ABHAYA C. NAYAK

FOUNDATIONAL BELIEF CHANGE

ABSTRACT. This paper is concerned with the construction of a base contraction (revision) operation such that the theory contraction (revision) operation generated by it will be fully AGM-rational. It is shown that the theory contraction operation generated by Fuhrmann's *minimal base contraction operation*, even under quite strong restrictions, fails to satisfy the "supplementary postulates" of belief contraction. Finally Fuhrmann's construction is appropriately modified so as to yield the desired properties. The new construction may be described as involving a modification of safe (base) contraction so as to make it maxichoice.

Key words. belief, change, contraction, revision, base, theory.

We often change our beliefs. We learn new things, occasionally things that conflict with our current beliefs. On such occasions new beliefs replace the old ones. It is as if this process is completed in two steps: (1) first we identify and throw out the beliefs that conflict with the new information *and then* (2) we accept the new information. In the literature (1) is referred to as the problem of "belief contraction", and (2) as the problem of "belief expansion". The combination of (1) followed by (2) is called "belief revision". Though this account of belief change is very intuitive,[1] its logic is not understood very well. If it is assumed that a rational epistemic (doxastic) agent wants to minimize unnecessary loss of information, then belief contraction becomes a very difficult problem.[2] This paper is about belief revision seen from a foundationalist perspective.

We show that in order to satisfy the Gärdenfors postulates for belief revision in a foundationalist framework, we need to radically revise Fuhrmann's [5] construction of a reject-set. This result makes an interesting connection between Nebel's [18] theory revision based on *maxichoice base revision* and his [20] *unambiguous partial meet revision*. This contraction operation also bridges the gap between Fuhrmann's [4, 5] minimal theory revision approach and Nebel's approach.

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1. BACKGROUND

We assume that the objects of beliefs are propositions, which may be represented as equivalence classes of sentences, or as sets of worlds. The believer's language $\mathscr{L}$ is assumed to contain the usual propositional connectives $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$. We represent the individual beliefs (propositions) by the lower-case Roman letters with or without subscripts and superscripts, and sets of beliefs (propositions) by upper-case Roman letters, again with or without subscripts and superscripts. *Cn* is used to denote the consequence operation of the believer's logic. The yielding relation $\vdash$ between a set of propositions A and a proposition a is defined in term of *Cn* as

$$A \vdash a \quad \text{iff} \quad a \in Cn(A)$$

We take *Cn* to be a mapping from sets of propositions to sets of propositions with the following properties: for all sets A and B of propositions,

$$A \subseteq Cn(A) \quad \text{(inclusion)}$$

$$Cn(A) = Cn(Cn(A)) \quad \text{(iteration, or idempotence)}$$

$$Cn(A) \subseteq Cn(B) \quad \text{whenever} \quad A \subseteq B \quad \text{(monotonicity).}$$

Furthermore we assume *Cn* to include tautological implication, to satisfy the rule of *introduction of disjunction in the premises*, i.e., if $x \in Cn(A \cup \{y\})$ and $x \in Cn(A \cup \{z\})$ then $x \in Cn(A \cup \{y \vee z\})$[3] and to be *compact*, i.e., if $x \in Cn(\Gamma)$ for a set of propositions, Γ, then there is a finite subset Γ^* of Γ such that x is in $Cn(\Gamma^*)$.

Throughout this paper K denotes the belief set (also called "knowledge corpus" or "theory") of our doxastic agent. K is always taken to be closed under *Cn*, i.e., $Cn(K) \subseteq K$, and usually consistent. We also assume that there is a *contextually fixed* set of basic beliefs that generates K. We take this set of basic beliefs to be finite, and it is denoted by B. Hence

DEFINITION 1.1. $K = Cn(B)$.

In other words, when we talk of a belief set K, we really talk of the pair $\langle K, B \rangle$ where B is the contextually determined base of K. When required, we use superscripts or subscripts on K or B to denote other belief sets or belief bases.

1.1. *The Gärdenfors Postulates*

By a *belief removal operation* we will understand an operation that effectively removes an undesirable element from a belief set. In the AGM system[4] a well-behaved belief removal operation is required to satisfy the following set of six "basic" postulates introduced by Gärdenfors. In these postulates (and later), for any belief x, K_x^- stands for the result of removing the belief x from the belief set K. Following Makinson [17], we call a belief removal operation $-$ that satisfies the first five postulates a *belief withdrawal operation*, and if $-$ satisfies all these six postulates, we call it a *belief contraction operation.*

(1–)	K_a^- is a theory if K is	(closure)
(2–)	$K_a^- \subseteq K$	(inclusion)
(3–)	If $K \nvdash a$ then $K_a^- = K$	(vacuity)
(4–)	If $\emptyset \nvdash a$ then $a \notin Cn(K_a^-)$	(success)
(5–)	If $Cn(\{a\}) = Cn(\{b\})$ then $K_a^- = K_b^-$	(preservation)
(6–)	$K \subseteq Cn((K_a^-) \cup \{a\})$ when K is a theory	(recovery)

Corresponding to the six basic postulates of contraction, there are the Gärdenfors postulates of revision. In these postulates K_x^* stands for the belief set that results from revising the existing belief set K by the new belief x, (i.e. suitably "adding" the new belief x to K). It must be noted that x is possibly inconsistent with K. In (5*) we use $K_\perp$ to denote the absurd belief set in $\mathscr{L}$.

(1*)	K_a^* is a theory
(2*)	$a \in K_a^*$
(3*)	$K_a^* \subseteq Cn(K \cup \{a\})$
(4*)	If $K \nvdash \neg a$ then $Cn(K \cup \{a\}) \subseteq K^*a$
(5*)	$K_a^* = K_\perp$ iff $\vdash \neg a$
(6*)	If $\vdash a \leftrightarrow b$, then $K_a^* = K_b^*$

Besides these basic postulates of contraction and revision, Gärdenfors has put forth a pair of supplementary postulates for each of these operations. They are:

(7–) $(K_a^-) \cap (K_b^-) \subseteq K_{(a\wedge b)}^-$ if K is a theory

(8–) If $a \notin K_{(a\wedge b)}^-$ then $K_{(a\wedge b)}^- \subseteq K_a^-$ for any theory K

(7*) $K_{(a\wedge b)}^* \subseteq Cn(K_a^* \cup \{b\})$ if K is a theory

(8*) If $\neg b \notin K_a^*$ then $Cn(K_a^* \cup \{b\}) \subseteq K_{(a\wedge b)}^*$

Motivations for these postulates can be found in [6].

As we indicated before, belief revision is normally seen as a sequential combination of two operations. When we come up with new observations, first we check whether they can be consistently accommodated into our current belief corpus. If it is so, we include the new data into our belief corpus and let the background logic take care of the rest. On the other hand, if the new data is not consistent with our current belief corpus, then we suitably contract the latter so as to make it consistent with the new data before we include the new data into it. This intuitive process is captured by the following identity named after Isaac Levi:

- Levi Identity: $K_a^* = Cn(K_{\neg a}^- \cup \{a\})$.

There are many ways of constructing the revision operations, the two best known among them being the "partial meet revision" and the "safe revision" operations.[5] In partial meet contraction a "choice function" γ picks out a suitable subset of $K \perp x$, which is the family of maximal subsets of K that fail to yield x. Then K_x^- is identified with the set of those beliefs that are common to all members of $\gamma(K \perp x)$. The revision operation is defined from this contraction operation via the Levi Identity. Here the crucial factor is what subset of $K \perp x$ the choice function γ picks out. It has been shown that, given that the background logic is supraclassical, every partial meet contraction (revision) operation satisfies the six basic postulates of contraction (revision). It has also been conversely shown that every contraction (revision) operation that satisfies the six basic postulates of contraction (revision) is a partial meet contraction (revision) operation. The more interesting case is when γ picks out the "best" elements of $K \perp x$. If there is a preference relation $\leq$ over 2^K and γ picks out the $\leq$-best elements of $K \perp x$ (i.e. γ is relational over K) then the (relational) partial meet contraction (revision) operation determined by γ will satisfy 7– (7*). If the preference relation in question is transitive, then γ is said to be transitively relational over K,[6] and the partial meet contraction (revision) operation determined by it is said to be a transitively relational partial meet contraction (revision) operation. It has been established that an operation on the belief set K is a transitively relational partial meet contraction (revision) operation if and only if it satisfies all the eight Gärdenfors postulates for contraction (revision). (For proof, see [1].)

Safe revision uses a method that is, so to say, the dual of that used in partial meet revision. Whereas partial meet revision uses as a tool the maximal subsets of K that fail to yield x, safe revision uses as a tool the minimal subsets of K that imply x. Let $<$ be a non-circular ordering[7] over K. Let $E'(x)$ be the set of minimal subsets of K that imply x and $R'(x)$ the set of $<$-minimal elements in the members of $E'(x)$. Then $K\backslash R'(x)$ is the subset of K whose members are *safe* with respect to x (not to be blamed for the implication of x). Accordingly, K_x^- is identified with $Cn(K\backslash R'(x))$ and revision is defined from it via the Levi Identity. It turns out that safe contraction satisfies the six basic postulates of contraction. Further conditions on the $<$-relation ensure that the supplementary postulates are satisfied. We must, however, remind that whereas partial meet revision uses a preference relation over subsets of a belief set, safe revision uses a preference relation over the elements of a belief set. (For details, see [1] or [3].)

1.2. *Belief Change: Coherence vs. Foundational Theory*

According to a dichotomy introduced by Gilbert Harman [8], the AGM approach is coherentist in character. In the AGM system, though the beliefs are graded according to their epistemic importance (or entrenchment), all beliefs are taken to be equally fundamental. Thus the distinction between the *basic* beliefs and *inferred* beliefs is allegedly obliterated.[8] On the other hand, Alchourrón and Makinson [2], André Fuhrmann [4, 5], Sven Ove Hansson [9, 10, 11, 13], David Makinson [17] and Bernhard Nebel [18, 19, 20] have emphasized the importance of keeping the basic beliefs separated from the inferred ones while updating beliefs. Fuhrmann, Hansson and Nebel have studied how the belief base is affected (or should be affected) when a belief set incorporates new information. In their works, the new belief set is determined by first determining what the new belief base would be. Following Harman's lead, we call this approach the foundationalist approach.

Suppose that K is a belief set and B a belief base such that $K = Cn(B)$. Let μ and ν be operations, respectively on B and K, such that $Cn(\mu(B,x)) = \nu(K,x)$. When the theory operation ν is thus determined by the base operation μ, we will call ν a foundational operation. Furthermore, if the foundational operation ν is a theory removal (revision) operation, we call the corresponding base operation μ a base removal (revision) operation.[9] When ν is determined by μ in the above manner, we will call ν a foundational removal (revision) operation. It must be noted that foundational operations are operations on theories or belief sets whereas base operations are operations on belief bases.

2. FOUNDATIONAL THEORY DYNAMICS

Corresponding to the two approaches to coherentist theory dynamics, there are two approaches to the foundationalist theory dynamics. Nebel and Hansson construct the base removal (revision) operations out of the maximal subsets of B that fail to entail the undesired belief a, and the foundational theory removal (revision) operations out of the base removal (revision) operations. Hence their approaches are related to the partial meet revision. Fuhrmann, on the other hand, constructs the base removal (revision) operations out of the minimal subsets of B that entail the undesired belief a, and the foundational theory removal (revision) operations out of such base operations. Hence his approach is closely related to the safe revision approach. Our account below is motivated by Fuhrmann's approach.

