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Assessment of economically optimum application rates of fertilizer N on the basis of response curves

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Abstract. N-response curves of numerous N-fertilizer trials with sugar beet and potato are described by a quadratic and a modified exponential equation. For both sugar beet and potato the modified exponential equation was much better than the quadratic equation when the residual sum of squares (RSS) was taken as a measure of the degree of fit. In order to take into account the few occasions when the quadratic model was superior, it is suggested that both models should be used for the data of each individual trial. The economically optimum application rate of fertilizer N is calculated on the basis of the best-fitting model. This procedure yielded optima which covered entire ranges of fertilizer-N levels tested: 0–250 kg ha^{-1} for sugar beet and 0–400 kg ha^{-1} for potato. The magnitude of the confidence intervals ($p > 95\%$) of the optimum N-fertilizer application rate frequently was very high. In 46% of the sugar beet trials and even in 60% of the potato trials it was higher than 300 kg ha^{-1} N. It is suggested that N-fertilizer recommendations be drawn up only with reliable optima.

Introduction

Fertilizer recommendations are usually based on results of field trials in which crop response to various rates of fertilizer application is determined. The response curve then provides for each trial the relationship between amount of fertilizer and crop yield. From this curve the economically optimum application rate of fertilizer, i.e. the minimum amount of fertilizer N needed for maximum financial yield, can be derived. Next, fertilizer recommendations can be made by correlating the optimum application rates of fertilizer in the various trials with one or more field characteristics, for instance the amount of nutrient which is already present in the soil and available to the crop, e.g. [33].

In the Netherlands numerous N-fertilizer trials have been conducted to establish the relationship between soil characteristics and the optimum application rate of N fertilizer in order to draw up N-fertilizer recommendations for sugar beet and potato. Although it was never stated explicitly, interpretations of the experimental results were based on hand-drawn curves [20, 26], as were the recommendations [4, 22]. In drawing curves by hand it was attempted to minimize the deviations of the points from the curve. An important prerequisite for this method of curve fitting was that the drawn curve should be in agreement with the general view on the shape of response curves. Details about the method are given by Visser [34, 35]. It is obvious that this method of curve fitting is rather subjective and time consuming. In the present paper results of the fertilizer trials are re-evaluated by determining the response curves on the basis of mathematical functions. Calculation of the curves is not only objective and rapid, but also makes it possible to assess more accurately the optimum application rate of fertilizer N and to state the reliability of the calculated optima.

Materials and methods

Experimental design

In cooperation with the Sugar Research Institute (IRS, Bergen op Zoom) and the Research Station for Arable Farming and Field Production of Vegetables (PAGV, Lelystad) 167 field trials with sugar beet (*Beta vulgaris* L.) and 99 field trials with potato (*Solanum tuberosum* L.) were conducted in the period 1973–1982 (Table 1). The trials were laid out scattered over the Netherlands.

The trials with sugar beet consisted of 24 plots: six application rates of fertilizer N (0, 40, 80, 120, 160, and 200 kg ha^{-1} or 0, 50, 100, 150, 200, and 250 kg ha^{-1}) in four replications. The trials with potato consisted of 21 plots: seven application rates of fertilizer N (0, 100, 150, 200, 250, 300, and 400 kg ha^{-1}) in three replications. With few exceptions plot size was $6 \times 25 = 150\,m^2$ for sugar beet and $6 \times 9 = 54\,m^2$ for potato. The fertilizer N was applied by hand as ammonium nitrate limestone as a single dressing in March or April. On some sugar beet trials the treatments 200 and 250 kg ha^{-1} were split into two application rates of fertilizer N: 150 + 50 and 150 + 100 kg ha^{-1}. After soil analysis each experimental field was uniformly fertilized with phosphate and potassium according to the recommendations. Weed-, disease- and pest-control measures were carried out by the farmer according to his normal practice. The cultivars mostly used were Monohil for sugar beet and Bintje for potato. Fresh root yield and sugar content of the roots were determined for each sugar beet plot, and fresh tuber yield was determined for each potato plot.

Table 1. Distribution of the trials over the years. Numbers are trial numbers.

Year	Sugar beet	Potato
1973		1– 5
1974	1– 8	6– 9
1975	9– 18	10–13
1976	19– 29	14–20
1977	30– 76	21–26
1978	77–120	27–33
1979	121–167	34–39
1980		40–65
1981		66–90
1982		91–99

Determination of optimum application rate of fertilizer N

In the Netherlands the price the farmer gets for his sugar beet depends not only on root yield, but also on sugar content of the roots. The price is based on roots with a sugar content of 16%. When the roots have a lower or a higher sugar content the price will be lower or higher, respectively. This means that the relationship between root yield and price may not be linear. Linearity is a prerequisite for calculating the economically optimum application rate of fertilizer N at various prices for the crop. The monetary ratio of fertilizer cost to crop value then can be varied simply by changing the slope of the line reflecting the cost of fertilizer [5]. When the relationship between crop yield and price is not linear the economically optimum application rate of fertilizer can only be assessed by determining response curves for financial yield at each crop price. In the present study linearity between root yield and price was introduced by adjusting data on root yields in such a way that they all pertain to roots with a sugar content of 16%. In doing so it was assumed that the adjustment per percent sugar deviating from 16% amounted to 8.5% of the price per tonne roots with a sugar content of 16%, as this was approximately the average value since 1980 [19]. So the measured root yield data (RY; in $\mathrm{t\,ha^{-1}}$) were converted into adjusted roots yields (ARY; in $\mathrm{t\,ha^{-1}}$) taking the measured sugar content (SC; in %) into account according to equation (1):

$$\mathrm{ARY} = \mathrm{RY} + \mathrm{RY}^*(\mathrm{SC} - 16) \times 0.085. \tag{1}$$

For potato an adjustment of tuber yield was not necessary, as the price the farmer gets for his ware potatoes is linearly related to tuber yield.

