Journal of Dynamics and Differential Equations, Vol. 3, No. 1, 1991

Linearization of Polynomial Flows and Spectra of Derivations[1]

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Received June 23, 1989

We show that a polynomial flow is a solution of a linear ordinary differential equation. From this, we draw conclusions about the possible dynamics of polynomial flows. We also show how point spectra of derivations associated with polynomial vector fields can be used to identify p-f vector fields.

KEY WORDS: Polynomial flows; linear ordinary differential equations; derivations.
AMS (MOS) SUBJECT CLASSIFICATIONS: Primary 34A05; Secondary 12H05, 13N05, 34A20, 34A30, 34C25, 34C35, 34D05, 58F13, 58F15, 58F22, 58F25.

1. INTRODUCTION

If one can turn a nonlinear differential equation into a linear one by means of a well-understood transformation, one can usually say much about the solutions of the original equation. Such is the case with the Korteweg-de Vries equation, which, for initial data that vanish sufficiently fast at infinity, can be linearized by inverse scattering techniques. In this paper we give a procedure for linearizing polynomial flows. We then investigate some of the facts that follow from this linearization. We also show how the point

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1040-7294/91/0100-0029$06.50/0

spectra of certain derivations can be used to determine whether a given polynomial vector field has a polynomial flow.

Consider the initial value problem

$$\dot{y}\left(\equiv \frac{dy}{dt}\right) = \mathbf{V}(y), \qquad y(0) = x \in \mathbb{F}^n \tag{1.1}$$

where $\mathbf{V}$ is a continuously differentiable vector field on $\mathbb{F}^n$ ($\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). Let ϕ: $\Omega \to \mathbb{F}^n$ be the (local) flow associated with (1.1), where Ω, an open subset of $\mathbb{R} \times \mathbb{F}^n$, is the maximal domain of ϕ. For each t in $\mathbb{R}$ let U^t be the set of all x in $\mathbb{F}^n$ such that (t, x) is in Ω. The flow ϕ is said to be a *polynomial flow* and $\mathbf{V}$ is said to be a *p-f vector field* if for each t in $\mathbb{R}$ the t-advance map ϕ^t: $U^t \to \mathbb{F}^n$ is polynomial. That is, if π_i: $\mathbb{R}^n \to \mathbb{R}$ is the projection map onto the ith coordinate, $\pi_i \circ \phi^t$ is polynomial for $i = 1,..., n$. Take the degree of a polynomial map P: $\mathbb{F}^n \to \mathbb{F}^n$ to be the maximum of the degrees of $\pi_i \circ P$ for $i = 1,..., n$.

If $\mathbf{V}$ is a linear vector field, say $\mathbf{V}(y) = Ay$, where A is an $n \times n$ matrix, we have $\phi(t, x) = e^{tA}x$. Hence linear vector fields are p-f vector fields. However, p-f vector fields are not restricted to linear ones. For example, even on $\mathbb{R}^2$ (henced also on $\mathbb{R}^n$ for any $n \geqslant 2$) there are p-f vector fields of all degrees (see Bass and Meisters, 1985, theorem 11.8; or Coomes, 1990a, Table I).

Polynomial flows were first discussed by Meisters (1982). A more thorough investigation is given by Bass and Meisters (1985). They show that

(1) if ϕ is a polynomial flow, there is a bound, valid for all t, on the degree of ϕ^t;

(2) p-f vector fields are polynomial:

(3) p-f vector fields have constant divergence; and

(4) p-f vector fields are *complete*—all solutions to (1.1) exist for all real time.

Coomes (1988, 1990b) investigates polynomial flows with complex initial conditions and time. First, he shows that a polynomial flow on $\mathbb{R}^n$ or $\mathbb{C}^n$ extends to a holomorphic function on $\mathbb{C} \times \mathbb{C}^n$ which satisfies the group property for complex time. That is, a polynomial flow ϕ extends to a holomorphic function Φ: $\mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\Phi(t, \Phi(s, x)) = \Phi(t + s, x), \qquad t, s \in \mathbb{C}, \qquad x \in \mathbb{C}^n$$

Second, he investigates power series forms of polynomial flows. Coomes (1988, 1990a) shows how one can use certain symmetries to show that the

Lorentz system does not have a polynomial flow. Meisters and Olech (1986) give a connection between polynomial flows and the Jacobian conjecture of algebraic geometry. See Bass *et al.* (1982) for an introduction to the Jacobian conjecture.

Our main result is that if ϕ is a polynomial flow, there is a linear Nth-order ordinary differential equation with constant coefficients

$$y^{(N)} - \sum_{j=0}^{N-1} c^j y^{(j)} = 0 \qquad (1.2)$$

such that for each x the solution ϕ_x of (1.1) satisfies (1.2). We also present an algebraic version of this result in theorem 3.1.

In Section 2 we recall a basic concept from commutative algebra, that of a derivation. We also discuss the connections among derivations, vector fields, and differentiation with respect to time. Next, in Section 3, we introduce some terminology from commutative algebra, examine the properties of derivations defined by p-f vector fields, and discuss spectra of polynomials with respect to a given derivation. Then, in Section 4, we discuss spectra of derivations, in particular, spectra of derivations defined by p-f vector fields. Next, in Section 5, we prove our main result, given general formulas for polynomial flows, and discuss the relationship between spectra and our formulas. Then, in Section 6, we discuss what our results imply about the possible dynamics of polynomial flows, addressing some of the questions asked by Meisters (1989). Next, in Section 7, we describe, in algebraic terms, the structure of invariant manifolds of p-f vector fields. Then, in Section 8, we show how to use our techniques to determine whether a given vector field has a polynomial flow. One of our examples in Section 8 answers a question posed by Coomes (1990a, question 4.1). Finally, in Section 9, we examine some similarities between our results and a result due to Weierstrass.

2. POLYNOMIAL VECTOR FIELDS AND DERIVATIONS

In this section we establish some notation and discuss the main tool of our paper, the concept of a derivation associated with a vector field. As the material we discuss here is basic, those familiar with the relationship between derivations and differentiation with respect to time may wish to skim this section.

Let $\mathbb{C}[X_1,..., X_n]$ denote the algebra over $\mathbb{C}$ (with respect to the usual addition, multiplication, and scalar multiplication) of polynomials over $\mathbb{C}$ in the n indeterminates $X_1,..., X_n$. Let $\mathbb{C}[Y]$ denote the algebra over $\mathbb{C}$

(again with respect to the usual addition, multiplication, and scalar multiplication) of polynomials over $\mathbb{C}$ in the indeterminate Y.

Let $\mathbf{V}$ be a polynomial vector field on $\mathbb{C}^n$. That is, $\mathbf{V} = (v_1, \ldots, v_n)^T$, where each v_j is in $\mathbb{C}[X_1, \ldots, X_n]$. Associated with $\mathbf{V}$ we have the initial value problem

$$\dot{y} = \mathbf{V}(y), \qquad y(0) = x \tag{2.1}$$

the flow $\phi: \Omega \to \mathbb{C}^n$ defined by (2.1), and the map $\mathscr{D}: \mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}[X_1, \ldots, X_n]$ given by

$$a \mapsto v_1 \partial_{X_1} a + \cdots + v_n \partial_{X_n} a$$

where ∂_{X_i} denotes partial differentiation with respect to X_i.

We have the following.

Definition 2.1. Let $\mathscr{B}$ be an algebra. A linear map $D: \mathscr{B} \to \mathscr{B}$ is called a *derivation* of $\mathscr{B}$ if for each pair a, b of elements of $\mathscr{B}$, we have

$$D(ab) = (Da)b + a(Db)$$

It follows, by induction, that for any derivation D of an algebra $\mathscr{B}$, for any integer $k \geqslant 0$, and for any pair a, b of elements of $\mathscr{B}$, we have

$$D^k(ab) = \sum_{i=0}^{k} \binom{k}{i} D^i a \cdot D^{k-i} b$$

Notice that $\mathscr{D}$ is a derivation of $\mathbb{C}[X_1, \ldots, X_n]$.

Let $\nabla: \mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}[X_1, \ldots, X_n]^n$ be given by

$$a \mapsto (\partial_{X_1} a, \ldots, \partial_{X_n} a)$$

Denote the derivation $\mathscr{D}$ by $\mathbf{V} \cdot \nabla$. This is natural since, for each a in $\mathbb{C}[X_1, \ldots, X_n]$ we have $\mathscr{D}a = (\nabla a)\mathbf{V}$, where the multiplication is the usual matrix multiplication.

Let D be any derivation of $\mathbb{C}[X_1, \ldots, X_n]$ and let $\mathbf{F} = (f_1, \ldots, f_n)^T$, where $f_i = DX_i$. Notice that D is the derivation associated with the polynomial vector $\mathbf{F}$. That is,

$$D = f_1 \partial_{X_1} + \cdots + f_n \partial_{X_n}.$$

Thus the map from polynomial vector fields on $\mathbb{C}^n$ to derivations of $\mathbb{C}[X_1, \ldots, X_n]$ given by $\mathbf{V} \mapsto \mathbf{V} \cdot \nabla$ is a one-to-one correspondence.

We distinguish between polynomials and polynomial functions. The connection between the two is provided by the evaluation map—for each

$x = (x_1, ..., x_n)^T$ in $\mathbb{C}^n$ define the map $E_x: \mathbb{C}[X_1, ..., X_n] \to \mathbb{C}$, which we call *evaluation at* x, by $a \mapsto a(x_1, ..., x_n)$.

We next need a class of functions for the components of solutions of (2.1) to live in. Let $C^\infty(\Omega, \mathbb{C})$ denote the algebra over $\mathbb{C}$ (with respect to point-wise addition, multiplication, and scalar multiplication) of infinitely differentiable functions from Ω to $\mathbb{C}$. If the map $\partial_t: C^\infty(\Omega, \mathbb{C}) \to C^\infty(\Omega, \mathbb{C})$ is given by differentiation with respect to the first component, ∂_t is a derivation of $C^\infty(\Omega, \mathbb{C})$.

Since $\mathscr{D}$ maps $\mathbb{C}[X_1, ..., X_n]$ to $\mathbb{C}[X_1, ..., X_n]$ and since $\mathbb{C}[X_1, ..., X_n]$ is a vector space, given any polynomial p in $\mathbb{C}[Y]$, we may define the map $p(\mathscr{D})$: $\mathbb{C}[X_1, ..., X_n] \to \mathbb{C}[X_1, ..., X_n]$ in the following way: If $p(Y) = \sum_{i=0}^{k} \alpha_i Y^i$, for each a in $\mathbb{C}[X_1, ..., X_n]$ we have

$$p(\mathscr{D})a = \sum_{i=0}^{k} \alpha_i \mathscr{D}^i a.$$

In a similar fashion we can define polynomials in ∂_t. We now give the connection between the derivations $\mathscr{D}$ and ∂_t.

Theorem 2.1. *Let* **V** *be a polynomial vector field, let* $\phi: \Omega \to \mathbb{C}^n$ *be the associated flow, let* $\mathscr{D}$ *be the associated derivation of* $\mathbb{C}[X_1, ..., X_n]$, *and let* p *be in* $\mathbb{C}[Y]$. *Then the following diagram commutes.*

$$\begin{array}{ccc} \mathbb{C}[X_1, ..., X_n] & \xrightarrow{p(\mathscr{D})} & \mathbb{C}[X_1, ..., X_n] \\ {\scriptstyle E_\phi}\downarrow & & \downarrow{\scriptstyle E_\phi} \\ C^\infty(\Omega, \mathbb{C}) & \xrightarrow{p(\partial_t)} & C^\infty(\Omega, \mathbb{C}) \end{array}$$

Proof. By the chain rule the diagram

$$\begin{array}{ccc} \mathbb{C}[X_1, ..., X_n] & \xrightarrow{\mathscr{D}} & \mathbb{C}[X_1, ..., X_n] \\ {\scriptstyle E_\phi}\downarrow & & \downarrow{\scriptstyle E_\phi} \\ C^\infty(\Omega, \mathbb{C}) & \xrightarrow{\partial_t} & C^\infty(\Omega, \mathbb{C}) \end{array}$$

commutes. Theorem 2.1 follows from this fact and the fact that E_ϕ is linear. ∎

3. PRELIMINARIES FROM COMMUTATIVE ALGEBRA

Many of the results in this section are well known but scattered in the literature. For the sake of completeness, we present all but the most basic facts. For the sake of brevity, we adopt the following conventions unless otherwise stated:

(1) rings are commutative with identity and modules are unitary, and

(2) vector spaces are vector spaces over $\mathbb{C}$ and algebras are algebras over $\mathbb{C}$.

Throughout this section $\mathscr{D}$ is a derivation of $\mathbb{C}[X_1,..., X_n]$. In the previous section we showed how to define the action of a polynomial in $\mathscr{D}$ on an element of $\mathbb{C}[X_1,..., X_n]$. Notice that with this action, $\mathbb{C}[X_1,..., X_n]$ becomes a $\mathbb{C}[\mathscr{D}]$-module. In a similar fashion, under the action of a polynomial in ∂_t on an element of the algebra $C^\infty(\mathbb{R}\times\mathbb{C}^n, \mathbb{C})$ of infinitely differentiable functions from $\mathbb{R}\times\mathbb{C}^n$ to $\mathbb{C}$, the set $C^\infty(\mathbb{R}\times\mathbb{C}^n, \mathbb{C})$ becomes a $\mathbb{C}[\partial_t]$-module.