2.1. *Minimal Theory Revision Based on a Choice Function*

Suppose that a doxastic agent is in a belief state represented by K with base B whereby he believes in the proposition a. Then he finds some contravening evidence so that he decides to dump his belief a. What does he do? He cannot dump a without making some corresponding adjustments in B, since $B \vdash a$. One might think that the obvious way is to find the smallest subset of B that yields a, and throw out the least important (whatever that may mean) members of that set. We can define this set as the set S such that

$$S \subseteq B \wedge a \in Cn(S) \wedge \forall_{S' \subset S}(a \notin Cn(S')).$$

But this procedure might completely fail, since there is no guarantee that S will be unique. Hence we define $E(a)$, the set of entailment sets of a as:

DEFINITION 2.1. $E(a) = \{S : S \subseteq B \wedge a \in Cn(S) \wedge \forall_{S' \subset S}(a \notin Cn(S'))\}$.

Thus, $E(a)$ is defined as the set of the inclusion-minimal subsets of B that entail a. Here we mention the following result that we will need later.[10]

OBSERVATION 2.1. *For any proposition a and any set of propositions S, if $a \in Cn(S)$, then there is a subset S^* of S such that $S^* \in E(a)$.*

It is clear that the basic beliefs to be discarded are those that are least important in some member of $E(a)$ or other. This leads Fuhrmann to define a relation, called *comparative retractability relation*, over B.[11] However, for more generality, we will assume that there is a choice function C defined over 2^B, which, from any subset of B picks out its most rejectable elements. We will call it a *rejector*.

DEFINITION 2.2. For any subset S of B, $C(S)$ is a subset of S.

Note that as the above definition stands, $C(S)$ is possibly empty even if S is non-empty. Later on we will require $C(S)$ to be non-empty if S is.

Now it is only natural to expect that in order to give up the belief a, we must reject the most rejectable members of different entailments sets in $E(a)$. Hence the reject-set $R_0(a)$ of a is defined as the set of all such basic beliefs:

DEFINITION 2.3. $R_0(a) = \{x : \exists_{S \in E(a)}(x \in C(S))\}$.

It may be noted that if there is more than one most rejectable member in a set in $E(a)$, then all such rejectable elements find their way to $R_0(a)$. Now we can define $\ominus_f$, corresponding to Fuhrmann's *minimal base contraction operation*, and the foundational theory removal operation $-_{fw}$, corresponding to Fuhrmann's *minimal theory contraction operation*, as follows:

DEFINITION 2.4. $B \ominus_f a = B \backslash R_0(a)$

DEFINITION 2.5. $K -_{fw} a = Cn(B \ominus_f a)$.

Similarly, we can define $\oplus_f$ and $+_f$, corresponding to Fuhrmann's *minimal base revision operation* and *minimal theory revision operation* respectively.

DEFINITION 2.6. $B \oplus_f a = (B \ominus_f \neg a) \cup \{a\}$

DEFINITION 2.7. $K +_f a = Cn(B \oplus_f a)$.

It can be easily verified that the foundational operation $-_{fw}$ satisfies (1–), (2–), (3–) and (5–). Furthermore, if the rejector C satisfies the following condition:

- *Success*: If $S \neq \emptyset$ then $C(S) \neq \emptyset$

then $-_{fw}$ satisfies (4–) too. The above condition on C is dubbed "Success" after the AGM name for (4–). That satisfaction of (4–) makes the imposition of *Success* on C almost necessary is shown by the theorem 4.1. However, even in presence of this condition on C, *Recovery* is not satisfied by $-_{fw}$. In other words, in presence *Success*, $-_{fw}$ is a withdrawal operation, but not a contraction operation. (Hence the subscript w in $-_{fw}$.)

We must emphasize that Fuhrmann, unlike us, does not take Cn to sanction the $\vee$-introduction in the premises. Either way, given that C satisfies *Success*, $-_{fw}$ is as well-behaved as Fuhrmann's *minimal theory contraction* is: both are only withdrawal operations. In the framework of [4] he shows the *minimal theory contraction operation* to violate (7–), but leaves it as an open problem whether (8–) is satisfied or not. In [5] he does not consider the question whether the supplementary postulates of contraction are satisfied by this operation. We will show in the sequel that $-_{fw}$ does not satisfy either (7–) or (8–), even when quite strong conditions are imposed on C. However, as we will see, the counter-examples in question will suggest a new way to construct a foundational contraction operation that satisfies *recovery*, (7–) and (8–).

2.2. *Recovery Regained*

Recovery requires that when we give up a belief x, we give up at most as much information as we would regain if we were to reintroduce x to our truncated belief set. The following example illustrates that this postulate is very appealing:

> I believe silicon based breast implants cause cancer, and hence, Ms. Maple, who is otherwise healthy but has such implants, is likely to get cancer. In the morning newspaper I read a news item that according to the Food and Drug Administration there is no reason to think that silicon based breast implants are dangerous. However, in the evening TV news it was announced that the news item in question was a misinformation campaign orchestrated by the breast implant industries.

After reading the newspaper in the morning, I give up the belief that silicon based breast implants cause cancer. However, I cannot consistently do that while believing that Ms. Maple, who is otherwise healthy, has such implants and will get cancer. So I give up the belief that Ms. Maple will get cancer. (Possibly I give up the belief that she has breast implants, too!) What should I do after listening to the evening news? Of course, if I do not take the evening news to be reliable, that is a different story. But, given that I accept the evening news, I have an epistemic obligation to recover the beliefs that I lost in the process – I should believe that Ms. Maple has breast implants, and that she is likely to get cancer. This is exactly what *recovery* mandates! Giving up recovery amounts to unnecessary and substantial loss in information.

However, Hansson [10, 11], Levi [14] and Niederée [21] have vehemently argued that *recovery* is very counterintuitive. Instead of taking up the issue here, we just note that it is not very difficult to revise the operation $-_{fw}$ so that the ensuing foundational removal operation will satisfy recovery. What we need to save recovery is to appropriately weaken the members of $R_0(a)$ instead of completely rejecting them. Toward this end we define a new base operation $\dot{-}_f$:

DEFINITION 2.8. $B \dot{-}_f a = (B \ominus_f a) \cup \{a \rightarrow x : x \in R_0(a)\}$

and accordingly a new foundational operation $-_{fc}$:

DEFINITION 2.9. $K -_{fc} a = Cn(B \dot{-}_f a)$.

These definitions are motivated by similar constructions by Nebel [18, 19]. Clearly, if K_a^- is equated with the $K -_{fc} a$, then it will satisfy the postulate of recovery. Furthermore, if C satisfies *Success*, then $-_{fc}$ will be a foundational *contraction* operation, and thereby, a partial meet contraction operation.

Interestingly, as Makinson [17] and Nebel [18, 19] observe in frameworks different from ours, if our interest is only in the *revision* of K, then it does not really matter whether we use $\ominus_f$ or $\dot{-}_f$ to generate the relevant foundational *revision* operation. We get the same operation in either case.

3. CONDITIONS ON THE REJECTOR

Choice functions like the rejector have been well studied in rational choice theory. Sten Lindström [15], in a completely different framework, has examined the consequences on belief revision and nonmonotonic reasoning of imposing different conditions on a choice function. The following conditions (except *Credulity* and β+) give a partial list of them. For all subsets X, Y of B,

(1)	If $X \neq \emptyset$, then $C(X) \neq \emptyset$	(Success)
(2)	If $x \in C(X)$ and $y \in C(X)$, then $x = y$	(Credulity)
(3)	$C(C(X)) = C(X)$	(Iteration)
(4)	If $C(X) \subseteq Y \subseteq X$, then $C(X) \subseteq C(Y)$	(Cut)
(5)	$C(X \cup Y) \subseteq C(X) \cup C(Y)$	(Distributivity)
(6)	$C(X) \cap Y \subseteq C(X \cap Y)$	(α, Chernoff)
(7)	If $X \subseteq Y$ and $C(X) \cap C(Y) \neq \emptyset$, then $C(X) \subseteq C(Y)$	(β, Sen)
(8)	If $X \subseteq Y$ and $X \cap C(Y) \neq \emptyset$, then $C(X) \subseteq C(Y)$	(β+)
(9)	If $C(X) \subseteq Y \subseteq X$, then $C(Y) \subseteq C(X)$	(Aizerman)
(10)	$C(C(X) \cup C(Y)) = C(X \cup Y)$	(Path Independence)
(11)	If $C(X) \cap Y \neq \emptyset$, then $C(X \cap Y) = C(X) \cap Y$	(Arrow).

Lindström calls *Success* "Consistency Preservation", but here it is called "Success" because, as mentioned earlier, it leads to the satisfaction of (4–) which is called "Success" by Gärdenfors. The second condition in the above list is called "Credulity" because it leads to what is called *credulous reasoning*, as opposed to *skeptical reasoning*, as defined in [24].

We will often use the following equivalent formulation of α given in [23]:

ALPHA. If S and T are subsets of B such that $S \subseteq T$, $x \in S$ and $x \in C(T)$, then $x \in C(S)$

Occasionally, we will refer to the former formulation of α as α_1 and the latter by α_2. It is easy to see that these two formulations are equivalent. To see that the earlier formulation entails the latter, suppose that $S \subseteq T$, $x \in C(T)$ and $x \in S$. Clearly then, $x \in C(T) \cap S$, from which, by the application of the former formulation it follows that $x \in C(T \cap S) = C(S)$. To see that the converse holds, suppose that $x \in C(X) \cap Y$. Clearly, $x \in C(X)$ and $x \in (X \cap Y) \subseteq X$. Applying the latter formulation we get $x \in C(X \cap Y)$.

The combination of *Success, Credulity* and α is rather special: as Theorem (3.1) shows, in presence of *Credulity* and *Success*, every other condition on C listed above is either equivalent to or entailed by α.

THEOREM 3.1. *In presence of Success and Credulity, condition α entails Iteration, Cut, Distributivity and β, and is equivalent to $\beta+$, Aizerman, Path Independence and Arrow.*

Hence, imposing these three conditions on a choice function amounts to imposing all of the eleven conditions above. It turns out that (see § 4.1) it is almost necessary to impose these conditions on C in order to construct a foundational removal operation which is fully "rational" in the AGM-sense. We will eventually show that if the rejector C satisfies the three conditions *Success*, *Credulity* and α, then a foundational contraction operation can be constructed based on $\ominus_f$ that satisfies (7–) and (8–).

4. THE SUPPLEMENTARY POSTULATES

The supplementary postulates (7–) and (8–) are rather strong conditions on a belief removal operation. This has led many writers like Fuhrmann

[4] and Levi [14] to reject these postulates. I, however, find these postulates very intuitive. (7–) looks impeccable. It essentially says that

- *If neither giving up* a *nor giving up* b *forces one to give up* c, *then one should not give up* c *by giving up the conjunction of* a *and* b.

The motivation for this condition should be obvious. In order to give up a conjunction it would suffice to give up one of the conjuncts. Since one should not be wasteful with information, one should not lose more information than what would suffice to ensure that the conjunction of a and b is given up. So, in either case, whether one loses a or one loses b in order to give up this conjunctive belief, one should still retain the belief c.

Similarly, (8–) is also extremely compelling. It may be paraphrased as:

- *If one gives up* a *by giving up the conjunction of* a *and* b, *and gives up* c *by giving up* a, *then one should give up* c *by giving up the conjunction of* a *and* b.

Suppose that you believe in both a and b. Now you are asked to give up your belief in $a \wedge b$, and you find yourself compelled to reject your belief in a. Why? Intuitively, it is necessary to get rid of one of the beliefs a and b in order to discard $a \wedge b$; and since you are discarding a willy-nilly, a must be at least as dispensable as b. Now that you discard a in order not to believe in $a \wedge b$, you must discard along with it the minimum that must be given up in order not to believe in a. You might discard more for some inexplicable reason, but it is the minimum that you must do. That is, you may not, while discarding $a \wedge b$, retain a belief which you must give up in order to dump a. That is exactly what postulate (8–) says: $K^-_{a \wedge b}$ has to be a subset of K^-_a.

