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Dimensions of Attractors for Discretizations for Navier-Stokes Equations

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In this paper, we discretize the 2-D incompressible Navier-Stokes equations with the periodic boundary condition by the finite difference method. We prove that with a shift for discretization, the global solutions exist. After proving some discrete Sobolev inequalities in the sense of finite differences, we prove the existence of the global attractors of the discretized system, and we estimate the upper bounds for the Hausdorff and the fractal dimensions of the attractors. These bounds are indepent of the mesh sizes and are considerably close to those of the continuous version.

KEY WORDS: Navier-Stokes equation; finite difference; attractor; Hausdorff dimension.

1. INTRODUCTION

In recent decades, great progress has been made in the research of the Navier-Stokes equations, especially in the 2-D case. Significant theory of dynamical properties of the 2-D incompressible Navier-Stokes equations can be found in [L2], [C-F1], [C-F2], [C-F-T], [T3], and the references listed therein. It is proved that there exist global attractors of finite Hausdorff and fractal dimensions. The bounds for the Hausdorff dimensions D of the global attractor of the 2-D incompressible Navier-Stokes equations have been improved to

$$D \leqslant cG^{2/3}((\log G)^{1/3}+1) \quad \text{in the periodic boundary condition case} \tag{1.1}$$

and

$$D \leqslant cG \qquad \text{in the Dirichlet boundary condition case,} \tag{1.2}$$

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where G is the Grashof number defined by

$$G = \frac{\|f\|_{L^2}}{\nu^2 \lambda_1} \tag{1.3}$$

where f is the volume force, ν is the viscosity constant and λ_1 is the smallest eigenvalue of the Stokes operator (see [T3]).

In this paper, we consider the discretization of the 2-D incompressible Navier-Stokes equations. We use the finite difference method. After discretizing all functions and differential operators involved in the Navier-Stokes equations by following the standard finite difference scheme, we obtain a system of finite dimensional ordinary differential equations simulating the continuous version of the Navier-Stokes equations on appropriate grids.

We concentrate on the long-time dynamical behavior of our discretized system, particularly on the existence of attractors and the estimates of their Hausdorff and fractal dimensions.

For this purpose, we follow the outline of arguments used for the study of continuous version of the Navier-Stokes equations. Technically, difficulties arise due to the discretized version of inequalities in the sense of finite difference.

Throughout this paper, we consider the discretizations for the 2-D incompressible Navier-Stokes equations with the *periodic boundary condition* only.

In Section 2, we discuss the basic properties of the discretization by using the finite difference method. Section 2.1 describes the discretization of the objects involved in the Navier-Stokes equations, such as functions, differential operators, integrals, norms, etc. In Section 2.2, we solve the spectrum of the discretized $-\Delta$ by using the finite difference method.

The existence of global solutions and global attractors are discussed in detail in Section 3. In the continuous version, the nonlinear term $B(u, u)$ of the Navier-Stokes equations satisfies

$$(B(u, v), v) = 0 \tag{1.4}$$

for both Dirichlet and periodic boundary conditions, which is extremely important for the global existence of solutions and for the study of the dimensions of the attractors, and

$$(B(u, u), -\Delta u) = 0 \tag{1.5}$$

for the periodic boundary condition only, which is used to improve the estimate of Hausdorff and fractal dimensions of the attractors. Yet for directly discretizing $B(u, u)$ of the continuous version by using the finite difference method, we have neither (1.4) not (1.5), due to the lack of a "product rule" for finite differences. For this reason, we make a "shift" of $B(u, u)$ so that the discretized version of (1.4) is true, which enables us to prove the global existence of the solutions of our discretized Navier-Stokes equations. Throughout this paper, we actually work on the "shifted" version of the discretized Navier-Stokes equations. We prove in Section 3 that the solutions have discretized L^2-absorbing properties, which implies the existence of attractors in the discretized L^2 sense. In the continuous version, since the spaces are of infinite dimension, both L^2- and H^1-absorbing properties are necessary to obtain the existence of L^2-attractors. In our finite dimensional discretized version, we need only the discretized L^2-absorbing property to obtain the global attractors in the discretized L^2-norm.

