Journal of Dynamics and Differential Equations, Vol. 4, No. 2, 1992

Inertial Manifolds and Inertial Sets for the Phase-Field Equations[1]

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The phase-field system is a mathematical model of phase transition, coupling temperature with a continuous order parameter which describes degree of solidification. The flow induced by this system is shown to be smoothing in $H^1 \times L^2$ and a global attractor is shown to exist. Furthermore, in low-dimensional space, the flow is essentially finite dimensional in the sense that a strongly attracting finite-dimensional manifold (or set) exists.

KEY WORDS: Parabolic; attractor; infinite-dimensional dynamical system; global existence and regularity.

0. INTRODUCTION

Starting from a Landau-Ginzburg free energy functional of the form

$$J(\phi) \equiv \int_\Omega [\xi^2 \, |\nabla\varphi|^2/2 + F(\phi)] \, dx$$

with double-well potential F, where the field ϕ is an order parameter representing local degree of solidification, one seeks an evolution equation for ϕ which will decrease $J(\phi)$. This view of phase transition was proposed by Halperin *et al.* [HHM] and Langer [L1, 2] and, later, by Collins and Levine [CL]. The potential F is temperature dependent so that the relative depth of the two wells, representing pure solid and pure liquid phases,

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changes with temperature. If the reduced temperature is denoted by u, then the usual choice for F, which ensures that solid is the preferred state for low temperatures and liquid for high temperatures, is given by

$$F(\phi) = \tfrac{1}{4}(\phi^2 - 1)^2 - 2u\phi$$

Here we have taken $u = 0$ to be the critical temperature for planar interfaces. At $u = 0$, $\phi = -1$ and $\phi = +1$ give the pure solid and liquid phases, respectively (see Fig. 1).

For the model to account for the latent heat released by freezing and subsequent conduction, an evolution equation for ϕ which decreases J for fixed temperature must be coupled with an evolution equation for u. The system devised by those mentioned previously is known as the *phase-field equations*:

$$(\mathrm{PF})\begin{cases} \tau\phi_t = \xi^2 \Delta\phi + \phi - \phi^3 + 2u \\ \left(u + \dfrac{l}{2}\phi\right)_t = K\Delta u \end{cases}$$

where τ is a relaxation time, ξ is a length scale, l is latent heat, and K is thermal diffusivity. A good description of the derivation of (PF) together with more sophisticated models which allow temperature-dependent latent heat, etc., is given by Penrose and Fife [PF] (see also [F]). They also show that these systems are thermodynamically consistent in the sense that entropy increases along trajectories of (ϕ, u).

Apart from the theoretical foundations being sound, computer simulations with the phase-field equations (see [K], for example) showing instability of moving planar interfaces and dendrite formation closely resemble physical experiments. Furthermore, recent analytical and formal asymptotic studies (see [AB], [BF], [C1–4], [CF], [F], and [FG], for

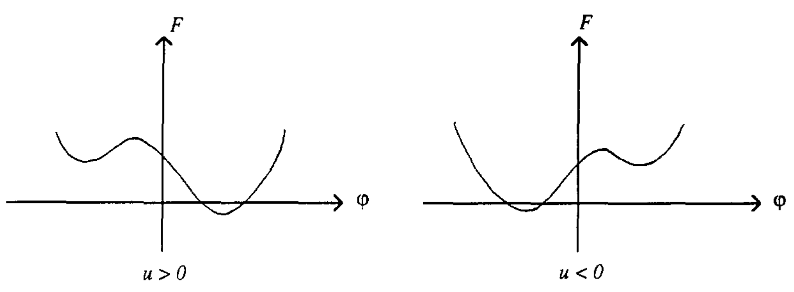

Fig. 1.

example) have predicted observed phenomena such as the Gibbs-Thompson relation, the spontaneous generation of phase interfaces, and subsequent coarsening. A rigorous analysis is far from complete, however. It is our intent to demonstrate that in one and two space dimensions, after a short time, the dynamics of (PF) are essentially governed by a finite system of ODEs. Granted, extremely complex behavior can be generated by finite-dimensional dynamical systems but we like to think that this, nevertheless, represents a significant simplification for a system of PDEs. When we say that the dynamics of (PF), together with appropriate boundary conditions, are essentially governed by a finite dimensional dynamical system, we are referring to the existence of an inertial manifold (or set). This is a finite dimensional manifold (or set) within the infinite dimensional state space which attracts all solutions to (PF) at an exponential rate (see [FST], [T]).

We wish to point out that a fundamental difficulty in dealing with the system (PF) is that it does not possess a maximum principle and only crude comparison results can be obtained. Furthermore, in its present form, regardless of which boundary conditions are imposed, the linearized operator is not self-adjoint and so does not fit the framework in [T] to produce inertial manifolds. For results in some non-self-adjoint cases the reader is referred to [M], [SY], and [Kw].

This paper is organized as follows: In Section 1 we consider the Dirichlet problem for (PF) in a bounded domain in $\mathbb{R}^n$ for $n \leqslant 3$. We show that positive semiorbits of (ϕ, u) are compact in $H^1 \times L^2$ and that the flow is smoothing. Furthermore, there exists a compact global attractor in $H^1 \times L^2$.

In Section 2, motivated by the results in [BF], we change variables in (PF), transforming it into a system with a self-adjoint linear part. We use a different change of variables from that given in [BF], which is more suitable for our choice of boundary conditions. We proceed then to demonstrate the existence of an inertial manifold in the case $n = 1$ and $n = 2$ with $\Omega = [0, L] \times [0, L]$, imposing Dirichlet boundary conditions on u and on ϕ.

In Section 3 we show that for a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \leqslant 3$, (PF) has an inertial set, that is, a positively invariant set of finite fractal dimension which attracts all solution at an exponential rate. The latter result relies on recent work by Eden *et al.* (see [EFNT 1, 2]).

Finally, we show that the previous results hold when u satisfies Neumann boundary conditions, provided that one restricts attention to fixed energy surfaces $\int_\Omega (u + (l/2)\phi)\, dx = \text{constant}$.

These energy surfaces are invariant under (PF) when zero-flux boundary conditions are imposed. This of course means that there is not a global attractor in the usual sense but the state space is foliated with invariant affine hyperplanes, each of which contains a compact attractor and an inertial manifold (or set).