Thus we see that the two supplementary postulates (7–) and (8–) are quite compelling. We will show in this section that if the foundational operations $-_{fw}$ and $-_{fc}$ are to satisfy the supplementary postulates (7–) and (8–), then it is almost necessary to impose *Credulity* and α on C. Furthermore, we also show that satisfaction of (4–) nearly necessitates the imposition of *Success* on C.

However, as it turns out, *Success*, *Credulity* and α are not enough to guarantee that either $-_{fw}$ or $-_{fc}$ will satisfy the supplementary postulates. We examine one more plausible-looking constraint F on C, but it leads to a triviality result. We argue in the next section (§ 5) that the construction of these removal operations is basically flawed and needs drastic repair.

Before we proceed any further, we need to define the concepts of *strong independence* and *partial independence* (a set of propositions being independent up to one of its subsets).

DEFINITION 4.1. A set of propositions S is (weakly) independent iff

$$\forall_{s \in S}[(S \setminus \{s\}) \nvdash s \text{ and } (S \setminus \{s\}) \nvdash \neg s].^{12}$$

DEFINITION 4.2. A set of propositions S is strongly independent iff

$$\forall_{\text{finite } S' \subseteq S}[(S \setminus S') \nvdash \vee(S')].^{13,14}$$

DEFINITION 4.3. T is weakly independent up to its subset S iff

$$\forall_{s \in S}[(T \setminus \{s\}) \nvdash s \text{ and } (T \setminus \{s\}) \nvdash \neg s].$$

DEFINITION 4.4. T is strongly independent up to its subset S iff

$$\forall_{\text{finite } S' \subseteq S}[(T \setminus S') \nvdash \vee(S')].$$

It is easily seen that the regular independence (call it "weak independence") is a special case of strong independence – namely, when S' ranges over those subsets of S that have less than two members. Note that in the two extreme cases – when S' is the null-set and when S' is S itself – strong independence degenerates to the conditions, respectively, that S be consistent, and that every member of S be informative. The set $\{p \vee q \vee r, p \vee q \vee s, p \vee r \vee s\}$ where $\{p, q, r, s\}$ is a partition, is consistent and weakly independent, but not strongly independent. On the other hand, $\{p \vee q \vee r \vee t, p \vee q \vee s \vee u, p \vee r \vee s \vee v\}$ is a consistent and strongly independent set of proposition, given that $\{p, q, r, s, t, u, v\}$ is a proper subset of a partition. In general, in order to construct a strongly independent set with n members we need a partition with at least 2^n members.

Another way of stating the difference between strong independence and weak independence is this: A set S is weakly independent iff S is consistent and every member s of S is such that $\neg s$ is consistent with $\wedge(S \setminus \{s\})$, i.e., if you replace any *one* member of S by its negation, you get a consistent set. On the other hand, S is strongly independent iff S is consistent and every subset S' of S is such that $\neg \vee (S')$ is consistent with $\wedge(S \backslash S')$, i.e., if you replace *any number* of members of S by their contradictories, you will still get a consistent set. This explication presupposes that S is finite.

In case S has just two members x and y, strong independence means that neither x nor its negation $\neg x$ follows either from y or from its negation $\neg y$. Thus strong independence is at least as strong as pair-wise independence.

4.1. *Why Constrain C?*

We have seen in § 3 that the combination of *Success*, *Credulity and* α is a very strong condition on C. Currently we give some results that motivate the imposition of this conditions on the rejector C.

THEOREM 4.1. *If a base B is independent up to its nonempty subset B_0, then, given that* (4–) *holds for an operator generated by C, C satisfies Success with respect to B_0; i.e., for all nonempty subsets B_1 of B_0, $C(B_1) \neq \emptyset$.*

THEOREM 4.2. *If a base B is independent up to its subset B_0, then, given that* (7–) *holds for an operator generated by C, α holds with respect to B_0; i.e., for all subsets B_1 and B_2 of B_0, if $B_1 \subseteq B_2$, $x \in B_1$ and $x \in C(B_2)$, then $x \in C(B_1)$.*

THEOREM 4.3. *If a base B is independent up to its subset B_0, then, given that* (8–) *holds for an operator generated by C, β+ holds with respect to B_0; i.e., for all subsets B_1 and B_2 of B_0, if $B_1 \subseteq B_2$, $x \in C(B_1)$ and $y \in B_1$, then $x \in C(B_2)$ if $y \in C(B_2)$.*

These results argue for the imposition of *Success*, α and β+, which are indeed reasonable conditions on the rejector C. However, many would have reservations against the imposition of *Credulity* on C. For instance, Pollock [22] has argued that though credulous reasoning is appropriate in *practical reasoning*, it is not suitable for *theoretical reasoning*. In the coherentist tradition of AGM and Levi, credulous reasoning is consciously avoided. Notwithstanding such reservations, as the following theorem (4.4) and its corollary.

THEOREM 4.4. *If B is independent up to its subset $\{x, y\}$ such that C cannot choose either of x or y over the other, then satisfaction of* (7–) *requires that $x \vee y \in Cn(B \backslash \{x, y\})$.*

COROLLARY 4.5. *If B is strongly independent up to its subset $\{x, y\}$ and* (7–) *holds, then either $C(x, y) = \{x\}$ or $C(x, y) = \{y\}$.*

show, if the agent manipulates her belief base in order to arrive at a new belief-corpus, credulous reasoning is almost necessary for (7–) and (8–) to be satisfied; that is, in a foundationalist framework, we have to accept something very close co *Credulity*.

Thus we see that there is good reason to impose the four conditions *Success*, α, $\beta+$ and *Credulity* on the rejector C. However, as we saw earlier (theorem 3.1), in presence of *Success* and *Credulity*, α and $\beta+$ are equivalent. Hence we will appropriate α, which is more popular and convenient, and assume henceforth that C satisfies the conditions *Success*, α and *Credulity*.[15]

4.2. *Success, α and Credulity are not Enough*

Unfortunately, even if the three conditions *Success*, α and *Credulity* are imposed on the rejector, neither the withdrawal operation $-_{fw}$ nor the contraction operation $-_{fc}$ satisfies either of the supplementary postulates (7–) or (8–), namely:

(7–) $\quad K_a^- \cap K_b^- \subseteq K_{(a \wedge b)}^-$

(8–) $\quad$ If $a \notin K_{(a \wedge b)}^-$, then $K_{(a \wedge b)}^- \subseteq K_a^-$.

We show this by presenting two counterexamples, one to (7–) and the other to (8–) that are similar to each other in an important way, and suggest that C needs to be further constrained.

EXAMPLE 1. The following example serves to show that conditions *Success*, α and *Credulity* on C are not sufficient to ensure the satisfaction of (7–) by $-_{fw}$ or $-_{fc}$. Let

$$B = \{a_1, b_1, a_2, b_2\}$$

$$a = a_1 \vee a_2$$

$$b = b_1 \vee b_2$$

where B is a strongly independent set of basic beliefs. Now simple computation will show that

$$E(a) = \{\{a_1\}, \{a_2\}\},$$
$$E(b) = \{\{b_1\}, \{b_2\}\}$$

and

$$E(a \wedge b) = \{\{a_1, b_1\}, \{a_1, b_2\}, \{a_2, b_1\}, \{a_2, b_2\}\}.$$

Now if we put $C(a_1, b_1) = a_1$,[16] $C(a_1, b_2) = a_1$, $C(a_2, b_1) = a_2$ and $C(a_2, b_2) = b_2$, then we get that

$$R_0(a) = \{a_1, a_2\},$$
$$R_0(b) = \{b_1, b_2\}$$

and

$$R_0(a \wedge b) = \{a_1, a_2, b_2\}.$$

Now then, $K -_{fw} a = Cn(B \backslash R_0(a)) = Cn(\{b_1, b_2\})$. Hence $a_2 \vee b_2$ is in $K -_{fw} a$. Similarly, $K -_{fw} b = Cn(\{a_1, a_2\})$ whereby $a_2 \vee b_2$ is in it. But $K -_{fw} (a \wedge b) = Cn(\{b_1\})$. Since B is strongly independent, $\{a_1, b_1\} \nvdash a_2 \vee b_2$, whereby $\{b_1\} \nvdash a_2 \vee b_2$. Hence $a_2 \vee b_2$ would not be in $K -_{fw} (a \wedge b)$, disproving postulate (7–). It may be noted that all the three conditions *Success*, α and *Credulity* are satisfied in this example.

Furthermore, it can be shown from this example that even if the foundational contraction operation $-_{fc}$ were used instead of $-_{fw}$, still (7–) will not be satisfied. Toward this end, we need to show that $\{b_1, a \wedge b \rightarrow a_1, a \wedge b \rightarrow a_2, a \wedge b \rightarrow b_2\} \nvdash a_2 \vee b_2$. For this, it will be sufficient to show that $X = \{b_1, a \wedge b \rightarrow a_1, a \wedge b \rightarrow a_2, a \wedge b \rightarrow b_2, \neg a_2, \neg b_2\}$ is consistent. Now, we know that $B = \{a_1, a_2, b_1, b_2\}$ is strongly independent, whereby, $Y = \{\neg a_1, \neg a_2, b_1, \neg b_2\}$ is consistent. However, since $\{\neg a_1, \neg a_2\} \vdash \neg(a \wedge b)$, $X \subseteq Cn(Y)$. Hence, X is consistent.

EXAMPLE 2. Similarly we can also show that *Success*, α and *Credulity* are not strong enough to ensure the satisfaction of postulate (8–). Let

$$B = \{a_1, a_2, b_1, b_2\}$$
$$a = (a_1 \wedge a_2) \vee (a_2 \wedge b_2)$$
$$b = (b_1 \wedge a_2) \vee (a_2 \wedge b_2)$$

where B is a strongly independent set of basic beliefs. Now simple computation will show that

$$E(a) = \{\{a_1, a_2\}, \{a_2, b_2\}\}$$

and

$$E(a \wedge b) = \{\{a_1, a_2, b_1\}, \{a_2, b_2\}\}$$

If we further stipulate that $C(a_1, a_2) = a_1$, $C(a_2, b_2) = a_2$ and $C(a_1, a_2, b_1) = b_1$, then we will see that

$$R_0(a) = \{a_1, a_2\}$$

and

$$R_0(a \wedge b) = \{a_2, b_1\}.$$

This shows that every member of $E(a)$ has a member in $R_0(a \wedge b)$, hence, $a \notin Cn(B \backslash R_0(a \wedge b))$. Furthermore, though $a_1 \in Cn(B \backslash R_0(a \wedge b))$, since B is strongly independent, $a_1 \notin Cn(B \backslash R_0(a))$ whereby $Cn(B \backslash R_0(a \wedge b)) \not\subseteq Cn(B \backslash R_0(a))$. Hence, $-_{fw}$ does not satisfy (8–).

Now, in order to show that $-_{fc}$ does not satisfy (8–) either, first we note that a is equivalent to $a_2 \wedge (a_1 \vee b_2)$ and $a \wedge b$ is equivalent to $a_2 \wedge (b_2 \vee (a_1 \wedge b_1))$. Accordingly, $K -_{fc} a = Cn(\{b_1, b_2, a \rightarrow a_1\})$ and $K -_{fc} (a \wedge b) = Cn(\{a_1, b_2, a \wedge b \rightarrow b_1\})$. Now, since B is strongly independent, $\{a_1, b_1, b_2\} \nvdash a_2$ whereby $a_2 \notin K -_{fc} (a \wedge b)$. However, a_2 is a consequence of a itself, hence $a \notin K -_{fc} (a \wedge b)$. Now, all we need to show is that $K -_{fc} (a \wedge b) \not\subseteq K -_{fc} a$. That is easy to show. Clearly $a_1 \in K -_{fc} (a \wedge b)$. However, since B is strongly independent, $X = \{b_1, b_2, \neg a_1, \neg a_2\}$ is consistent. However, since $\{\neg a_2\} \vdash \neg a$, $K -_{fc} a \subseteq Cn(X)$ whereby $(K -_{fc} a) \cup \{\neg a_1\}$ is consistent. Hence $a_1 \notin K -_{fc} a$.

