2}t}(x)]^2\right)\\
&\quad=-2\int_0^t ds\,\mathsf{E}\left(M^{-1}\sum_{j=1}^{M}[u_s^\varepsilon(\varepsilon y_j)-v_s^\varepsilon(j)]\right.\\
&\qquad\left.\times[\sigma'(u_s^\varepsilon(\varepsilon y_j))-\sigma'(v_s^\varepsilon(j))]\right)+W(\varepsilon, l)
\end{aligned} \tag{A.17}$$

Since σ' is increasing, the integrand is nonnegative. Therefore

$$\mathsf{E}\left(\varepsilon\sum_x[h_t(\varepsilon x)-\varepsilon\phi_{\varepsilon^{-2}t}(x)]^2\right)\leqslant W(\varepsilon, l) \tag{A.18}$$

Thus we only have to show that $W(\varepsilon, l)$ vanishes as we first let $\varepsilon\to 0$ and then $l\to\infty$. For this purpose we need three estimates.

(i) (*Lattice approximation*) We have the following result.

Lemma 2. There is a constant b such that

$$|u_t^\varepsilon(x)-u_t(x)|\leqslant b\varepsilon \tag{A.19}$$

for all $x\in[1, 0]$.

Proof. Ref. 39. ∎

By Lemma 2

$$\lim_{\varepsilon\to 0}\int_0^1 dx\,[h_t^\varepsilon(x)-h_t(x)]^2=0 \tag{A.20}$$

Therefore it suffices to consider the limit $\varepsilon\to 0$ in (A.12).

(ii) (*A priori bounds*) Since the $\eta(x)$'s are unbounded, we must make sure that they do not become too large.

Lemma 3. Let $|u_0| \leqslant u_{\max}$. Let f be an increasing function of $|\eta|$ in the sense that $f(|\eta|) \leqslant f(|\eta'|)$ if $|\eta| \leqslant |\eta'|$, i.e., $|\eta(x)| \leqslant |\eta'(x)|$ for all x. Then

$$\mathsf{E}(f(|\eta_t|)) \leqslant \langle f(|\eta|)\rangle(u_{\max}) \tag{A.21}$$

Proof. We follow ref. 6 and compare the process $|\eta_t|$ with a standard zero-range process ζ_t. To do so, we have to realize both processes on the same probability space. Let $\lambda_t(x, \pm)$ be independent Poisson processes with intensity 1/2. Let us fix one particular realization. For the ζ_t process, if a "mark" for $\lambda_t(x, +)$ $[\lambda_t(x, -)]$ appears, then a particle is transferred from x to $x+1$ $[x-1]$. The mark is ignored if $\zeta(x)=0$ at that time. The η_t history is constructed correspondingly with the additional rule that an A particle $[\eta(x)>0]$ and a B particle $[\eta(x)<0]$ annihilate each other on site and instantaneously. Clearly, if for the initial configuration $|\eta| \leqslant \zeta$, then $|\eta_t| \leqslant |\zeta_t|$ for all $t>0$. Therefore, if initially ζ has the same distribution as $|\eta|$, then

$$\mathsf{E}(f(|\eta_t|)) \leqslant \mathsf{E}(f(|\zeta_t|)) \tag{A.22}$$

By assumption, the initial measure for ζ is stochastically smaller than $\langle\cdot\rangle(u_{\max})$. By the monotonicity of ζ_t and the invariance of $\langle\cdot\rangle(u_{\max})$ we conclude

$$\mathsf{E}(f(\zeta_t)) \leqslant \langle f(|\eta|)\rangle(u_{\max}) \quad \blacksquare \tag{A.23}$$

Lemma 3 implies the uniform bound

$$\mathsf{E}(|\eta_t(x)|) \leqslant u_{\max} \tag{A.24}$$

and allows us to replace in (A.12) $\eta_{\varepsilon^{-2}t}(x)$ and $|\eta_{\varepsilon^{-2}t}(x)|$ by their cutoff version $h_R(\eta(x))$, resp. $h_R(|\eta(x)|)$, with $h_R(n)=n$ for $|n| \leqslant R$, $h_R(n)=0$ for $|n|>R$, at the expense of an error exponentially small in R.

(iii) (*One block estimate*) Let μ_t^ε be the distribution of $\eta_{\varepsilon^{-2}t}$ and let $\bar{\mu}^\varepsilon$ be its space-time average, i.e.,

$$\bar{\mu}^\varepsilon(f) = \frac{1}{t}\int_0^t ds \frac{1}{N}\sum_{x=1}^{N} \mu_s^\varepsilon(\tau_x f) \tag{A.25}$$

with τ_x the shift by x and f a bounded function on Z^N.