1. ABSORBING SET AND GLOBAL ATTRACTOR

In this section we are going to prove that for the following phase-field problem,

$$\tau\phi_t = \xi^2 \Delta\phi + \phi - \phi^3 + 2u \tag{1.1}$$

$$u_t + \frac{l}{2}\phi_t = K\Delta u \tag{1.2}$$

$$u|_\Gamma = u_\Gamma(x), \qquad \phi|_\Gamma = \phi_\Gamma(x) \tag{1.3}$$

$$u|_{t=0} = u_0(x), \qquad \phi|_{t=0} = \phi_0(x) \tag{1.4}$$

for given functions $u_\Gamma(x)$ and $\phi_\Gamma(x)$ and for all $u_0(x)$, $\phi_0(x)$ in certain Sobolev spaces, there exists an absorbing set and a global attractor. We first prove the following global existence and uniqueness results.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \leqslant 3)$ be a bounded domain with smooth boundary Γ and let $u_\Gamma(x)$ and $\phi_\Gamma(x)$ be given smooth functions of x on Γ. Suppose $u_0(x) \in L^2(\Omega)$, $\phi_0(x) \in H^1(\Omega)$, satisfying the compatibility condition $\gamma_0(\phi_0) = \phi_\Gamma$. Then problem (1.1)–(1.4) admits a unique global solution, $\phi \in C(\mathbb{R}^+; H^1)$, $u \in C(\mathbb{R}^+, L^2)$ for any $T > 0$, $\phi_t \in L^2([0, T], L^2)$, $\phi \in L^2([0, T], H^2)$. Moreover, u and $\phi \in C^\infty((0, \infty), C^\infty(\Omega))$ and the orbit $t \in [\varepsilon, +\infty) \to (\phi(\cdot, t), u(\cdot, t))$ is compact in $H^1 \times L^2$ for any $\varepsilon > 0$.*

Remark. The restriction $n \leqslant 3$ is not necessary, and for general n the solution $(\phi(\cdot, t), u(\cdot, t)) \in (H^1 \cap L^4) \times L^2$. For existence and uniqueness of solutions, we need only the boundary data u_Γ and ϕ_Γ to be in the trace class $H^{1/2}(\Gamma)$. The corresponding regularity of the solution is as expected.

Proof. The global existence and uniqueness of a smooth solution have been proved in [EZ] for $(\phi_0, u_0) \in H^2(\Omega) \times H^2(\Omega)$. Moreover, without loss of generality, taking $u_\Gamma = \phi_\Gamma = 0$, we find

$$\int_\Omega \left(\frac{\xi^2}{2}|\nabla\phi|^2 + \frac{1}{4}\phi^4 - \frac{1}{2}\phi^2 + \frac{2}{l}u^2\right) dx + \tau\int_0^t \|\phi_t\|^2\, dt + \int_0^t \frac{4K}{l}\|\nabla u\|^2\, dt$$

$$= \int_\Omega \left(\frac{\xi^2}{2}|\nabla\phi_0|^2 + \frac{1}{4}\phi_0^4 - \frac{1}{2}\phi_0^2 + \frac{2}{l}u_0^2\right) dx, \quad \forall t > 0 \tag{1.5}$$

Then the usual compactness argument yields the global existence and uniqueness for $(\phi_0, u_0) \in H^1 \times L^2$. Moreover, the identity (1.5) still holds. To prove the compactness of the orbit $t \in [\varepsilon, +\infty) \to (\phi(\cdot, t), u(\cdot, t))$ we need the following lemma.

Lemma 1.2. *Suppose $f \in L^2([0, T]; L^2)$, $u_0 \in L^2(\Omega)$. Then the following problem,*

$$u_t - \Delta u = f \text{ in } \Omega \times (0, T) \tag{1.6}$$

$$u|_\Gamma = 0 \text{ on } \Gamma \times (0, T) \tag{1.7}$$

$$u|_{t=0} = u_0(x) \text{ in } \Omega \tag{1.8}$$

admits a unique solution, $u \in C([0, T]; L^2) \cap L^2([0, T]; H_0^1)$. Moreover, $u \in C((0, T]; H_0^1) \cap L^2([\varepsilon, T]; H^2)$, $u_t \in L^2([\varepsilon, T]; L^2)$ for any $\varepsilon > 0$,

$$\|u(t)\|_{H^1}^2 \leqslant \frac{2}{t} \|u_0\|^2 + 2 \int_0^t \|f\|^2 \, dt, \qquad \forall t > 0 \tag{1.9}$$

Furthermore, if $f_t \in L^2((0, T], L^2)$, then $u_t \in C((0, T]; H_0^1)$, and $u \in C((0, T]; H^2)$

$$\|u_t(t)\|_{H^1}^2 \leqslant \frac{16}{t^3} \|u_0\|^2 + \frac{4}{t} \|f(0)\|^2 + 4 \int_0^t \|f_t\|^2 \, dt, \qquad \forall t > 0 \tag{1.10}$$

We postpone the proof of Lemma 1.2.

Once we have Lemma 1.2, it follows from Eq. (1.2) that $u \in C((0, T]; H^1)$ and

$$\|u(t)\|_{H^1} \leqslant C_\varepsilon, \qquad \forall t \in [\varepsilon, T] \tag{1.11}$$

It turns out from (1.2) and the regularity results (see Theorem II.3.3 in [T]) that we have

$$u_t \in L^2([\varepsilon, T]; L^2) \qquad \text{for} \quad \varepsilon > 0 \tag{1.12}$$

Thus, Eq. (1.1) can be viewed as

$$\tau \phi_t = \xi^2 \Delta \phi + f \tag{1.13}$$

with

$$f \in L^2([\varepsilon, T]; L^2), \qquad f_t \in L^2([\varepsilon, T]; L^2)$$

Applying Lemma 1.2 again, we conclude

$$\phi_t \in C([\varepsilon, T], H^1), \qquad \phi \in C([\varepsilon, T]; H^2) \tag{1.14}$$

$$\|\phi(t)\|_{H^2} \leqslant C_\varepsilon, \qquad \forall t \in [2\varepsilon, T] \tag{1.15}$$

By the usual bootstrap argument, we get the $C^\infty(\Omega \times (0, +\infty))$ regularity results. The compactness of the orbit $t \in (\varepsilon, +\infty) \to (\phi(\cdot, t), u(\cdot, t))$ in $H^1 \times L^2$ follows from (1.11), (1.15), and the uniform a priori estimates given in [EZ]. Thus the proof of Theorem 1.1 is completed. We now give the proof of Lemma 1.2.

Proof of Lemma 1.2. The existence and uniqueness of solution in the space $u \in C([0, T], L^2) \cap L^2([0, T]; H_0^1)$ are well-known (for instance, see Theorem II.3.1 in [T]; see also [H] and [P]). Therefore, we need only to prove (1.9) and (1.10). Similar estimates can be found in [H] but for later use we include the details of the proof.

Let u_1 be the solution to the problem

$$u_t - \Delta u = f \tag{1.16}$$

$$u|_\Gamma = 0 \tag{1.17}$$

$$u|_{t=0} = 0 \tag{1.18}$$

and u_2 be the solution to the problem

$$u_t - \Delta u = 0 \tag{1.19}$$

$$u|_\Gamma = 0 \tag{1.20}$$

$$u|_{t=0} = u_0(x) \tag{1.21}$$

By uniqueness we have

$$u = u_1 + u_2 \tag{1.22}$$

Applying the regularity result to u_1 (see Theorem II.3.3 in [T]), we have $u_1 \in C([0, T]; H_0^1) \cap L^2([0, T], H^2)$, $u_{1_t} \in L^2([0, T]; L^2)$. Moreover,

$$\|u_1(t)\|^2 \leqslant \int_0^t \|f\|^2 \, dt, \qquad \forall t \geqslant 0 \tag{1.23}$$

Since $-\Delta$ is a symmetric operator with the domain $D(A) = H^2 \cap H_0^1$ dense in $L^2(\Omega)$ by a well-known result in semigroup theory [P] we have

$$u_2 \in C^j((0, \infty); D(A^k)), \qquad \forall j, k \geqslant 0 \tag{1.24}$$

Multiplying Eq. (1.19) by u and u_t, respectively, and integrating yields

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2+\|\nabla u\|^2=0, \qquad \forall t>0 \tag{1.25}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\|u_t\|^2=0, \qquad \forall t>0 \tag{1.26}$$