Nitrogen response curves are described by a quadratic or an exponential function.

The quadratic function has the form of

$$y = \beta_0 + \beta_1 N + \beta_2 N^2 \tag{2}$$

where y is yield in $\mathrm{t\,ha^{-1}}$, N is applied fertilizer N in $\mathrm{kg\,ha^{-1}}$ and β_0, β_1, and β_2 are coefficients which are estimated from the experimental data. The expected quadratic response curve is described by

$$\hat{y} = b_0 + b_1 N + b_2 N^2 \tag{3}$$

where $\hat{y}$ is the expected yield and b_0, b_1, and b_2 are unbiased estimations of β_0, β_1, and β_2. The coefficients b_0, b_1, and b_2 are calculated by linear regression analysis.

The optimum of the expected quadratic response curve is reached when

$$\frac{\partial \hat{y}}{\partial N} = b_1 + 2b_2 N = P \tag{4}$$

where P equals ratio of the cost of 1 kg fertilizer to the price of 1 tonne crop yield. This means that the economically optimum application rate of fertilizer

N, N_{op}, in $\mathrm{kg\,ha^{-1}}$, is

$$N_{op} = \frac{P - b_1}{2b_2}. \tag{5}$$

The exponential function is modified by the addition of a linear term to allow for decreasing yields at nitrogen levels in excess of the level for maximum yield. The function has the form of

$$y = \beta_0 + \beta_1 N + \beta_2 e^{aN} \tag{6}$$

and the expected modified exponential response curve is described by

$$\hat{y} = b_0 + b_1 N + b_2 e^{aN}. \tag{7}$$

The meaning of y, $\hat{y}$, N, β_0, β_1, β_2, b_0, b_1, and b_2 corresponds to that in equation (2) and equation (3) and a is a constant which was predetermined to avoid non-linear regression analysis. The constant a was set equal to one of nine predetermined values defined arbitrarily by

$$a = 4 \times \ln(c)/N_{max} \tag{8}$$

where c has values of 0.03, 0.05, 0.10, 0.20, 0.35, 0.60, 0.90, 1.50, and 2.70 and N_{max} is the highest level of fertilizer N applied in $\mathrm{kg\,ha^{-1}}$. The constant a is expressed in units of N_{max} (equation 8) to be able to apply equation (7) to both sugar beet and potato (and various other crops) with different levels of fertilizer N. The values of c cover a wide range of values of a. For each value of a the expected response curve was calculated. The response curve which yielded the lowest residual sum of squares (RSS) was considered to be the curve of best fit. In doing so one degree of freedom was taken into account. From equation (7) it can be deduced that the economically optimum application rate of fertilizer N, N_{op}, is

$$N_{op} = \frac{\ln \dfrac{P - b_1}{ab_2}}{a} \tag{9}$$

where P has the same meaning as in equation (4).

In the Appendix the deduction of the confidence limits ($p > 95\%$) of the economically optimum application rate of fertilizer N is described.

Results

Optimum application rate of fertilizer N

For each of the 167 sugar beet and the 99 potato trials the optimum application rate of fertilizer N was calculated both on the basis of the quadratic and the modified exponential response curve. The monetary ratio of fertilizer cost to crop value was fixed at 0.0125 for sugar beet and at 0.0075 for potato (a monetary ratio of 0.0075 for potato means for instance that the cost of 1 kg fertilizer N is DFL 1.50 and the price of 1 tonne tubers is Dfl. 200.00). These values are currently valid in the Netherlands [24].

When the quadratic response curve was used, the optima of 23 sugar beet and 13 potato trials fell outside the range of tested levels of fertilizer N. When the modified exponential response curve was used this phenomenon occurred in 34 sugar beet trials and 23 potato trials. These trials were omitted when the optima were compared because the optima were derived after extrapolation, which may considerably decrease reliability.

Figure 1, which gives frequency distributions of the calculated optima, shows rather large differences among optima calculated on the basis of quadratic and modified exponential response curves. The modified exponential response curve led to a higher proportion of optima at low fertilizer-N levels and to a more uniform distribution of the optima over the range of fertilizer-N levels tested. The modified exponential curve may have its optimum at low fertilizer-N levels, whereas the quadratic curve must be symmetrical around its maximum, which may lead to higher optima (Fig. 2).