In Section 4 we give upper bounds for the Hausdorff and fractal dimensions of the discretized L^2-attractors. In Section 4.1, different versions of discretized Sobolev embedding inequalities are developed, which are used to study the dimensions. In Section 4.2, we prove the discretized version (in the sense of finite difference method) of the Lieb-Thirring inequality, the continuous version of which improves the bounds of Hausdorff and fractal dimensions of the attractors from exponential in G to polynomial in G [G is as in (1.3)]. In Section 4.3, we obtain a bound of the Hausdorff dimension

$$D' \leqslant c'G' \tag{1.6}$$

where G' is the discretized analogue of G in (1.3). This corresponds to (1.2), the estimate for the continuous version in the case of the Dirichlet boundary conditions. Due to lack of the discretized version of (1.5), we do not have (1.1) in the discretized version. However, our bounds for the discretized version are very close to those for the continuous version, and more importantly, *there is limit of the mesh size beyond which the finer discretization does not give more valuable information of the dynamics.*

In other words, by taking formal finite difference discretization for the 2-D incompressible Navier-Stokes equations, with a modification of the nonlinear term, we obtain attractors similar to those of the classical Navier-Stokes equations. Improving the mesh size does not change the estimate for the dimensions of these attractors.

We also prove the existence of the discretized H^1-attractors in the Appendix. We study the discrete 2-D Fourier transform in Section A.1, and we prove a discretized interpolation inequality in Sections A.1 and A.2, which plays an important role for discretized H^1-absorbing property. The discretized H^1-attractors are proved in Section A.3. In Section A.4, a simplified proof of the results discussed in the Appendix is given.

It is worth pointing out that the discretized L^2-attractor and the discretized H^1-attractor are geometrically the same.

2. PRELIMINARIES

Consider the 2-D incompressible Navier-Stokes equations,

$$\begin{cases} \rho\left(\dfrac{\partial U}{\partial t}+(U\cdot\nabla)\,U\right)-\nu\Delta U+\nabla P=f \\ \operatorname{div} U=0 \\ \displaystyle\int_\Omega U\,dx=0 \end{cases} \tag{2.1}$$

where $\Omega=(0,1)\times(0,1)$, $U(x,t)=(U_1(x,t),U_2(x,t))\colon \Omega\times(0,1)\to\mathbf{R}^2$, $P(x,t)\colon \Omega\times(0,1)\to\mathbf{R}$, $f\in L^2(\Omega)^2$, $\nu>0$, and $\mathscr{D}(-\Delta)=H^2_{\text{per}}(\Omega)^2$.

Note that in (2.1) the unknowns are $U(x,t)\in\mathbf{R}^2$ and $P(x,t)\in\mathbf{R}$, which stand for the velocity of the particle and the pressure at the position x and at the time t, respectively, ρ is the density, and ν is the kinematic viscosity. For details, see, for example, [T3].

Remark 2.1. We may assume without loss of generality that $\rho=1$. If this is the case, Eq. (2.1) is called the *nondimensional form* of the Navier-Stokes equation. Throughout this paper, we always assume that $\rho=1$. $\square$

In this section, we first discretize the spatial variables of (2.1) and differential operators by using the finite difference method. Then we study the preliminary properties of the discretized $-\Delta$ and various norms in $\mathbf{R}^k$.

2.1. Discretization for the Navier-Stokes Equations

Let $m\in\mathbf{N}$, $h=1/m$. We approximate a function $U(x)=(U_1(x_1,x_2), U_2(x_1,x_2))\colon \Omega\to\mathbf{R}^2$ by $u=u_{k,ij}$:

$$u_{k,ij}=U_k(ih,jh)=U_k\left(\frac{i}{m},\frac{j}{m}\right)\qquad (k=1,2,\ 1\leqslant i,j\leqslant m) \tag{2.2}$$

For a fixed k, we can think of $u_{k,\dots}$ as a matrix of size $m\times m$ or a vector in $\mathbf{R}^{m^2}$.