Multiplying (1.26) by t and then adding to (1.25) yields

$$\frac{1}{2}\frac{d}{dt}(t\|\nabla u\|^2)+t\|u_t\|^2+\frac{1}{2}\frac{d}{dt}\|u\|^2+\frac{1}{2}\|\nabla u\|^2=0 \tag{1.27}$$

Integrating with respect to t gives

$$t\|\nabla u(t)\|^2+\|u(t)\|^2\leqslant\|u_0\|^2 \tag{1.28}$$

$$\|\nabla u(t)\|^2\equiv\|u(t)\|^2_{H^1}\leqslant\frac{1}{t}\|u_0\|^2 \tag{1.29}$$

Adding (1.29) with (1.23) results in (1.9). Similarly, since $(u_1)_t$ satisfies

$$u_t-\Delta u=f_t \tag{1.30}$$

$$u|_\Gamma=0 \tag{1.31}$$

$$u|_{t=0}=f(x,0) \tag{1.32}$$

we have

$$\|(u_1)_t(t)\|^2_{H^1}\leqslant\frac{2}{t}\|f(0)\|^2+2\int_0^t\|f_t\|^2\,dt \tag{1.33}$$

For u_2 we have

$$\|(u_2)_t(t)\|^2_{H^1}\leqslant\frac{1}{(t/2)}\left\|(u_2)_t\left(\frac{t}{2}\right)\right\|^2 \tag{1.34}$$

Noticing that $\|(u_2)(t)\|^2$ is decreasing with respect to t, we have, by integrating (1.27) with respect to t,

$$\frac{t^2}{2}\|u_t(t)\|^2+\frac{1}{2}\|u(t)\|^2\leqslant\frac{1}{2}\|u_0\|^2 \tag{1.35}$$

$$\|u_t(t)\|^2\leqslant\frac{1}{t^2}\|u_0\|^2 \tag{1.36}$$

Using (1.34), (1.36), we find

$$\|(u_2)_t(t)\|_{H^1}^2 \leqslant \frac{1}{(t/2)} \left\|(u_2)_t\left(\frac{t}{2}\right)\right\|^2 \leqslant \frac{8}{t^3}\|u_0\|^2 \tag{1.37}$$

Thus, (1.10) follows from (1.34), (1.36).

In what follows, we prove the existence of an absorbing set. We first use translation of u and ϕ to make the boundary condition homogeneous. Let $\bar{u}$, $\bar{\phi}$ be harmonic functions satisfying on the boundary Γ

$$\bar{u}|_\Gamma = u_\Gamma(x), \qquad \bar{\phi}|_\Gamma = \phi_\Gamma(x) \tag{1.38}$$

We introduce new unknown functions

$$v = u - \bar{u}, \qquad \psi = \phi - \bar{\phi} \tag{1.39}$$

Then ψ and v satisfy

$$\tau\psi_t = \xi^2 \Delta\psi + (\psi + \bar{\phi}) - (\psi + \bar{\phi})^3 + 2(v + \bar{u}) \tag{1.40}$$

$$v_t + \frac{l}{2}\psi_t = K\Delta v \tag{1.41}$$

$$\psi|_\Gamma = v|_\Gamma = 0 \tag{1.42}$$

$$\psi|_{t=0} = \psi_0(x) \equiv \phi_0(x) - \bar{\phi}(x), \qquad v|_{t=0} = u_0(x) - \bar{u}(x) = v_0(x) \tag{1.43}$$

To prove the existence of an absorbing set for ϕ and u, we need only to prove the existence of absorbing set for ψ and v.

Multiplying (1.40) by ψ_t and (1.41) by $(4/l)v$, and adding and integrating with respect to x, yields

$$\frac{d}{dt}\int_\Omega \left(\frac{\xi^2}{2}|\nabla\psi|^2 + \frac{1}{4}(\psi + \bar{\phi})^4 - \frac{1}{2}(\psi + \bar{\phi})^2 + \frac{2}{l}v^2 - 2\psi\bar{u}\right)dx$$
$$+ \tau\|\psi_t\|^2 + \frac{4K}{l}\|\nabla v\|^2 = 0 \tag{1.44}$$

Let

$$V(t) = \int_\Omega \left(\frac{\xi^2}{2}|\nabla\psi|^2 + \frac{1}{4}(\psi + \bar{\phi})^4 - \frac{1}{2}(\psi + \bar{\phi})^2 + \frac{2}{l}v^2 - 2\psi\bar{u}\right)dx \tag{1.45}$$

It is easy to see from the expresion for $V(t)$ that the boundedness of $V(t)$ from above implies the boundedness of $\|\psi\|_{H^1}^2 + \|v\|^2$. Therefore, we need only to prove that $\limsup_{t\to\infty} V(t) \leqslant C$, independent of ψ_0 and v_0. Multiplying (1.40) by ψ and integrating with respect to x yields

$$\int_\Omega [\xi^2 |\nabla\psi|^2 + (\psi+\bar{\phi})^4 - \bar{\phi}(\psi+\bar{\phi})^3 - (\psi+\bar{\phi})^2 + \bar{\phi}(\psi+\bar{\phi}) - 2(\psi+\bar{\phi})v + 2\bar{\phi}v - 2(\psi+\bar{\phi})\bar{u} + 2\bar{\phi}\bar{u} + \tau\psi_t(\psi+\bar{\phi}) - \tau\psi_t\bar{\phi}]\, dx = 0 \tag{1.46}$$

By the Young inequality $ab \leqslant (a^p/p) + (b^q/q)$, we easily get from (1.46)

$$\int_\Omega [\xi^2 |\nabla\psi|^2 + \frac{1}{2}(\psi+\bar{\phi})^4 - (\psi+\bar{\phi})^2 - 4\psi\bar{u}]\, dx \leqslant 2\tau \|\psi_t\|^2 + \varepsilon \|v\|^2 + C_\varepsilon \tag{1.47}$$

with ε being an arbitrary constant and $C_\varepsilon > 0$ a constant depending only on ε, $\bar{\phi}$, $\bar{u}$.

By the Poincare inequality, we have

$$\|v\|^2 \leqslant C \|\nabla v\|^2 \tag{1.48}$$

with $C>0$ depending only on the domain Ω. Dividing (1.47) by 2 and choosing $\varepsilon = 8K/Cl$, then adding with (1.44) yields

$$\frac{dV}{dt} + V(t) \leqslant C' \tag{1.49}$$

with $C'>0$ depending only on $\bar{\phi}$, $\bar{u}$. It follows from (1.49) that

$$V(t) \leqslant e^{-t} V(0) + C' \tag{1.50}$$

Notice that

$$V(0) = \int_\Omega \left(\frac{\xi^2}{2} |\nabla\psi_0|^2 + \frac{1}{4}\phi_0^4 - \frac{1}{2}\phi_0^2 + \frac{2}{l} v_0^2 - 2\psi_0 \bar{u} \right) dx \tag{1.51}$$

is bounded if $\|\psi_0\|_{H^1}^2 + \|v_0\|^2$ is bounded. The inequality (1.50) implies the existence of an absorbing set.

We now have the following theorem.