To determine which of the two models was the most suitable, the residual sum of squares (RSS) of each curve was taken as a measure of best fit. For both sugar beet and potato the average RSS was lower in the modified exponential model than in the quadratic model (Table 2). In 95% of the sugar beet trials and 92% of the potato trials RSS was lower in the modified exponential model. These results show that with few exceptions the modified exponential model fitted the data best. Figure 3 shows the frequency distributions of the optima as determined with the curve of best fit, either the quadratic one or the modified

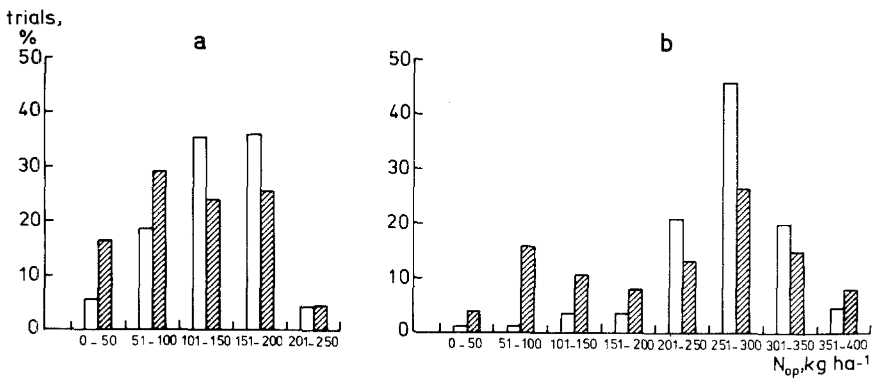

Fig. 1. Optimum application rate of fertilizer N (N_{op}) for sugar beet (a) and potato (b). N_{op} was determined on the basis of the quadratic response curve (white bars) and on the basis of the modified exponential response curve (dashed bars).

(a) Sugar beet. Number of trials: quadratic response curve = 144, modified exponential response curve = 133; monetary ratio = 0.0125.

(b) Potato. Number of trials: quadratic response curve = 86, modified exponential response curve 76; monetary ratio = 0.0075.

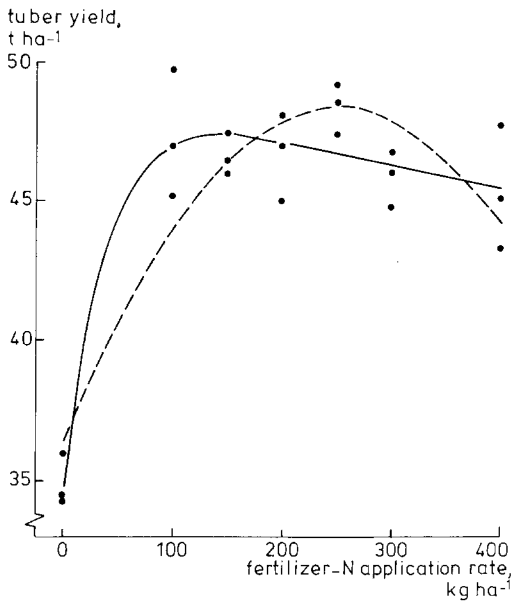

Fig. 2. Example of N-response curves for potato. Trial 60.
--- = quadratic response curve; — = modified exponential response curve.

Table 2. Average residual sum of squares (RSS) after fitting the quadratic and the modified exponential model. Sugar beet: 167 trials. Potato: 99 trials.

Model	Average RSS ($t^2 ha^{-2}$)	
	Sugar beet	Potato
Quadratic	358	206
Modified exponential	339	191

exponential one. As in most trials the modified exponential curve is the best fitting, it is obvious that the frequency distributions in Fig. 3 very much resemble those of the modified exponential curve in Fig. 1.

Confidence intervals ($p > 95\%$) of optimum application rate of fertilizer N

For each calculated optimum application rate of fertilizer N the confidence interval was determined to distinguish between the degrees of reliability of the optima obtained.

Figure 4 shows the frequency distribution of the magnitude of the confidence intervals associated with the optima taken from Fig. 3. The optimum application rate of fertilizer N for sugar beet generally had a much narrower confidence interval than that for potato. For instance, 39% of the sugar beet trials had a

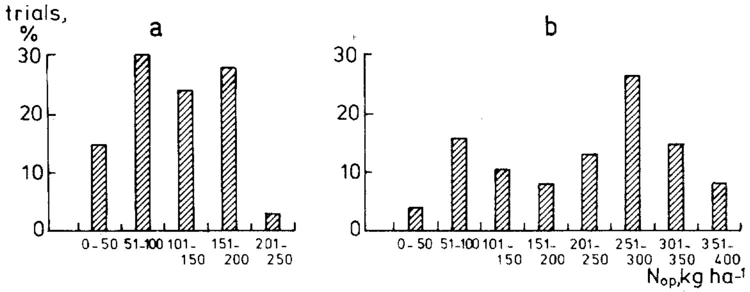

Fig. 3. Optimum application rate of fertilizer N (N_{op}) for sugar beet (a) and potato (b). N_{op} at each trial was determined on the basis of the best fitting response curve: quadratic or modified exponential. Sugar beet: 133 trials; monetary ratio = 0.0125. Potato: 76 trials; monetary ratio = 0.0075.

confidence interval of less than 100 kg ha^{-1}, whereas this was the case in only 26% of the potato trials. Apparently, the sugar beet trials yielded much more reliable results. It is striking that the confidence interval for the optimum application rate of fertilizer N was often more than 300 kg ha^{-1}. This occurred in 46% of the sugar beet trials and in 60% of the potato trials (Fig. 4). Obviously these optima cannot be used for developing fertilizer recommendations.