For convenience, we reorder the subscripts of components of any $v\in\mathbf{R}^{m^2}$ row by row:

$$v=(v_{11},v_{12},\cdots,v_{1m},v_{21},v_{22},\cdots,v_{2m},\cdots,v_{m1},v_{m2},\cdots,v_{mm})^{tr} \tag{2.3}$$

where "*tr*" is the transpose operation for matrixes, so that v is a column vector.

To distinguish such an ordering from that of vectors in the 2-D Euclidean spaces, we define

$$\mathbf{M} = \{v \in \mathbf{R}^{m^2} \mid \text{subscripts of components of } v \text{ are ordered as in (2.3)}\} \tag{2.4}$$

Remark 2.2. The only difference between $\mathbf{M}$ and $\mathbf{R}^{m^2}$ is the order of subscripts of components. Elements in $\mathbf{M}$ can be treated as vectors in $\mathbf{R}^{m^2}$. Globally, $\mathbf{M}$ has the induced linear structure, inner product, and so forth. An element $v \in \mathbf{M}_h$ can be treated as an approximation of a function $V: \Omega \to \mathbf{R}$ as shown in (2.2), in which the subscript (i, j) corresponds to the space position (ih, jh). □

Remark 2.3. Since we consider periodic boundary conditions only, we extend the indexes of any $v \in \mathbf{M}$ by periodicity:

$$v_{ij} = v_{(i \bmod_+ m), (j \bmod_+ m)} \qquad (i, j \in \mathbf{Z}) \tag{2.5}$$

where

$$k \bmod_+ m = \begin{cases} m & \text{if } k \text{ is a multiple of } m \\ k \bmod m & \text{otherwise} \end{cases} \qquad \square \tag{2.6}$$

We define the operators D_1, D_2, A, DIV, and D as the finite difference discretizations of $\partial/\partial x_1$, $\partial/\partial x_2$, $-\Delta$, div, and ∇ of the continuous version as follows.

For any $z \in \mathbf{M}$, we denote by $z_{ij} (1 \leqslant i, j \leqslant m)$ the (i, j)-th component of z, whose order of subscripts is given by (2.3). Then, for $v \in \mathbf{M}$, we define the linear operators $D_1, D_2, A: \mathbf{M} \to \mathbf{M}$ by

$$(D_1 v)_{ij} = m(v_{ij} - v_{i-1,j}) \qquad (1 \leqslant i, j \leqslant m) \tag{2.7}$$

$$(D_2 v)_{ij} = m(v_{ij} - v_{i,j-1}) \qquad (1 \leqslant i, j \leqslant m) \tag{2.8}$$

and

$$(Av)_{ij} = m^2(4v_{ij} - v_{i+1,j} - v_{i-1,j} - v_{i,j+1} - v_{i,j-1}) \qquad (1 \leqslant i, j \leqslant m) \tag{2.9}$$

We define the linear operator DIV: $\mathbf{M} \times \mathbf{M} \to \mathbf{M}$ for any $w = (w_1, w_2)^{\mathrm{tr}} \in \mathbf{M} \times \mathbf{M}$ by

$$\mathrm{DIV} w = D_1 w_1 + D_2 w_2 \qquad (1 \leqslant i, j \leqslant m) \tag{2.10}$$

and we define the linear operator $D\colon \mathbf{M} \to \mathbf{M} \times \mathbf{M}$ for any $v \in \mathbf{M}$ by

$$Dv = \begin{pmatrix} D_1 v \\ D_2 v \end{pmatrix} \tag{2.11}$$

Remark 2.4. Equations (2.7)–(2.11) define the operators D_1, D_2, A, DIV, and D in components. By (2.3) and Remark 2.2, any $v \in \mathbf{M}$ can still be considered as a usual vector in $\mathbf{R}^{m^2}$. So the above operators can be expressed as matrixes under the natural basis of $\mathbf{M}$. □

Without any confusion, we use the same symbol for a linear operator as well as its matrix representation. Then D_1, D_2, A, are matrixes of size $m^2 \times m^2$, which are given as follows.