Theorem 1.3. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary Γ. Suppose $u_\Gamma(x)$, $\phi_\Gamma(x)$ are given smooth functions. Then the semigroup $S(t)$ associated with the system (1.1)–(1.4) possesses a maximal attractor $\mathscr{A}$ which is bounded in $H^2(\Omega) \times H^2(\Omega)$, compact and connected in $H^1(\Omega) \times L^2(\Omega)$, and attracts the bounded sets of $H^1(\Omega) \times L^2(\Omega)$.*

Proof. The semigroup $S(t)$ associated with the system (1.1)–(1.4) is defined as follows:

$$S(t)\colon (\phi_0, u_0) \in H^1 \times L^2 \to (\phi(\cdot, t), v(\cdot, t)) \tag{1.52}$$

Since $\phi_t \in L^2(\mathbb{R}^+, L^2(\Omega))$, $u \in L^2([0, T]; H^1(\Omega))$ for any $T > 0$ as proved in Theorem 1.1, then $f \equiv \phi - \phi^3 + 2u \in L^2([0, T]; L^2)$, $g \equiv (l/2)\phi_t \in L^2([0, T]; L^2)$ immediately imply that $S(t)$ is continuous in $H^1(\Omega) \times L^2(\Omega)$ for $t \geqslant 0$. Theorem 1.1 also claims that for any $\varepsilon > 0$ and any bounded set $B \subset H^1 \times L^2$, $\bigcup \{S(t)B\colon t \geqslant \varepsilon\}$ is relatively compact in $H^1 \times L^2$. The existence of an absorbing set has been proved in the above. Thus the conclusion of this theorem follows from Theorem I.1.1 in [T].

Remark. If ϕ satisfies homogeneous Neumann instead of Dirichlet boundary conditions, the previous proofs are easily modified to again deduce the existence of a compact attractor.

2. INERTIAL MANIFOLDS

In this section we discuss the inertial manifold of semigroup $S(t)$ associated with the system (1.40)–(1.43) instead of (1.1)–(1.4) in one and two space dimension, and in the next section we also discuss the existence of an inertial set, a notion recently introduced, and studied by Eden *et al.* (see [EFNT 1, 2] and [EMN]).

We first discuss the system (1.40)–(1.43) is one space dimension.

Since the phase-field equations (1.40)–(1.41) are not a diagonal parabolic system, if we put them into the abstract framework of first-order evolution equations

$$\frac{du}{dt} + Au + F(u) = 0 \tag{2.1}$$

by subtracting (1.41) from (1.40) times $-(l/2\tau)$, then the operator is not self-adjoint. But the existing theory for inertial manifolds (see [T]) usually requires that A be a self-adjoint operator. In what follows we use a technique similar to that in [BF] to reduce the problem to one with A being self-adjoint. Dividing (1.40) by τ we obtain

$$\psi_t = \frac{\xi^2}{\tau} \Delta\psi + \frac{1}{\tau} [(\psi + \bar{\phi}) - (\psi + \bar{\phi})^3 + 2v + 2\bar{u}] \tag{2.2}$$

Multiplying (2.2) by $-(l/2)$ and then adding to (1.41) yields

$$v_t = K \Delta v - \frac{l\xi^2}{2\tau} \Delta\psi - \frac{l}{2\tau} [(\psi + \bar{\phi}) - (\psi + \bar{\phi})^3 + 2v + 2\bar{u}] \tag{2.3}$$

Since $-\Delta$ defined on $H^2 \cap H_0^1 \subset L^2(\Omega)$ is a positive definite operator, we can write $-\Delta$ as

$$-\Delta = A^2 \tag{2.4}$$

where A is a self-adjoint positive definite operator. It can be given explicity by

$$Au = \sum_{n=1}^{\infty} \lambda_n^{1/2}(u, u_n)u_n, \qquad \forall u \in D(A) = H_0^1 \tag{2.5}$$

with u_n being normalized eigenfunctions of $-\Delta$ associated with eigenvalues λ_n and (u, u_n) being the inner product in L^2. Also,

$$A^{-1}u = \sum_{n=1}^{\infty} \lambda_n^{-1/2}(u, u_n)u_n \tag{2.6}$$

Let

$$a = \frac{2}{\sqrt{l}\,\xi}, \qquad e = aA^{-1}v \tag{2.7}$$

Then (2.2) becomes

$$\psi_t = \frac{\xi^2}{\tau}\Delta\psi + \frac{\sqrt{l}\,\xi}{\tau}Ae + f_1(\psi) \tag{2.8}$$

$$f_1 = \frac{1}{\tau}\left[(\psi + \bar{\phi}) - (\psi + \bar{\phi})^3 + 2\bar{u}\right] \tag{2.9}$$

Acting on Eq. (2.3) with aA^{-1} yields

$$e_t = K\Delta e + \frac{\sqrt{l}\,\xi}{\tau}A\psi + f_2(\psi) - \frac{l}{\tau}e \tag{2.10}$$

with

$$f_2(\psi) = \frac{-\sqrt{l}}{\tau\xi}A^{-1}\left[(\psi + \bar{\phi}) - (\psi + \bar{\phi})^3 + 2\bar{u}\right] \tag{2.11}$$

Then the system (2.8), (2.10) can be written as

$$\frac{dU}{dt} + \mathscr{A}U = R(U) \tag{2.12}$$

with

$$U=\begin{pmatrix}\psi\\ e\end{pmatrix},\qquad \mathscr{A}=\begin{pmatrix}-\dfrac{\xi^2}{\tau}\Delta & -\dfrac{\sqrt{l}\,\xi}{\tau}\Delta\\ -\dfrac{\sqrt{l}\,\xi}{\tau}\Delta & -K\Delta+\dfrac{l}{\tau}I\end{pmatrix},\qquad R=\begin{pmatrix}f_1\\ f_2\end{pmatrix} \tag{2.13}$$

and initial condition

$$U(0)=U_0\equiv\begin{pmatrix}\psi_0\\ aA^{-1}v_0\end{pmatrix} \tag{2.14}$$

Here $\mathrm{dom}(\mathscr{A})=H^2\cap H_0^1\times H^2\cap H_0^1$.

It is easy to see from the expression for R that the system (1.40)–(1.43) is equivalent to the system (2.12)–(2.14) in the sense that if (ψ, u) is a solution to the system (1.40)–(1.43), then (ψ, e) is a solution to (2.12)–(2.14), and vice versa.

In what follows, we study the dynamical system (2.12)–(2.14) instead of the system (1.40)–(1.43). Theorem 1.3 shows, by the equivalence of the two systems mentioned above, that the semigroup operator $S(t)$ associated with the system (2.12)–(2.14) possesses a global (maximal) attractor which is bounded in $H^2\times H^3$, compact and connected in $H_0^1\times H_0^1$, and attracts the bounded sets of $H_0^1(\Omega)\times H_0^1(\Omega)$. Consider $\mathscr{A}$ as an operator with domain $H^2\cap H_0^1\times H^2\cap H_0^1$ in $L^2\times L^2$. Then it is easy to see that $\mathscr{A}$ is self-adjoint. Also,

$$\begin{aligned}(\mathscr{A}U, U)&=\left(-\frac{\xi^2}{\tau}\Delta\psi,\psi\right)-2\frac{\sqrt{l}\,\xi}{\tau}(\Delta\psi, e)+\left(-K\Delta e+\frac{l}{\tau}e, e\right)\\
&=\frac{\xi^2}{\tau}\sum_{n=1}^{\infty}\lambda_n(\psi,u_n)^2-2\frac{\sqrt{l}\,\xi}{\tau}\sum_{n=1}^{\infty}\lambda_n^{1/2}(\psi,u_n)(e,u_n)\\
&\quad+\frac{l}{\tau}\sum_{n=1}^{\infty}(e,u_n)^2+K\sum_{n=1}^{\infty}\lambda_n(e,u_n)^2\\
&\geqslant\begin{cases}K\displaystyle\sum_{n=1}^{\infty}\lambda_n(e,u_n)^2 & \text{if } e\neq 0\\ \dfrac{\xi^2}{\tau}\displaystyle\sum_{n=1}^{\infty}\lambda_n(\psi,u_n)^2 & \text{if } e=0\end{cases}\end{aligned} \tag{2.15}$$

Thus $\mathscr{A}$ is a positive definite operator.