At this point it seems that we have two modules over different rings. This is not the case. In $\mathbb{C}[\mathscr{D}]$ we think of $\mathscr{D}$ as a indeterminate and in $\mathbb{C}[\partial_t]$ we think of ∂_t as an indeterminate. It is only because of the naturalness of applying $\mathscr{D}$ to $\mathbb{C}[X_1,..., X_n]$ and ∂_t to $C^\infty(\mathbb{R}\times\mathbb{C}^n, \mathbb{C})$ that we choose not to think of them both as $\mathbb{C}[Y]$-modules. In this section we examine the algebraic structure of these modules with particular interest in the case where $\mathscr{D}$ is defined by a p-f vector field.

Recall (see, for example, Hungerford, 1974; Atiyah and Macdonald, 1969) the following facts and definitions. Let R be a ring and let $\mathscr{A}$ be an R-module. For each a in $\mathscr{A}$ let $\operatorname{Ann}(a)=\{p\in R\colon pa=0\}$. The set $\operatorname{Ann}(a)$ is called the *annihilator* of a in R. Notice that $\operatorname{Ann}(a)$ is an ideal of R. If $\operatorname{Ann}(a)$ is not the zero ideal, a is called *a torsion element.* Let $T(\mathscr{A})$ be the set of all torsion elements of $\mathscr{A}$. The set $T(\mathscr{A})$ is a submodule of $\mathscr{A}$. The submodule $T(\mathscr{A})$ is called the *torsion submodule* of $\mathscr{A}$.

If $\mathscr{D}$ is a derivation of $\mathbb{C}[X_1,..., X_n]$, we take $\mathbb{C}[X_1,..., X_n]$ to be a $\mathbb{C}[\mathscr{D}]$-module in the sense described above unless otherwise stated. In this case we denote the torsion submodule of $\mathbb{C}[X_1,..., X_n]$ by $T(\mathscr{D})$.

Proposition 3.1. *Let $\mathscr{D}$ be derivation of $\mathbb{C}[X_1,..., X_n]$ and let a be in $\mathbb{C}[X_1,..., X_n]$. Then the following are equivalent.*

(1) The element a is a torsion element.

(2) The submodule $\mathbb{C}[\mathscr{D}]$ a is a finite dimensional vector space (with respect to the usual addition and scalar multiplication).

(3) The submodule $\mathbb{C}[\mathscr{D}]a$ is contained in a $W\subset\mathbb{C}[X_1,..., X_n]$, that is a finite dimensional vector space.

Proof. (1) $\Rightarrow$ (2). Let $p=\sum_{i=0}^{k}\alpha_i\mathscr{D}^i$ be a nonzero element of

Ann(a). Let U be the set of linear combinations over $\mathbb{C}$ of a, $\mathscr{D}a,\ldots,\mathscr{D}^{k-1}a$. Then U is a finite-dimensional vector space. Notice that $U \subset \mathbb{C}[\mathscr{D}]a$.

Let f be in $\mathbb{C}[\mathscr{D}]$. Then, by the division algorithm, there exist polynomials q and r in $\mathbb{C}[\mathscr{D}]$ with $\deg r < k$ such that $f = pq + r$. Thus

$$\begin{aligned} fa &= (pq+r)a \\ &= q(pa) + ra \\ &= ra \end{aligned}$$

Notice that ra is in U. Hence $\mathbb{C}[\mathscr{D}]a \subset U$ and thus $\mathbb{C}[\mathscr{D}]a = U$.

(2) $\Rightarrow$ (3). Take $W = \mathbb{C}[\mathscr{D}]a$.

(3) $\Rightarrow$ (1). There exists an integer k such that $a, \mathscr{D}a,\ldots,\mathscr{D}^k a$ are linearly dependent over $\mathbb{C}$. Thus there exist scalars $\alpha_0,\ldots,\alpha_k$ in $\mathbb{C}$, not all zero, such that

$$\alpha_0 a + \alpha_1 \mathscr{D}a + \cdots + \alpha_k \mathscr{D}^k a = 0$$

Notice that $p = \sum_{i=0}^{k} \alpha_i \mathscr{D}^i$ is a nonzero element of Ann(a). ∎

Corollary 3.1. *Let $\mathscr{D}$ be a derivation of $\mathbb{C}[X_1,\ldots,X_n]$. Then the torsion submodule $T(\mathscr{D})$ is subalgebra of $\mathbb{C}[X_1,\ldots,X_n]$. Furthermore, $T(\mathscr{D})$ contains the constant polynomials.*

Proof. Notice that

$$\mathscr{D}1 = \mathscr{D}(1^2) = 1(\mathscr{D}1) + (\mathscr{D}1)1 = \mathscr{D}1 + \mathscr{D}1$$

Thus $\mathscr{D}1 = 0$. Hence for any constant a we have

$$\mathscr{D}a = \mathscr{D}(a1) = a\mathscr{D}1 = 0$$

Thus $T(\mathscr{D})$ contains all the constant polynomials.

Since $T(\mathscr{D})$ is a submodule of $\mathbb{C}[X_1,\ldots,X_n]$, it is a vector space. It remains to show that $T(\mathscr{D})$ is closed under multiplication. Let a and b be in $T(\mathscr{D})$. By proposition 3.1 we have that $\mathbb{C}[\mathscr{D}]a$ and $\mathbb{C}[\mathscr{D}]b$ are finite-dimensional vector spaces. Suppose they are spanned by $a_1,\ldots,a_{m_1}$ and $b_1,\ldots,b_{m_2}$, respectively. Let W be the vector space spanned by $\{a_i b_j\colon 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2\}$. For any nonnegative integer k we have

$$\mathscr{D}^k(ab) = \sum_{i=0}^{k} \binom{k}{i} \mathscr{D}^i a \cdot \mathscr{D}^{k-i} b$$

Thus $\mathscr{D}^k(ab)$ is in W for each k. It follows that given any polynomial f in

$\mathbb{C}[\mathscr{D}]$ we have that fab is in W. Hence $\mathbb{C}[\mathscr{D}]ab \subset W$. Notice that W is a finite-dimensional vector space. Thus, by proposition 3.1, ab is in $T(\mathscr{D})$. ▌

We allow operators such as ∇, ∂_t, $\mathscr{D}$, and E_x to act component-wise on vector-valued objects. Define the vector of indeterminates $X = (X_1, ..., X_n)$. Then, for example, $\mathscr{D}X = (\mathscr{D}X_1, ..., \mathscr{D}X_n)$.

We now discuss a connection between our development and polynomial flows. Let $\mathbf{V}$ be a polynomial vector field on $\mathbb{C}^n$. Define the *Maclaurin coefficients* δ_k of $\mathbf{V}$ by

$$\delta_0 = X^T$$

$$\delta_{k+1} = (\mathbf{V} \cdot \nabla)\delta_k, \qquad k = 0, 1, 2, ...$$

Notice that each δ_k is polynomial. Coomes (1990b, theorem 3.1) shows that the flow ϕ associated with $\mathbf{V}$ is given by

$$\phi(t, x) = \sum_{k=0}^{\infty} \frac{E_x \delta_k}{k!} t^k,$$

on some neighborhood of $\{0\} \times \mathbb{C}^n$ in $\mathbb{C} \times \mathbb{C}^n$. Coomes also shows (1990b, theorem 4.1) that a polynomial vector field on $\mathbb{R}^n$ has a polynomial flow if and only if there is a bound on the degree of the Maclaurin coefficients. That is, $\mathbf{V}$ is a p-f vector field if and only if there exists an integer d such that $\deg \delta_k \leqslant d$ for all $k \geqslant 0$ and all $i = 1, ..., n$. The assumption that the vector field be on $\mathbb{R}^n$ is not used in the proof of Coomes (1190b, theorem 4.1). The proof works equally well for polynomial vector fields on $\mathbb{C}^n$. In the sequel, we take Coomes' (1990b) theorem 4.1 to be valid for polynomial vector fields on $\mathbb{C}^n$. Notice that if $\mathscr{D}$ is the operator $\mathbf{V} \cdot \nabla$, the Maclaurin coefficients are given by $\delta_k = \mathscr{D}^k X^T$ for all $k \geqslant 0$.

The following theorem is a cornerstone in our development.

Theorem 3.1. *Let* $\mathbf{V}$ *be a polynomial vector field on* $\mathbb{C}^n$ *and let* $\mathscr{D}$ *be the associated derivation of* $\mathbb{C}[X_1, ..., X_n]$. *Then the following are equivalent.*

(1) The vector field $\mathbf{V}$ *is a p-f vector field.*

(2) The torsion submodule $T(\mathscr{D})$ *is* $\mathbb{C}[X_1, ..., X_n]$.

(3) There exists a nonzero polynomial p *in* $\mathbb{C}[\mathscr{D})$ *such that* $pX_i = 0$ *for* $i = 1, ..., n$.

Proof. (1) ⇒ (2). By Coomes' (1990b) theorem 4.1, there is a bound d on the degrees of $\mathscr{D}^j X_i$ valid for $j \geqslant 0$ and for $i = 1, ..., n$. The set $W = \{w \in \mathbb{C}[X_1, ..., X_n]$: $\deg w \leqslant d\}$ is a finite-dimensional vector space. We have

$\mathscr{D}^j X = (\mathscr{D}^j X_1, \ldots, \mathscr{D}^j X_n) \in W^n$ for all $j \geqslant 0$. Since W^n is a finite-dimensional vector space, there exists an integer k and scalars $\alpha_0, \ldots, \alpha_k$, not all zero, such that

$$\alpha_0 X + \alpha_1 \mathscr{D}X + \cdots + \alpha_k \mathscr{D}^k X = 0$$

If p in $\mathbb{C}[\mathscr{D}]$ is given by $p = \sum_{i=0}^{k} \alpha_i \mathscr{D}^i$, then p is nonzero and $pX_i = 0$ for $i = 1, \ldots, n$. Thus $X_1, \ldots, X_n$ are all in $T(\mathscr{D})$. Since $T(\mathscr{D})$ is a subalgebra of $\mathbb{C}[X_1, \ldots, X_n]$ that contains the constants, we have $T(\mathscr{D}) = \mathbb{C}[X_1, \ldots, X_n]$.

(2) ⇒ (3). For each i, let p_i be a nonzero element of $\mathrm{Ann}(X_i)$. Notice that $p = p_1 \cdots p_n$ is a nonzero element of $\bigcap_{i=1}^{n} \mathrm{Abb}(X_i)$. Hence $pX_i = 0$ for $i = 1, \ldots, n$.

(3) ⇒ (1). Without loss of generality, p is monic of degree k, say $p = \mathscr{D}^k - \sum_{i=0}^{k-1} c_i \mathscr{D}^i$. Let $U \subset \mathbb{C}[X_1, \ldots, X_n]^n$ be the finite-dimensional subspace spanned by $\{X, \mathscr{D}X, \ldots, \mathscr{D}^{k-1}X\}$. Notice that if $u \in U$, then $\mathscr{D}u \in U$. For if $u = \beta_0 X + \beta_1 \mathscr{D}X + \cdots + \beta_{k-1} \mathscr{D}^{k-1} X$, then

$$\begin{aligned}\mathscr{D}u &= \beta_0 \mathscr{D}X + \beta_1 \mathscr{D}^2 X + \cdots + \beta_{k-1} \mathscr{D}^k X \\ &= \beta_0 \mathscr{D}X + \beta_1 \mathscr{D}^2 X + \cdots + \beta_{k-1} \sum_{i=0}^{k-1} c_i \mathscr{D}^i X \\ &\in U\end{aligned}$$

It follows that if $u \in U$, then $\mathscr{D}^j u \in U$ for all $j \geqslant 0$.

Notice that there is a bound on the degrees of components of elements of U; the bound is the largest of the degrees of the components of $X, \mathscr{D}X, \ldots, \mathscr{D}^{k-1}X$. Hence the Maclaurin coefficients $\delta_j = \mathscr{D}^j X^T$ of $\mathbf{V}$ are of bounded degree. By Coomes' (1990b), $\mathbf{V}$ is a p-f vector field. ▮

Let R be a ring and let $\mathrm{Spec}(R)$ denote the set of prime ideals of R. Let $\mathscr{A}$ be an R-module and let a be in the torsion submodule $T(\mathscr{A})$. Define

$$\mathrm{Spec}(a) = \{P \in \mathrm{Spec}(R) : P \supset \mathrm{Ann}(a)\}.$$

We call $\mathrm{Spec}(R)$ the *spectrum of R* and $\mathrm{Spec}(a)$ the *spectrum of a*.

The set $\mathrm{Spec}(\mathbb{C}[Y])$ can be identified with $\mathbb{C}$ ini the following way: identify λ with the prime ideal P_λ generated by $Y - \lambda$. Thus we identify the spectra of torsion elements with subsets of $\mathbb{C}$. If $\mathscr{A}$ us a $\mathbb{C}[Y]$-module and a is a torsion element of $\mathscr{A}$, the ideal $\mathrm{Ann}(a)$ is generated by a single non-zero monic polynomial p. Let

$$p = \prod_{i=1}^{m} (Y - \lambda_i)^{l_i}$$

where each l_i is a positive integer. Then the prime ideal P_λ contains Ann(a) if and only if $Y-\lambda$ divides p if and only if λ is one of $\lambda_1,..., \lambda_m$. That is, λ is in the spectrum of a if and only if λ is a root of the monic generator of Ann(a). Since any two generators of Ann(a) differ only by a factor of a constant, any two generators of Ann(a) have the same set of root. Thus the spectrum of a is the set of roots of any generator of Ann(a).

The following theorem gives a *spectral decomposition* for torsion elements.