$$D_1 = m \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_1 \end{pmatrix}_{m^2 \times m^2} \tag{2.12}$$

here D_1 is expressed by $m \times m$ blocks of size $m \times m$, the blocks on the main diagonal are all d_1, all other blocks are 0, where, in (2.12), d_1 is a block of size $m \times m$ given by

$$d_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & -1 \\ -1 & 1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdot & \vdots \\ \vdots & \cdot & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & -1 & 1 \end{pmatrix}_{m \times m} \tag{2.13}$$

where the entries of d_1 on the main diagonal are all 1, the entries on the diagonal below the main diagonal and the entry in the upper-right corner are -1, and 0 elsewhere;

$$D_2 = m \begin{pmatrix} I & 0 & \cdots & \cdots & -I \\ -I & I & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdot & \vdots \\ \vdots & \cdot & \ddots & I & 0 \\ 0 & \cdots & \cdots & -I & I \end{pmatrix}_{m^2 \times m^2} \tag{2.14}$$

where I is the unit matrix of size $m \times m$; here D_2 is expressed by $m \times m$ blocks of size $m \times m$, the blocks of D_2 on the main diagonal are all I, the blocks on the diagonal below the main diagonal and the block in the upper-right corner are $-I$, and 0 blocks elsewhere;

$$A = m^2 \begin{pmatrix} a & -I & 0 & \cdots & \cdots & \cdots & -I \\ -I & a & -I & \cdots & \cdots & \cdots & 0 \\ 0 & -I & a & \ddots & \cdot & \cdot & \vdots \\ \vdots & \cdot & \ddots & \ddots & \ddots & \cdot & \vdots \\ \vdots & \cdot & \cdot & \ddots & a & -I & 0 \\ 0 & \cdots & \cdots & \cdots & -I & a & -I \\ -I & \cdots & \cdots & \cdots & 0 & -I & a \end{pmatrix}_{m^2 \times m^2} \tag{2.15}$$

Here A is expressed by $m \times m$ blocks of size $m \times m$, the blocks on the main diagonal are all a, the blocks on the diagonals above and below the main diagonal and that in the upper-right and lower-left corners are all $-I$, and all other blocks are 0 matrix, where, in (2.15), a is a block of size $m \times m$ given by

$$a = \begin{pmatrix} 4 & -1 & 0 & \cdots & \cdots & \cdots & -1 \\ -1 & 4 & -1 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 4 & \ddots & \cdot & \cdot & \vdots \\ \vdots & \cdot & \ddots & \ddots & \ddots & \cdot & \vdots \\ \vdots & \cdot & \cdot & \ddots & 4 & -1 & 0 \\ 0 & \cdots & \cdots & \cdots & -1 & 4 & -1 \\ -1 & \cdots & \cdots & \cdots & 0 & -1 & 4 \end{pmatrix}_{m \times m} \tag{2.16}$$

where the entries on the main diagonal are all 4, the entries on the diagonals above and below the main diagonal are all -1, the entries in the upper-right and lower-left corners are -1, and all other entries are 0.

DIV is a matrix of size $m^2 \times 2m^2$, which is defined by

$$\mathrm{DIV} = (D_1 \quad D_2) \tag{2.17}$$

D is a matrix of size $2m^2 \times m^2$, which is defined by

$$D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \tag{2.18}$$

where (2.18) is understood as

$$Dv = \begin{pmatrix} D_1 v \\ D_2 v \end{pmatrix} \qquad (\forall v \in \mathbf{M}) \tag{2.19}$$

As an easy consequence of the definitions of A and D, one can check that

$$D^{tr}D = A \tag{2.20}$$

Now we turn to the discretization of the Navier-Stokes equations (2.1). We consider discretizations in $\mathbf{M} \times \mathbf{M}$. If we treat any $v \in \mathbf{M} \times \mathbf{M}$ as a step function approximating a function $V: \Omega \to \mathbf{R}^2$ as in (2.2), then the integral of V over Ω is naturally discretized by

$$\int_\Omega V\,dx \doteq \begin{pmatrix} \dfrac{1}{m^2} \displaystyle\sum_{i,j=1}^{m} v_{1,ij} \\ \dfrac{1}{m^2} \displaystyle\sum_{i,j=1}^{m} v_{2,ij} \end{pmatrix} \in \mathbf{R}^2 \tag{2.21}$$