Theorem 2.1. *Let $n=1$, $(\Omega=(0,L))$. Then system (2.12)–(2.14) possesses an inertial manifold of the form given by Theorem VIII.3.2 in*

([T], p. 436) in $D(\mathscr{A}^{1/2})=H_0^1\times H_0^1$. This implies that the system (1.40)–(1.43) admits an inertial manifold in $H_0^1\times L^2$.

Proof. It remains to prove that R is a bounded mapping from $D(\mathscr{A}^\alpha)$ into $D(\mathscr{A}^\alpha)$ (taking $\alpha=\frac{1}{2}$, $\gamma=0$ in [T]) and R is locally Lipschitz and, also, to prove that the spectral gap condition is satisfied.

For $\psi\in H_0^1$, $n=1$, by Sobolev's imbedding theorem, f_1, f_2 are bounded mappings from $H_0^1\to H_0^1$. It is easy to see that f_1 is locally Lipschitz from H_0^1 to H_0^1. To prove f_2 is also locally Lipschitz, since A^{-1} is a bounded operator from L^2 to H_0^1, we need only to consider the term $A^{-1}[(\psi+\bar{\phi})^3]$. By (2.6) we have

$$\|A^{-1}[(\psi_1+\bar{\phi})^3]-A^{-1}[(\psi_2+\bar{\phi})^3]\|_{H^1}=\|(\psi_1+\bar{\phi})^3-(\psi_2+\bar{\phi})^3\|$$
$$\leqslant C_M\|\psi_1-\psi_2\|_{H^1} \qquad \text{if} \quad \|\psi_1\|_{H^1}\leqslant M, \quad \|\psi_2\|_{H^1}\leqslant M \tag{2.16}$$

C_{M} being a constant depending on M.

The spectral gap condition is the condition that the spectrum of $\mathscr{A}$ lies outside a sufficiently large interval of the positive real axis. We show that there are arbitrarily large gaps in the spectrum of $\mathscr{A}$.

For $\Omega=(0, L)$, we look for the eigenvalue λ and the associated eigenfunction such that $(\psi, e)\in H^2\cap H_0^1\times H^2\cap H_0^1$

$$\begin{pmatrix} -\dfrac{\xi^2}{\tau}\Delta & -\dfrac{\sqrt{l}\,\xi}{\tau}A \\ -\dfrac{\sqrt{l}\,\xi}{\tau}A & -K\Delta+\dfrac{l}{\tau}\end{pmatrix}\begin{pmatrix}\psi\\ e\end{pmatrix}=\lambda\begin{pmatrix}\psi\\ e\end{pmatrix} \tag{2.17}$$

We rewrite the equations separately:

$$-\frac{\xi^2}{\tau}\Delta\psi-\frac{\sqrt{l}\,\xi}{\tau}Ae=\lambda\psi \tag{2.18}$$

$$-\frac{\sqrt{l}\,\xi}{\tau}A\psi-K\Delta e+\frac{l}{\tau}e=\lambda e \tag{2.19}$$

Acting with A on (2.18), using (2.4), replacing $A\psi$ by the one in (2.9), we get

$$\frac{K\xi}{\sqrt{l}}\Delta^2 e+\left(\frac{\lambda\xi}{\sqrt{l}}+\frac{K\tau}{\sqrt{l}\,\xi}\lambda\right)\Delta e+\left(\frac{\lambda^2\tau}{\sqrt{l}\,\xi}-\frac{\sqrt{l}}{\xi}\lambda\right)e=0 \tag{2.20}$$

The normalized eigenfunctions, which are also the eigenfunctions of $-\Delta$ on $H^2\cap H_0^1$, are

$$e_n=\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, \qquad n=1, 2, \ldots, \tag{2.21}$$

The corresponding eigenvalues $\lambda = \tilde{\lambda}_n$ satisfy

$$\left(\frac{\lambda^2 \tau}{\sqrt{l}\,\xi} - \frac{\lambda\sqrt{l}}{\xi}\right) - \lambda\left(\frac{K\tau + \xi^2}{\sqrt{l}\,\xi}\right)\lambda_n + \frac{K\xi}{\sqrt{l}}\lambda_n^2 = 0 \tag{2.22}$$

where $\{\lambda_n\}^\infty$ are the eigenvalues of $-\Delta$ on $H^2 \cap H_0^1$, which is this case are given by $\lambda_n = (n\pi/L)^2$, $n = 1, 2, \ldots$.

Thus, the eigenvalues $\{\tilde{\lambda}_n\}_{n=1}^\infty$ are given by two forms:

$$\begin{aligned} \lambda_n^+ &= a\lambda_n + b + \sqrt{(a\lambda_n + b)^2 - c^2\lambda_n^2} \\ \lambda_n^- &= a\lambda_n + b - \sqrt{(a\lambda_n + b)^2 - c^2\lambda_n^2} \end{aligned} \tag{2.23}$$

where $a = (K\tau + \xi^2)/2\tau$, $b = (l/2\tau)$, and $c = \sqrt{(K/\tau)}\,\xi$. Note that $a^2 \geqslant c^2$. We find

$$\begin{aligned} d_n^+ &\equiv \lambda_{n+1}^+ - \lambda_n^+ = (\lambda_{n+1} - \lambda_n)[a + \alpha_n] \\ d_n^- &\equiv \lambda_{n+1}^- - \lambda_n^- = (\lambda_{n+1} - \lambda_n)[a - \alpha_n] \end{aligned} \tag{2.24}$$

where $\alpha_n \to \sqrt{a^2 - c^2}$ as $n \to \infty$.

It follows that $d_n^+ \geqslant d_n^- \to \infty$ since $\lambda_{n+1} - \lambda_n \to \infty$ as $n \to \infty$. For fixed N, we define the *gap* at λ_N^- to be the maximum of $(\lambda_N^- - \mu)$ and $(\nu - \lambda_N^-)$, where $\mu(\nu)$ is the largest (smallest) eigenvalue of $\mathscr{A}$ less (greater) than λ_N^-. Let $K = K(N)$ be defined by

$$\lambda_K^+ \leqslant \lambda_N^- < \lambda_{K+1}^+$$

Then

$$\text{either} \quad \mu = \lambda_K^+ \quad \text{or} \quad \mu = \lambda_{N-1}^-$$

and

$$\text{either} \quad \nu = \lambda_{K+1}^+ \quad \text{or} \quad \nu = \lambda_{N+1}^-$$

It follows that the gap at λ_N^- is at least

$$d_N \equiv \min\{d_N^-, d_{N-1}^-, \tfrac{1}{2}d_K^+\} \tag{2.25}$$

Clearly $K = K(N) \to \infty$ and hence $d_N \to \infty$ as $N \to \infty$. Thus, the spectral gap conditions (3.7) and (3.51) in [T] (pp. 423, 435) are satisfied and the proof of the Theorem is complete.