Theorem 3.2. *Let $\mathscr{A}$ be a $\mathbb{C}[Y]$-module and let a torsion element of $\mathscr{A}$. Then a decomposes uniquely as a sum (the spectral decomposition)*

$$a = \sum_{\lambda \in \mathrm{Spec}(a)} a_\lambda$$

of torsion elements satisfying $\mathrm{Spec}(a_\lambda) = \{\lambda\}$. *Furthermore, there exist polynomials p_λ in $\mathbb{C}[Y]$ such that $p_\lambda a = a_\lambda$.*

Proof. Let $p = \prod_{i=1}^{m} (Y-\lambda_i)^{l_i}$ be the monic generator of Ann(a). If $a=0$, then $p=1$, $\mathrm{Spec}(a) = \varnothing$, and the result follows vacuously.

Assume that $a \neq 0$. We first prove the existence of the spectral decomposition. We have $m \geqslant 1$. Without loss of generality, we may assume that each l_j is positive and that λ_j are distinct. By the argument above, $\mathrm{Spec}(a) = \{\lambda_1,..., \lambda_m\}$. Let

$$q_j = \prod_{\substack{i=1 \\ i \neq j}}^{m} (Y-\lambda_i)^{l_i}$$

The polynomials $q_1,..., q_m$ are relatively prime. Hence there exist polynomials $r_1,..., r_m$ in $\mathbb{C}[Y]$ such that $r_1 q_1 + \cdots + r_m q_m = 1$. Let $a_{\lambda_j} = r_j q_j a$. Notice that $a_{\lambda_1} + \cdots + a_{\lambda_m} = a$. Also notice that λ_j is not a root of r_j (else λ_j would be a root of each of $r_1 q_1,..., r_m q_m$ and hence a root of 1). Thus p does not divide $r_j q_j$, which implies that $a_{\lambda_j} \neq 0$. Further,

$$\begin{aligned} (Y-\lambda_j)_{l_j} a_{\lambda_j} &= (Y-\lambda_j)^{l_j} r_j q_j a \\ &= r_j p a \\ &= 0 \end{aligned}$$

Hence $(Y-\lambda_j)^{l_j}$ is in Ann(a_{λ_j}). Thus the monic generator of Ann(a_{λ_j}) is a nonunit that divides $(Y-\lambda_j)^{l_j}$. Hence $\mathrm{Spec}(a_{\lambda_j}) = \{\lambda_j\}$. This proves existence of the spectral decomposition.

We now prove uniqueness of the decomposition. Let

$$a = \sum_{i=1}^{m} a'_{\lambda_j}$$

be any spectral decomposition of a. We have

$$\begin{aligned} a - a &= 0 \\ &= \sum_{i=1}^{m} (a_{\lambda_i} - a'_{\lambda_i}) \end{aligned}$$

Fix j between 1 and m. For $i \neq j$ we have that λ_i is a root of q_j. Thus for each $i \neq j$ between 1 and m there exists a positive integer n_i such that

$$q_j^{n_i}(a_{\lambda_i} + a'_{\lambda_i}) = 0$$

Letting $n = \max_{i \neq j}\{n_i\}$, we have

$$\begin{aligned} q_j^n 0 &= 0 \\ &= \sum_{i=1}^{m} q_j(a_{\lambda_j} - a'_{\lambda_j}) \\ &= q_j^n(a_{\lambda_j} - a'_{\lambda_j}) \end{aligned}$$

This implies that the spectrum of $a_{\lambda_j} - a'_{\lambda_j}$ is contained in the roots $\{\lambda_1, ..., \lambda_m\} \setminus \{\lambda_j\}$ of q_j. Notice that there exists a positive integer k such that

$$(Y - \lambda_j)^k (a_{\lambda_j} - a'_{\lambda_j}) = 0$$

Thus the spectrum of $a_{\lambda_j} - a'_{\lambda_j}$ is contained in $\{\lambda_j\}$. Therefore $\mathrm{Spec}(a_{\lambda_j} - a'_{\lambda_j}) = \varnothing$. Notice that the only torsion element with empty spectrum is zero (see proposition 3.2 for a proof). Thus $a_{\lambda_j} = a'_{\lambda_j}$. Since j was arbitrary, we have uniqueness of the spectral decomposition.

To prove the second part of the theorem, take $p_{\lambda_j} < r_j q_j$ for $j = 1, ..., m$. ∎

The following is a technical lemma needed in the sequel.

Lemma 3.1. *Let $\mathscr{D}$ be a derivation of $\mathbb{C}[X_1, ..., X_n]$ and let a and b be in $\mathbb{C}[X_1, ..., X_n]$. Suppose that $(\mathscr{D} - \lambda)^{i+1} a = (\mathscr{D} - \mu)^{j+1} b = 0$ for some pair i, j of nonnegative integers and some pair λ, μ of scalars. Then*

(1) $$(\mathscr{D} - (\lambda + \mu))^{i+j} ab = \frac{(i+j)!}{i!\, j!} (\mathscr{D} - \lambda)^i a \cdot (\mathscr{D} - \mu)^j b,$$

(2) $$(\mathscr{D} - (\lambda + \mu))^{i+j+1} ab = 0$$

Proof. (1) We induct on $m=i+j$. If $m=0$, then $i=j=0$ and the result follows. Suppose that the result holds for some $m\geqslant 0$. Let $i+j=m+1$. We calculate

$$(\mathscr{D}-(\lambda+\mu))^{m+1}ab$$
$$=(\mathscr{D}-(\lambda+\mu))^m((\mathscr{D}-\lambda)a\cdot b)+(\mathscr{D}-(\lambda+\mu))^m(a\cdot(\mathscr{D}-\mu)b) \quad (31)$$

If $i=0$, then $j\geqslant 1$. By (3.1) and by our induction hypothesis applied to a and $(\mathscr{D}-\mu)b$, we have

$$\begin{aligned}(\mathscr{D}-(\lambda+\mu))^{m+1}ab&=(\mathscr{D}-(\lambda+\mu))^m(a\cdot(\mathscr{D}-\mu)b)\\&=\frac{(i+j-1)!}{i!(j-1)!}(\mathscr{D}-\lambda)^ia\cdot(\mathscr{D}-\mu)^jb\\&=\frac{(j-1)!}{(j-1)!}a\cdot(\mathscr{D}-\mu)b\\&=\frac{(i+j)!}{i!j!}(\mathscr{D}-\lambda)^ia\cdot(\mathscr{D}-\mu)^jb\end{aligned}$$

and the induction step follows. A similar argument proves the induction step when $j=0$. If both i and j are positive, our induction hypothesis together with (3.1) implies that

$$\begin{aligned}(\mathscr{D}-(\lambda+\mu))^{m+1}ab&=\frac{(i-1+j)!}{(i-1)!j!}(\mathscr{D}-\lambda)^ia\cdot(\mathscr{D}-\mu)^jb\\&\quad+\frac{(i+j-1)!}{i!(j-1)!}(\mathscr{D}-\lambda)^ia\cdot(\mathscr{D}-\mu)^jb\\&=\frac{(i+j)!}{i!j!}(\mathscr{D}-\lambda)^ia\cdot(\mathscr{D}-\mu)^jb\end{aligned}$$

By induction, the result follows.

(2) This follows directly from (1). ∎

Let $\mathscr{D}$ be a derivation on $\mathbb{C}[X_1,\dots,X_n]$ and let

$$\mathscr{S}(\mathscr{D})=\bigcup_{a\in T(\mathscr{D})}\operatorname{Spec}(a)$$

We call the set $\mathscr{S}(\mathscr{D})$ the *spectrum* of $\mathscr{D}$. This name is just since $\mathscr{S}(\mathscr{D})$ is the point spectrum of the linear map $\mathscr{D}$.

A subset M of an abelian group $\mathscr{G}$ is called a *lattice* if, whenever λ and

μ are in M, we have $\lambda+\mu$ is in M. If N is any subset of $\mathscr{G}$, we can form the set Lat(N) of all finite sums of elements of N. Notice that the set Let(N) is a lattice. We call the set Lat(N) the *lattice generated by the set N*.

Let M and N be two subsets of $\mathbb{C}$. Define the set $M+N$ in the usual way. An element ξ in the sum $M+N$ is called an *extremal point* (with respect to M and N) if ξ can be written uniquely as $\xi=\lambda+\mu$ with $\lambda \in M$ and $\mu \in N$.

Example 3.1. Let $M=\{-1, 1, 2\}$ and let $N=\{0, 2, 5\}$. Then $M+N=\{-1, 1, 2, 3, 4, 6, 7\}$ and the set of extremal points of $M+N$ is $\{-1, 2, 3, 6, 7\}$. The element 1 is not an extremal point since $2= -1+2= 1+0$.

The following proposition summarizes the basic properties of spectra. As always, if a is a torsion element of a module, Spec(a) denotes the spectrum of a.

Proposition 3.2. *Let $\mathscr{D}$ be a derivation of $\mathbb{C}[X_1,..., X_n]$, let a and b be torsion elements of $\mathbb{C}[X_1,..., X_n]$, and let p be in $\mathbb{C}[\mathscr{D}]$. Then the following hold.*

(1) The spectrum of a is the set of roots of any generator of the annihilator of a.

(2) A torsion element is zero if and only if it has an empty spectrum.

(3) If λ is in the spectrum of $\mathscr{D}$, there exists a nonzero elements a of $\mathbb{C}[X_1,..., X_n]$ such that $(\mathscr{D}-\lambda)a=0$.

(4) The spectrum of $\mathscr{D}$ is a lattice.

(5) If $pa=0$, the spectrum of a contained in the roots of p.

(6) The spectrum of pa is contained in the spectrum of a.

(7) If p does not vanish on the spectrum of a, the spectrum of pa is equal to the spectrum of a.

(8) The spectrum of $a+b$ is contained in Spec(a) $\cup$ Spec(b).

(9) The spectrum of ab is contained in Spec(a) + Spec(b).

(10) If ξ is an extremal point of Spec(a) + Spec(b), *then ξ is in the spectrum of ab.*

Proof. (1) The proof of this precedes theorem 3.2.

(2) Let a be a torsion element and let q be the monic generator of Ann(a). The torsion element a has an empty spectrum if and only if q has no roots if and only if q is 1 if and only if $a=1a=0$.

(3) There exists a torsion element b with λ in its spectrum and thus,

by theorem 3.2, there exists a torsion element b_λ with $\operatorname{Spec}(b_\lambda)=\{\lambda\}$. The monic generator of $\operatorname{Ann}(b_\lambda)$ must be $(\mathscr{D}-\lambda)^k$ for some positive integer k. Let $a=(\mathscr{D}-\lambda)^{k-1}b$.

(4) Let λ and μ be in $\mathscr{S}(\mathscr{D})$. Then, by (3), there exist nonzero torsion elements a and b such that $(\mathscr{D}-\lambda)a=(\mathscr{D}-\mu)b=0$. Notice that $ab\neq 0$ and, by lemma 3.1, $(\mathscr{D}-(\lambda+\mu))ab=0$. Thus $\lambda+\mu$ is in $\mathscr{S}(\mathscr{D})$.

(5) The polynomial p is in $\operatorname{Ann}(a)$. Thus, if q is a generator of $\operatorname{Ann}(a)$, then q divides p. Hence the spectrum of a, which by (1) is the set of roots of q, is contained in the set of roots of p.

(6) Let q be a generator of $\operatorname{Ann}(a)$. Then $qpa=0$. Hence, by (5), the spectrum of pa is contained in the set of roots of q. That is, $\operatorname{Spec}(pa)\subset \operatorname{Spec}(a)$.

(7) Let r be a generator of $\operatorname{Ann}(pa)$. Then rp is in $\operatorname{Ann}(a)$. Thus if λ is in $\operatorname{Spec}(a)$, then λ is a root of rp. Since λ is not a root of p, ikt must be a root of r. Therefore λ is in $\operatorname{Spec}(pa)$. That is, $\operatorname{Spec}(a)\subset\operatorname{Spec}(pa)$. By (6), we have $\operatorname{Spec}(a)=\operatorname{Spec}(pa)$.

(8) Let q generate $\operatorname{Ann}(a)$ and let r generate $\operatorname{Ann}(b)$. Notice that $pq(a+b)=0$. Thus, by (5), the spectrum of $a+b$ is contained in the set of roots of pq. Notice that the set of roots of pq is $\operatorname{Spec}(a)\cup\operatorname{Spec}(b)$.

(9) Let $a=\sum_{\lambda\in\operatorname{Spec}(a)} a_\lambda$ and $b=\sum_{\mu\in\operatorname{Spec}(b)} b_\mu$ be spectral decompositions for a and b. Then

$$ab=\sum_{\lambda\in\operatorname{Spec}(a)}\ \sum_{\mu\in\operatorname{Spec}(b)} a_\lambda b_\mu .$$

For each λ in $\operatorname{Spec}(a)$, let $(\mathscr{D}-\lambda)^{n_\lambda+1}$ generate $\operatorname{Ann}(a_\lambda)$. Similarly, for each μ in $\operatorname{Spec}(b)$, let $(\mathscr{D}-\mu)^{m_\mu+1}$ generate $\operatorname{Ann}(b_\mu)$. Then the n_λ's are all nonnegative by lemma 3.1,

$$(\mathscr{D}-(\lambda+\mu))^{n_\lambda+m_\mu+1}a_\lambda b_\mu=0$$

Since $a_\lambda b_\mu\neq 0$, we have that $\operatorname{Spec}(a_\lambda b_\mu)=\{\lambda+\mu\}$ by (8), we have

$$\operatorname{Spec}(ab)\subset\bigcup_{\lambda\in\operatorname{Spec}(a)}\ \bigcup_{\mu\in\operatorname{Spec}(b)}\{\lambda+\mu\}=\operatorname{Spec}(a)+\operatorname{Spec}(b)$$

(10) In the proof of (9) we showed that we may write ab as the double sum

$$ab=\sum_{\lambda\in\operatorname{Spec}(a)}\ \sum_{\mu\in\operatorname{Spec}(b)} a_\lambda b_\mu$$

where $\operatorname{Spec}(a_\lambda b_\mu) = \{\mu + \mu\}$. Let $\lambda + \mu$ be an extremal point of $\operatorname{Spec}(a) + \operatorname{Spec}(b)$. Then, by (8), we have

$$\operatorname{Spec}(ab - a_\lambda b_\mu) \subset (\operatorname{Spec}(a) + \operatorname{Spec}(b)) \setminus \{\lambda + \mu\}$$

Let p be a generator of $\operatorname{Ann}(ab - a_\lambda b_\mu)$. Then p does not vanish at $\lambda + \mu$. Notice that $pab = pa_\lambda b_\mu$. Thus

$$\begin{aligned} \{\lambda + \mu\} &= \operatorname{Spec}(a_\lambda b_\mu) \\ &= \operatorname{Spec}(pa_\lambda b_\mu) \\ &= \operatorname{Spec}(pab) \\ &\subset \operatorname{Spec}(ab) \quad \blacksquare \end{aligned}$$

The following example shows that if $\lambda + \mu$ is not an extremal point of $\operatorname{Spec}(a) + \operatorname{Spec}(b)$, then $\lambda + \mu$ need not be in the spectrum of ab. That is, the containment in part 9 of proposition 3.2 may be strict.