Hence we define *the average* of $v \in \mathbf{M} \times \mathbf{M}$ by

$$\bar{v} = \frac{1}{m^2} \sum_{i,j=1}^{m} v_{1,ij} \begin{pmatrix} e \\ 0 \end{pmatrix} + \frac{1}{m^2} \sum_{i,j=1}^{m} v_{2,ij} \begin{pmatrix} 0 \\ e \end{pmatrix} \in \mathbf{M} \times \mathbf{M} \tag{2.22}$$

where $e = (1, 1, ..., 1)^{tr} \in \mathbf{M}$.

We define *the weighted norm* $\|\cdot\|_0$ in $\mathbf{M} \times \mathbf{M}$ by

$$\|v\|_0 = \left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} |v_{k,ij}|^2 \right)^{1/2} \tag{2.23}$$

for any $v \in \mathbf{M} \times \mathbf{M}$, which corresponds to the discretized L^2-norm.

Similarly, we define *the weighted norm* $\|\cdot\|_0$ in $\mathbf{M}$ by

$$\|w\|_0 = \left(\frac{1}{m^2} \sum_{i,j=1}^{m} |w_{ij}|^2 \right)^{1/2} \tag{2.24}$$

for any $w \in \mathbf{M}$.

As indicated in (2.2), we sample the function $U: \Omega \to \mathbf{R}^2$ by

$$u = u_{k,ij} \qquad (k = 1, 2,\ 1 \leqslant i, j \leqslant m) \tag{2.25}$$

and similarly we sample the other unknown $P: \Omega \to \mathbf{R}$ by

$$p = p_{ij} \qquad (1 \leqslant i, j \leqslant m) \tag{2.26}$$

Finally, we sample the known function $f: \Omega \to \mathbf{R}^2$ by

$$\Gamma = \Gamma_{k,ij} \qquad (k = 1, 2,\ 1 \leqslant i, j \leqslant m) \tag{2.27}$$

To characterize the fact that $f \in L^2(\Omega)^2$ in the continuous version (2.1), we make the following hypothesis

$$\|\Gamma\|_0^2 = \frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} |\Gamma_{k,ij}|^2 \leqslant C^2 \tag{H}$$

where C is a constant independent of m.

Remark 2.5. The hypothesis (H) indicates that the discretized L^2-norm of Γ is uniformly bounded with respect to m. □

With the above definitions, we rewrite the discretized version of (2.1) as

$$\begin{cases} \dfrac{du}{dt} + (u \cdot D)\, u + \nu A u + Dp = \Gamma \\ \mathrm{DIV} u = 0 \\ \bar{u} = 0 \end{cases} \tag{2.28}$$

where $u = u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \mathbf{M} \times \mathbf{M}$, $u_k(t) = (u_{k,ij}(t)) \in \mathbf{M}$ $(k = 1, 2)$, $p = (p_{ij}(t)) \in \mathbf{M}$, $\Gamma = (\Gamma_{k,ij}) \in \mathbf{M} \times \mathbf{M}$, and

$$(u \cdot D)\, u = (u_1 D_1 + u_2 D_2)\, u = \begin{pmatrix} u_1 D_1 u_1 + u_2 D_2 u_1 \\ u_1 D_1 u_2 + u_2 D_2 u_2 \end{pmatrix} \in \mathbf{M} \times \mathbf{M} \tag{2.29}$$

$$Au = \begin{pmatrix} Au_1 \\ Au_2 \end{pmatrix} \in \mathbf{M} \times \mathbf{M} \tag{2.30}$$

$$\mathrm{DIV} u = D_1 u_1 + D_2 u_2 \in \mathbf{M} \tag{2.31}$$

and $\bar{u}$ is the average of u defined as in (2.22).