For $n = 2$ and $\Omega = [0, L]^2$, using a result in number theory (see [R]), we have the following theorem.

Theorem 2.2. *Let $n = 2$, $(\Omega = [0, L]^2)$. Then system (2.12)–(2.14) possesses an inertial manifold of the form given by Theorem VIII.3.2 in [T]* *in* $D(\mathscr{A}) = H^2 \cap H_0^1 \times H^2 \cap H_0^1$. *This implies that the system (1.40)–(1.43) admits an inertial manifold in* $H^2 \cap H_0^1 \times H_0^1$.

Proof. By Sobolev's imbedding theorem, $H^2(n=2)$ is continuously imbedded in $C(\Omega)$. Therefore, R is a bounded mapping from $D(\mathscr{A})$ into $D(\mathscr{A})(\alpha = 1, \gamma = 0)$. The same argument as in Theorem 2.1 yields that R is also locally Lipschitz. The spectrum is still given by (2.23) but λ_n is now the nth eigenvalue of $-\Delta$ with domain

$$H^2([0, L]^2) \cap H_0^1([0, L]^2)$$

These eigenvalues have the form

$$\left(\frac{\pi}{L}\right)^2 (i^2 + j^2) \qquad \text{with } i \text{ and } j \text{ integers} \tag{2.26}$$

and a result in number theory (see [R]) then implies the existence of $\beta > 0$ such that

$$\lambda_{n+1} - \lambda_n > \beta \log n \qquad \text{as} \quad n \to \infty \tag{2.27}$$

As before, the spectral gap condition is satisfied and so the proof is complete.

Remark. It is clear that this approach will fail for Ω a cube in $\mathbb{R}^3$ since the set of integers expressible as the sum of three squares has uniformly bounded gaps.

Remark. If we have the Neumann boundary condition for ϕ and the Dirichlet boundary condition for u,

$$u|_\Gamma = u_\Gamma(x), \qquad \left.\frac{\partial \phi}{\partial n}\right|_\Gamma = 0 \tag{1.3$'$}$$

instead of both Dirichlet boundary conditions (1.3), then the theorems on the existence of absorbing set and the global attractor still hold. But the above symmetrized method fails and the existence of an initial manifold remains open.

3. INERTIAL SET

We can see from the above that the gap condition imposed severe restrictions on the domain in order to obtain the existence of an inertial manifold. Recently, Eden *et al.* (see [EFNT1] and [EFNT2]) introduced the notion of the inertial set, which is defined to be a set of finite fractal dimension that attracts all solutions at an exponential rate. More precisely, let H be a separable Hilbert space and B a compact subset of H. Let $\{S(t)\}_{t\geqslant 0}$ be a nonlinear continuous semigroup that leaves the set B invariant. Let $\mathscr{S}$ be the global attractor for $\{S(t)\}_{t\geqslant 0}$ on B. Let us now recall the definition of inertial set (see [EMN], [EFNT1,2]).

Definition 3.1. A set M is called an inertial set for $(\{S(t)_{t\geqslant 0}, B\})$ if (i) $\mathscr{S}\subseteq M\subseteq B$, (ii) $S(t)M\subseteq M$ for every $t\geqslant 0$, (iii) for every $u_0\in B$, $\text{dist}_H(S(t)u_0, M)\leqslant C_1 e^{-C_2 t}$ for all $t\geqslant 0$, where C_1 and C_2 are independent of u_0, and (iv) M has finite fractal dimension, $d_f(M)$.

Definition 3.2. A continuous semigroup $\{S(t)\}_{t\geqslant 0}$ is said to satisfy the squeezing property on B if there exists $t_*>0$ such that $S_*=S(t_*)$ satisfies: there exists an orthogonal projection P of rank N_0 such that if for every u and v in B satisfying

$$\|P(S_*u-S_*v)\|_H\leqslant \|(I-P)(S_*u-S_*v)\|_H \tag{3.1}$$

then

$$\|S_*u-S_*v\|_H\leqslant \tfrac{1}{8}\|u-v\|_H \tag{3.2}$$

In [EMN] and [EFNT1,2] the following result has been established.

Theorem 3.1. *If $(\{S(t)\}_{t\geqslant 0}, B)$ satisfies the squeezing property on B and if $S_*=S(t_*)$ is Lipschitz on B with Lipschitz constant L, then there exists an inertial set M for $(\{S(t)\}_{t\geqslant 0}, B)$ such that*

$$d_f(M)\leqslant N_0\max\{1, \ln(16L+1)/\ln 2\} \tag{3.3}$$

and

$$\text{dist}_H(S(t)B, M)\leqslant C_1\exp\{(-C_2/t_*)t\} \tag{3.4}$$

In what follows, we are going to prove that for the system (1.1)–(1.4) and for general smooth domain Ω ($n\leqslant 3$), there exists an inertial set.

As in Section 2, instead of system (1.1)–(1.4), we consider system (2.12)–(2.14). We notice that the squeezing property implies the Lipschitz condition on the map $(t, u_0)\in[0, t_*]\times B\to S(t)u_0$ in the norm of H.

In what follows we take the product space $L^2\times L^2$ as H. We also take the product space $H^2\cap H_0^1\times H^2\cap H_0^1$ as E.

Theorem 3.2. *Let $\Omega\subset\mathbb{R}^n$ ($n\leqslant 3$) be a bounded domain with smooth boundary Γ. Let $U_0=(\psi_0, e_0)\in H^2\cap H_0^1\times H^2\cap H_0^1$. The system (2.12)–(2.14) admits a global solution $(\psi, e)\in C(\mathbb{R}^+, H^2\cap H_0^1\times H^2\cap H_0^1)\cap$*

$C^1(\mathbb{R}^+, L^2 \times L^2)$. *Moreover, there exists an absorbing set* B *in* $E = H^2 \cap H_0^1 \times H^2 \cap H_0^1$.

Proof. Since $n \leqslant 3$, by Sobolev's imbedding theorem R is locally Lipschitz on E. Thus, the local existence follows from standard results from semigroup theory. To prove the global existence it suffices to have uniform a priori E-norm estimates for (ψ, e), i.e., $H^2 \cap H_0^1 \times H_0^1$ norm for (ψ, v) for the system (1.40)–(1.43), which have already been proved in [EZ]. Thus the global existence and uniqueness follow. To prove the existence of an absorbing set B, it suffices to prove that there exists an absorbing set of (ψ, v) in $H^2 \cap H_0^1 \times H_0^1$ for the system (1.40)–(1.43). Multiplying (1.41) by v_t and integrating with respect to x yields

$$\frac{K}{2}\frac{d}{dt}\|\nabla v\|^2 + \|v_t\|^2 = -\frac{l}{2}\int_\Omega \psi_t v_t \, dx \leqslant \frac{1}{2}\|v_t\|^2 + \frac{l^2}{8}\|\psi_t\|^2 \tag{3.5}$$

$$\frac{K}{2}\frac{d}{dt}\|\nabla v\|^2 + \frac{1}{2}\|v_t\|^2 \leqslant \frac{l^2}{8}\|\psi_t\|^2 \tag{3.6}$$