Example 3.2. Let $\mathscr{D}$ be the derivation of $\mathbb{C}[X_1, X_2]$ associated with the vector field $\mathbf{V} = (1, X_2)^T$. Let $a = X_1 - X_2$ and $b = X_1 + X_2$. Then $\operatorname{Spec}(a) = \operatorname{Spec}(b) = \{0, 1\}$. We have $ab = X_1^2 - X_2^2$. Notice that $\operatorname{Spec}(ab) = \{0, 2\}$. Hence 1 is an element of $\operatorname{Spec}(a) + \operatorname{Spec}(b)$ but not of $\operatorname{Spec}(ab)$.

We now discuss the relationship between algebraic operations in $T(\mathscr{D})$ and multiplicities of roots of generators of annihilators. Let $a \in T(\mathscr{D})$ and let p generate $\operatorname{Ann}(a)$. Define $\operatorname{ord}_\lambda(a)$ to be one less than the multiplicity of λ as a root of p. That is,

$$\operatorname{ord}_\lambda(a) = \sup\{m \in \mathbb{Z} : m \geqslant -1 \text{ and } (\mathscr{D} - \lambda)^{m+1} \text{ divides } p\}$$

We call $\operatorname{ord}_\lambda(a)$ the *order opf* λ with respect to a.

If $\sum_{\lambda \in \operatorname{Spec}(a)} a_\lambda$ is a spectral decomposition for a, then $\operatorname{ord}_\lambda(a) = \operatorname{ord}_\lambda(a_\lambda)$ for each $\lambda \in \operatorname{Spec}(a)$. To see this, notice that since

$$pa = \sum_{\lambda \in \operatorname{Spec}(a)} pa_\lambda = 0$$

we must have $pa_\lambda = 0$ for each λ. Furthermore, for a particular $\lambda \in \operatorname{Spec}(a)$ and with $q_\lambda = p/(\mathscr{D} - \lambda)$, we have

$$q_\lambda a = q_\lambda a_\lambda \neq 0$$

That is, each generator of $\operatorname{Ann}(a_\lambda)$ divides p but not q_λ. Thus $\operatorname{ord}_\lambda(a) = \operatorname{ord}_\lambda(a_\lambda)$.

Proposition 3.3. *Let $\mathscr{D}$ be a derivation of $\mathbb{C}[X_1,\ldots,X_n]$, let a and b be torsion elements of $\mathbb{C}[X_1,\ldots,X_n]$, and let λ be a complex number. Then*

(1) $\operatorname{ord}_\lambda(a+b) \leqslant \max\{\operatorname{ord}_\lambda(a), \operatorname{ord}_\lambda(b)\}$,

(2) *if* $\operatorname{ord}_\lambda(a) < \operatorname{ord}_\lambda(b)$, *then* $\operatorname{ord}_\lambda(a+b) = \operatorname{ord}_\lambda(b)$, *and*

(3) *if* $\lambda \in \operatorname{Spec}(a)$, $\mu \in \operatorname{Spec}(b)$, *and* $\lambda+\mu$ *is an extremal point of* $\operatorname{Spec}(a)+\operatorname{Spec}(b)$, *then* $\operatorname{ord}_{\lambda+\mu}(ab) = \operatorname{ord}_\lambda(a) + \operatorname{ord}_\mu(b)$.

Proof. Let

$$q = \prod_{v \in \operatorname{Spec}(a) \cup \operatorname{Spec}(b)} (\mathscr{D} - v)^{\max\{\operatorname{ord}_v(a), \operatorname{ord}_v(b)\}+1}$$

and let $\sum_{v \in \operatorname{Spec}(a)} a_v$ and $\sum_{v \in \operatorname{Spec}(b)} b_v$ be spectral decompositions for a and b, respectively.

(1) We have $qa = qb = 0$. Hence $q(a+b) = 0$ and therefore any generator of $\operatorname{Ann}(a+b)$ divides q. This proves (1).

(2) By (1), we have $\operatorname{ord}_\lambda(a+b) \leqslant \operatorname{ord}_\lambda(b)$ and

$$\operatorname{ord}_\lambda(b) = \operatorname{ord}_\lambda(a+b+(-a)) \leqslant \max\{\operatorname{ord}_\lambda(a+b), \operatorname{ord}_\lambda(a)\}$$

Hence we must have $\operatorname{ord}_\lambda(b) \leqslant \operatorname{ord}_\lambda(a+b)$ and (2) follows.

(3) Let $\sum_{v \in \operatorname{Spec}(ab)} c_v$ be a spectral decomposition for ab. Since $\lambda+\mu$ is an extremal point of $\operatorname{Spec}(a)+\operatorname{Spec}(b)$, by proposition 3.2 we have $\lambda+\mu \in \operatorname{Spec}(ab)$. Notice that $c_{\lambda+\mu} = a_\lambda b_\mu$. By lemma 3.1,

$$(\mathscr{D} - (\lambda+\mu))^{\operatorname{ord}_\lambda(a)+\operatorname{ord}_\mu(b)} c_{\lambda+\mu} \neq 0$$
$$(\mathscr{D} - (\lambda+\mu))^{\operatorname{ord}_\lambda(a)+\operatorname{ord}_\mu(b)+1} c_{\lambda+\mu} = 0$$

That is,

$$\operatorname{ord}_{\lambda+\mu}(ab) = \operatorname{ord}_{\lambda+\mu}(c_{\lambda+\mu}) = \operatorname{ord}_\lambda(a) + \operatorname{ord}_\mu(b) \quad \blacksquare$$

4. THE SPECTRUM OF A DERIVATION

Let $\mathscr{D}$ be a derivation of $\mathbb{C}[X_1,\ldots,X_n]$. In this section we show that the spectrum of $\mathscr{D}$ is contained in the spectrum of another, in some cases, simpler derivation. In the case where $\mathscr{D}$ is defined by a p-f vector field with a fixed point, we show that the spectrum of $\mathscr{D}$ can be expressed in terms of the eigenvalues of the linearization of the vector field about the fixed point.

We first construct a generalization of degree. Let $\mathbb{N}$ denote the non-negative integers and let X denote $(X_1,\ldots,X_n)$. For each $r = (r_1,\ldots,r_n)$ in $\mathbb{N}^n$

we define the monomial $X^r = X_1^{r_1} \cdots X_n^{r_n}$. Notice that we may write any polynomial a in $\mathbb{C}[X_1, ..., X_n]$ as

$$a = \sum_{r \in \mathbb{N}^n} c_r(a) X^r$$

where each $c_r(a)$ is a scalar uniquely defined by a. For each polynomial a, only finitely many of the $c_r(a)$'s are nonzero.

Let $\omega = (\omega_1, ..., \omega_n)$ be a vector with nonnegative real entries. For a nonzero monomial cX^r, define

$$\text{wght}_\omega(cX^r) = \langle r, \omega \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. Also, define

$$\text{wght}_\omega(0) = -\infty$$

For any polynomial a in $\mathbb{C}[X_1, ..., X_n]$ define

$$\text{wght}_\omega(a) = \max\{\text{wght}_\omega(c_r(a) X^r) : r \in \mathbb{N}^n\}$$

(notice that this maximum is taken over a finite set).

We call $\text{wght}_\omega(a)$ the *weight of the polynomial a with respect to* ω. Notice that if a and b are in $\mathbb{C}[X_1, ..., X_n]$, then

(1) $\text{wght}_\omega(ab) = \text{wght}_\omega(a) + \text{wght}_\omega(b)$;

(2) $\text{wght}_\omega(a+b) \leqslant \max\{\text{wght}_\omega(a), \text{wght}_\omega(b)\}$;

(3) if $\text{wght}_\omega(a) \neq \text{wght}_\omega(b)$, then $\text{wght}_\omega(a+b) = \max\{\text{wght}_\omega(a), \text{wght}_\omega(b)\}$; and

(4) if $\omega = (1, ..., 1)$, then $\text{wght}_\omega(a)$ os the degree of a.

Let h be in $\mathbb{R}$. The polynomial a in $\mathbb{C}[X_1, ..., X_n]$ is said to be *ω-homogeneous of weight h* if a is a sum of monomials, each of weight h. Take zero to be ω-homogeneous of all weights. Let $\mathscr{H}_h$ be the set of all polynomials that are ω-homogeneous of weight h. Notice that $\mathscr{H}_h$ is a vector space and that only countably many $\mathscr{H}_h$'s are nonzero. Furthermore, notice that

$$\mathbb{C}[X_1, ..., X_n] = \bigoplus_{h \in \mathbb{R}} \mathscr{H}_h$$

as a vector space. For each pair h, h' in $\mathbb{R}$ let $\mathscr{H}_h \mathscr{H}_{h'} = \{aa' : a \in \mathscr{H}_h, a' \in \mathscr{H}_{h'}\}$. Then

$$\mathscr{H}_h \mathscr{H}_{h'} \subset \mathscr{H}_{h+h'}$$

Let $L\colon \mathbb{C}[X_1,\ldots,X_n] \to \mathbb{C}[X_1,\ldots,X_n]$ be a linear map. Suppose that there exists a real number μ such that for each h in $\mathbb{R}$ we have

$$L(\mathscr{H}_h) \subset \mathscr{H}_{h+\mu}$$

Then L is said to be *ω-homogeneous of weight* μ. Notice the following facts about ω-homogeneous linear maps from $\mathbb{C}[X_1,\ldots,X_n]$ to $\mathbb{C}[X_1,\ldots,X_n]$.

(1) If $L \neq 0$ is an ω-homogeneous linear map, its weight is unique.

(2) If L_1 and L_2 are ω-homogeneous linear maps of weight μ, so is $L_1 + L_2$.

(3) If L_1 and L_2 are ω-homogeneous linear maps of weights μ_1 and μ_2, respectively, $L_1 \circ L_2$ is ω-homogeneous of weight $\mu_1 + \mu_2$.

If $a \in \mathbb{C}[X_1,\ldots,X_n]$ is ω-homogeneous of weight h, the operator $a\partial_{X_i}$ is ω-homogeneous of weight $h - \omega_i$. Thus any derivation $\mathscr{D}$ of $\mathbb{C}[X_1,\ldots,X_n]$ can be written

$$\mathscr{D} = \mathscr{D}_1 + \cdots + \mathscr{D}_k$$

where $\mathscr{D}_i$ is ω-homogeneous of weight h_i and $h_1 < h_2 < \cdots < h_k$.

Theorem 4.1. *Let $\mathscr{D}$ be a derivation of $\mathbb{C}[X_1,\ldots,X_n]$ and let ω be an n-vector with nonnegative real entries. Suppose $\mathscr{D} = \mathscr{D}_0 + \cdots + \mathscr{D}_k$, where $\mathscr{D}_i$ is ω-homogeneous of weight h_i, $0 = h_0 < h_1 < \cdots < h_k$. Then the spectrum of $\mathscr{D}$ is contained in the spectrum of $\mathscr{D}_0$.*

Proof. Let $\lambda \in \mathscr{S}(\mathscr{D})$. Choose a nonzero element $a \in \mathbb{C}[X_1,\ldots,X_n]$ such that $(\mathscr{D} - \lambda)a = 0$. Write a as the sum

$$a = a_0 + \cdots + a_m$$

where a_i is a nonzero polynomial ω-homogeneous of weight α_i and $\alpha_0 < \alpha_1 < \cdots < \alpha_m$. We have

$$(\mathscr{D} - \lambda)a = (\mathscr{D}_0 + \cdots + \mathscr{D}_k - \lambda)(a_0 + \cdots + a_m)$$
$$= \mathscr{D}_0 a_0 - \lambda a_0 + b$$

where b is a sum of monomials each of weight greater than α_0. Notice that both $\mathscr{D}a_0$ and λa_0 are ω-homogeneous of weight α_0. Thus, since $(\mathscr{D} - \lambda)a = 0$, we have $\mathscr{D}_0 a_0 - \lambda a_0 = 0$ and $b = 0$. Thus $\lambda \in \mathscr{S}(\mathscr{D}_0)$.

We need the following technical lemma in the sequel.