4.3. *No Further Constraints on C*

Let us consider the example (2). Analysis of this counterexample shows that its success mostly depends on three facts:

- $\{a_1, a_2\} \in E(a)$
- $\{a_2, b_2\} \in E(a)$

and

- though a_2 is in both these sets, it is chosen by C from one only.

This is really ironical, since if a_2 were chosen from both, we would lose less information. It is interesting to note that the counterexample to (7−) shares this feature. For example, both the sets $\{a_2, b_1\}$ and $\{a_2, b_2\}$ with the common element a_2 are in $E(a \wedge b)$, but whereas this common element a_2 is chosen from the former, it is not chosen from the latter. Once we assume that a_2 is chosen from both of these sets, the counterexample is blocked, since in that case b_2 would still be in $K^-_{(a \wedge b)}$ to yield $a_2 \vee b_2$.

Fuhrmann comes pretty close to this point when he considers and rejects as too strong the following principle as a guide to forming a reject-set (see [4] p. 118):

(*) For each pair of entailment sets (S, S') such that $S \cap S' \neq \emptyset$, consider only sentences in $S \cap S'$ as candidates for rejection.

The following related constraint on C appears to be more promising:

$$(F) \qquad \forall_{S,S' \in E(a)} \forall_{x \in S \cap S'} (x \in C(S) \leftrightarrow x \in C(S')).$$

It is easily seen that acceptance of F can block the counterexamples to (7−) and (8−). In fact, it can be shown that *Success*, α, *Credulity* and F together can ensure the satisfaction of these supplementary postulates. However, the following triviality result

OBSERVATION 4.1. *No choice function C can simultaneously satisfy the three conditions Success, Credulity and F if B has a strongly independent subset containing more than two elements.*

shows that F is too strong a condition, and that we must find alternative ways of satisfying the supplementary postulates.

5. REVISION BASED ON A NEW REJECT-SET

The triviality result in the last section shows that if the foundational belief revision is determined by $-_{fc}$ or $-_{fw}$, then the epistemic system will occasionally crash: for there may not be any rational way of contracting (withdrawing) certain beliefs from the belief corpus. Though it is a very discomforting result, it is not surprising. *Success*, *Credulity* and F together imply that for any member S of $E(a)$, there is only one member of S in $R_0(a)$. That would obviously be impossible if $E(a)$ has more members than $\cup E(a)$ has, or, as in the relevant part of the case considered in the proof of observation (4.1), $E(a)$ and $\cup E(a)$ have the same number of members, but not all members of $E(a)$ are singletons. Since we cannot avoid such logical facts, we must weaken at least one of the three conditions *Success*, *Credulity* and F.

The case of *Success* is rather uncontroversial. The motivation for the conditions, *Credulity* and F, is to minimize the loss of information. If we have to give up a, we must give up at least one, and preferably exactly one element from each member of $E(a)$. From this vantage point *Credulity* looks reasonable. Furthermore, theorem (4.4) and its corollary provide strong support for *Credulity*. On the other hand, F imposes the condition that exactly one element from each member of $E(a)$ is ultimately rejected. But we have seen that it is often impossible to satisfy this condition. So the moral is to satisfy F *to the extent it is possible.*

5.1. *The New Reject-Set*

Let us again look at the example (2) for guidance. We notice that the sets $\{a_1, a_2\}$ and $\{a_2, b_2\}$ are in $E(a)$, and that a_1 is chosen from the former whereas a_2 is chosen from the latter. We argued that if a_2 were chosen from both, then the loss of information will be minimized while satisfying our purpose whence we considered imposing the condition F. But the point is, in the example in question, there was unnecessary loss of information not because F was violated, *but because even if we had not rejected a member of* $R_0(a)$, *namely* a_1, *still we could have successfully given up* a. Hence instead of putting an extra condition like F, we should try to modify the way the reject-set is constructed. The reject-set $R(a)$ that we are looking for should not contain an element such that if we reject only the rest we could give up a. In other words, $B \setminus R(a)$ should be an inclusion-maximal subset of B that fails to entail a.

DEFINITION 5.1. For all sets X and proposition a,

$$X \perp a = \{Y \subseteq X : a \notin Cn(Y) \wedge$$
$$\wedge \forall_{Y' \subseteq X}(Y \subset Y' \rightarrow a \in Cn(Y'))\}.$$

Thus $X \perp a$ (pronounced "X less a") is the set of maximal subsets of X

that fail to entail a. Note that X is any arbitrary set, hence it could be a belief base or a belief set (theory). Going back to our present concern, then it seems that if $R(a)$ is the set of elements to be discarded from B, then $B \backslash R(a)$ should be a member of $B \perp a$. We have noticed that *Success*, *Credulity* and α are not jointly sufficient to ensure that $B \backslash R_0(a)$ is a member of $B \perp a$. Hence we need to identify a suitable subset $R(a)$ of $R_0(a)$ such that $B \backslash R(a) \in B \perp a$.

We suggest the following method of constructing $R(a)$. First look at the set of elements of $R_0(a)$ that can be retained without yielding a and identify its least rejectable member, say x_1. Then we consider if $R_1(a) = R_0(a) \backslash \{x_1\}$ is the right reject-set $R(a)$. If $B \backslash R_1(a) \notin B \perp a$, then we continue repeating the above process till we get $R_n(a) \in B \perp a$ and identify it with $R(a)$. This step-wise process of identifying $R(a)$ is captured by the following definition:[17]

DEFINITION 5.2.

$$R_0(a) = \{x : \exists_{S \in E(a)} x \in C(S)\}$$

$$R_{i+1}(a) = \begin{cases} R_i(a) & \text{if } \forall_{x \in R_i(a)} (B \backslash R_i(a)) \cup \{x\} \vdash a \\ R_i(a) \backslash \{y\} & \text{otherwise} \end{cases}$$

where $y \in R_i(a)$ is such that

(*i*) $(B \backslash R_i(a)) \cup \{y\} \nvdash a$

and

(*ii*) $\forall_{z \in R_i(a)} ((B \backslash R_i(a)) \cup \{z\} \nvdash a \rightarrow z \in C(y, z))$;

$$R(a) = R_n(a) \text{ such that } R_n(a) = R_{n+1}(a).$$

It is easily seen that $R(a)$ is well defined: since B is finite, there will be some n or other such that $R_n(a) = R_{n+1}(a)$. Accordingly, we define

DEFINITION 5.3. $B \ominus_m a = B \backslash R(a)$

DEFINITION 5.4. $B \dot{-}_m a = (B \backslash R(a)) \cup \{a \rightarrow x : x \in R(a)\}$.

Since $B \ominus_m a$ is a member of $B \perp a$, we call $\ominus_m$ a maxichoice base withdrawal operation, and correspondingly $\dot{-}_m$ a maxichoice base contraction operation. It remains to be seen whether any of these operations can generate the desired foundational operation.

5.2. $B\backslash R(a)$ *Induces the Desired Operation*

There is a sense in which the propositions that do not occur in any member of $E(a)$ are not relevant to the entailment of a. Hence $\cup E(a)$ contains all the a-relevant propositions. The following observation shows that if we consider the sub-base of B that contains all and only a-relevant propositions, taking out members of $R(a)$ from it leaves us with one of its maximal subset that fail to entail a.

OBSERVATION 5.1. $\cup E(a)\backslash R(a) \in \cup E(a) \perp a$

Furthermore, $\cup E(a)\backslash R(a)$ is not just any member of $\cup E(a) \perp a$, it is the latter's most preferred element in the following sense: all members of $\cup E(a) \perp a$ other than $\cup E(a)\backslash R(a)$ retain a more rejectable element at the cost of rejecting a less rejectable element. That it is so is shown by the corollary to theorem (5.1). Let us first make this notion of preference in question more precise:

DEFINITION 5.5. $\sqsubseteq$ is a preference relation over 2^B such that for all subsets X and Y of B, $X \sqsubseteq Y$ iff $\forall_{x\in X\backslash Y}\exists_{y\in Y\backslash X} C(x,y) = x$.

This definition says that X is at most as preferred as Y just in case for every x in X that is not in Y, there is a member y in Y that is not in X such that of the two elements, x and y, the former is more reject worthy. The following observation shows that $\sqsubseteq$ is transitive over 2^B.

OBSERVATION 5.2. *Given Success, Credulity and* α, $\sqsubseteq$ *is transitive over* 2^B.

In the proof of the above, we use the following fact: since every member X of 2^B is finite, there is x_0 in X such that $C(x, x_0) = x$ if x is in X. We can construct it by successively eliminating the chosen elements first from X, then from $X\backslash C(X), \ldots$ The finiteness of X and α will ensure the desired property of this element. We call it *the least rejectable element of* X. In general, we will denote the least rejectable element of a set X by $C_{min}(X)$.

The following theorem (5.1) says that if some member S of $\cup E(a) \perp a$ contains an element x that is not retained in $\cup E(a)\backslash R(a)$, then there is an entailment-set X in $E(a)$ with x as its most rejectable element such that it has at least one other element x' which S fails to retain, yet is retained by $\cup E(a)\backslash R(a)$. As a corollary to this theorem, we get that $\cup E(a)\backslash R(a)$ is the most $\sqsubseteq$-preferable element of $\cup E(a) \perp a$. It may be helpful to note that the set $S\backslash(\cup E(a)\backslash R(a))$ used in the statement of the theorem (5.1) is identical to the set $S \cap R(a)$.

THEOREM 5.1. *Let $S \in \cup E(a) \perp a$ and $x \in S\backslash(\cup E(a)\backslash R(a))$. Then, given that Success, Credulity and α are satisfied by C,*
$\exists_{X \in E(a)}(C(X) = x \wedge X \not\subseteq S \cup R(a))$.

COROLLARY 5.2. *If C satisfies the conditions Success, Credulity and α, then for all S in $\cup E(a) \perp a$, $S \sqsubseteq \cup E(a)\backslash R(a)$.*

As you might have already anticipated, the analogous result holds for members of $B \perp a$. Theorems (5.3) and (5.4) together show that $B\backslash R(a)$ is the unique, most $\sqsubseteq$-preferred element of $B \perp a$.

OBSERVATION 5.3. *For all S in $B \perp a$ and S' in $\cup E(a) \perp a$,*

- *(a)* $(B\backslash \cup E(a)) \subseteq S$.
- *(b.1)* $S' \cup (B\backslash \cup E(a))$ *is in* $B \perp a$ and
  (b.2) $S\backslash(B\backslash \cup E(a))$ *is in* $\cup E(a) \perp a$.
- *(c) If a particular member X of $\cup E(a) \perp a$ is included in S, then S is unique; i.e., S is the only member of $B \perp a$ that includes X.*

THEOREM 5.3. *Given that C satisfies Success, Credulity and α, for all S in $B \perp a$, $S \sqsubseteq B\backslash R(a)$.*

THEOREM 5.4. *Given that C satisfies Success, Credulity and α, if $S \in B \perp a$ is such that for all $S' \in B \perp a$, $S' \sqsubseteq S$, then S is unique; i.e., S is the only member of $B \perp a$ such that $S' \sqsubseteq S$ for all $S' \in B \perp a$.*

These results suggest that all the members of $K \perp a$ that contain $B\backslash R(a)$ might be the (only) best elements of $K \perp a$ in the relevant sense. Hence we define:

DEFINITION 5.6. $\gamma_C(K \perp a) = \{M \in (K \perp a) : B\backslash R(a) \subseteq M\}$

It is clear that if there is a relation $\mathscr{R}$ over 2^K such that γ_C picks out the $\mathscr{R}$-best elements of $K \perp a$ then the partial meet contraction operation generated by it will satisfy the postulate (7−) and further more, if $\mathscr{R}$ is transitive over 2^K then (8−) will also be satisfied by this contraction operation. Hence we extend $\sqsubseteq$ to 2^K in the following manner:

DEFINITION 5.7. For all subsets X and Y of K,

$$X \sqsubseteq Y \text{ iff } X \cap B \sqsubseteq Y \cap B.$$

It can be easily verified $\sqsubseteq$ is transitive over 2^K. Finally, theorem (5.5) shows that $\gamma_C(K \perp a)$ is really the set of $\sqsubseteq$-best elements of $K \perp a$.