The absolute values of the confidence intervals of all sugar beet and potato trials, including those which were omitted in Fig. 1, are shown in Figs 5 and 6. In these figures only those parts of the confidence intervals are shown which lay within the relevant range of applied fertilizer N, viz., 0–250 kg ha^{-1} for sugar beet and 0–400 kg ha^{-1} for potato. Where no confidence interval is indicated the

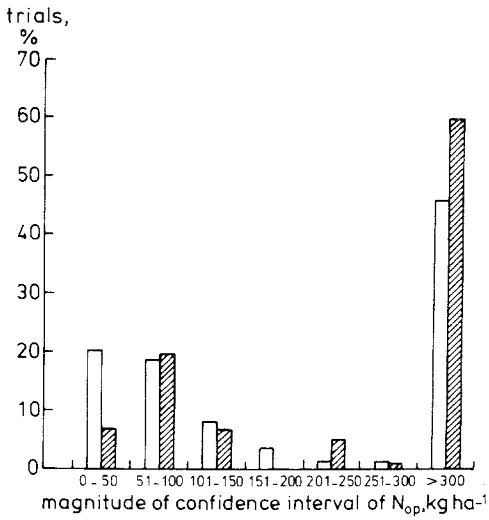

Fig. 4. Magnitude of confidence interval for optimum application rate of fertilizer N (N_{op}) for sugar beet (white bars) and potato (dashed bars). N_{op} at each trial was determined on the basis of the best fitting response curve: quadratic or modified exponential. Sugar beet: 133 trails; monetary ratio = 0.0125. Potato: 76 trials; monetary ratio = 0.0075.

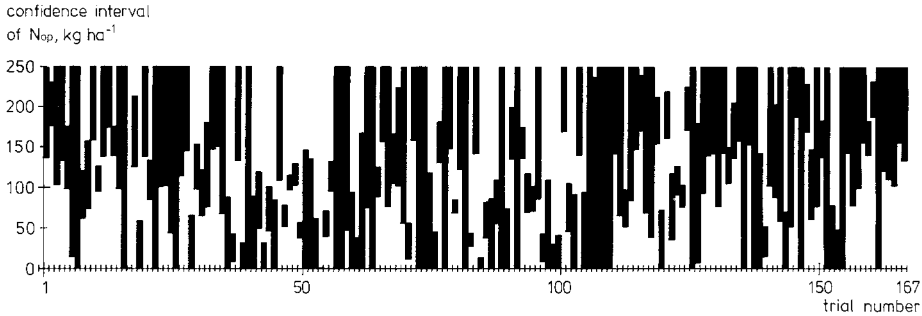

Fig. 5. Confidence interval (black bars) for optimum application rate of fertilizer N (N_{op}) for sugar beet. See Table 1 for trial numbers. Only those parts of the confidence intervals are shown which lay within the relevant range of applied fertilizer N: 0–250 kg ha^{-1}. N_{op} at each trial was determined on the basis of the best fitting response curve: quadratic or modified exponential. Number of trials = 167; monetary ratio = 0.0125.

confidence interval lay completely outside the range of the fertilizer-N levels tested. Figures 5 and 6 show that within the range of fertilizer-N levels tested the confidence intervals of the optima were also generally wide.

Discussion

Crop response to fertilizer N can be described by polynomials, inverse polynomials, exponentials, and split lines [38]. Comparison of a variety of models to response of cereals to fertilizer N showed that the model of two straight lines performed best [7, 30]. However, in the present study, which aims at describing responses of sugar beet and potato to N fertilizer, this model has not been used. The sharp break in the slope does not permit determination of the dependence of optimum N-fertilizer application rate on the monetary ratio of fertilizer cost to crop value. The model also requires many coefficients and there is an element of subjectivity in assessing the values. Moreover, the model does not appear to provide a sound basis for biological interpretation [10]. In the present paper responses of sugar beet and potato to N fertilizer are described by means of a quadratic and a modified exponential equation. The quadratic model was chosen because it has been widely used for various crops [1, 2, 13, 14, 16, 18, 23, 25, 27, 28, 29]. The modified exponential model was chosen because it was preferred to various other models in recent wheat research [10]. This was done because the model could describe the general form of N-response curves very well and required little computing time. To avoid non-linear regression analysis, which needs much computing time, George [10] and Sylvester-Bradley et al. [31] assumed a fixed value for the exponential coefficient of the modified exponential equation. In the present paper, however, the exponential coefficient first was fixed at several values to get more flexibility in the shape of the curve. Next the value which yielded the lowest residual sum of squares (RSS) was considered to be the best.