Lemma 4.1. *Let $\mathscr{D}$, ω, $\mathscr{D}_i$, and h_i be as in theorem 4.1. Let*

$X=(X_1,\ldots, X_n)$ and let p be a polynomial in $\mathbb{C}[Y]$. Then the operator $p(\mathscr{D})$ decomposes as

$$p(\mathscr{D})=\Xi_0+\cdots+\Xi_m$$

with each Ξ_i ω-homogeneous of weight μ_i, $0=\mu_0<\mu_1<\cdots<\mu_m$. Furthermore, $\Xi_0=p(\mathscr{D}_0)$ and if $p(\mathscr{D})X=0$, then $\Xi_iX=0$ for $i=1,\ldots, m$.

Proof. Recall that the composition of two ω-homogeneous linear maps is an ω-homogeneous linear map. Recall, also, that the weight of the composition is the sum of the weights of the factors. Repeated application of this fact shows that for each nonnegative integer k, the map $\mathscr{D}^k$ can be written as a sum of ω-homogeneous linear maps each of weight at least zero. Furthermore, the term in the sum with weight zero must be $\mathscr{D}_0^k$. The decomposition for p follows, as does the fact that $\Xi_0=p(\mathscr{D}_0)$.

If $p(\mathscr{D})X=0$, then $p(\mathscr{D})X_j=\Xi_0X_j+\cdots+\Xi_mX_j=0$. But Ξ_iX_j is ω-homogeneous of weight $\mu_i+\omega_j$. Since the μ_i's are all distinct, each $\Xi_iX_j=0$. That is, $\Xi_iX=0$ for $i=1,\ldots, m$. ∎

For the remainder of this section we consider the case of $\omega=(1,\ldots, 1)$. That is, we consider the case where wght_ω is degree. For this choice of ω, we drop the ω-prefix from ω-homogeneous. We say a polynomial mapping $P\colon \mathbb{C}^n\to\mathbb{C}^n$ is homogeneous of degree k if each component of P is polynomial and homogeneous of degree k.

Theorem 4.2. *Let $\mathbf{V}$ be a polynomial vector field on $\mathbb{C}^n$ and let $\mathscr{D}=\mathbf{V}\cdot\nabla$. Suppose $\mathbf{V}(0)=0$. Then let*

$$\mathbf{V}=H_1+\cdots+H_n$$

where H_i is homogeneous of degree i and let M be the set of eigenvalues of the linear map $x\mapsto E_xH_1$. Then the spectrum of $\mathscr{D}$ is contained in the lattice generated by $M\cup\{0\}$.

Remark. It is sometimes useful to not that $\text{Lat}(M\cup\{0\})=\text{Lat}(M)\cup\{0\}$.

Proof. We have $\mathscr{D}=H_1\cdot\nabla+\cdots+H_n\cdot\nabla$. Notice that $H_i\cdot\nabla$ is a homogeneous derivation of weight $i-1$. Thus, by theorem 4.1, the spectrum of $\mathscr{D}$ is contained in the spectrum of $\mathscr{D}_0=H_1\cdot\nabla$.

If c is any constant, $\mathscr{D}_0c=0$. Hence zero is in the spectrum of $\mathscr{D}_0$. We identify H_1 with the matrix of the linear transformation $x\mapsto E_xH_1$. That is,

$\mathscr{D}_0 = (H_1 X) \cdot \nabla$. A calculation shows that $\mathscr{D}_0^k X = H_1^k$ for each nonnegative integer k. Hence, if p in $\mathbb{C}[Y]$ is the minimal polynomial of H_1, then

$$p(\mathscr{D}_0) X = p(H_1) X = 0$$

Thus the spectrum of each $X_1, \ldots, X_n$ with respect to the derivation $\mathscr{D}_0$ is contained in M. Since every element of $\mathbb{C}[X_1, \ldots, X_n]$ is a finite sum of products of constants and $X_1, \ldots, X_n$, repeated application of parts (8) and (9) of proposition 3.2 proves that $\mathscr{S}(\mathscr{D}_0)$ is contained in $\mathrm{Lat}(M \cup \{0\})$. By theorem 4.1, we have $\mathscr{S}(\mathscr{D}) \subset \mathscr{S}(\mathscr{D}_0)$. ∎

In the case where $\mathbf{V}$ is a p-f vector field, we can prove a stronger version of theorem 4.2.

Theorem 4.3. *Let* $\mathbf{V}$ *be a p-f vector field on* $\mathbb{C}^n$ *and let* $\mathscr{D} = \mathbf{V} \cdot \nabla$. *Suppose* $\mathbf{V}(0) = 0$. *Then let*

$$\mathbf{V} = H_1 + \cdots + H_n$$

where H_i *is homogeneous of degree* i *and let* M *be the set of eigenvalues of the linear map* $x \mapsto E_x H_1$. *Then the spectrum of* $\mathscr{D}$ *is the lattice generated by* $M \cup \{0\}$.

Proof. By theorem 4.2, we have $\mathscr{S}(\mathscr{D}) \subset \mathrm{Lat}(M \cup \{0\})$. Let p be a generator of $\bigcap_{i=1}^n \mathrm{Ann}(X_i)$. Recall, by theorem 3.1, that $p \neq 0$. Notice that each root of p is in $\mathscr{S}(\mathscr{D})$. As in the proof of theorem 4.2, we identify H_1 with its matrix. By lemma 4.1 we have

$$p(H_1) X = p(H_1 \cdot \nabla) X = 0$$

Thus $p(H_1) = 0$ and hence each eigenvalue of H_1 must be a root of p. That is, $M \subset \mathscr{S}(\mathscr{D})$. Since $\mathscr{D} 1 = 0$, zero is in $\mathscr{S}(\mathscr{D})$. Since $\mathscr{S}(\mathscr{D})$ is a lattice, $\mathrm{Lat}(M \cup \{0\}) \subset \mathscr{S}(\mathscr{D})$. Hence $\mathrm{Lat}(M \cup \{0\}) = \mathscr{S}(\mathscr{D})$. ∎

Corollary ***4.1.*** *Let* $\mathbf{V}$ *be a p-f vector field on* $\mathbb{C}^n$ *and let* $\mathscr{D} = \mathbf{V} \cdot \nabla$. *Suppose that* $\mathbf{V}(x_0) = 0$ *for some* $x_0 \in \mathbb{C}^n$. *Let* M *be the set of eigenvalues of* $\nabla \mathbf{V}(x_0)$. *Then the spectrum of* $\mathscr{D}$ *is the lattice generated by* $M \cup \{0\}$.

Proof. The map E_{X+x_0} is an algebra automorphism of $\mathbb{C}[X_1, \ldots, X_n]$. Let $\mathbf{V}' = E_{X+x_0} \mathbf{V}$ and let $\mathscr{D}' = \mathbf{V}' \cdot \nabla$. Then for each $a \in \mathbb{C}[X_1, \ldots, X_n]$ and each polynomial $p \in \mathbb{C}[Y]$ we have

$$p(\mathscr{D}')(E_{X+x_0} a) = E_{X+x_0} p(\mathscr{D}) a \tag{4.1}$$

Since $T(\mathscr{D}) = \mathbb{C}[X_1, \ldots, X_n]$, we have $T(\mathscr{D}') = \mathbb{C}[X_1, \ldots, X_n]$ and thus, by theorem 3.1, the vector field $\mathbf{V}'$ is a p-f vector field. Furthermore, it follows from (4.1) that $\mathscr{S}(\mathscr{D}') = \mathscr{S}(\mathscr{D})$. The set M is the set of eigenvalues of $\nabla\mathbf{V}'(0)$. By theorem 4.3, the result follows. ∎

5. A POLYNOMIAL FLOW AS A SUBSYSTEM OF A LINEAR FLOW

Throughout this section, let $\mathbf{V}$ be a p-f vector field, $\mathscr{D} = \mathbf{V} \cdot \nabla$ be the associated derivation of $\mathbb{C}[X_1,..., X_n]$, and ϕ be the associated flow. In this section we show that ϕ satisfies a linear homogeneous ordinary differential equation with constant coefficients. We also derive a formula for ϕ.

In Sections 2, 3, and 4 we approached the subject from an algebraic viewpoint. In this section we begin our departure from that approach and instead concentrate on the initial value problem

$$\dot{y} = \mathbf{V}(y), \qquad y(0) = x \in \mathbb{C}^n \tag{5.1}$$

and its solutions. This is not to say that our previous work is superfluous. The tools developed in the previous sections provide natural and elegant proofs for the results throughout the remainder of the paper.

Theorem 3.1 provides us with the foundations for our development in this section. This theorem states that there is a nonzero polynomial in $\mathbb{C}[\mathscr{D}]$ that annihilates each of $X_1,..., X_n$. Let $p(\mathscr{D})$ be the monic generator of $\bigcap_{i=1}^{n} \operatorname{Ann}(X_i)$. We write

$$p(\mathscr{D}) = \mathscr{D}^N - \sum_{j=0}^{N-1} c_j \mathscr{D}^j$$

for some set of scalars $c_0,..., c_{N-1}$. Notice that we must have $N \geqslant 1$. We now have the following.

Theorem 5.1. *Let* $\mathbf{v}$, ϕ, p, *and* $c_0,..., c_{N-1}$ *be as above. Then* ϕ *satisfies the linear homogeneous Nth-order ordinary differential equation*

$$y^{(N)} - \sum_{j=0}^{N-1} c_j y^{(j)} = 0 \tag{5.2}$$

That is, for each $x \in \mathbb{C}^n$, *the solution* ϕ_x *of (5.1) satisfies (5.2).*

Proof. By theorems 2.1 and 3.1 we have

$$\begin{aligned} p(\partial_t)\phi &= p(\partial_t)\, E_\phi X^T \\ &= E_\phi\, p(\mathscr{D}) X^T \\ &= 0 \quad ∎ \end{aligned}$$

The remainder of this section is devoted to describing the flow ϕ. Let A be the matrix

$$\begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ c_0 I & c_1 I & c_2 I & \cdots & c_N - 1 I \end{pmatrix}$$

where I denotes the $n \times n$ identity matrix. Let $F: \mathbb{C}^n \to \mathbb{C}^{nN}$ be the injective polynomial map given by

$$x \mapsto (c^T, E_x \mathscr{D} X, \dots, E_x \mathscr{D}^{N-1} X)^T$$

Let $\pi: \mathbb{C}^{nN} \to \mathbb{C}^n$ be given by

$$(x_1, \dots, x_{nN})^T \mapsto (x_1, \dots, x_n)^T$$

We have the following.

Theorem 5.2. *Let* **V**. ϕ, A, F, *and* π *be as above. Then for each* $x \in \mathbb{C}^n$, *the map* F *sends the solution of (5.1) to the solution of*

$$\dot{z} = Az, \qquad z(0) = F(x) \tag{5.3}$$

That is, for each $x \in \mathbb{C}^n$, *the function* $F \circ \phi_x$ *satisfies (5.3). Furthermore,* π *is a left inverse of* F.

Proof. For $j = 0, 1, \dots, N-1$, let $\psi_j = \partial_t^j \phi$. Then

$$\begin{aligned} F(\phi) &= ((\phi)^T, E_\phi \mathscr{D} X, \dots, E_\phi \mathscr{D}^{N-1} X)^T \\ &= ((\phi)^T, \ (\partial_t \phi)^T, \dots, (\partial_t^j \phi)^T)^T \\ &= ((\psi_0)^T, (\psi_1)^T, \dots, (\psi_{N-1})^T)^T \end{aligned}$$

Notice that

$$\partial_t \psi_j = \psi_{j+1}, \qquad j = 0, 1, \dots, N-2$$

and that

$$\partial_t \psi_{N-1} = \partial_t^N \phi = \sum_{j=0}^{N-1} c_j \partial_t^j \phi = \sum_{j=0}^{N-1} c_j \psi_j$$

That is,

$$\partial_t (F \circ \phi) = AF \circ \phi$$

Furthermore, $F(\phi(0, x)) = F(x)$. Thus F sends solutions of (5.1) to solutions of (5.3). Notice that π is a left inverse of F. ▌

Corollary 5.1. *Let* $\mathbf{V}$, ϕ, A, F, *and* π *be as above. Then*

$$\phi(t, x) = \pi e^{At} F(x)$$

Proof. This theorem follows from theorem 5.1 and the theory of linear ordinary differential equations. ▌

Corollary 5.2. *Let* $\mathbf{V}$ *and* ϕ *be as above. Then there exists a positive integer* m, *nonnegative integers* l_i *for* $i = 1, \ldots, m$, *distinct complex numbers* λ_i *for* $i = 1, \ldots, m$, *and polynomial functions* $a_{ij}: \mathbb{C}^n \to \mathbb{C}^n$ *for* $i = 1, \ldots, m$ *and* $j = 0, \ldots, l_i$ *with* $a_{il_i} \neq 0$ *such that*

$$\phi(t, x) = \sum_{i=1}^{m} \sum_{j=0}^{l_i} a_{ij}(x)\, t^j e^{\lambda_i t}/j! \qquad (5.4)$$

Furthermore, if $\mathscr{D} = \mathbf{V} \cdot \nabla$, *the monic generator of* $\bigcap_{i=1}^n \operatorname{Ann}(X_i)$ *is* $\prod_{i=1}^m (\mathscr{D} - \lambda_i)^{l_i}$.

Proof. Equation (5.4) follows from corollary 5.1. It remains to show that $p(\mathscr{D}) = \prod_{i=1}^m (\mathscr{D} - \lambda_i)^{l_i}$ is the monic generator of $\bigcap_{i=1}^n \operatorname{Ann}(X_i)$. Notice that $E_\phi\, p(\mathscr{D}) X^T = p(\partial_t)\phi = 0$. Hence $p(\mathscr{D})X = 0$ and thus $p(\mathscr{D}) \in \bigcap_{i=1}^n \operatorname{Ann}(X_i)$. If $q(\mathscr{D})$ is a factor of $p(\mathscr{D})$ and $\deg q(\mathscr{D}) < \deg p(\mathscr{D})$, then $q(\partial_t)\phi \neq 0$. Thus $E_{\phi q}(\mathscr{D}) X \neq 0$. Since E_ϕ is linear, $q(\mathscr{D}) X \neq 0$. Thus $q(\mathscr{D})$ is not in $\bigcap_{i=1}^n \operatorname{Ann}(X_i)$. Hence $p(\mathscr{D})$ must be a generator of $\bigcap_{i=1}^n \operatorname{Ann}(X_i)$. Notice that $p(\mathscr{D})$ is monic. ▌

The following is a technical lemma which will be used in the sequel.