LEMMA 5.1. *Let $D \subseteq K$ such that $a \notin Cn(D)$. Then there is a superset D^* of D such that $D^* \in K \perp a$.*

THEOREM 5.5. $\{X \in K \perp a : (B \backslash R(a)) \subseteq X\} = \{X \in K \perp a : \forall_{X' \in K \perp a} X' \sqsubseteq X\}$

Since the extended $\sqsubseteq$ is transitive over 2^K, it follows that γ_C is a transitively relational choice function, and hence the contraction operation generated by it will be a transitively relational partial meet contraction operation and will satisfy all Gärdenfors postulates of belief contraction, including the supplementary ones. Furthermore the revision operation associated with it will satisfy all the Gärdenfors postulates of revision.

5.3. *Rejecting by Saving*

We have seen that $B \backslash R(a)$ is the best element of $B \perp a$ we were looking for. Note that we get this element by strategically rejecting the "worst elements" from B so that with minimal loss of information we succeed in removing a from $K = Cn(B)$. We may wonder whether there might be another way of arriving at $B \backslash R(a)$ by similarly saving the "best elements" of B as much as possible so long as the elements thus saved do not yield a. We recall that for any nonempty subset B' of B, we refer to the least rejectable element of B' by $C_{min}(B')$. We will denote $C_{min}(B)$ by b_1 and $C_{min}(B \backslash \{b_1, \ldots, b_i\})$ by b_{i+1}. Now, we inductively define $B \uparrow a$ as follows:

DEFINITION 5.8.

$$b_{i+1} \in B \uparrow a$$
$$\text{iff} \begin{cases} \{b_1\} \nvdash a & \text{if } i = 0 \\ (\{b_1, \ldots, b_i\} \cap B \uparrow a) \cup \{b_{i+1}\} \nvdash a & \text{otherwise} \end{cases}$$

intuitively, $B \uparrow a$ is formed in a step-wise manner: we first begin with $C_{min}(B)$. It will be in $B \uparrow a$ if and only if it does not imply a. Then we consider $C_{min}(B \backslash \{C_{min}(B)\})$. If it, together with $C_{min}(B)$ in case the latter is in $B \uparrow a$, does not imply a, then it is in $B \uparrow a$; otherwise it is not in $B \uparrow a$. Then we consider the least rejectable element of $B \backslash \{C_{min}(B), C_{min}(B \backslash \{C_{min}(B)\})\}$, and so on. Clearly $B \uparrow a$ is an element of $B \perp a$. The following observation shows that $B \uparrow a$ is a $\sqsubseteq$-best element of $B \perp a$.

OBSERVATION 5.4. *For all members S of $B \perp a$, $S \sqsubseteq B \uparrow a$.*

Given the uniqueness of the $\sqsubseteq$-best elements of $B \perp a$ (theorem 5.4), it follows that $B \uparrow a$ is identical with $B \backslash R(a)$.

5.4. *Base After the Update*

Now the question arises, Do we have a base contraction operation associated with this belief contraction operation generated by γ_C? for it is also desirable to identify the new base after the belief update is completed. Theorem (5.6) shows that $\dot{-}_m$ does the trick: we can profitably take $B \dot{-}_m a$ to be the new base after removing a from $K = Cn(B)$.

LEMMA 5.2. *Let $S \subseteq K$ such that $S \nvdash a$. Then*

$$\cap\{X \in K \perp a : S \subseteq X\} = Cn(S \cup (K \cap Cn(\{\neg a\})))$$

For its proof, see Nebel's proof of lemma 13 in [18].

THEOREM 5.6. $\cap\gamma_C(K \perp a) = Cn(B \dot{-}_m a)$

Thus the foundational contraction operation generated by the maxichoice base contraction operation $\dot{-}_m$ is a transitively relational partial meet contraction operation and hence will satisfy all the eight postulates of contraction.[18] It can be easily shown that the corresponding base revision operation $\dot{+}_m$

DEFINITION 5.9. $B \dot{+}_m a = (B \dot{-}_m \neg a) \cup \{a\}$

will generate a foundational revision operation which, being a transitively relational partial meet revision operation, will satisfy all the eight postulates of revision. Hence $B \dot{+}_m a$ will be the new base after revising K by a.

We may mention that the foundational revision operation corresponding to $\ominus_m$ will be identical with the one generated by $\dot{+}_m$. Hence, if we are interested only in belief revision, it does not really matter whether we use $\ominus_m$ or $\dot{-}_m$ to generate the foundational revision operation in question. We however leave open the question whether the foundational withdrawal operation generated by $\ominus_m$ will satisfy the supplementary postulates of contraction.

6. SUMMARY AND OUTLOOK

We started with an intuitive way of revising a belief base B in face of new information $\neg a$: first minimally contract B (if necessary) so that the contracted base no longer implies a, and then add this new information to this contracted base. The basic idea behind this removal of a was to delete from the base B those elements that are least desirable in a-entailing inclusion minimal subsets of B. For this purpose we assumed a choice function C over 2^B that chooses the least desirable elements of any set of basic beliefs. In sections § 2 and § 3 we showed that this process, in general, fails to meet the demands of rationality on three counts: (1) it violates "recovery", i.e., by re-believing the proposition willfully discarded, it was not possible to recover all the information lost in the process, (2) it violates (7–) which maintains that in giving up a conjunction of two propositions we should not lose more information than what we lose by giving up the two conjuncts in parallel, and (3) it violates (8–), which says that if one must give up the belief x in order to give up $x \wedge y$, then in order to give up this conjunctive belief $x \wedge y$ one must give up what it is necessary to give up in order to give up x.

In § 2.2 we noted that we can save recovery if, instead of completely rejecting the required basic beliefs, we weaken them in an appropriate manner. In § 4 we discussed the need of three conditions on the choice function C, namely, *Success*, α and *Credulity*. We saw that though these conditions are almost necessary, they are not sufficient to ensure the satisfaction of the supplementary postulates. This result suggested that the set of beliefs not rejected (weakened) should be an inclusion maximal subset of B that fails to imply a. In § 5 we suggested how to construct this reject-set from $R_0(a)$ by successively removing the "good" elements from the latter, and then showed that if we appropriately weaken the base B with respect to the members of this reject-set, then we satisfy all the Gärdenfors postulates of belief contraction. It follows that the corresponding revision operation will satisfy all the Gärdenfors postulates of belief revision too.

The suggested revision procedure, however, has the following limitations.

(1) The three conditions *Success*, *Credulity* and (α) constitute a very strong set of conditions on the rejector – they entail all the other eight conditions that we mentioned in § 3. However, we cannot easily weaken these conditions since the construction of the reject-set $R(a)$ presupposes that the rejector satisfies *Success* and *Credulity*. Hence a more versatile way of constructing the reject-set is called for, one that does not presuppose any condition of the rejector C, but is capable of generating the reject-set $R(.)$ that we have studied in the special case when C satisfies *Success*, (α) and *Credulity*. It might prove valuable to study foundational belief revision in such a general framework.

(2) It cannot handle repeated belief changes, since the choice function used is defined for the old base only. It would be interesting to see if we can generalize our approach using super rejectors analogous to the *super selectors* of Hansson [12].

(3) If the belief set $Cn(\{a \wedge b\})$ with base $\{a \wedge b\}$ is contracted by a we essentially get $Cn(\{a \rightarrow b\})$ as the new belief set. In case a and b are independent, we lose more than we bargained for: intuitively, we should get $Cn(\{b\})$ as the new belief set. This suggests that the ideal solution may be more complicated than our solution.

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APPENDIX

Proof of Observation 2.1. Let $a \in Cn(S)$. Since Cn is compact, there is a finite subset, say S', of S such that $a \in Cn(S')$. It will be sufficient to show that S' has a subset S^* such that $S^* \in E(a)$. We do that by applying strong induction on the size of S'.

Assume that if S' has less than m members, then it has a member of $E(a)$ as its subset. We need to show that if S' has m members, then there is a subset S^* of S' such that $S^* \in E(a)$. Let then S' have m members. Either S' itself is the required S^* or it is not. The first case is trivial. As to the second case, suppose that S' is not the required S^*. Then there is an $(m-1)$-membered proper subset S'' of S' such that $a \in Cn(S'')$. Hence, by the inductive hypothesis, there is a subset S^* of S'' such that $S^* \in E(a)$. But S^* is a subset of S'. **QED**

Proof of Theorem 3.1. α implies *Iteration, Cut and Distributivity* even in absence of *Success* and *Credulity* (see [15]). Clearly $\beta+$ implies β. We give the proofs of the rest below. When the sets in question are empty, the proofs become trivial; hence proofs of only the principal cases are given.

Assume that C satisfies *success* and *Credulity*.

(i) We show that α_2 is equivalent to $\beta+$. For left to right, suppose that $C(X) = \{x\}$, $y \in X$, $X \subseteq Y$ and $C(Y) = \{y\}$.[19] Now, by α, $y \in C(X)$ and by *Credulity*, $x = y$. Hence $x \in C(Y)$. For right to left, suppose that $X \subseteq Y$, $x \in X$ and $C(Y) = \{x\}$. Let $C(X) = \{y\}$. Then, by $\beta+$, $y \in C(Y)$. Now, by *Credulity*, $x = y$, hence $x \in C(Y)$.

(ii) It is to be shown that α_2 is equivalent to *Aizerman*. For left to right, assume that $C(X) \subseteq Y \subseteq X$. By α, we get $C(X) \subseteq C(Y)$, wherefrom, by *Credulity* we get $C(Y) \subseteq C(X)$. For right to left, assume that $S \subseteq T$, $x \in C(T)$ and $x \in S$. By *Credulity* we get $C(T) \subseteq S \subseteq T$, hence by *Aizerman* we get $C(S) \subseteq C(T)$. By *Credulity*, then $C(S) = C(T)$ whereby $x \in C(S)$.

(iii) We show that α_2 is equivalent to *Path Independence*. Left to right: First we show that $C(X \cup Y) \subseteq C(C(X) \cup C(Y))$. Let $x \in C(X \cup Y)$. Then, x must be either in X or in Y. However, since X and Y are subsets of $X \cup Y$, by α, either $x \in C(X)$ or $x \in C(Y)$, whereby $x \in C(X) \cup C(Y)$. Furthermore, $C(X) \cup C(Y) \subseteq X \cup Y$, hence, by α, we get the desired result that $x \in C(C(X) \cup C(Y))$. From $C(X \cup Y) \subseteq C(C(X) \cup C(Y))$, by *Credulity*, we get that $C(X \cup Y) = C(C(X) \cup C(Y))$. Right to Left: Suppose that $S \subseteq T$, $x \in S$ and $x \in C(T)$. Since $T = S \cup (T \backslash S)$, by *Path Independence* we get $x \in C(S) \cup C(T \backslash S)$. Since $x \in S$, $x \notin T \backslash S$, hence x must be in $C(S)$.