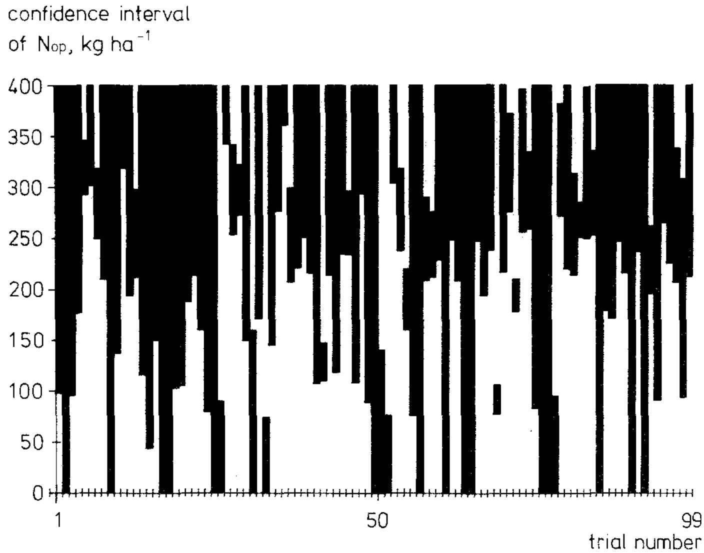

Fig. 6. Confidence interval (black bars) for optimum application rate of fertilizer N (N_{op}) for potato. See Table 1 for trial numbers. Only those parts of the confidence intervals are indicated which lay within the relevant range of applied fertilizer N: 0–400 kg ha^{-1}. N_{op} at each trial was determined on the basis of the best fitting response curve: quadratic or modified exponential. Number of trials = 99; monetary ratio = 0.0075.

Optimum application rate of fertilizer N

Use of quadratic or modified exponential N-response curves for sugar beet and potato resulted in rather large differences in the calculated optimum application rates of fertilizer N (Fig. 1). The optima were generally higher in the quadratic model. When the residual sum of squares, after fitting the models, was taken as a measure of best fit, the modified exponential model performed much better than the quadratic model (previous section on optimum application rate of fertilizer N). A disadvantage of the quadratic model is that the curve must be symmetrical around its maximum. This may lead to over-estimation of the maximum yield and under-estimation of yields at low and high fertilizer-N levels (Fig. 2). Obviously the modified exponential curve is more flexible in fitting the data points and thus gives better results. This conclusion agrees with results of Boyd et al. [7] and Sparrow [30].

A point that deserves attention is that the optimum application rate of fertilizer N for sugar beet was seldom higher than 200 kg ha^{-1}, whereas for potato it was usually higher than 200 kg ha^{-1} (Fig. 3). The N uptake by both crops, however, is more or less the same: 200–300 kg ha^{-1} [3, 9, 11, 12, 15, 17]. This apparent contrast may be explained by the ability of sugar beet to exploit the soil more effectively, probably due to a better root system [32]. Moreover, the growth period of sugar beet is about two months longer, which allows the crop to take up more soil nitrogen.

Reliability of optimum application rate of fertilizer N

In Western Europe N-fertilizer recommendations for arable crops are usually based on the relationship between soil mineral N and the optimum application rate of N fertilizer [4, 6, 8, 21, 22, 26, 36, 37]. It is striking in these papers that no attention was paid to the reliability of the optimum N application rates used for establishing the recommendations. The implication is that the recommendations were established with reliable and less reliable data which were counted equally. Confidence limits for optimum N application rates for sugar beet and potato, respectively, were calculated in the present study. Figures 4–6 show that the confidence intervals can frequently be very wide, especially for potato. The magnitude of the confidence interval depends on the variability between yields per replicate and on the shape of the curve, i.e. the significance of the coefficients b_1 and b_2 in equations (3) and (7). Confidence intervals for optimum N application rates are wide when differences among replications are large and/or the curves are flat and do not decline significantly beyond their maximum. Variability among replications depends on the homogeneity of the experimental field. Flat curves may occur when levels of soil mineral N are high before fertilizer is applied [26]. Incidence of diseases may also lead to flat curves, as shown by Dilz et al. [8]; crops not protected from diseases needed less fertilizer N. Extreme weather conditions, e.g. drought in the early stages of growth (esp. in potato) as well as heavy rainfall after fertilizer application may also decrease crop response to N fertilizer. An important reason why potato has a higher proportion of (very) wide confidence intervals for the optimum N-fertilizer application rate than sugar beet is that the N-response curves of potato generally do not decline beyond their maximum. Applying too much fertilizer N to potato seldom had a distinctly negative effect on tuber yield. The value of b_1 in equations (3) and (7) is therefore seldom significant. The value of b_1 for sugar beet is more likely to be significant as the yield, in this case root yield adjusted to a sugar content of 16%, generally decreases when too much fertilizer N is applied due to the negative relationship between amount of fertilizer N and sugar content.

N-fertilizer recommendations

In the foregoing it is shown that the modified exponential model yielded much better results than the quadratic model. This would suggest that the quadratic model should not be used and that the modified exponential model should be preferred. However, to take account of the (few) times that the quadratic model

Table 3. Average optimum application rate of fertilizer N (N_{op}) for sugar beet and potato.