Before we start to study the discretized Navier-Stokes equation (2.28), we define some more metric structures in $\mathbf{M}$.

In $\mathbf{M}$, define *the first-order difference seminorm* $|\cdot|_1$ by

$$|v|_1 = \|Dv\|_0 = \left(\sum_{i,j=1}^{m} \left(|v_{ij} - v_{i-1,j}|^2 + |v_{ij} - v_{i,j-1}|^2\right) \right)^{1/2} \tag{2.32}$$

the second-order difference semi norm $|\cdot|_2$ by

$$|v|_2 = \|Av\|_0 = \left(m^2 \sum_{i,j=1}^{m} (4v_{ij} - v_{i+1,j} - v_{i-1,j} - v_{i,j+1} - v_{i,j-1})^2 \right)^{1/2} \tag{2.33}$$

and *the max norm* $\|\cdot\|_\infty$ by

$$\|v\|_\infty = \max_{1 \leqslant i,j \leqslant m} |v_{ij}| \tag{2.34}$$

For any $v = v_{ij}$, $w = w_{ij} \in \mathbf{M}$, the usual Euclidean inner product $(\cdot, \cdot)$ is given by

$$(v, w) = \sum_{i,j=1}^{m} v_{ij} w_{ij} \tag{2.35}$$

We define *the weighted inner product* $\langle \cdot, \cdot \rangle$ in $\mathbf{M}$ by

$$\langle v, w \rangle = \frac{1}{m^2} (v, w) = \frac{1}{m^2} \sum_{i,j=1}^{m} v_{ij} w_{ij} \tag{2.36}$$

If v and w are interpreted as approximations of V, $W: \Omega \to \mathbf{R}$, then $\langle v, w \rangle$ can be interpreted as the discretized inner product in $L^2(\Omega)$, i.e.,

$$\int_{\Omega} VW \, dx \doteq \langle v, w \rangle \tag{2.37}$$

We can rewrite (2.20) in terms of inner products. We have, for v, $w \in \mathbf{M}$,

$$(Dv, Dw) = (v, Aw) = (Av, w), \qquad \langle Dv, Dw \rangle = \langle v, Aw \rangle = \langle Av, w \rangle \tag{2.38}$$

and

$$\|Dv\|_0^2 = \langle Dv, Dv \rangle = \langle v, Av \rangle \tag{2.39}$$

Remark 2.6. Equations (2.32)–(2.39) can be standardly extended to the product space as in (2.23), i.e., for v, $w \in \mathbf{M} \times \mathbf{M}$. ▯

2.2. Eigenvalues of A

To study the dynamical behaviors of (2.28), we study the eigenvalues of the linear operator $A: \mathbf{M} \to \mathbf{M}$ as in (2.9), or, equivalently, of the matrix A of size $m^2 \times m^2$ as in (2.15).

Lemma 2.1. *A is a symmetric matrix, so it can be diagonalized.* 0 *is an eigenvalue of A with the corresponding eigenvector*

$$e = (e_{ij}) \in \mathbf{M}, \qquad \textit{where} \qquad e_{ij} = 1 \qquad (1 \leqslant i, j \leqslant m) \tag{2.40}$$

Proof. It is a direct consequence of (2.15) and (2.16). ▯

In [Y], the eigenvalues of discretized one-dimensional $-\Delta$ are solved. By separating variables, we can also find the eigenvalues of the discretized high-dimensional $-\Delta$.