Lemma 4. Let $g: \mathbf{Z} \to \mathbf{R}$ be bounded and let $\hat{g}(u) = \sum_n p^{(u)}(n)\, g(n)$. Then

$$\lim_{l \to \infty} \limsup_{\varepsilon \to 0} \bar{\mu}^\varepsilon \left(\left| \frac{1}{l} \sum_{x=1}^{l} g(\eta(x)) - \hat{g}\left(\frac{1}{l} \sum_{x=1}^{l} \eta(x) \right) \right| \right) = 0 \tag{A.26}$$

Proof. Let f be a bounded local function. Then

$$\varepsilon^2 \frac{1}{N} \sum_{x=1}^{N} [\mu_t^\varepsilon(\tau_x f) - \mu_0^\varepsilon(\tau_x f)] = t \bar{\mu}^\varepsilon(Lf) \tag{A.27}$$

and therefore

$$\lim_{\varepsilon \to 0} \bar{\mu}^\varepsilon(Lf) = 0 \tag{A.28}$$

Thus the limit points of $\bar{\mu}^\varepsilon$ restricted to the interval $[1,..., l]$ are stationary measures for the η_t process in the interval $[1,..., l]$ with closed ends. These are $\prod_{x=1}^{l} p^{(1)}(\eta(x))$ conditioned on $\sum_{x=1}^{l} \eta(x) = n$ for $n \geqslant 0$ and $\prod_{x=1}^{l} p^{(-1)}(\eta(x))$ conditioned on $\sum_{x=1}^{l} \eta(x) = n$ for $n \leqslant 0$. The limit $l \to \infty$ in (A.26) follows then from the law of large numbers. ∎

To complete our argument, we have to estimate the error $W(\varepsilon, l)$. We just consider the fourth term on the right-hand side of (A.12) with corresponding error term $W_4(\varepsilon, l)$, the other terms being similar. We have

$$W_4(\varepsilon, l) = \left| \int_0^t ds \left\{ \varepsilon \sum_{x=1}^{N} u_s^\varepsilon(\varepsilon x)\, \mu_s^\varepsilon(g(\eta(x)) \right. \right. \tag{A.30}$$

$$\left. \left. - \frac{1}{M} \sum_{j=1}^{M} u_s^\varepsilon(\varepsilon y_j)\, \mu_s^\varepsilon\left(\hat{g}\left(\frac{1}{l} \sum_{x \in B_j} \eta(x) \right) \right) \right\} \right| \tag{A.31}$$

with g bounded. By adding and subtracting we obtain

$$W_4(\varepsilon, l) \leqslant \int_0^t ds\, \varepsilon \sum_{x=1}^{N} |u_s^\varepsilon(\varepsilon x) - \bar{u}_s^\varepsilon(\varepsilon x)| + (\sup_x |u_s^\varepsilon(\varepsilon x)|)$$

$$\times \int_0^t ds \frac{1}{M} \sum_{j=1}^{M} \mu_s^\varepsilon\left(\left| \frac{1}{l} \sum_{x \in B_j} g(\eta(x)) - \hat{g}\left(\frac{1}{l} \sum_{x \in B_j} \eta(x) \right) \right| \right) \tag{A.31}$$

with $\bar{u}_s^\varepsilon(\varepsilon x) = u_s^\varepsilon(\varepsilon y_j)$ for $x \in B_j$. By Lemma 2, the first term vanishes as $\varepsilon \to 0$. $|u_s^\varepsilon|$ is bounded by $u_{\max}$. The second term in (A.31) is not quite of the desired form. There is a time average, but the space average is only over multiples of l. We repeat then the estimate leading to (A.17) only with

the partition $\{B_j, j=1,..., M\}$ shifted by y units, $y=1,..., l-1$. This leads again to (A.31) with the correspondingly shifted partition. Summing up, we are led to an expression of the form (A.26), which by Lemma 4 vanishes in the limit $\varepsilon \to 0$, $l \to \infty$.