Crop	Trials	Number of trials	N_{op} (kg ha^{-1})		
			Average	Lowest	Highest
Sugar beet	All trials*	133	111	3	243
	Trials with narrow confidence interval**	63	114	34	210
Potato	All trials*	76	219	1	395
	Trials with narrow confidence interval**	25	254	87	353

* Trials in which N_{op} lay within the range of fertilizer-N levels tested.
**Trials in which N_{op} lay within the range of fertilizer-N levels tested and in which the magnitude of the confidence interval of N_{op} did not exceed 150 kg ha^{-1} N.

is superior to the modified exponential model, it is proposed that both models be used for the data of each individual trial. The optimum application rate of fertilizer N and its confidence interval are then calculated on the basis of the best-fitting model using the least RSS as the criterion. This procedure yields for each trial a calculated optimum application rate of N fertilizer (at the current monetary ratio of fertilizer cost to crop value) and a measure of its reliability. Next, N-fertilizer recommendations can be established by using the reliable optima only. Finally, the recommendations developed in this way can be tested by checking whether the recommended N-fertilizer rate for each trial lies within the confidence interval of the calculated optimum. When the checks give satisfactory results the recommendations can be passed on to the farmer.

Care should be exercised to draw up recommendations only with the use of optima having narrow confidence intervals. It is quite possible that these optima lie within a limited range and do not reflect real variation. Low optima can be underestimated, because they are usually derived from flat curves and have, therefore, wide confidence intervals. Table 3 indicates that this possibly happened to potato (compare the average value of the optimum N-application rate in all trials with that in the trials with a narrow confidence interval). Notwithstanding these underestimations the optima with narrow confidence intervals covered a broad range of fertilizer-N levels (Table 3), thus enabling the establishment of fertilizer recommendations based on reliable data.

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References

1. Abshahi A, Hills FJ and Broadbent FE (1984) Nitrogen utilization by wheat from residual sugarbeet fertilizer and soil incorporated sugarbeet tops. Agron J 76:954–958
2. Adams RM, Farris PJ and Halvorson AD (1983) Sugar beet N fertilization and economic optima: Recoverable sucrose vs. root yield. Agron J 75:173–176
3. Armstrong MJ, Milford GFJ, Pocock TO, Last PJ and Day W (1986) The dynamics of nitrogen uptake and its remobilization during the growth of sugar beet. J Agric Sci Cambridge 107: 145–154
4. Bakker Y, Withagen L and Wijnen G (1981) De nieuwe richtlijn voor de stikstofbemesting van suikerbieten. Bedrijfsontwikkeling 12:383–385
5. Barber SA (1973) The changing philosophy of soil test interpretations. In: Soil testing and plant analysis. Rev. edn., pp 201–211. Madison, Wisc: Soil Science Society of America
6. Boon R (1981) Stikstofadvies op basis van profielanalyse voor wintergraan en suikerbieten op diepe leem- en zandleemgronden. Pedologie 21: 347–363
7. Boyd DA, Yuen LTK and Needham P (1976) Nitrogen requirement of cereals. 1. Response curves. J Agric Sci Cambridge 87:149–162.
8. Dilz K, Darwinkel A, Boon R and Verstraeten LMJ (1982) Intensive wheat production as related to nitrogen fertilisation, crop protection and soil nitrogen: Experience in the Benelux. Proc Fertiliser Society London, No. 211:93–124
9. Draycott AP (1972) Sugar-Beet Nutrition. London: Applied Science Publishers LTD
10. George BJ (1983) Design and interpretation of nitrogen response experiments. In: The nitrogen requirement of cereals. RB 385, pp 133–149. London: Min Agric Food Fish
11. Greenwood DJ, Draycott A, Last PJ and Draycott AP (1984) A concise simulation model for interpreting N-fertilizer trials. Fert Res 5:355–369
12. Greenwood DJ, Neeteson JJ and Draycott A (1985) Response of potatotes to N fertilizer: Quantitative relations for components of growth. Plant Soil 85:163–183
13. Haby VA, Simons C, Stauber MS, Lund R and Kresge PO (1983) Relative efficiency of applied N and soil nitrate for winter wheat production. Agron J 75:49–52
14. Hargrove WL, Touchton JT and Johnson JW (1983) Previous crop influence on fertilizer nitrogen requirements for double-cropped wheat. Agron J 75:855–859
15. Harris PM (1978) Mineral nutrition. In: Harris PM (ed.) The potato crop. The scientific basis for improvement, pp 195–243. London: Chapman and Hall
16. Hesterman OB, Shaeffer CC, Barnes DK, Lueschen WE and Ford JH (1986) Alfalfa dry matter and nitrogen production, and fertilizer nitrogen response in legume-corn rotations. Agron J 78:19–23
17. Hoffmann F and Ulbrich B (1986) Stickstoffaufnahme und -verwertung durch Zuckerrüben. Arch Acker-Pflanzenbau Bodenk 30:283–291
18. Holbrook JR, Osborne JD and Ridgman WJ (1982) An attempt to improve yield and quality of direct-drilled winter wheat grown continuously. J Agric Sci Cambridge 99:163–172
19. LEI (1985) Landbouwcijfers. 's Gravenhage: Landbouw-Economisch Instituut/Centraal Bureau voor de Statistiek
20. Neeteson JJ (1985) Effectiveness of the assessment of nitrogen fertilizer requirement for potatoes on the basis of soil mineral nitrogen. In: Neeteson JJ and Dilz K (eds) Assessment of nitrogen fertilizer requirement, pp 15–24. Haren (Gn): Inst Soil Fertility
21. Neeteson JJ and Smilde KW (1983) Correlative methods of estimating the optimum nitrogen fertilizer rate for sugar beet as based on soil mineral nitrogen at the end of the winter period. Proc symp 'Nitrogen and Sugar Beet', pp 409–421. Brussels: IIRB
22. Neeteson JJ, Wijnen G and Zandt PA (1984) Nieuwe stikstofbemestingsadviezen voor aardappelen. Bedrijfsontwikkeling 15:331–333
23. Ogunlela VB, Lombin GL and Abed SM (1982) Growth response, yield and yield components of upland cotton (Gossypium hirsutum L.) as affected by rates and time of nitrogen application in the Nigerian savannah. Fert Res 3:399–409