(iv) We show that α_1 is equivalent to *Arrow*. Right to left is trivial. For left to right, assume that $C(X) \cap Y \neq \emptyset$. Then by *Credulity* we get $C(X) \cap Y = C(X)$. Now, by α, $C(X) \subseteq C(X \cap Y)$ and then by *Credulity*, $C(X) = C(X \cap Y)$. **QED**

Proof of Theorem 4.1. Assume that (4–) holds. Let a base B be independent up to its subset B_0. Consider the removal of $\wedge(B_1)$ from $Cn(B)$ for any nonempty subset B_1 of B_0. Clearly, $E(\wedge(B_1))$ has only one member, namely B_1. Hence, whether we use $-_{fw}$ or $-_{fc}$, by (4–), $C(B_1)$ must be a nonempty subset of B_1. **QED**

Proof of Theorem 4.2. Assume that (7–) holds. Let B be independent up to its subset B_0. Let B_1 and B_2 be subsets of B_0 such that $B_1 \subseteq B_2$, $x \in B_1$ and $x \in C(B_2)$. Define $a = \wedge(B_1)$ and $b = \wedge(B_2 \backslash B_1)$. Since B is independent up to B_0, we get $E(a) = \{B_1\}$, $E(b) = \{B_2 \backslash B_1\}$ and $E(a \wedge b) = \{B_2\}$. Accordingly, $x \in R_0(a \wedge b)$ but $x \notin R_0(b)$. Hence, invoking the partial independence of B, we get that $x \notin K -_{fw} (a \wedge b)$ but $x \in K -_{fw} b$. Since $x \notin K -_{fw} (a \wedge b)$, by (7–), $x \notin (K -_{fw} a) \cap (K -_{fw} b)$. However x is already in $K -_{fw} b$. Hence, surely, $x \notin K -_{fw} a$ whereby $x \in R_0(a) = C(B_1)$.

In order to see that the theorem holds even if $-_{fc}$ were used instead of

$-_{fw}$, we note that $K -_{fc} (a \wedge b) = K -_{fw} (a \wedge b)$ and $K -_{fc} a = K -_{fw} a$. This is so because (1) since $a \wedge b$ is logically equivalent to $\wedge(B_2)$, $a \wedge b \rightarrow y$, for any $y \in R_0(a \wedge b) = C(B_2)$, is a tautology, and (2) since a is defined as $\wedge(B_1)$, $a \rightarrow y$, for any $y \in R_0(a) = C(B_1)$, is a tautology. **QED**

Proof of Theorem 4.3. Assume that (8–) holds. Let B be independent up to its subset B_0. Let B_1 and B_2 be subsets of B_0 such that $B_1 \subsetneq B_2$, $x \in C(B_1)$, $y \in B_1$ and $y \in C(B_2)$. Define $a = \wedge(B_1)$ and $b = \wedge(B_2 \backslash B_1)$. Since B is independent up to B_0, we get $E(a) = \{B_1\}$, $E(b) = \{B_2 \backslash B_1\}$ and $E(a \wedge b) = \{B_2\}$. Accordingly, $x \in R_0(a)$ and $y \in R_0(a \wedge b)$. Since B is independent up to B_0 and $y \in C(B_2)$, it follows that $y \notin K -_{fw} (a \wedge b)$. However, since $y \in B_1$, $\{a\} \vdash y$. Hence $a \notin K -_{fw} (a \wedge b)$. By (8–), then, $K -_{fw} (a \wedge b) \subseteq K -_{fw} a$. We further note that since $x \in R_0(a)$, $x \notin K -_{fw} a$. Hence $x \notin K -_{fw} (a \wedge b)$. Since $x \in B_2 \in E(a \wedge b)$, it follows that $x \in R_0(a \wedge b) = C(B_2)$.

Using the same argument as in the proof of the theorem 4.2, we can show that this theorem holds even if $-_{fc}$ were used instead of $-_{fw}$. **QED**

Proof of Theorem 4.4. Let B be independent up to $\{x, y\}$ such that $C(x, y) = \{x, y\}$. Due to this independence, $x \in (B \backslash R_0(y))$ and $y \in (B \backslash R_0(x))$. Hence $x \vee y \in Cn(B \backslash R_0(x)) \cap Cn(B \backslash R_0(y)) = K -_{fw} x \cap K -_{fw} y$. So, by (7–), $x \vee y \in K -_{fw} (x \wedge y) = Cn(B \backslash R_0(x \wedge y))$. Now, clearly the set $\{x, y\}$ is in $E(x \wedge y)$. Since $C(x, y) = \{x, y\}$, surely $\{x, y\} \subseteq R_0(x \wedge y)$, so that $B \backslash R_0(x \wedge y) \subseteq B \backslash \{x, y\}$. Hence $x \vee y \in Cn(B \backslash \{x, y\})$.

If we use $-_{fc}$ instead of $-_{fw}$, the result is no different. Clearly $x \vee y$ is in both $K -_{fc} x$ and $K -_{fc} y$. Furthermore, since $(B \dot{-}_f (x \wedge y)) \backslash (B \ominus_f (x \wedge y))$ is a set of tautologies, namely $\{x \wedge y \rightarrow x, x \wedge y \rightarrow y\}$, we get that $K -_{fc} (x \wedge y) = K -_{fw} (x \wedge y)$, wherefore $x \vee y$ is in $Cn(B \backslash \{x, y\})$. **QED**.

Proof of Corollary 4.5. Let the antecedents hold. By the definition, $C(x, y)$ must be one of the sets $\{x\}$, $\{y\}$ or $\{x, y\}$. Suppose, for *reductio*, that $C(x, y) = \{x, y\}$. Hence by theorem (4.4), $x \vee y \in B \backslash \{x, y\}$. But this is impossible if B is strongly independent up to $\{x, y\}$. **QED**.

Proof of Observation 4.1. Let B contain a strongly independent subset

$\{x, y, z\}$. Let $a = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$. Hence $E(a) \supseteq \{\{x, y\}, \{y, z\}, \{z, x\}\}$. Consider $C(x, y)$. By *Success* and *Credulity*, it has only one element. Let $C(x, y) = x$. (The other case is similar.) Then, by F, $x \in C(z, x)$. By another application of *Credulity* we get that $C(z, x) = x$. Again, by *Success* and *Credulity*, either $C(y, z) = y$ or $C(y, z) = z$. In either case F is violated; in the first case since y is also in $\{x, y\}$ but not chosen from it, in the latter because z is also in $\{z, x\}$, but not chosen from it. **QED**.

Proof of Observation 5.1. $\cup E(a) \backslash R(a) \subseteq B \backslash R(a)$. Hence $\cup E(a) \backslash R(a) \nvdash a$. Let $x \in R(a)$. It would suffice to show that $(\cup E(a) \backslash R(a)) \cup \{x\} \vdash a$. Now $(B \backslash R(a)) \cup \{x\} \vdash a$. Hence there exists a set S in $E(a)$ such that $S \subseteq (B \backslash R(a)) \cup \{x\} = (\cup E(a) \backslash R(a)) \cup ((B \backslash \cup E(a)) \backslash R(a))$. Furthermore, since $S \in E(a)$, $S \cap (B \backslash \cup E(a)) = \emptyset$. Hence it follows that $S \subseteq (\cup E(a) \backslash R(a)) \cup \{x\}$ whereby $(\cup E(a) \backslash R(a)) \cup \{x\} \vdash a$. **QED**.

Proof of Observation 5.2. In this proof we will utilize the following property of C:

- If $x \in C(x, y)$ and $y \in C(y, z)$, then, $x \in C(x, z)$, given that C satisfies α and $\beta+$.

The proof is quite easy. Suppose that $x \in C(x, y)$ and $y \in C(y, z)$. Then, by $\beta+$, $x \in C(x, y, z)$ wherefore, by α, $x \in C(x, z)$. Of course, if C also satisfies *Credulity*, then we can strengthen it to: *If* $x \in C(x, y)$ *and* $y \in C(y, z)$, *then* $x = C(x, z)$. We will refer to this property as the "transitivity" of C.

Now, Suppose that X, Y and Z are subsets of B and that $X \sqsubseteq Y$ and $Y \sqsubseteq Z$. It would suffice to show that $X \sqsubseteq Z$. Let x be an arbitrary element in $X \backslash Z$. We need to show that there is an element z in $Z \backslash X$ such that $C(x, z) = x$. Let $C_{min}(X \backslash Z) = x_0$. Now, either x_0 is in Y or not.

In the first case, suppose that x_0 is in Y. Then x_0 is in $Y \backslash Z$. Hence, by hypothesis, there exists z_0 in $Z \backslash Y$ such that $C(x_0, z_0) = x_0$. Now, either z_0 is in X or not. If not, then by the transitivity of C, z_0 is the required element z. On the other hand, if z_0 is in X, then $z_0 \in (X \cap Z) \backslash Y \subseteq X \backslash Y$. Let z_1 be the least rejectable element of $(X \cap Z) \backslash Y$. Hence $C(z_0, z_1) = z_0$. Again, by hypothesis, there exists y_0 in $Y \backslash X$ such that $C(z_1, y_0) = z_1$. If y_0 is in Z then, by the transitivity of C, it is the desired element z. On the other hand, if y_0 is not in Z, then it is in $Y\backslash(X \cup Z)$. Let y_1 be the least rejectable element of $Y\backslash(X \cup Z)$. Since y_1 is in $Y\backslash Z$ there exists z_2 in $Z\backslash Y$ such that $C(y_1, z_2) = y_1$. Now, the following argument shows that z_2 is not in X. Suppose that z_2 is in X. Then z_2 is in $(X \cap Z)\backslash Y$. Now, by hypothesis, $C(z_1, z_2) = z_2$. Hence, by the transitivity of C, $C(z_1, y_1) = y_1$. Furthermore, $C(z_1, y_0) = z_1$ and $C(y_0, y_1) = y_0$. Again, by the transitivity of C, $C(z_1, y_1) = z_1$. Hence, by *Credulity*, $y_1 = z_1$. But that is impossible, since y_1 and z_1 are, respectively, in the two disjoint sets $Y\backslash(X \cup Z)$ and $(X \cap Z)\backslash Y$. Hence z_2 is not in X, whereby it, being in $Z\backslash X$, is the desired element z.

The second case is similar. Suppose that x_0 is not in Y. Then there is a y_0 in $Y\backslash X$ such that $C(x_0, y_0) = x_0$. If y_0 is in Z then it is the desired element. On the other hand, if y_0 is not in Z, then consider the least rejectable element y_1 of the set $Y\backslash(X \cup Z)$ to which y_0 belongs. Let z_0 be in $Z\backslash Y$ such that $C(y_1, z_0) = y_1$. Now, either z_0 is in X or not. If z_0 is not in X, then it is the desired element. On the other hand, if z_0 is in X, then it is in $(X \cap Z)\backslash Y$. Let z_1 be the least rejectable element of $(X \cap Z)\backslash Y$. Now, z_1 is in $X\backslash Y$. Hence, let y_2 be in $Y\backslash X$ such that $C(z_1, y_2) = z_1$. Now, using an argument similar to the one we used in the first case, we can show that y_2 must be in Z, i.e., in $(Y \cap Z)\backslash X$. Hence it is the desired element. **QED**.

Proof of Theorem 5.1. Assume (1) and (2):

(1) $\quad S \in \cup E(a) \perp a$

(2) $\quad x \in S\backslash(\cup E(a)\backslash R(a))$.

Furthermore, for *reductio*, suppose that

(3) $\quad \forall_{X \in E(a)}(C(X) = x \rightarrow \forall_{x' \in X}(x' \notin S \rightarrow x' \in R(a)))$.

We conclude the proof by establishing the following two opposing claims that follow from assumptions (1)–(3):

Claim (1): $\exists_{X \in E(a)}(\{x\} = X \cap R(a))$

Claim (2): $\forall_{X \in E(a)}(x \in X \rightarrow \{x\} \subset X \cap R(a))$.

In order to establish claim (1), first we observe that for every $y \in R(a)$ there exists a set $Y \in E(a)$ such that $Y \cap R(a) = \{y\}$. (Suppose otherwise. Let $y_0 \in R(a)$ such that $Y \cap R(a) = \{y_0\}$ for no $Y \in E(a)$.