APPENDIX B. GINZBURG–LANDAU EFFECTIVE INTERFACE MODEL

We consider the Ginzburg–Landau model of Section 6. The dynamics is governed by Eq. (6.4). The potential V is even, bounded from below as $V(\phi) \geqslant c\,|\phi|^{1+\delta}$ for suitable c, $\delta > 0$, and twice differentiable. As reference "plane" we choose the box $\Lambda(\varepsilon) = [1,..., N]^d \subset \mathbf{Z}^d$ with periodic boundary conditions, $N = [\varepsilon^{-1}]$, $[q]$ denoting the integer part of q. To smoothen somewhat our notation, we take $d+1$ as the dimension of physical space and set $\beta = 1$.

We have no proof of the mean curvature equation. However, the strategy used for the zero-temperature Ising model seems to be promising. We would like to compare then the height process with the solution of the mean curvature equation in the L^2 norm. When carrying out this program one notices that the real gaps relate to the equilibrium measures. Therefore the purpose of this Appendix is to provide a list of missing properties—in the hope of motivating further research on massless Gibbs fields.

If V is strictly convex and $V(\phi) \geqslant c\phi^2$ with $c > 0$, then the equilibrium measures satisfy the Brascamp–Lieb inequality.[40] It provides bounds on correlations in terms of a Gaussian measure. Thus we may as well assume these stronger properties for the potential V.

According to Section 6, the mean curvature equation reads

$$\frac{\partial}{\partial t} h_t = \sum_{\alpha,\beta=1}^{d} \sigma_{\alpha\beta}(\nabla h_t) \frac{\partial^2}{\partial x_\alpha \partial x_\beta} h_t \tag{B.1}$$

on the torus $[0, 1]^d$. For this equation to be meaningful we need the following result.

Property 1. The surface free energy σ defined in (6.11) and the "pressure" p defined in (6.14) exist, are Legendre transforms of each other, and have some smoothness as a function of u, resp. λ.

Let the scaled height process be defined by

$$\phi_t^\varepsilon(x) = \varepsilon \phi_{\varepsilon^{-2}t}(\varepsilon^{-1}x) \tag{B.2}$$

$x \in \varepsilon \Lambda(\varepsilon)$. In the obvious way we regard ϕ_t^ε as a piecewise constant function on $[0, 1]^d$. Let h_t be the solution of (B.1) with initial datum h_0. We would like to prove that

$$\lim_{\varepsilon \to 0} \mathsf{E}\left(\int_{[0,1]^d} d^d x \, [h_t(x) - \phi_t^\varepsilon(x)]^2 \right) = 0 \tag{B.3}$$

for $t > 0$, if so at time $t = 0$.

We discretize Eq. (B.1) as

$$\frac{\partial}{\partial t} h_t^\varepsilon(x) = -\sum_{\alpha=1}^{d} \nabla_\alpha^{\varepsilon *} \sigma_\alpha(\nabla^\varepsilon h_t^\varepsilon(x)) \tag{B.4}$$

$x \in \Lambda(\varepsilon)$. Here $\sigma_\alpha = \partial\sigma/\partial u_\alpha$, $\nabla_\alpha^\varepsilon f(x) = \varepsilon^{-1}(f(x + \varepsilon e_\alpha) - f(x))$ with e_α the αth unit vector, and $\nabla^{\varepsilon *}$ denotes the adjoint gradient operator. By a straightforward computation

$$\begin{aligned}
&\mathsf{E}\left(\varepsilon^d \sum_{x \in \Lambda(\varepsilon)} [h_t^\varepsilon(\varepsilon x) - \varepsilon \phi_{\varepsilon^{-2} t}(x)]^2 \right) \\
&\quad = \mathsf{E}\left(\varepsilon^d \sum_{x \in \Lambda(\varepsilon)} [h_0(\varepsilon x) - \varepsilon \phi_0(x)]^2 \right) \\
&\qquad -2 \int_0^t ds \, \varepsilon^d \sum_{x \in \Lambda(\varepsilon)} \Bigg[\sum_{\alpha=1}^{d} \{ \sigma_\alpha(\nabla^\varepsilon h_s^\varepsilon(\varepsilon x)) \, \nabla_\alpha^\varepsilon h_s(\varepsilon x) \\
&\qquad - \sigma_\alpha(\nabla^\varepsilon h_s^\varepsilon(\varepsilon x)) \, \mathsf{E}(\nabla_\alpha \phi_{\varepsilon^{-2} s}(x)) \\
&\qquad - \nabla_\alpha^\varepsilon h_s^\varepsilon(\varepsilon x) \, \mathsf{E}(V'(\nabla_\alpha \phi_{\varepsilon^{-2} s}(x))) + \mathsf{E}(\nabla_\alpha \phi_{\varepsilon^{-2} s}(x) \, V'(\nabla_\alpha \phi_{\varepsilon^{-2} s}(x))) \} - 1 \Bigg]
\end{aligned} \tag{B.5}$$