24. PAGV (1984) Kwantitatieve Informatie. Publ No 16. Lelystad: PAGV
25. Pal UR, Murari K and Malik HS (1984) Yield response of sorghum cultivars to inorganic nitrogen fertilizer. J Agric Sci Cambridge 102:7–10
26. Ris J, Smilde KW and Wijnen G (1981) Nitrogen fertilizer recommendations for arable crops as based on soil analysis. Fert Res 2:21–32
27. Sahota TS and Singh M (1984) Relative efficiency of N fertilizers as influenced by N-serve in the potato crop. Plant Soil 79:143–152
28. Sanford JO and Hairston JE (1984) Effects of N fertilization on yield, growth and extraction of water by wheat following soybeans and grain sorghum. Agron J 76:623–626
29. Sharma RC (1985) Nitrogen management of potatoes in the presence of farmyard manure and PK fertilizer on acid hill soil of Shimla. J Agric Sci Cambridge 107:15–19
30. Sparrow PE (1979) Nitrogen response curves of spring barley. J Agric Sci Cambridge 92:307–317
31. Sylvester-Bradley R, Dampney PMR and Murray AWA (1983) The response of wheat to nitrogen. In: The nitrogen requirement of cereals. RB 385, pp 151–174. London: Min Agric Food Fish
32. Van Noordwijk M, Floris J and De Jager A (1985) Sampling schemes for estimating root density distribution in cropped fields. Neth J Agric Sci 33:241–261
33. Van der Paauw F (1954) Evaluation of methods of soil testing by means of field experiments. Intern Soc Soil Sci. Joint Meeting, Dublin 1952. Transactions I. pp 207–221.
34. Visser WC (1942) Over de bruikbaarheid van de grafisch-statistische bewerkingstechniek. Landbkd Tijdschr 54:403–416
35. Visser WC (1943) Over eenige grondslagen van het grafisch-statistisch onderzoek. Landbkd Tijdschr 55:163–170
36. Wehrmann J and Scharpf HC (1979) Der Mineral-stickstoffgehalt des Bodens als Massstab für den Stickstoffdüngerbedarf (Nmin-Methode). Plant Soil 52:109–126
37. Wehrmann J and Scharpf HC (1980) Der Mineralstickstoffgehalt des Bodens als Grundlage der Stickstoffdüngung bei Zuckerrüben. Proc 43th Winter-congress, pp 327–341. Brussels: IIRB
38. Wood J (1980) The mathematical expression of crop response to inputs. In: Physiological aspects of crop productivity, pp 263–271. Berne: Int Potash Inst

Appendix

Calculation of the confidence interval for the optimum application rate of fertilizer

1. The quadratic response function

The economic response μ_x is given by:

$$\mu_x = \beta_0 + (\beta_1 - P)\cdot x + \beta_2\cdot x^2 + e_r, \qquad (1)$$

in which

β_i = parameter to be estimated from the experimental data,

x = fertilization,

P = price ratio of one unit of μ per unit of area to one unit of x per unit of area,

e_r = normally distributed residual error.

The expected economic response $\hat{\mu}_x$ is given by:

$$\hat{\mu}_x = b_0 + (b_1 - P)\cdot x + b_2\cdot x^2, \tag{2}$$

in which

b_i = unbiased estimate of β_i calculated by linear regression analysis.

Consider:

$$\Phi_x \equiv \frac{\partial \mu_x}{\partial_x} = (\beta_1 - P) + 2\cdot\beta_2\cdot x \tag{3}$$

μ_x has an extreme for $x = x_e$ if Φ_x is not significantly different from zero.
It then follows that Φ_x is normally distributed and μ_{x_e} is an extreme if:

$$-t^{\gamma/2}_{n-3}\sqrt{\mathrm{var}(\hat{\Phi}_{x_e})} \leqslant \hat{\Phi}_{x_e} \leqslant t^{\gamma/2}_{n-3}\sqrt{\mathrm{var}(\hat{\Phi}_{x_e})} \rightleftharpoons \tag{4}$$

$$\hat{\Phi}^2_{x_e} \leqslant [t^{\gamma/2}_{n-3}]^2\,\mathrm{var}(\hat{\Phi}_{x_e}) \tag{5}$$

in which

$\hat{\Phi}_{x_e}$ = unbiased estimated of Φ_{x_e},

$\mathrm{var}(\hat{\Phi}_{x_e})$ = variance of $\hat{\Phi}_{x_e}$,

$t^{\gamma/2}_{n-3}$ = student-t value at confidence level $\gamma/2$ with $n - 3$ degrees of freedom,

n = number of experimental observations.

p, β, b, and C are defined as follows:

$$p \equiv (0, 1, 2\cdot x)' \tag{6}$$

$$\beta \equiv (\beta_0, \beta_1, \beta_2)' \tag{7}$$

$$b \equiv (b_0, b_1, b_2)' \tag{8}$$

$$C \equiv \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix} \tag{9}$$

in which

C = the estimated covariance matrix

and

c_{ij} = the estimated covariance of b_i and b_j.