Lemma 5.1. *Let* $\mathbf{V}$, $\mathscr{D}$, ϕ *be as above. Let* m, l_i, *and* a_{ij} *be as in corollary 5.2. Then there exist polynomials* $p_i(Y) \in \mathbb{C}[Y]$ *for* $i = 1, \ldots, m$ *and* $q_{ik}(Y) \in \mathbb{C}[Y]$ *for* $i = 1, \ldots, m$ *and for* $k = 0, \ldots, l_i$ *such that*

(1) $p_i(\partial_t)\phi = E_\phi\, p_i(\mathscr{D}) X^T = \sum_{j=0}^{l_i} a_{ij}(x)\, t^j e^{\lambda_i t}/j!$,

(2) $q_{ik}(\partial_t)\phi = E_\phi q_{ik}(\mathscr{D}) X^T = \sum_{j=0}^{l_i - k} a_{ij+k}(x)\, t^j e^{\lambda_i t}/j!$, *and*

(3) $(q_{ik}(\partial_t)\phi)|_{t=0} = E_x q_{ik}(\mathscr{D}) X^T = a_{ik}(x)$.

Proof. Notice that Eq. (5.4) gives a spectral decomposition for ϕ as an element of the $\mathbb{C}[\partial_t]$-module $C^\infty(\mathbb{C} \times \mathbb{C}^n, \mathbb{C}^n)$. Hence, by theorem 3.2, there exist polynomials $p_i(\partial_t)$ in $\mathbb{C}[\partial_t]$ such that

$$p_i(\partial_t)\phi = \sum_{j=0}^{l_i} a_{ij}(x)\, t^j e^{\lambda_i t}/j!, \qquad i = 1, \ldots, m$$

But we also have

$$p_i(\partial_t)\phi = p_i(\partial_t)\, E_\phi X^T = E_\phi\, p_i(\mathscr{D})X^T$$

Next, notice that for $i = 1,..., m$ and for $k = 0,..., l_i$ we have

$$\begin{aligned}(\partial_t - \lambda_i)^k\, p_i(\partial_t)\phi &= \sum_{j=k}^{l_i} a_{ij}(x)\, t^{j-k} e^{\lambda_i t}/(j-k)! \\ &= \sum_{j=0}^{l_i - k} a_{ij+k}(x)\, t^j e^{\lambda_i t}/j! \qquad (5.5)\end{aligned}$$

Let $q_{ik}(\partial_t) = (\partial_t - \lambda_i)^k\, p_i(\partial_t)$. Then (2) follows from the fact that

$$q_{ik}(\partial_t)\phi = Aq_{ik}(\partial_t)\, E_\phi X^T = E_\phi q_{ik}(\mathscr{D})X^T \qquad (5.6)$$

Part (3) follows from setting $t = 0$ in Eqs. (5.5) and (5.6). ▌

Using the notation of corollary 5.2, let

$$\phi_{\lambda_i}(t, x) = \sum_{j=0}^{l_i} a_{ij}(x)\, t^j e^{\lambda_i t}/j!, \qquad i = 1,..., m$$

Lemma 5.1 shows that we may "isolate" ϕ_{λ_i} in ϕ by either

(1) applying a polynomial $p_i(\partial_t)$ in the different operator ∂_t to the flow ϕ or

(2) evaluating the polynomial function $p_i(\mathscr{D})X^T$ at ϕ.

Part (1) is not surprising. Part (2) allows us to isolate ϕ_{λ_i} without differentiating.

We now present some examples to illustrate the results in this section.

Example 5.1. Consider the initial value problem

$$\dot{y}_1 = 0 \qquad y_1(0) = x_1$$

$$\dot{y}_2 = q(y_1) \qquad y_2(0) = x_2$$

where $q(Y) \in \mathbb{C}[Y]$ is monic of degree greater than one. In this case the associated derivation $\mathscr{D} = q_1(X_1)\partial_{X_2}$. Hence $\mathscr{D}^2 X = 0$ and $\mathscr{D}^2$ generates $\mathrm{Ann}(X_1) \cap \mathrm{Ann}(X_2)$. We have

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, since $A^2=0$,

$$\begin{aligned}\phi(t,x)&=\pi e^{At}F(x)\\&=\pi\begin{pmatrix}1&0&t&0\\0&1&0&t\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\0\\q(x_1)\end{pmatrix}\\&=(x_1,x_2+tq(x_1))^T\end{aligned}$$

In general, the matrix A is similar ini appearance to the matrix

$$B=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\&\vdots&&\ddots&\vdots\\0&0&0&\cdots&1\\c_0&c_1&c_2&\cdots&c_{N-1}\end{pmatrix}$$

which is the matrix one obtains when converting a linear Nth-order ordinary differential equation into a first-order system. We wish to make precise the connection between A and B. Let $\mathrm{Mat}_m(\mathbb{C})$ denote the set of all $m\times m$ matrices with complex entries. We take $\mathrm{Mat}_m(\mathbb{C})$ to be an algebra (over $\mathbb{C}$) in the usual sense. Define the map Λ: $\mathrm{Mat}_n(\mathbb{C})\to\mathrm{Mat}_{nN}(\mathbb{C})$ via

$$\begin{pmatrix}a_{11}&\cdots&\alpha_{1n}\\\vdots&\ddots&\vdots\\\alpha_{n1}&\cdots&\alpha_{nn}\end{pmatrix}\mapsto\begin{pmatrix}\alpha_{11}I&\cdots&\alpha_{1n}I\\\vdots&\ddots&\vdots\\\alpha_{n1}I&\cdots&\alpha_{nn}I\end{pmatrix}$$

where I is the $n\times n$ identity matrix. Notice that Λ is a linear map and a monomorphism of rings. Since $\mathrm{Mat}_n(\mathbb{C})$ and $\mathrm{Mat}_{nN}(\mathbb{C})$ are finite dimensional as vector spaces over $\mathbb{C}$, the map Λ is continuous with respect to any pair of norm-induced topologies on $\mathrm{Mat}_n(\mathbb{C})$ and $\mathrm{Mat}_{nN}(\mathbb{C})$. We have $A=\Lambda(B)$. From the facts outlined above, we have $e^{At}=\Lambda(e^{Bt})$.

Example 5.2. Consider the initial value problem

$$\dot{y}_1=\alpha m y_1+y_2^m \qquad y_1(0)=x_1$$
$$\dot{y}_2=\alpha u_2 \qquad y_2(0)=x_2$$

where $m=2,3,4,\ldots$. In this case the associated derivation $\mathscr{D}=(\alpha m X_1+X_2^m)\partial_{X_1}+\alpha\, X_2\partial_{X_2}$. Hence

$$\mathscr{D}^3X=\alpha^2m^2X-\alpha^2m(m+2)\,\mathscr{D}X\;+(2m+1)\,\mathscr{D}^2X$$

and $\mathscr{D}^3-(2m+1)\mathscr{D}^2+\alpha^2m(m+2)\mathscr{D}-\alpha^3m^2$ generates $\mathrm{Ann}(X_1)\cap\mathrm{Ann}(X_2)$. We have

$$B=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha^3m^2 & -\alpha^2m(m+2) & \alpha(2m+1) \end{pmatrix}$$

Thus if

$$B_1=\begin{pmatrix} m^2 & -2m/\alpha & 1/\alpha^2 \\ \alpha m^2 & -2m & 1/\alpha \\ \alpha^2m^2 & -2\alpha m & 1 \end{pmatrix}$$

$$B_2=\begin{pmatrix} 1-2m & 2m/\alpha & -1/\alpha^2 \\ -\alpha m^2 & m^2+1 & -1/\alpha \\ -\alpha^2m^2 & 2\alpha m & m^2-2m \end{pmatrix}$$

$$B_3=\begin{pmatrix} \alpha m(m-1) & 1-m^2 & (m-1)/\alpha \\ \alpha^2m^2(m-1) & \alpha m(1-m^2) & m(m-1) \\ \alpha^3m^3(m-1) & \alpha^2m^2(1-m^2) & \alpha m^2(m-1) \end{pmatrix}$$

then

$$e^{Bt}=\frac{1}{(m-1)^2}\left(e^{\alpha t}B_1+e^{\alpha mt}B_2+te^{\alpha mt}B_3\right)$$

We have

$$F(x_1,x_2)=(x_1,x_2,\alpha mx_1+x_2^m,\alpha x_2,\alpha^2m^2x_1+2\alpha mx_2^m,\alpha^2x_2)^T$$

and therefore

$$\begin{aligned}\phi(t,x)&=\pi\Lambda(e^{Bt})\,F(x_1,x_2)\\&=((x_1+x_2^mt)e^{\alpha mt},x_2e^{\alpha t})^T\end{aligned}$$

6. DYNAMICS OF POLYNOMIAL FLOWS

Bass and Meisters (1985) show that on $\mathbb{R}^2$ (and on $\mathbb{C}^2$) any polynomial flow can, via a polyomorphic change of coordinates, be put into one of a list of well-understood "normal forms." This normal forms theorem is one of the most beautiful parts of the theory of polynomial flows. The global dynamics of the normal forms are easily understood, hence the possible global phase portraits of polynomial flows in two dimensions are well understood.

On $\mathbb{R}^n$ for $n\geqslant 3$, however, there is no known normal forms theorem. Up to now little has been proven about the dynamics of polynomial flows in more than two dimensions. Three facts that are well established are as follows:

(1) polynomial flows are complete;

(2) the divergence of a p-f vector field is constant; and

(3) if a p-f vector field has an attracting fixed point, it is a global attractor.

See Bass and Meisters (1985) for proofs of (1) and (2). See Meisters and Olech (1986) for a proof of (3). Meisters and Olech (1988) also contains results relevant to global asymptotic stability of polynomial vector fields on the plane.

Meisters (1989) and Coomes (1988) have asked questions about the dynamics of polynomial flows. We can now draw a clearer, albeit, not complete, picture of the dynamics of polynomial flows in dimensions greater than two.

Time t is usually taken to be a real parameter when one considers dynamics. Hence in this section t is real. Recall that since a p-f vector field $\mathbf{V}$ on $\mathbb{R}^n$ is polynomial, it extends to a vector field on $\mathbb{C}^n$. Furthermore, the flow associated with $\mathbf{V}$ extends to the flow of the extension of $\mathbf{V}$ to $\mathbb{C}^n$ [see Coomes (1990b) for a proof]. Thus we consider only p-f vector fields on $\mathbb{C}^n$.

Let $\mathbf{V}$ be a p-f vector field on $\mathbb{C}^n$ and let F be as in Section 5. Notice that F is a polynomial embedding of $\mathbb{C}^n$ into a higher dimensional space. Recall that F sends solutions of

$$\dot{y} = \mathbf{V}(y) \tag{6.1}$$

to solutions of a linear first-order system. It follows immediately that the orbits of (6.1) have to behave like orbits of linear systems. A few special specific instances are treated in the theorem below.

Theorem 6.1. *Let* $\mathbf{V}$ *be a p-f vector field on* $\mathbb{C}^n$ *and let* ϕ *be the associated flow. Then the following apply.*

(1) The vector field $\mathbf{V}$ *has no homoclinic orbits and*

(2) no heteroclinic orbits.

(3) Given $x \in \mathbb{C}^n$, *if* ϕ_x *is bounded for all* t, *there exist a positive integer* m, *real numbers* ω_i *and* θ_i *for* $i = 1, \ldots, m$, *and vectors* $a_i \in \mathbb{C}^n$ *for* $i = 0, \ldots, m$, *such that*

$$\phi_x(t) = a_0 + a_1 \sin(\omega_1 t + \theta_1) + \cdots + a_m \sin(\omega_m t + \theta_m) \tag{6.2}$$

Proof. Let F and π be as Section 5. As mentioned above, for each

$x \in \mathbb{C}^n$, the function $F \circ \phi_x$ is a solution of a linear first-order system. Hence $F \circ \phi_x$ cannot be a homoclinic [hereroclinic] solution. Notice that if ϕ_x is a homoclinic [heteroclinic] solution, so is $F \circ \phi_x$. Thus ϕ_x can be neither a homoclinic nor a heteroclinic solution. This proves (1) and (2).

To prove (3), notice that if ϕ_x is bounded for all t, so is $F \circ \phi_x$. Thus, since $F \circ \phi_x$ is a solution of a linear first-order system, there exists an integer m, real numbers ω_i and θ_i for $i = 1,..., m$, and vectors $a_i' \in \mathbb{C}^{nN}$ for $i = 0,..., m$, such that

$$F(\phi_x(t)) = a_0' + a_1' \sin(\omega_1 t + \theta_1) + \cdots + a_k' \sin(\omega_m t + \theta_m)$$

Thus $\phi_x = \pi \circ F \circ \phi_x$ is of the form 6.2. ∎

How many isolated fixed points (that is, isolated from other fixed points) can a p-f vector field have? This question is unanswered. Here are the cases where the answer is known: in one dimension the answer is at most one. (Recall that all one-dimensional p-f vector fields are of the form $\alpha y + \beta$, where α and β are scalars.) From the normal forms theorem of Bass and Meisters, it follows that in two dimensions the answer is at most one. If x_0 is an attracting fixed point of a p-f vector field on $\mathbb{R}^n$ or $\mathbb{C}^n$, then x_0 is a global attractor. Hence a p-f vector field with an attracting fixed point has only one fixed point. It also follows, by reversing time, that a p-f vector field with a repelling fixed point has only one fixed point.