***Lemma* 2.2.** *For any* $0 \leqslant k,\ l \leqslant m-1$ *and* $1 \leqslant i, j \leqslant m$, *define*

$$g(k; i) = \begin{cases} 1 & \text{if } \ k=0 \\ \sin\left(\dfrac{2k\pi}{m}\cdot i\right) & \text{if } \ 1 \leqslant k \leqslant \left\lfloor \dfrac{m}{2} \right\rfloor \\ \cos\left(\dfrac{2(m-k)\pi}{m}\cdot i\right) & \text{if } \ \left\lfloor \dfrac{m}{2} \right\rfloor + 1 \leqslant k \leqslant m-1 \end{cases} \tag{2.41}$$

where $\lfloor s \rfloor$ *is the largest integer not greater than* s, *and define* $e(k, l) \in \mathbf{M}$ *by*

$$e(k, l)_{ij} = g(k; i) \cdot g(l; j) \tag{2.42}$$

Then for $0 \leqslant k,\ l \leqslant m-1$, *we have*

$$Ae(k, l) = 4m^2 \left(\sin^2 \frac{k\pi}{m} + \sin^2 \frac{l\pi}{m} \right) e(k, l) \tag{2.43}$$

Proof. First we see that for all $0 \leqslant k,\ l \leqslant m-1$,

$$e(k, l)_{ij} = e(k, l)_{(i \bmod_+ m),\ (j \bmod_+ m)} \qquad (\forall i, j \in \mathbf{Z})$$

Hence we can use (i, j) in (2.42) as their natural indexes for $e(\cdot, \cdot)$. The definition of $e(k, l)$ obeys (2.5).

We consider the case $1 \leqslant k, l \leqslant \lfloor m/2 \rfloor$ only. In other cases, the lemma can be proved similarly.

Denote by $\alpha = 2k\pi/m$ and $\beta = 2l\pi/m$. Then by (2.9), for any $1 \leqslant k$, $l \leqslant m-1$,

$$\begin{aligned} \frac{1}{m^2} Ae(k, l)_{ij} &= 4 \sin i\alpha \sin j\beta - \sin(i+1)\,\alpha \sin j\beta - \sin(i-1)\,\alpha \sin j\beta \\ &\quad - \sin i\alpha \sin(j+1)\,\beta - \sin i\alpha \sin(j-1)\,\beta \\ &= 4 \left(\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} \right) \sin i\alpha \sin j\beta \\ &= 4 \left(\sin^2 \frac{k\pi}{m} + \sin^2 \frac{l\pi}{m} \right) e(k, l)_{ij} \end{aligned}$$

Therefore (2.43) is true in the case $1 \leqslant k, l \leqslant \lfloor m/2 \rfloor$. $\square$

Combining Lemmas 2.1 and 2.2, we obtain complete information about the eigenvalues and eigenvectors of A.

Theorem 2.1. *The eigenvectors of A are given by $e(k, l)$ as in (2.42) and (2.41), and the corresponding eigenvalues are*

$$\lambda(k, l) = 4m^2 \left(\sin^2 \frac{k\pi}{m} + \sin^2 \frac{l\pi}{m} \right) \qquad (k, l = 0, 1, 2, \ldots, m-1). \quad \square$$

Corollary 2.2. *Denote by*

$$e^{\perp} = \{ v \in \mathbf{M} \mid \langle v, e \rangle = 0 = (v, e) \} \tag{2.44}$$

where $e \in \mathbf{M}$ is as in (2.40). Then

$$Av = 0 \qquad (\forall v \in \operatorname{span}\{e\}) \tag{2.45}$$

and if $m \geqslant 2$, we have

$$\langle Av, v \rangle = \|Dv\|_0^2 \geqslant 4m^2 \sin^2 \frac{\pi}{m} \|v\|_0^2 \geqslant 16 \|v\|_0^2 \qquad (\forall v \in e^{\perp}) \tag{2.46}$$

Proof. Since $\sin x/x$ is decreasing on $(0, \pi/2)$, we know that $4\mathrm{m}^2 \sin^2(\pi/m) \geqslant 16$. $\square$

Remark 2.7. Note that A is the discretization of $-\Delta$ on the square $\Omega = (0, 1)^2$ with periodic boundary condition. In this section we found all eigenvalues and eigenvectors of A. We can discretize $-\Delta$ on any cube $\prod_{i=1}^{n} (a_i, b_i)$ of any size and of any dimension n, with periodic boundary condition in a similar way. Following the arguments in this section, we can also find the eigenvalues and eigenvectors of discretized n-dimensional $-\Delta$. Furthermore, our arguments can also be used to find the eigenvalues and eigenvectors of the discretized $-\Delta$ (of any dimension n) with Dirichlet boundary condition. $\square$