The right side of (B.5) involves only the height differences

$$\eta(b) = \eta(x, y) = \phi(y) - \phi(x), \qquad b = (x, y), \qquad |x - y| = 1 \tag{B.6}$$

residing on the directed bonds of the lattice. Note that $\eta(b) = -\eta(-b)$, $-b = (y, x)$. Being the gradient of ϕ, the rotation of the vector field η has to vanish. More explicitly

$$\sum_{b \in \mathscr{C}} \eta(b) = 0 \tag{B.7}$$

for any closed loop $\mathscr{C}$ in Λ. Equivalently

$$\text{(P)}: \quad \sum_{b \in \mathscr{P}} \eta(b) = 0 \qquad \text{for every plaquette } \mathscr{P}$$

provided Λ is simply connected. The space of height difference configurations is constrained by (P). The dynamics of the height differences is governed by the stochastic differential equation

$$\frac{d}{dt}\eta_t(x, y) = -\sum_{e,|e|=1}^{*} \{V'(\eta_t(y, y+e)) - V'(\eta_t(x, x+e))\} + \sqrt{2}\,(\dot{w}_t(y) - \dot{w}_t(x)) \tag{B.8}$$

$\sum^*$ means that the sum is restricted to run only over those bonds such that both their endpoints are in Λ. The unique invariant measure for (B.8) is then

$$Z^{-1} \prod_b d\eta(b)\, e^{-V(\eta(b))} \delta(\mathrm{rot}\,\eta) \tag{B.9}$$

with $\delta(\mathrm{rot}\,\eta)$ as short hand for the set of linear constraints (P).

(i) (*Local ergodicity*) We follow the standard argument, e.g., in the form given in ref. 28, and consider the space-time averaged local measure for the height differences in the limit $\varepsilon \to 0$ (possibly along subsequences). Let us call the limit measure μ. This is a measure on the space of height differences on the lattice $\mathbf{Z}^d$ satisfying the set of local constraints (P). By construction μ satisfies (a) an entropy bound, (b) translation invariance, and (c) stationarity.

This last item has to be spelled out. We consider a box Λ with bordering sites $\partial\Lambda$ and denote by Λ^b the set of bonds such that at least one of their two endpoints is in Λ. Then in terms of the conditional measure stationarity reads

$$\int (L_\Lambda f)\, \mu(d\eta(b), b \in \Lambda^b \mid \eta(b), b \in (\Lambda^b)^c) = 0 \tag{B.10}$$

for all strictly local, smooth functions f. Here L_Λ is defined through the Dirichlet form

$$-(f, L_\Lambda f) = Z^{-1} \int \prod_{b \in \Lambda^b} d\eta(b)\, e^{-V(\eta(b))} \delta(\mathrm{rot}\,\eta) \times \sum_{x \in \Lambda} \sum_{e,|e|=1} [\partial f / \partial \eta_{(x,x+e)}]^2 \tag{B.11}$$

(B.10) has a unique solution. It is written down more easily for the height variables. Fixing the $\eta(b)$'s with b in the complement of Λ^b determines the

heights $\psi(x)$, $x \in \partial\Lambda$, up to a global constant. Given $\psi(x)$, $x \in \partial\Lambda$, the unique solution to (B.10) is

$$Z^{-1}(\psi) \exp\Big[- \sum_{\langle x, y\rangle, x, y \in \Lambda} V(\phi(x) - \phi(y)) - \sum_{\langle x, y\rangle, x \in \Lambda, y \in \partial\Lambda} V(\phi(x) - \psi(y)) \Big] \prod_{x \in \Lambda} d\phi(x) \tag{B.12}$$

which are the usual DLR equations.

Property 2. Let μ be a probability measure for the height differences $\eta(b)$, $b \in (\mathbf{Z}^d)^b$, satisfying the local linear constraints (P). μ is assumed to have a bounded free energy per unit volume, to be translation invariant, and to be ergodic with respect to translations. The conditional measures of μ satisfy the DLR equations (B.12) (which are to be read for height differences in the obvious way). Then μ is unique and given as the infinite-volume limit of (B.12) with boundary conditions $\psi(x) = u \cdot x$, $x \in \partial\Lambda$, $u_\alpha = \mu(\eta(0, e_\alpha))$, $\alpha = 1, \ldots, d$.