Then no member Y of $E(a)$ is a subset of $(B\backslash R(a)) \cup \{y_0\}$ whereby $(B\backslash R(a)) \cup \{y_0\} \nvdash a$, which is prohibited by the definition (5.2)!) However, it follows from the assumptions (1) and (2) that $x \in R(a)$. Hence there is a set $X \in E(a)$ such that $X \cap R(a) = \{x\}$. This establishes claim (1).

We establish claim (2) in two steps. In the first step, we show that $\forall_{X \in E(a)}((x \in X \wedge C(X) = x) \rightarrow \{x\} \subset X \cap R(a))$. In the second step we show that $\forall_{X \in E(a)}((x \in X \wedge C(X) \neq x) \rightarrow \{x\} \subset X \cap R(a))$.

Step 1: Take an arbitrary member X of $E(a)$ such that $x \in X$ and $C(X) = x$. Since $x \in R(a)$, clearly $\{x\} \subseteq X \cap R(a)$. Since $x \in X$, there is $x' \in X$ such that $x' \notin S$. (If there is no such $x' \in X$, then $X \subseteq S$ whereby $S \vdash a$; but since $S \in \cup E(a) \vdash a$, $S \nvdash a$.) Hence, by assumption (3), $x' \in R(a)$ whereby $\{x, x'\} \subseteq X \cap R(a)$. Since $x \in S$ but $x' \notin S$, surely $x \neq x'$. Hence, $\{x\} \subset X \cap R(a)$.

Step 2: Take an arbitrary member X of $E(a)$ such that $x \in X$ and $C(X) \neq x$. Let $X_1, \ldots, X_m$ be all the members of $E(a)$ that have x as a member such that for all $i \leq m$, $C(X_i) = x_i \neq x$. Let $R(a) = R_n(a)$.

First we show by induction that $\{x_1 \ldots x_m\} \subseteq R_j(a)$ for all $j \leq n$. Clearly $\{x_1 \ldots x_m\} \subseteq R_0(a)$. For the inductive step, suppose that $\{x_1 \ldots x_m\} \subseteq R_j(a)$ for some $j < n$. We need to show that $\{x_1 \ldots x_m\} \subseteq R_{j+1}(a)$. Consider an arbitrary element x_i of the set $\{x_1, \ldots, x_m\}$. It will be sufficient to show that $x_i \in R_{j+1}(a)$. Now, either $(B\backslash R_j(a)) \cup \{x_i\} \vdash a$ or not. The first case is trivial, considering the definition (5.2). As to the second case, let $(B\backslash R_j(a)) \cup \{x_i\} \nvdash a$. Now, the following argument shows that $(B\backslash R_j(a)) \cup \{x\} \nvdash a$:

> Since $x \in R(a) \subseteq R_j(a), x \in R_j(a)$. However, $x \in X_i$ for all $i \leq m$. Hence, every set $X_i \in E(a)$ such that $x \in X_i$ and $C(X_i) \neq x$ has at least one of its members (namely x_i) besides x present in $R_j(a)$. Similarly, it follows from step (1) that every member $X' \in E(a)$ such that $x \in X'$ and $C(X') = x$ has at least one of its members besides x in $R_j(a)$. Thus every member of $E(a)$ that has x as one of its members has one of its members besides x in $R_j(a)$ whereby $(B\backslash R_j(a)) \cup \{x\} \nvdash a$.

Thus $\{x, x_i\} \subseteq R_j(a)$, $(B\backslash R_j(a)) \cup \{x\} \nvdash a$ and $(B\backslash R_j(a)) \cup \{x_i\} \nvdash a$.

Furthermore, $C(x, x_i) = x_i$ (note that though $x \in X_i$, $C(X_i) = x_i \neq x$). Hence, by the definition (5.2), $x_i \in R_{j+1}(a)$.

We have seen that $\{x_1, \ldots, x_m\} \subseteq R_j(a)$ for all $j \leq n$. Since $R(a) = R_n(a)$, it follows that $\{x_1, \ldots, x_m\} \subseteq R(a)$. Now, since $X \in E(a)$, $x \in X$ and $C(X) \neq x$, for some $k \leq m$, $X = X_k$ and $C(X) = x_k$. Hence $\{x, x_k\} \subseteq X \cap R(a)$ whereby $\{x\} \subset X \cap R(a)$. This completes step (2) and thereby establishes claim (2). **QED**.

Proof of Corollary 5.2. We need to show that if C satisfies the conditions *Success*, *Credulity* and α, then for all S in $\cup E(a) \perp a$, if x is in S but not in $\cup E(a) \backslash R(a)$, then there is y in $\cup E(a) \backslash R(a)$ but not in S such that $x \in C(x, y)$. This easily follows from the above theorem (5.1). Suppose that S is in $\cup E(a) \perp a$, x is in S but not in $\cup E(a) \backslash R(a)$. Then by the theorem (5.1), there exists $X \in E(a)$ such that $C(X) = x$ and some member x' of X is such that $x' \notin S$ and $x' \notin R(a)$. Now, since $x' \notin R(a)$ but $x' \in X \in E(a)$, clearly $x' \in (\cup E(a) \backslash R(a)$. Also, $x' \notin S$. Furthermore, since both x and x' are in x and $C(X) = x$, by condition α, $C(x, x') = x$. Hence x' is the desired y. **QED**.

Proof of Observation 5.3. (a) Suppose to the contrary that S is in $B \perp a$ but $(B \backslash \cup E(a)) \nsubseteq S$. Then $S \cup (B \backslash \cup E(a)) \vdash a$. In that case, there is a proposition x such that $S \vdash x$ and $(B \backslash \cup E(a)) \vdash (x \rightarrow a)$. Furthermore, $S \nvdash (x \rightarrow a)$. Now take a minimal subset S_1 of S such that $S_1 \vdash x$. (Existence of such a subset is guaranteed by observation (2.1).) Now, $S_1 \cup (B \backslash \cup E(a)) \vdash a$. Hence, there is a subset S_2 of $S_1 \cup (B \backslash \cup E(a))$ such that S_2 is in $E(a)$. Now, since $S_1 \nvdash a$, the set S_2 must contain at least one member of $B \backslash \cup E(a)$. But since S_2 is a member of $E(a)$, it does not contain any member of $B \backslash \cup E(a)$. Contradiction.

(b) Suppose that $S \in (B \perp a)$ and $S' \in (\cup E(a) \perp a)$.

(1) Suppose for *reductio* that $S' \cup (B \backslash \cup E(a)) \notin B \perp a$. Then $S' \cup (B \backslash \cup E(a)) \vdash a$, whereby, for some x, $S' \vdash x$ and $B \backslash \cup E(a) \vdash (x \rightarrow a)$. Let S_1 be a minimal subset of S' such that $S_1 \vdash x$. Then there is a subset S_2 of $S_1 \cup (B \backslash \cup E(a))$ such that S_2 is in $E(a)$. But, in order to entail a, S_2 must contain at least one member of $B \backslash \cup E(a)$. But that is impossible since S_2 is in $E(a)$.

(2) Surely $S \backslash (B \backslash \cup E(a)) \nvdash a$. Suppose for *reductio* that $S \backslash (B \backslash \cup E(a)) \notin \cup E(a) \perp a$. Then clearly there is a subset of S'' of

$\cup E(a)$ such that $S \backslash (B \backslash \cup E(a)) \subset S''$ and $S'' \nvdash a$. Now, since $\cup E(a)$ is finite, we can construct $S_1 \supset S \backslash (B \backslash \cup E(a))$ such that $S_1 \in \cup E(a) \perp a$. Now, $S \subset (S_1 \cup (B \backslash \cup E(a)))$ and by (b-1) above, $S_1 \cup (B \backslash \cup E(a)) \in B \perp a$. But this is impossible since $S \in B \perp a$.

(c) Let $X \in \cup E(a) \perp a$. Let S_1 and S_2 be members of $B \perp a$ such that X is included in both of them. Clearly $X \subseteq S_{i(=1,2)} \backslash (B \backslash \cup E(a))$. Furthermore, by (b-2) above, $S_{i(=1,2)} \backslash (B \backslash \cup E(a))$ is in $\cup E(a) \perp a$. However, X itself is a member of $\cup E(a) \perp a$. Hence $X = S_{i(=1,2)} \backslash (B \backslash \cup E(a))$. It follows from it that $S_{i(=1,2)} \subseteq X \cup (B \backslash \cup E(a))$. Now, by (b-1) above, $X \cup (B \backslash \cup E(a))$ is a member of $B \perp a$. Since S_1 and S_2 are also members of $B \perp a$ and are its subsets, clearly they are identical with it, whereby $S_1 = S_2$. **QED.**

Proof of Theorem 5.3. Given that C satisfies *Success*, *Credulity* and α, we show that for all S in $B \perp a$, if x is in S but not in $B \backslash R(a)$, then there is y in $B \backslash R(a)$ but not in S such that $C(x, y) = x$. Let x be in S but not in $B \backslash R(a)$. Clearly, by observation (5.3a), $B \backslash \cup E(a)$ is a subset of both S and $B \backslash R(a)$. Hence x is in $\cup E(a)$, whereby in $S \cap \cup E(a)$. Now

$$\begin{aligned} \cup E(a) \cap (B \backslash R(a)) &= (\cup E(a) \cap B) \backslash (\cup E(a) \cap R(a)) \\ &= \cup E(a) \backslash R(a). \end{aligned}$$

Furthermore, by observation (5.3b.2) $S \cap \cup E(a) = S \backslash (B \backslash \cup E(a))$ is in $\cup E(a) \perp a$. Hence, by corollary (5.2), there exists y in $\cup E(a) \backslash R(a)$ such that y is not in $S \cap \cup E(a)$ and $C(x, y) = x$. Now, $y \notin (S \cap \cup E(a)) \cup (B \backslash \cup E(a))$. Again, since $S \cap \cup E(a)$ is in $\cup E(a) \perp a$, it follows by observation (5.3b.2) that $(S \cap \cup E(a)) \cup (B \backslash \cup E(a))$ is in $B \perp a$. Since $S \cap \cup E(a)$ is also included in S and S is in $B \perp a$, it follows by observation (5.3c) that S is identical with $(S \cap \cup E(a)) \cup (B \backslash \cup E(a))$, whereby y is not in S. Clearly y is in $B \backslash R(a)$. **QED.**

Proof of Theorem 5.4. Given that C satisfies *Success*, *Credulity* and α, we show that if $S \in B \perp a$ is such that for all $S' \in B \perp a$, if x is in $S' \backslash S$ there is $y \in S \backslash S'$ such that $C(x, y) = x$, then S is unique. Let S_1 and S_2 be two members of $B \perp a$ such that for all S' in $B \perp a$, if $x \in S' \backslash S_{i(=1,2)}$ then there is y in $S_{i(=1,2)} \backslash S'$ such that $C(x, y) = x$. Suppose that S_1 and S_2 are different. Then, given *Credulity* and α it follows that there is an element

s_2 in $S_2 \backslash S_1$ such that for all x in $S_1 \backslash S_2$, $C(x, s_2) = x$ and there is s_1 in $S_1 \backslash S_2$ such that for all y in $S_2 \backslash S_1$, $C(s_1, y) = y$. Then, by *Credulity*, $s_1 = s_2$. But by assumption, s_1 and s_2 are in two disjoint sets. Hence, by reductio, $S_1 = S_2$. **QED**.