Differentiating (1.40) with respect to t, then multiplying it by ψ_t and integrating with respect to x, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\|\psi_t\|^2 + \xi^2\|\nabla\psi_t\|^2 + 3\int_\Omega (\psi + \bar{\phi})^2 \psi_t^2 \, dx = \|\psi_t\|^2 + 2\int_\Omega v_t \psi_t \, dx$$

$$\leqslant \frac{1}{2}\|v_t\|^2 + 3\|\psi_t\|^2 \tag{3.7}$$

Adding (3.7) and (3.6) yields

$$\frac{K}{2}\frac{d}{dt}\|\nabla v\|^2 + \frac{\tau}{2}\frac{d}{dt}\|\psi_t\|^2 + \xi^2\|\nabla\psi_t\|^2 \leqslant \left(3 + \frac{l^2}{8}\right)\|\psi_t\|^2 \tag{3.8}$$

Multiplying (3.8) by a small positive number $\delta > 0$ specified later and adding it to (1.44) yields

$$\frac{d}{dt}\left[V(t) + \frac{\delta K}{2}\|\nabla v\|^2 + \frac{\delta\tau}{2}\|\psi_t\|^2\right] + \tau\|\psi_t\|^2 + \frac{4K}{l}\|\nabla v\|^2 + \delta\xi^2\|\nabla\psi_t\|^2$$

$$\leqslant \delta\left(3 + \frac{l^2}{8}\right)\|\psi_t\|^2 \tag{3.9}$$

Applying Young's inequality and Poincare's inequality yields

$$V(t) \leqslant \frac{\tau}{4}\|\psi_t\|^2 + \frac{2K}{l}\|\nabla v\|^2 + C \tag{3.10}$$

We choose

$$\delta = \frac{\tau}{4(3 + l^2/8)} \tag{3.11}$$

and add (3.10) to (3.9) to obtain

$$\frac{d}{dt}\left[V(t) + \frac{\delta K}{2}\|\nabla v\|^2 + \frac{\delta\tau}{2}\|\psi_t\|^2\right] + \frac{\tau}{2}\|\psi_t\|^2 + \frac{2K}{l}\|\nabla v\|^2 + V(t) \leqslant C \tag{3.12}$$

Let

$$C_0 = \min\left(1, \frac{4}{\delta l}, \frac{1}{\delta}\right) \tag{3.13}$$

It follows from (3.12) that

$$\frac{d}{dt}\left[V(t) + \frac{\delta K}{2}\|\nabla v\|^2 + \frac{\delta\tau}{2}\|\psi_t\|^2\right] + C_0\left[V(t) + \frac{\delta K}{2}\|\nabla v\|^2 + \frac{\delta\tau}{2}\|\psi_t\|^2\right] \leqslant C \tag{3.14}$$

which results in

$$V(t) + \frac{\delta K}{2}\|\nabla v\|^2 + \frac{\delta\tau}{2}\|\psi_t\|^2$$
$$\leqslant \left(V(0) + \frac{\delta K}{2}\|\nabla v_0\|^2 + \frac{\delta\tau}{2}\|\psi_t(0)\|^2\right)\cdot e^{-C_0 t} + C \tag{3.15}$$

Since

$$\|\psi_t(0)\|^2 \leqslant \tilde{C} \tag{3.16}$$

with $\tilde{C}$ being a positive constant depending on $\|\psi_0\|_{H^2}$, and $\|v_0\|_{L^2}$, (3.15) implies that $\|v(t)\|_{H^1}$ and $\|\psi(t)\|_{H^1}$, $\|\psi_t\|_{L^2}$ is absorbed in a bounded set. By Eq. (1.40) we obtain the existence of an absorbing set of (ψ, v) in $H^2 \cap H_0^1 \times H_0^1$. This gives the existence of an absorbing set B for (ψ, e) in E.

To apply Theorem 3.1, we have to verify that $(\{S(t)\}_{t \geqslant 0}, B)$ satisfies the squeezing property.

Let U and $\bar{U}$ be two solutions of (2.12)–(2.14) and

$$\bar{V} = U - \bar{U} \tag{3.17}$$

Then $\bar{V}$ satisfies

$$\frac{d\bar{V}}{dt} + \mathscr{A}\bar{V} = R(U) - R(\bar{U}) \tag{3.18}$$

$$\bar{V}(0) = \bar{V}_0 \tag{3.19}$$

The self-adjoint positive definite operator $\mathscr{A}$ is given by (2.13), which has relabeled eigenvalues $\lambda^{(n)}$ $(n=1,...)$ satisfying

$$\lambda^{(n)} \to +\infty \qquad (n \to +\infty) \tag{3.20}$$

Let V_n be the corresponding eigenvector functions, i.e., $\mathscr{A}V_n = \lambda^{(n)}V_n$. Let $H_N = \operatorname{span}\{V_1,..., V_N\}$ and $P_N: H \to H_N$, the orthogonal projection onto H_N, and $Q_N = I - P_N$. Let

$$W = Q_N \bar{V} \tag{3.21}$$

Then by (3.18)–(3.19) we have

$$\begin{aligned} &\frac{dW}{dt} + \mathscr{A}W = Q_N(R(U) - R(\bar{U})) \\ &W(0) = Q_N \bar{V}_0 \end{aligned} \tag{3.22}$$

Let $V = H_0^1 \times H_0^1$, then multiplying (3.22) by W^T and integrating with respect to x yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|W\|_H^2 + \|W\|_V^2 &\leqslant \|W\|_H \|Q_N(R(U) - R(\bar{U}))\|_H \\ &\leqslant \frac{1}{(\lambda^{(N+1)})^{1/2}} \|W\|_H \|R(U) - R(\bar{U})\|_V \end{aligned} \tag{3.23}$$

When U_0, $\bar{U}_0 \in B$, by Theorem 3.2 we have

$$\|U(t)\|_E \leqslant C, \qquad \|\bar{U}(t)\|_E \leqslant C, \qquad \forall t \geqslant 0 \tag{3.24}$$

with $C > 0$ a constant depending on B. From the expression for R and Sobolev's imbedding theorem, we have

$$\|R(U) - R(\bar{U})\|_V \leqslant \tilde{C} \|U - \bar{U}\|_V \tag{3.25}$$

with $\tilde{C} > 0$ a constant depending only on B. From (3.23) we have that

$$\frac{1}{2}\frac{d}{dt}\|W\|_H^2 + \lambda^{(N+1)}\|W\|_H^2 \leqslant \frac{\lambda^{(N+1)}}{2}\|W\|_H^2 + \frac{\tilde{C}^2}{2(\lambda^{(N+1)})^2}\|U - \bar{U}\|_V^2 \tag{3.26}$$

Applying Gronwall's inequality to (3.26) yields

$$\|W(t)\|_H^2 \leqslant e^{-t\lambda^{(N+1)}}\|W(0)\|_H^2 + \frac{\tilde{C}^2}{(\lambda^{(N+1)})^2}\int_0^t \|U - \bar{U}\|_V^2\, dt \tag{3.27}$$

On the other hand, from (3.18) it follows that

$$\frac{1}{2}\frac{d}{dt}\|\bar{V}\|_H^2+\|\bar{V}\|_V^2\leqslant\|\bar{V}\|_H\,\|R(U)-R(\bar{U})\|_H\leqslant C_1\|\bar{V}\|_H^2 \tag{3.28}$$