(ii) (*Local averaging*) Following the strategy in the proof of Theorem 1, we should average locally in (B.5). Let $\langle \cdot \rangle(u)$ denote the expectation with respect to the equilibrium measure μ of Proposition 2. Then we need

Property 3.

(a) $$\langle \eta(0, e_\alpha) \rangle(u) = u_\alpha \tag{B.13}$$

(b) $$\langle V'(\eta(0, e_\alpha)) \rangle(u) = \sigma_\alpha(u) \tag{B.14}$$

(c) $$\Big\langle \sum_{\alpha=1}^{d} \eta(0, e_\alpha)\, V'(\eta(0, e_\alpha)) - 1 \Big\rangle(u) = \sum_{\alpha=1}^{d} u_\alpha \sigma_\alpha(u) \tag{B.15}$$

Formally (a)–(c) are easily verified. (a) holds by definition. (b) is shown in (7.6). For (c) we use the grand-canonical measure [compare with (6.14)]. We introduce the slope chemical potential λ and use that $\lambda_\alpha = \sigma_\alpha$ in the infinite-volume limit. Therefore we have to show that the left side of (B.15) equals $u \cdot \lambda$. We have

$$\frac{1}{|\Lambda|} \sum_{\langle x, y\rangle, x, y \in \Lambda} (\phi(y) - \phi(x))\, V'(\phi(y) - \phi(x)) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sum_{\langle x, y\rangle, x, y \in \Lambda} \phi(x)\, V'(\phi(x) - \phi(y)) \tag{B.16}$$

Therefore, up to boundary terms,

$$\left\langle \sum_{\alpha=1}^{d} \eta(0, e_\alpha)\, V'(\eta(0, e_\alpha)) \right\rangle - 1$$

$$= -\frac{1}{Z} \int \prod_{x \in \Lambda} d\phi(x)\, \delta(\phi(0)) \exp\left[\sum_{\alpha=1}^{d} \sum_{x \in \Lambda, x+e_\alpha \in \Lambda} \lambda_\alpha \eta(x, x+e_\alpha) \right]$$

$$\times \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \phi(x) \frac{\partial}{\partial \phi(x)} \exp\left[- \sum_{\langle x, y \rangle, x, y \in \Lambda} V(\phi(x) - \phi(y)) \right] - 1$$

$$= \frac{1}{|\Lambda|} \left\langle \sum_{\alpha=1}^{d} \sum_{x \in \Lambda, x+e_\alpha \in \Lambda} \lambda_\alpha \eta(x, x+e_\alpha) \right\rangle = \lambda \cdot u \tag{B.17}$$

We partition $\Lambda(\varepsilon)$ into boxes B_j of side length l, $j = 1, \ldots, M = (N/l)^d$. Let $u_t^\varepsilon(j)$ be the value of $\nabla^\varepsilon h_t^\varepsilon$ in the middle of the jth box and let

$$v_t^\varepsilon(j) = l^{-d} \sum_{x \in B_j} \nabla \phi_{\varepsilon^{-2} t}(x) \tag{B.18}$$

be the box averaged height. Then, up to an error from the local averaging, the time integral in (B.5) becomes

$$-2 \int_0^t ds\, M^{-1} \sum_{j=1}^{M} \mathsf{E}\left(\sum_{\alpha=1}^{d} [u_{s\alpha}^\varepsilon(j) - v_{s\alpha}^\varepsilon(j)][\sigma_\alpha(u_s^\varepsilon(j)) - \sigma_\alpha(v_s^\varepsilon(j))] \right)$$

$$= -2 \int_0^t ds\, M^{-1} \sum_{j=1}^{M} \int_0^1 d\lambda\, \mathsf{E}\left(\sum_{\alpha, \beta=1}^{d} [u_{s\alpha}^\varepsilon(j) - v_{s\alpha}^\varepsilon(j)] \right.$$

$$\left. \times \sigma_{\alpha\beta}(v_s^\varepsilon(j) + \lambda(u_s^\varepsilon(j) - v_s^\varepsilon(j)))[u_{s\beta}^\varepsilon(j) - v_{s\beta}^\varepsilon(j)] \right) \leqslant 0 \tag{B.19}$$

since $\{\sigma_{\alpha\beta} \geqslant 0\}$ as a $d \times d$ matrix.

We conclude that the central open problem is Property 2. In a somewhat disguised form, this is the uniqueness problem for the massless Gibbs measures of effective interface models.

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