It then follows that:

$$\hat{\Phi}_{x_e} = \widehat{(p, \beta)} = (p, b) = (b_1 - P) + 2\cdot b_2\cdot x_e \tag{10}$$

and

$$\mathrm{var}(\hat{\Phi}_{x_e}) = \mathrm{var}((p, b)) = p'\cdot C\cdot p = c_{11} + 4\cdot x_e\cdot c_{12} + 4\cdot x_e^2\cdot c_{22} \tag{11}$$

in which

$\widehat{(p, \beta)}$ = unbiased estimate of (p, β).

Combining the equations (5), (10) and (11) yields:

$$Z(x_e) \leqslant 0 \tag{12}$$

$$Z(x) \equiv a_2 \cdot x^2 + a_1 \cdot x + a_0 \tag{13}$$

$$a_2 \equiv 4 \cdot b_2^2 - 4 \cdot \tilde{c}_{22} \tag{14}$$

$$a_1 \equiv 4 \cdot (b_1 - P) \cdot b_2 - 4 \cdot \tilde{c}_{12} \tag{15}$$

$$a_0 \equiv (b_1 - P)^2 - \tilde{c}_{11} \tag{16}$$

$$\tilde{c}_{ij} \equiv c_{ij} \cdot [t_{n-3}^{\gamma/2}]^2 \tag{17}$$

$$D \equiv a_1^2 - 4 \cdot a_2 \cdot a_0, \tag{18}$$

in which

D = the discriminant of $Z(x) = 0$.

In Williams (1959, p. 111) the equation

$$Z(x) = 0 \tag{19}$$

is given for calculating the limits of the confidence interval (for $P = 0$).

By evaluating the sign of $Z(x)$ a γ confidence interval of x_e can be established (equation 12). The sign of $Z(x)$ is both dominated by the sign of a_2 and that of D. If $Z(x_{\min}) = 0$ and $Z(x_{\max}) = 0$ and $x_{\min} < x_{\max}$ (this means that $D > 0$), then

$$x_{\min} \leqslant x_e \leqslant x_{\max} \quad \text{only when} \quad a_2 > 0 \tag{20}$$

and

$$x_e \leqslant x_{\min} \quad \text{and} \quad x_e \geqslant x_{\max} \quad \text{only when} \quad a_2 < 0. \tag{21}$$

Only the values of x_e belonging to β_2 values which are < 0 ($\partial^2\mu/\partial x^2 < 0$, convex response function) are optimum x values (those fertilizer applications give a maximum financial return, i).

It should also be noted that

$$a_2 > 0 \rightleftharpoons \beta_2 \neq 0 \quad \text{(at confidence level } \gamma\text{, ii)}$$

(cf equation 14).

From (i), (ii) and equation (20) it can be concluded that a closed interval of optimum x values is established only when β_2 is significantly < 0.

2. The modified exponential response function

The economic response is given by:

$$\mu_x = \beta_0 + (\beta_1 - P) \cdot x + \beta_2 \cdot e^{a \cdot x} + e_r \tag{22}$$

in which a is a constant and P, e_r, β_i and μ_x have the same meaning as in equation (1).

When the same procedure as for the quadratic response function is followed it can be shown that μ_x has an extreme for $x = x_e$ when:

$$E(z_e) \leqslant 0 \tag{23}$$

$$E(z) \equiv e_2 \cdot z^2 + e_1 \cdot z + e_0 \tag{24}$$

$$z \equiv a \cdot \mathrm{e}^{a \cdot x} \tag{25}$$

$$e_2 \equiv b_2^2 - \tilde{c}_{22} \tag{26}$$

$$e_1 \equiv 2 \cdot (b_1 - P) \cdot b_2 - 2 \cdot \tilde{c}_{12} \tag{27}$$

$$e_0 \equiv (b_1 - P)^2 - \tilde{c}_{11} \tag{28}$$

$$D \equiv e_1^2 - 4 \cdot e_0 \cdot e_2 \tag{29}$$

in which

D = the discriminant of $E(z) = 0$

and $\tilde{c}_{ij}$ has the same meaning as in equation (17).

By evaluating the sign of $E(z)$ a γ confidence interval of x_e can be established. Again only the values of x_e belonging to negative β_2 values are optimum x values ($\partial^2 \mu_x / \partial x^2 < 0$). Because of the transformation of x_e to z_e a closed interval of z_e could lead to an interval of x_e in which the limits are exclusive (that means that infinity is included in the interval).

A closed interval of optimum x values is established only when β_2 is significantly negative and the sign of $(P - \beta_1)$ is significantly opposite to that of a.

Ref: Williams EJ (1959) Regression analysis. New York: John Wiley.