Proof of Lemma 5.1.[20] Assume the antecedents. If $a \notin K$, then the solution is trivial: we identify D^* with K itself. For the principal case, let W be a maximally consistent set of propositions such as $D \subseteq W$ and $\neg a \in W$. Now, either $W \subseteq K$ or not. If $W \subseteq K$ (this is the case if K is inconsistent), W itself is the required D^*. On the other hand, if $W \nsubseteq K$ (i.e. K is consistent), consider $W \cap K$. Clearly $a \notin Cn(W \cap K)$. Let $x \in K \backslash W$. Since W is maximally consistent, $\neg x$ is in W, whereby $\neg x \vee a$ is in it. Furthermore, since a is in K, surely $\neg x \vee a$ is in K too. Hence $\neg x \vee a$ is in $W \cap K$, whereby $a \in Cn((W \cap K) \cup \{x\})$. Hence $W \cap K$ is the desired D^*. **QED**.

Proof of Theorem 5.5. (*Left to right*) Let $X \in K \perp a$ such that $B \backslash R(a) \subseteq X$. Let $X' \in K \perp a$. It suffices to show that $X' \sqsubseteq X$, i.e. $B \cap X' \sqsubseteq B \cap X$.

First of all, we claim that $B \cap X = B \backslash R(a)$. Right to left is trivial (surely $B \backslash R(a) \subseteq B$ and $B \backslash R(a) \subseteq X$.) Now we show that $B \cap X \subseteq B \backslash R(a)$. Suppose that $x \notin B \backslash R(a)$. We need to show that $x \notin B \cap X$. Since $x \notin B \backslash R(a)$, either $x \notin B$ or $x \in R(a)$. The first case is trivial. As to the second case, suppose that $x \in R(a)$. It will be sufficient to show that $x \notin X$. Suppose, to the contrary, that $x \in X$. Then $B \backslash R(a) \cup \{x\} \subseteq X$. But, since $x \in R(a)$, by the definition (5.2), $a \in Cn((B \backslash R(a)) \cup \{x\}) \subseteq Cn(X)$. However, since $X \in K \perp a$, $X \nvdash a$. Contradiction!

We further note that since $X' \cap B \nvdash a$ and $X' \cap B \subseteq B$ there is $S \in B \perp a$ such that $X' \cap B \subseteq S$. Clearly, $S \sqsubseteq B \backslash R(a)$. It follows then that since $X' \cap B$ is a subset of S, $(X' \cap B) \sqsubseteq B \backslash R(a)$.

(*Right to left*) Let $X \in K \perp a$ such that for all $X' \in K \perp a$, $X' \sqsubseteq X$. It suffices to show that $B \backslash R(a)$ is a subset of X. By lemma (5.1), for all S' in $B \perp a$ there is X' in $K \perp a$ such that $S' \subseteq X'$. Hence, for all S' in $B \perp a$, $S' \sqsubseteq X \cap B$. Furthermore, since $X \cap B \subseteq B$ and $X \cap B \nvdash a$ surely there is $S \in B \perp a$ such that $X \cap B \subseteq S$. Hence there is $S \in B \perp a$ such that for all $S' \in B \perp a$ we have $S' \sqsubseteq X \cap B \subseteq S$. In particular, $S \sqsubseteq X \cap B \subseteq S$, from which it easily follows that $X \cap B = S$. Hence $S \in B \perp a$ is such that for all $S' \in B \perp a$, $S' \sqsubseteq S$. But then by theorem (5.4), S is unique and by theorem (5.3) is $B \backslash R(a)$. Hence $B \backslash R(a) = X \cap B \subseteq X$. **QED**.

Proof of Observation 5.4. Let $S \in B \perp a$. Assume that x is in S but not in $B \uparrow a$. Surely there is $k \geq 1$ such that $x = b_k$. Hence $(B \uparrow a \cap \{b_1, \ldots, b_{k-1}\}) \cup \{b_k\} \vdash a$. However, since $b_k \in S \in B \perp a$, clearly there is $j < k$ such that $b_j \in B \uparrow a$ and $b_j \notin S$. Since $C(b_j, b_k) = b_k, b_j$ is the desired element. **QED**.

Proof of Theorem 5.6.

$$\begin{aligned} \cap\gamma C(K \perp a) &= \cap\{X \in K \perp a : B \backslash R(a) \subseteq X\} \\ &= Cn(B \backslash R(a) \cup (Cn(B) \cap Cn(\{\neg a\}))) \\ &\qquad \text{(by lemma (5.2))} \\ &= Cn(B \backslash R(a) \cup Cn((B \cap \{\neg a\}) \cup \\ &\quad \cup \{x \vee y : x \in (B \backslash \{\neg a\}), y \in (\{\neg a\} \backslash B)\} \\ &= Cn((B \backslash R(a)) \cup \{\neg a \vee x : x \in B\}) \\ &\qquad \text{(since } \neg a \notin B) \\ &= Cn((B \backslash R(a)) \cup \{a \rightarrow x : x \in R(a)\}) \\ &= Cn(B \dot{-}_m a). \ \textbf{QED}. \end{aligned}$$

NOTES

[1] Not that everybody agrees with it. Isaac Levi's [14] account of "coerced contraction" and Hansson's [10] account of "external revision" presuppose that first new information is added to the knowledge corpus, and then, if lacking, consistency is restored in it by selective deletion.

[2] See especially [16] for a very good discussion of this problem.

[3] Here we differ from Fuhrmann who, unlike us, does not assume that Cn satisfies the $\vee$-introduction in the premises, or that it includes tautological implication.

[4] After the authors of [1].

[5] For partial meet revision see [1] or [6]; for safe revision see [1], [3] or [6].

[6] γ is transitively relational over K just in case there is a transitive relation $\leq$ over 2^K such that, irrespective of what x is, a member A' of $K \perp x$ is in $\gamma(K \perp x)$ iff $A \leq A'$ for all $A \in K \perp x$. Intuitively speaking, the transitive relation $\leq$ orders subsets of K according to the degree they are worth keeping, and γ picks out the best such elements in $K \perp x$.

[7] The relation $<$ over a set A is said to be non-circular if and only if for no members $a_1, a_2, \ldots, a_{n(n \geq 1)}$ of A does $a_1 < a_2 < \ldots < a_n < a_1$ hold. This condition can be equivalently reformulated as the minimality condition that for all $a_1, \ldots, a_n \in A$ there exists $i \leq n$ such that for no $j \leq n$, $a_j < a_i$.

[8] See [7] for an interesting discussion on the distinction between the foundationalist vs. coherentist theories of belief change. Gärdenfors argues to the effect that though a belief set itself does not distinguish between the basic beliefs and the inferred beliefs, the epistemic entrenchment ordering associated with the belief set may implicitly contain some of the relevant information; hence the distinction between the basic beliefs and the inferred beliefs is not completely obliterated in the coherentist framework.

[9] We must note that we are deviating from the intuitive notion of contraction here. For instance, suppose that $B = \{a, a \leftrightarrow b\}$, $K = Cn(\{a, b\})$ and $x = b \rightarrow a$. Let $\nu(K, x) = Cn(\{b\})$ and let $\mu(B, x) = \{b\}$. Clearly ν can be taken to be a theory contraction operation on K, hence, according to this convention, μ is a base contraction operation on B. However, many like Isaac Levi would refuse to call μ a contraction operation on B simply because $\mu(B, x) \not\subseteq B$.

[10] We give the proofs of this and all other subsequent claims in the appendix at the end of the paper.

[11] In [4] he allows two or more members of B to be equally retractable, but in [5] he avoids such ties by the condition that comparative retractability is an acyclic relation. However, in both of these works he allows members of B to be incomparable under the comparative retractability relation.

[12] Note that the second clause is automatically satisfied if S is consistent, and that we have assumed that the base B is consistent.

[13] $\vee(X)$ and $\wedge(X)$ stand, respectively, for the mutual disjunction and the mutual conjunction of the members of X. As usual, $\vee(\emptyset)$ is taken to be the "falsity" F and $\wedge(\emptyset)$ is taken to be the "truth" T. Definitions (4.2) and (4.4) must presuppose that S' is finite, but this restriction hardly limits the power of these definitions. This can be shown by semantic considerations. Suppose we define $\vee(X)$ to be satisfiable just in case at least one member of X is satisfied in some model. Then we can talk of the satisfiability of $\vee(X)$ where X is arbitrarily large. Given some reasonable properties (like compactness) of $\models$, it can be shown that $\forall_{S' \subseteq S}[(S \setminus S') \not\models \vee(S')]$ just in case $\forall_{\text{finite}\, S' \subseteq S}[(S \setminus S') \not\models \vee(S')]$.

[14] One might think that we would need extra clauses in definitions (4.2) and (4.4) corresponding to the second clauses of definitions (4.1) and (4.3) in order to deal with a possibly inconsistent set S. But such contingencies are already taken care of: definition (4.2) implies that if S is strongly independent, then $(S \setminus \emptyset) \not\vdash \vee(\emptyset)$; i.e., if S is strongly independent, then S is consistent; similarly, definition (4.4) implies that if T is strongly independent to any of its subset S, then T is consistent.

[15] There is an interesting connection between the choice function that we propose to use and the (total) preference ordering that Nebel [18] uses. Suppose we define a preference ordering $\leq_C$ over B induced by C in the following manner:

$$x \leq_C y \quad \text{iff} \quad \exists_{S \subseteq B}(y \in S \wedge x \in C(S)).$$

Now, given *Success*, *Credulity* and α, we can show that $\leq_C$ is a total ordering over B. Furthermore, since in presence of *Success* and *Credulity*, α is equivalent to $\beta+$, the choice function C is "normal"; i.e. if we define a choice function $C_\leq$ induced by the preference relation $\leq$ in the manner:

$$\text{For all subsets } S \text{ of } B, \quad C_\leq(S) = \{x : \forall_{y \in S} x \leq y\}$$

then $C(S) = C_{\leq C}(S)$. This makes our approach all the more interesting. See Amartya Sen's [23] for more on it.
[16] Henceforth we use the convention of dropping the brackets in $C(\{\ldots\})$ and use $C(\ldots)$ instead.
[17] This definition of a reject-set might look rather ad-hoc. But, as we show in § 5.3, (observation 5.4), a more intuitive looking construction of a reject-set turns out to be equivalent to this.
[18] Readers acquainted with [18, 20] will notice some similarity between Nebel's results and ours. Nebel shows that given a total ordering $\leq'$ over B, we can have a transitive ordering $\sqsubseteq'$ over 2^K:

$$X \sqsubseteq' Y \quad \text{iff} \quad \forall_{x \in (B \cap X) \setminus (B \cap Y)} \exists_{y \in (B \cap Y) \setminus (B \cap X)} x \leq' y$$

and that there is exactly one member S of $B \perp a$ such that the $\sqsubseteq'$-best elements of $K \perp a$ are those elements of $K \perp a$ that include S. But he does not identify that interesting element of $B \perp a$. Using a choice function instead of a total ordering over B, we have identified the corresponding member of $B \perp a$ that does the trick. In his more recent paper [20] Nebel has developed the concept of *prioritized epistemic relevance*, that he introduced in [19]. He has shown that given an *unambiguous epistemic relevance ordering* over the belief base B (which amounts to a total ordering over B), the prioritized base revision operation (called *unambiguous partial meet revision*) satisfies all the eight revision postulates of Gärdenfors. However, he does not make any connection between the unambiguous partial meet revision of [20] and the maxichoice base revision of [18]. Interestingly, it turns out that the unambiguous partial meet revision corresponds to the foundational revision based on $B \uparrow a$. Hence, observation (5.4) shows that the unambiguous partial meet revision is really the foundational revision based on $\dot{+}_m$ in a different guise, and that the comparative retractability relation of Fuhrmann is intimately connected with the prioritized epistemic relevance of Nebel.
[19] By *Success* and *Credulity*, $C(\Gamma)$ is always a singleton (unless Γ is empty). Hence, in presence of this condition, we will often write $C(\Gamma) = \gamma$ instead of $C(\Gamma) = \{\gamma\}$, if there is no fear of confusion.
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Knowledge System Group,
Basser Dept. of Computer Science,
University of Sydney,
NSW 2006, Australia