Theorem 6.2. *Let* **V** *be a p-f vector field on* $\mathbb{C}^n$, *let* $\mathscr{D} = \mathbf{V} \cdot \nabla$, *and let* $p(\mathscr{D})$ *be a generator* of $\bigcap_{i=1}^n \operatorname{Ann}(X_i)$. *Suppose that zero is not a root of* $p(\mathscr{D})$. *Then* **V** *has at most one fixed point.*

Proof. Without loss of generality we may take $p(\mathscr{D}) = \mathscr{D}^N - \sum_{j=0}^{N-1} c_j \mathscr{D}^j$, where $N \geq 1$. Recall (theorem 5.2) that the map F as defined in Section 5 sends solutions of (6.1) to solutions of

$$\dot{z} = Az \tag{6.3}$$

where

$$A = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ c_0 I & c_1 I & c_2 L & \cdots & c_{N-1} I \end{pmatrix}$$

If zero is not a root of $p(\mathscr{D})$, then $c_0 \neq 0$. In this case A is nonsingular and hence (6.3) has a unique fixed point. Since F is injective and sends fixed points of $\mathbf{V}$ to fixed points of (6.3), the vector field $\mathbf{V}$ has at most one fixed point. ▮

Let M be an open subset of $\mathbb{C}^n$, let $\mathbf{F}$ be a vector field on M, and let ψ be its local flow. We say that a nonconstant differentiable function $f: M \to \mathbb{C}$ is an *integral* of $\mathbf{F}$ if $\partial_t(f \circ \psi)$ vanishes on the natural domain of ψ.

Corollary 6.1. *Let $\mathbf{V}$ be a p-f vector field on $\mathbb{C}^n$ with a fixed point at zero. Suppose that $\mathbf{V}$ has no polynomial integrals. Then zero is the only fixed point of $\mathbf{V}$.*

Proof. Suppose that $\mathbf{V}$ has more than one fixed point. Let $\mathscr{D} = \mathbf{V} \cdot \nabla$. By theorem 6.2, zero is a root of the monic generator $p(\mathscr{D})$ of $\bigcap_{i=1}^{n} \operatorname{Ann}(X_i)$. Thus for some nonzero polynomial $p'(\mathscr{D})$ we have

$$p(\mathscr{D})X = \mathscr{D}p'(\mathscr{D})X = 0$$

Notice that for some i, the polynomial $p'(\mathscr{D})$ is not in $\operatorname{Ann}(X_i)$. That is, $p'(\mathscr{D})X_i \neq 0$. Since $\mathbf{V}$ has a fixed point at zero, $E_0 p'(\mathscr{D})X_i = 0$. Thus $p'(\mathscr{D})X_i$ is not constant. However,

$$\partial_t(E_\phi p'(\mathscr{D})X_i) = E_\phi \mathscr{D}p'(\mathscr{D})X_i = 0$$

so $p'(\mathscr{D})X_i$ is a polynomial integral of $\mathbf{V}$. By the contrapositive, the theorem holds. ▮

The converse of theorem 6.2 is not true as the following example shows.

Example 6.1. Consider the initial value problem

$$\dot{y} = y_1 + 7y_2^2 + 12y_2 y_1^3 + 36y_2^3 y_1^2 + 36y_1 y_2^5 + 12y_2^7, \qquad y_1(0) = x_1$$
$$\dot{y}_2 = -6y_1^3 - 18y_1^2 y_2^2 - 18y_1 y_2^4 - 6y_2^6 - 3y_2 \qquad y_2(0) = x_2$$

with polynomial flow

$$\begin{aligned} y_1(t) = {} & (-x_1^6 - 6x_2^3x_1^2 - 6x_2^5x_1 - 2x_2x_1^3 - 20x_1^3x_2^6 - 15x_1^2x_2^8 - 6x_1^5x_2^2 \\ & - 15x_1^4x_2^4 - 2x_2^7 - x_2^{12} - 6x_1x_2^{10} - x_2^2)e^{-6t} \\ & + 2x_1^6 + 30x_1^2x_2^8 + 12x_1x_2^{10} + 2x_2^7 + 2x_2^{12} + 2x_2x_1^3 + 6x_2^3x_1^2 + 6x_2^5x_1 \\ & + 12x_1^5x_2^2 + 30x_1^4x_2^4 + 40x_1^3x_2^6 + (x_1 + x_2^2)e^t \\ & + (-15x_1^4x_2^4 - x_2^{12} - 6x_1^5x_2^2 - 20x_1^3x_2^6 - x_1^6 - 6x_1x_2^{10} - 15x_1^2x_2^8)e^{6t} \end{aligned}$$

$$y_2(t) = (x_1 + x_1^3 + 3x_1^2x_2^2 + 3x_1x_2^4 + x_2^6)e^{-3t} + (-x_2^6 - x_1^3 - 3x_1^2x_2^2 - 3x_1x_2^4)e^{3t}$$

The vector field has a hyperbolic, hence isolated fixed point at the origin. It follows from the normal forms theorem of Bass and Meisters that the origin is the only fixed point of the vector field. Notice that if $\mathscr{D}$ is the derivation associated with this vector field, the monic generator of $\mathrm{Ann}(X_1) \cap \mathrm{Ann}(X_2)$ is

$$p(\mathscr{D}) = (\mathscr{D}+6)(\mathscr{D}+3)\,\mathscr{D}(\mathscr{D}-1)(\mathscr{D}-3)(\mathscr{D}-6)$$

which has zero as a root.

7. STABLE AND UNSTABLE MANIFOLDS

In this section we show that the stable and unstable manifolds of fixed points of p-f vector fields are algebraic varieties. We also show that attracting and repelling fixed points of p-f vector fields are hyperbolic.

Let $\mathbf{V}$ be a p-f vector field on $\mathbb{C}^n$ and let ϕ be the associated flow. We take

$$\phi(t, x) = \sum_{i=1}^{m} \sum_{j=0}^{l_i} a_{ij}(x)\, t^j e^{\lambda_i t}/j!$$

as in corollary 5.2. Recall that each a_{il_i} is nonzero.

For each $x \in \mathbb{C}^n$ define

$$W^s(x) = \{y \in \mathbb{C}^n\colon \|\phi_y(t) - \phi_x(t)\| \to 0 \text{ as } t \to \infty\}$$

and

$$W^u(x) = \{y \in \mathbb{C}^n\colon \|\phi_y(t) - \phi_x(t)\| \to 0 \text{ as } t \to -\infty\}$$

where $\|\cdot\|$ denotes the usual Euclidean norm. We call $W^s(x)$ [respectively, $W^u(x)$] the *stable* [respectively, *unstable*] *manifold of* x. We do not concern ourselves with the question of whether the sets $W^s(x)$ and $W^u(x)$ are submanifolds of $\mathbb{C}^n$.

For each set $S \subset \mathbb{C}[X_1, \ldots, X_n]$, let $V(S) = \{x \in \mathbb{C}^n\colon E_x f = 0$ for every $f \in S\}$. We call $V(S)$ ther *affine variety* defined by S or simply an affine variety. Let $\mathscr{R}(\mu)$ denote the real part of the complex number μ.

Theorem 7.1. *Let $\mathbf{V}$, ϕ, m, l_i, λ_i, and a_{ij} be as above. Let $x_0 \in \mathbb{C}^n$ be a fixed point of $\mathbf{V}$. Define sets U, S, and $C(x_0)$ of polynomials as follows:*

$$U = \{\textit{components of } a_{ij}\colon \mathscr{R}(\lambda_i) > 0,\ 0 \leqslant j \leqslant l_i\}$$

$$S = \{\textit{components of } a_{ij}\colon \mathscr{R}(\lambda_i) < 0,\ 0 \leqslant j \leqslant l_i\}$$

$$C(x_0) = \{\textit{components of } a_{ij} - a_{ij}(x_0)\colon \mathscr{R}(\lambda_i) = 0,\ 0 \leqslant j \leqslant l_i\}$$

Then the stable manifold of x_0 *is the affine variety defined by* $U \cup C(x_0)$ *and the unstable manifold of* x_0 *is the affine variety defined by* $S \cup C(x_0)$. *That is,*

$$W^s(x_0) = V(U \cup C(x_0))$$
$$W^u(x_0) = V(S \cup C(x_0))$$

Remark. Let $\mathscr{D} = \mathbf{V} \cdot \nabla$ and let $q_{ik}(\mathscr{D})$ be as lemma 5.1. Since x_0 is a fixed point of $\mathbf{V}$ and

$$E_{\phi_{x_0}(t)}\, q_{ik}(\mathscr{D}) X^T = \sum_{j=0}^{l_i - k} a_{ij+k}(x_0)\, t^j e^{\lambda_i t}/j!$$

we have $a_{ik}(x_0) = 0$ if $\lambda_i \neq 0$ or $k \geqslant 1$. Furthermore, we have $a_{i0}(x_0) = x_0$ if $\lambda_i = 0$.

Proof. Since for each y in $\mathbb{C}^n$ we have

$$\phi_y(t) - x_0 = \sum_{i=1}^{m} \sum_{j=0}^{l_i} (a_{ij}(y) - a_{ij}(x_0))\, t^j e^{\lambda_i t}/j!$$

we have $V(U \cup C(x_0)) \subset W^s(x_0)$. Let y be in $W^u(x_0)$, let $\mathscr{D} = \mathbf{V} \cdot \nabla$, and let $p_i(\mathscr{D})$ be as in lemma 5.1. Then

$$(E_{\phi_y(t)} - E_{x_0})\, p_i(\mathscr{D}) X^T = \sum_{j=0}^{l_i} (a_{ij}(y) - a_{ij}(x_0))\, t^j e^{\lambda_i t}/j! \tag{7.1}$$

Since $\phi_y(t) \to x_0$ as $t \to \infty$, the right-hand side of (7.1) goes to zero as $t \to \infty$. The left-hand side of 7.1 goes to zero as t goes to infinity if and only if $\mathscr{R}(\lambda_i) < 0$ or $a_{ij}(y) - a_{ij}(x_0) = 0$ for $j = 0,..., l_i$. Hence y is in $V(U \cup C(x_0))$. Thus $W^s(x_0) \subset V(U \cup C(x_0))$ and therefore $W^s(x_0) = V(U \cup C(x_0))$. A similar argument shows that $W^u(x_0) = V(S \cup C(x_0))$. ∎

Meisters and Olech (1986) show that an attracting fixed point of a p-f vector field is globally attracting. We can add some information, about the nature of attracting fixed points of p-f vector fields. We say a fixed point x_0 of a C^1 vector field $\mathbf{F}$ is *hyperbolic* if all the eigenvalues of the matrix $\nabla \mathbf{F}(x_0)$ have negative real parts.

Theorem 7.2. *Let* x_0 *in* $\mathbb{C}^n$ *be an attracting fixed point of the p-f vector field* $\mathbf{V}$. *Then* x_0 *is hyperbolic.*

Proof. Without loss of generality $x_0 = 0$. Notice that $W^s(0) = \mathbb{C}^n$. Let U and $C(0)$ be as in theorem 7.1. By that lemma, either each polynomial in $U \cup C(0)$ is zero or $U \cup C(0)$ is empty. Since each a_{il_i} is nonzero, we must have that $U \cup C(0)$ is empty. That is, $\mathscr{R}(\lambda_i) < 0$ for each λ_i. Let $\mathscr{D} = \mathbf{V} \cdot \nabla$. By corollary 5.2, $p(\mathscr{D}) = \prod_{i=1}^{m} (\mathscr{D} - \lambda_i)^{l_i}$ is the generator of $\bigcap_{i=1}^{n} \operatorname{Ann}(X_i)$. By lemma 4.1 with $\omega = (1, \dots, 1)$, we have that $\nabla \mathbf{V}(0)$ satisfies $p(\mathscr{D})$. That is, $p(\nabla \mathbf{V}(0)) = 0$. Hence all the eigenvalues of $\nabla \mathbf{V}(0)$ are roots of $p(\mathscr{D})$. Thus all the eigenvalues of $\nabla \mathbf{V}(0)$ have negative real parts. ▌

The following example shows that a fixed point of a vector field can be attracting without being hyperbolic.

Example 7.1. The vector field

$$\dot{y} = -y^3$$

has a nonhyperbolic fixed point at $y = 0$. Yet if $y(0) = x$, then $y(t) = x(1 + 2x^2 t)^{-1/2}$, hence $y = 0$ is an attracting fixed point.

8. THE IDENTIFICATION PROBLEM

Which vector fields have polynomial flows? In one dimension this question is easily answered; the one-dimensional p-f vector fields are exactly those of the form $\mathbf{V}(y) = \alpha y + \beta$, where α and β are scalars. In higher dimensions, however, this question has not been definitively answered. We use results in this paper to give new insights into this identification problem. We give a new proof that the Lorenz system does not have a polynomial flow and we prove that the Maxwell-Bloch system does not have a polynomial flow.

Coomes (1988, 1990a, b) that the Lorentz system

$$\left.\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy \end{aligned}\right\} (x, y, z) \in \mathbb{R}^3, \quad \sigma, \rho, \beta > 0 \tag{8.1}$$

(see, for example, Lorentz, 1963; Sparrow, 1982; Guckenheimer and Holmes, 1983) does not have a polynomial flow. In fact, Coomes shows that (8.1) does not have a polynomial flow when the only restriction on parameters is $\sigma \neq 0$. For the sake of illustrating our techniques, we prove, using spectral methods, that (8.1) does not have a polynomial flow. One could argue that since, for certain parameter choices, the flow behaves "chaotically," the Lorenz system cannot be embedded in a linear system and hence does not have a polynomial flow for those parameter choices. However, our methods are algebraic and do not take dynamics into account.