Applying Gronwall's inequality to (3.28) yields

$$\int_0^t \|\bar{V}\|_V^2\,dt\leqslant\frac{1}{2}e^{2C_1t}\|\bar{V}(0)\|_H^2 \tag{3.29}$$

Inserting (3.29) into (3.27), we obtain

$$\begin{aligned}\|W(t)\|_H^2&\leqslant e^{-\lambda^{(N+1)}t}\|W(0)\|_H^2+\frac{1}{2}\frac{\tilde{C}^2}{(\lambda^{(N+1)})^2}e^{2C_1t}\|\bar{V}(0)\|_H^2\\&\leqslant\left(e^{-\lambda^{(N+1)}t}+\frac{1}{2}\frac{\tilde{C}^2}{(\lambda^{(N+1)})^2}e^{2C_1t}\right)\|\bar{V}(0)\|_H^2\end{aligned} \tag{3.30}$$

from which the squeezing property follows. Indeed, we choose

$$t_*=\frac{6\ln 2}{\lambda^{(1)}} \tag{3.31}$$

and we choose N_0 such that when $N\geqslant N_0$,

$$\lambda^{(N+1)}\geqslant\frac{\tilde{C}}{8\sqrt{2}}e^{C_1t_*} \tag{3.32}$$

Thus if

$$\|P_{N_0}\bar{V}(t_*)\|_H\leqslant\|Q_{N_0}\bar{V}(t_*)\|_H \tag{3.33}$$

then

$$\|\bar{V}(t_*)\|_H^2\leqslant 2\,\|Q_{N_0}\bar{V}(t_*)\|_H^2 \tag{3.34}$$

Also, from (3.30)

$$\|V(t_*)\|_H^2\leqslant\tfrac{1}{64}\|\bar{V}(0)\|_H^2 \tag{3.35}$$

that is,

$$\|\bar{V}(t_*)\|_H\leqslant\tfrac{1}{8}\|\bar{V}(0)\|_H \tag{3.36}$$

which implies the squeezing property.

Applying Theorems 3.1, we have proved the following.

Theorem 3.3. *Let $\Omega \subseteq \mathbb{R}^n$ $(n \leqslant 3)$ be a bounded domain with smooth boundary Γ. Then the system (2.12)–(2.14) [accordingly, the system (1.40)–(1.43)] has an inertial set M in $H^2 \cap H_0^1 \times H^2 \cap H_0^1$ [accordingly, an inertial set $\tilde{M}$ in $H^2 \cap H_0^1 \times H_0^1$]. Moreover, (3.3), (3.4) hold, where t_* and N_0 are given by (3.31), (3.32).*

4. THE ENERGY CONSERVING SYSTEM

Here we consider the case where temperature satisfies homogeneous Neumann boundary conditions. There is an important difference between this situation and that discussed previously. In this case there is no bounded absorbing set for initial data varying throughout the whole space. This is because Eq.(1.2) and the boundary conditions imply

$$\int_\Omega \left(u + \frac{l}{2}\phi\right) dx = \int_\Omega \left(u_0 + \frac{l}{2}\phi_0\right) dx \qquad \text{for} \quad t \geqslant 0 \tag{4.1}$$

This is not as problematic as it appears, however. The energy conservation property (4.1) just means that all evolution takes place in an affine hyperplane, and so to understand the dynamics we can consider each of these invariant hyperplanes separately.

We replace (1.3) by

$$\frac{\partial u}{\partial n} = 0, \qquad \frac{\partial \phi}{\partial n} = 0, \qquad \text{or} \quad \phi = \phi_\Gamma(x) \text{ on } \Gamma \tag{1.3''}$$

We change variables by writing

$$v = u + \frac{l}{2}\phi - c_0 \tag{4.2}$$

where

$$c_0 = \frac{1}{|\Omega|} \int_\Omega \left(u_0 + \frac{l}{2}\phi_0\right) dx \tag{4.3}$$

and work in $L^2 \times \bar{L}^2$, where

$$\bar{L}^2 = \left\{ v \in L^2 : \int_\Omega v = 0 \right\}$$

Spaces $\bar{H}^k = H^k \cap \bar{L}^2$ and $H_N^k \cap \bar{L}^2$ are also used, where the subscript N refers to the weak homogeneous Neumann boundary condition being satisfied. Note that we have a Poincare inequality for $v \in \bar{H}_N^1$.

When ϕ satisfies the Neumann boundary condition, Eq. (1.44) becomes

$$\frac{d}{dt}\int_{\Omega}\left(\xi^2|\nabla\phi|^2+\frac{1}{4}\phi^4-\frac{1}{2}\phi^2+\frac{2}{l}\left(v-\frac{l}{2}\phi+c_0\right)^2\right)dx$$

$$+\tau\|\phi_t\|^2+\frac{4K}{l}\left\|\nabla\left(v-\frac{l}{2}\phi\right)\right\|^2=0 \tag{4.4}$$

In the same way as before, this yields an absorbing set in $H^1\times\bar{L}^2$ for any fixed c_0. Similarly, the proof of Theorem 3.2 may be modified to obtain the existence of an absorbing set in $H^2_N\times\bar{H}^1$ for fixed c_0. The usual argument demonstrates the existence, for fixed c_0, of a global attractor which is compact in $H^1\times\bar{L}^2$ (see Theorem 1.3).

To obtain inertial manifolds (or sets) for the system

$$\tau\phi_t=\xi^2\,\Delta\phi+\phi-\phi^3+2v-l\phi+2c_0 \tag{4.5}$$

$$v_t=K\,\Delta v-\frac{Kl}{2}\,\Delta\phi \tag{4.6}$$

with $N-N$ boundary conditions, we again change variables to produce a self-adjoint linear part. Let

$$A^2=-\Delta \qquad \text{with domain } \bar{H}^2_N \text{ in } \bar{L}^2 \tag{4.7}$$

Then A is positive definite on its domain, $\bar{H}^1$. Let

$$b=\frac{2}{\sqrt{Kl}} \qquad \text{and} \qquad e=bA^{-1}v \tag{4.8}$$

then the system (4.5), (4.6) becomes

$$\tau\phi_t=\xi^2\,\Delta\phi+\sqrt{Kl}\,A^{-1}(-\Delta)e+f(\phi) \tag{4.9}$$

$$e_t=K\Delta e+\sqrt{Kl}\,A^{-1}(-\Delta)\phi \tag{4.10}$$

where $f(\phi)=(1-l)\phi-\phi^3+2c_0$.

System (4.9)–(4.10) with initial data $(\phi_0, bA^{-1}v_0)$ is equivalent to the original system in $H^2_N\times\bar{H}^2_N$, as can be seen by the existence and uniqueness of solutions. Furthermore, this modified system is in a form such that inertial manifolds and inertial sets can be shown to exist in the appropriate $H^k_N\times\bar{H}^k_N$ space, depending on $n\leqslant 3$. If ϕ satisfies Dirichlet boundary conditions, then the symmetrization fails and the existence of an inertial manifold remains an open question. Allowing c_0 to vary in $\mathbb{R}$ certainly changes the set of equilibria for (PF) with (1.3)″ and, hence, the dynamics. However, state space should have a global, finite-dimensional, attracting manifold for (u, ϕ) foliated with the invariant affine planes. Locally, this is the case. An interesting question is how the inertial manifolds may change when c_0 passes through a critical value.

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Printed in Belgium