In our discussion the symbols x, y, and z play a dual role—we think of them both as functions of t and as indeterminates. Proposition 3.2 is used extensively.

Example 8.1. The Lorenz system (8.1) does not have a polynomial flow.

Proof. Let $\mathbf{V}$ be the vector field of (8.1) and let $\mathscr{D} = \mathbf{V} \cdot \nabla$. Since zero is a fixed point of (8.1), by theorem 4.2 the spectrum $\mathscr{S}(\mathscr{D})$ is contained in the lattice generated by zero and the eigenvalues of

$$\nabla \mathbf{V}(0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

which are

$$-\beta, \qquad (-\sigma - 1 \pm \sqrt{(\sigma-1)^2 + 4\rho\sigma})/2$$

Thus $\mathscr{S}(\mathscr{D}) \subset \mathbb{R}$.

By way of contradiction, suppose that the Lorenz system has a polynomial flow. Then by theorem 3.1, x, y, and z are torsion elements of the $\mathbb{C}[\mathscr{D}]$-module $\mathbb{C}[x, y, z]$. By the argument above, the spectrum of each of x, y, and z is contained in $\mathbb{R}$. Let $x_{\max}$ and $x_{\min}$ denote the largest and smallest elements of Spec(x), respectively. Similar definitions hold for $y_{\max}$, $y_{\min}$, $z_{\max}$, and $z_{\min}$.

Since $\mathscr{D}x = \sigma(y - x)$, we have

$$(\mathscr{D}/\sigma + 1)x = y \tag{8.2}$$

Since $\mathscr{D}y = \rho x - y - xz$, we have

$$xz = -(\mathscr{D} + 1)y + \rho x = (\rho - (\mathscr{D} + 1)(\mathscr{D}/\sigma + 1))x \tag{8.3}$$

Thus $\mathrm{Spec}(xz) \subset \mathrm{Spec}(x)$. Since both $x_{\min} + z_{\min}$ and $x_{\max} + z_{\max}$ are extremal elements of $\mathrm{Spec}(x) + \mathrm{Spec}(z)$, they both lie in $\mathrm{Spec}(xz)$ and hence

$$x_{\min} + z_{\min} \geqslant x_{\min} \tag{8.4}$$

$$x_{\max} + z_{\max} \leqslant x_{\max} \tag{8.5}$$

Together, Eqs. (8.4) and (8.5) imply

$$0 \leqslant z_{\min} \leqslant z_{\max} \leqslant 0$$

That is $\mathrm{Spec}(z) = \{0\}$.

Since $\mathscr{D}z = -\beta z + xy$, we have

$$xy = (\mathscr{D} + \beta)z \tag{8.6}$$

which implies that $\text{Spec}(xy) \subset \text{Spec}(z)$. Since $x_{\min} + y_{\min}$ and $x_{\max} + y_{\max}$ are both extremal elements of $\text{Spec}(x) + \text{Spec}(y)$, they both lie in $\text{Spec}(xy)$ and hence

$$0 \leqslant x_{\min} + y_{\min} \leqslant x_{\max} + y_{\max} \leqslant 0$$

Thus $\text{Spec}(x)$ and $\text{Spec}(y)$ must both be singletons. Furthermore, if $\text{Spec}(x) = \{\xi\}$, then $\text{Spec}(y) = \{-\xi\}$. By (8.2), we have $\text{Spec}(y) \subset \text{Spec}(x)$. Thus $\xi = -\xi$ and hence $\xi = 0$. Therefore

$$\text{Spec}(x) = \text{Spec}(y) = \text{Spec}(z) = \{0\}$$

We now argue using multiplicities (see proposition 3.3). From (8.2) we have

$$\text{ord}_0(x) = \text{ord}_0((\mathscr{D}/\sigma + 1)x) = \text{ord}_0(y) = \theta$$

From (8.6) we have

$$\text{ord}_0(z) = \text{ord}_0((\mathscr{D} + \beta)z) = \text{ord}_0(x) + \text{ord}_0(y) = 2\theta$$

From (8.3) we have

$$3\theta = \text{ord}_0(x) + \text{ord}_0(z) = \text{ord}_0((\rho - (\mathscr{D} + 1)(\mathscr{D}/\sigma + 1))x) \leqslant \text{ord}_0(x) = \theta$$

Since $\theta \geqslant 0$ and $3\theta \leqslant \theta$, we must have $\theta = 0$. But this implies that

$$\mathscr{D}x = \mathscr{D}y = \mathscr{D}z = 0$$

which is a contradiction. Thus (8.1) does not have a polynomial flow. ∎

While the proof of example 8.1 was a new proof of a known fact, the following result is new. Coomes (1990a, question 4.1) posed the questions, "Does the system

$$\left.\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x + \varepsilon z \\ \dot{z} &= -\varepsilon y \\ \dot{\varepsilon} &= \vartheta \\ \dot{\vartheta} &= -\varepsilon + \beta(x + \varepsilon z) \end{aligned}\right\} (x, y, z, \varepsilon, \vartheta) \in \mathbb{R}^5, \quad \beta \in \mathbb{R} \tag{8.7}$$

which is complete and has a polynomial vector field with constant divergence, have a polynomial flow?" [See Coomes (1990a) for a proof that this system is complete.] We show that (8.7) does not have a polynomial flow when $\beta \neq 0$.

This system, which was shown to one of the authors by Darrel Holm, arises in nonlinear optics. It is a reduction of the coupled Maxwell-Bloch equations. See Allen and Eberly (1987) or Nath and Ray (1987) for more details.

Example 8.2. The Maxwell-Bloch system 8.7 does not have a polynomial flow if $\beta \neq 0$.

Remark. Attempts to prove this result using the techniques of Coomes (1988, 1990a, b) have not been successful. That is, a proof of example 8.2 that is similar in structure to either of Coomes's proofs that the Lorentz system does not have a polynomial flow has not been found.

Coomes's proofs are based on *p-symmetries* of vector fields; a polynomial map $P: \mathbb{C}^n \to \mathbb{C}^n$ is said to be a p-symmetry of the vector field $\mathbf{V}$ if P has a polynomial inverse and

$$\nabla P(x)\, \mathbf{V}(x) = \mathbf{V}(P(x)), \qquad x \in \mathbb{C}^n \tag{8.8}$$

If $\mathbf{V}$ is a p-f vector field, all t-advance maps associated with $\mathbf{V}$ are p-symmetries of $\mathbf{V}$. If one can say a sufficient amount about the p-symmetries of a given vector field, one can decide whether it has a polynomial flow. Such is the case with the Lorentz system.

A key step in Coomes' proof that (8.1) does not have a polynomial flow is his lemma 3.3 (1990a). This lemma shows that if $P = (p, q, r)^T$ is a p-symmetry of (8.1), by (8.8) we have the three equations

$$\sigma(y-x)\,p_x + (\rho x - y - xz)\,p_y + (xy - \beta z)\,p_z + \sigma p = \sigma q$$
$$\sigma(y-x)q_x + (\rho x - y - xz)q_y + (xy - \beta z)q_z + q = p(\rho - r)$$
$$\sigma(y-x)r_x + (\rho x - y - xz)r_y + (xy - \beta z)r_z + \beta r = pq$$

which show that $\deg p + \deg r \leqslant \deg q + 1$ and $\deg p + \deg q \leqslant \deg r + 1$. From these two inequalities, the fact that q can be written in terms of p, and the fact that P is invertible, one concludes that $\deg p = 1$, and $\deg q$, $\deg r \leqslant 2$. That is, there is a bound on the degree of p-symmetries of (8.1).

Now consider a similar approach to the Maxwell-Bloch system. Let $P = (p, q, r, s, u)^T$ be a p-symmetry of (8.7) and let $i = \deg p$, $j = \deg q$,

$k = \deg r$, $l = \deg s$, and $m = \deg u$. The possible inequalities one can derived from (8.8) in this case are

$$\begin{aligned} j &\leqslant i+1 \\ i &\leqslant \max\{k+l, j+1\} \\ k+l &\leqslant \max\{i, j+1\} \\ j+l &\leqslant k+1 \\ m &\leqslant l+1 \\ i &\leqslant \max\{k+l, m+1\} \\ k+l &\leqslant \max\{i, m+1\} \end{aligned} \tag{8.9}$$

Notice that for any positive integer n, the choice $i = 3n$, $j = n$, $k = 2n$, $l = n$, and $m = n$ satisfy all the inequalities in (8.9). That is, we cannot get a bound on the degree of the p-symmetries of (8.7) using arguments analogous to those in the proof of Coomes' (1990a) lemma 3.3. A more intricate degree argument may be successful.

Proof of Example 8.2. Let $\mathbf{V}$ be the vector field of (8.7). Notice that each point on the z-axis (the set $\{(x, y, z, \varepsilon, \vartheta)^T \in \mathbb{R}^5 \colon x = y = \varepsilon = \vartheta = 0\}$) is a fixed point of $\mathbf{V}$. Let $x_0 = (0, 0, \alpha, 0, 0)^T$.

By way of contradiction, assume that (8.7) has a polynomial flow. Let $\mathscr{D} = \mathbf{V} \cdot \nabla$. By corollary 4.1, the spectrum $\mathscr{S}(\mathscr{D})$ is the lattice generated by zero and the eigenvalues of

$$\nabla \mathbf{V}(x_0) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \beta & 0 & 0 & \alpha\beta - 1 & 0 \end{pmatrix}.$$

which for $\alpha\beta \geqslant 0$ are 0, $(\sqrt{\alpha\beta} \pm i\sqrt{4 - \alpha\beta})/2$, and $(-\sqrt{\alpha\beta} \pm i\sqrt{4 - \alpha\beta})/2$. Thus $\alpha = 0$ implies that $\mathscr{S}(\mathscr{D}) = \operatorname{Lat}\{0, \pm i\}$ and hence each eigenvalue of $\mathscr{D}$ is purely imaginary. But $\alpha\beta > 0$ implies that $\mathscr{S}(\mathscr{D}) = \operatorname{Lat}\{0, (\sqrt{\alpha\beta} \pm i\sqrt{4 - \alpha\beta})/2, (-\sqrt{\alpha\beta} \pm i\sqrt{4 - \alpha\beta})/2\}$ and hence some eigenvalues of $\mathscr{D}$ have nonzero real parts. This is a contradiction. Hence (8.7) does not have a polynomial flow. ∎

We thank Gary H. Meisters for reading the first draft of this paper and for making many helpful suggestions.

9. ADDENDUM

The referee pointed out that there are similarities between the results in this paper and a classical result, which Markushevich (1966) attributes to Weierstrass, on function equations. We say that a function $f: \mathbb{C} \to \mathbb{C}$ *obeys an algebraic addition theorem* if there exists a nonzero polynomial a in $\mathbb{C}[X_1, X_2, X_3]$ such that

$$a(f(t+s), f(t), f(s)) = 0, \qquad t, s \in \mathbb{C}$$

We have the following

Theorem 9.1 (Weierstrass). *Suppose that an entire function $f: \mathbb{C} \to \mathbb{C}$ obeys an algebraic addition theorem. Then either*

$$f(t) = a_0 + a_1 t + \cdots + a_n t^n, \qquad t \in \mathbb{C}$$

or

$$f(t) = a_0 + [a_1 \cos(\alpha t) + b_1 \sin(\alpha t)] + [a_2 \cos(2\alpha t) + b_2 \sin(2\alpha t)] + \cdots + [a_n \cos(n\alpha t) + b_n \sin(n\alpha t)], \qquad t \in \mathbb{C}$$

for some integer $n \geq 0$ and some set of complex scalars α, $a_1, ..., a_n$, $b_1, ..., b_n$.

Remark. See Markushevich (1966) for a proof. See Painlevé (1903) for an extension of this result to systems.

Suppose that $a = X_1 - b$, where b is in $\mathbb{C}[X_2, X_3]$. In this case, the proof of theorem 9.1 reduces to solving linear first-order ordinary differential equations.

Lemma 9.1. *Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$, let $f: \mathbb{F} \to \mathbb{F}$ be a function whose range contains a nonempty open set, and let b be a polynomial in two variables with coefficients in $\mathbb{F}$. Suppose that*

$$f(t+s) = b(f(t), f(s)) \qquad t, s \in \mathbb{F}$$

Then b is of degree one in each of its variables. Furthermore, there exist scalars $\alpha \neq 0$ and β such that if $g = \alpha f + \beta$, either

$$g(t+s) = g(t)\, g(s), \qquad t, s \in \mathbb{F} \tag{9.1}$$

or

$$g(t+s) = g(t) + g(s), \qquad t, s \in \mathbb{F} \tag{9.2}$$

Remark. Aczél (1966, Section 2.2.4) proves this result with $\mathbb{F} = \mathbb{R}$ and f a nonconstant continuous function. However, Aczél's proof works under the hypotheses of lemma 9.1.

If, in addition to satisfying the hypotheses of lemma 9.1, the function f is entire, the function g is also entire. Thus we may differentiate either (9.1) or (9.2) with respect to s and set $s=0$ to obtain a linear first-order ordinary differential equation for g. In either case the differential equation can easily be solved. It follows that $g(t)=e^{\lambda t}$ or $g(t)=\lambda t$ for some nonzero scalar λ. Since $f=(g-\beta)/\alpha$, theorem 9.1 follows.

We thank the referee for bringing this result to our attention.

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