3. GLOBAL EXISTENCE AND GLOBAL ATTRACTORS

Consider the discretized Navier-Stokes equation given in Section 2 [see (2.28)]:

$$\begin{cases} \dfrac{du}{dt} + (u \cdot D)\, u + \nu A u + Dp = \Gamma \\ \mathrm{DIV} u = 0 \\ \bar{u} = 0 \end{cases} \tag{3.1}$$

where $u \in \mathbf{M} \times \mathbf{M}$ and $p \in \mathbf{M}$ are unknowns, and $\Gamma \in \mathbf{M} \times \mathbf{M}$ is fixed and satisfies (H).

To study the discretized Navier-Stokes equation, this setting has some technical problems. In this section, we first modify our discretized model (3.1) so that the new model is essentially the same as the original one (in an infinitesimal sense). Then we prove that the weighted norm of the solution of our new model is bounded, which implies the global existence of solutions and the existence of the $\|\cdot\|_0$-attractors. We prove in the Appendix that the first-order difference seminorm of the solution also has the absorbing property, which implies the existence of the $\|\cdot\|_1$-attractors.

3.1. Shifted Version of the Discretized Navier-Stokes Equations

Let us first define some shifting operators in $\mathbf{M}$ and $\mathbf{M} \times \mathbf{M}$.

Define $\tau_1 : \mathbf{M} \to \mathbf{M}$ by

$$(\tau_1 v)_{ij} = v_{i-1,j} \qquad (1 \leqslant i, j \leqslant m) \tag{3.2}$$

and $\tau_2 : \mathbf{M} \to \mathbf{M}$ by

$$(\tau_2 v)_{ij} = v_{i,j-1} \qquad (1 \leqslant i, j \leqslant m) \tag{3.3}$$

for $v \in \mathbf{M}_h$, and define $\tau : \mathbf{M} \times \mathbf{M} \to \mathbf{M} \times \mathbf{M}$ by

$$\tau w = \tau \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \tau_1 \omega_1 \\ \tau_2 w_2 \end{pmatrix} \tag{3.4}$$

for $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbf{M} \times \mathbf{M}$.

The operators τ_1, τ_2, and τ have inverses given by

$$(\tau_1^{-1} v)_{ij} = v_{i+1,j} \qquad (1 \leqslant i, j \leqslant m) \tag{3.5}$$

$$(\tau_2^{-1} v)_{ij} = v_{i,j+1} \qquad (1 \leqslant i, j \leqslant m) \tag{3.6}$$

and

$$\tau^{-1} w = \tau^{-1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \tau_1^{-1} w_1 \\ \tau_2^{-1} w_2 \end{pmatrix} \tag{3.7}$$

With the above definitions, we can compose, associate, and commute linear operators in the standard way:

$$\tau_l (D_k v) = (\tau_l D_k)\, v = (D_k \tau_l)\, v = D_k (\tau_l v) \qquad (k, l = 1, 2, \qquad \forall v \in \mathbf{M})$$

We also use the following composition and commutation of linear operators:

$$(\tau^{-1} D)\, v = (D \tau^{-1})\, v = \begin{pmatrix} D_1 \tau_1^{-1} \\ D^2 \tau_2^{-1} \end{pmatrix} v = \begin{pmatrix} D_1 \tau_1^{-1} v \\ D_2 \tau_2^{-1} v \end{pmatrix} \qquad (\forall v \in \mathbf{M}) \tag{3.8}$$

and for any $w \in \mathbf{M} \times \mathbf{M}$, we define

$$((\tau w) \cdot D)\, w = \begin{pmatrix} [(\tau_1 w_1)\, D_1 + (\tau_2 w_2)\, D_2]\, w_1 \\ [(\tau_1 w_1)\, D_1 + (\tau_2 w_2)\, D_2]\, w_2 \end{pmatrix} \in \mathbf{M} \times \mathbf{M}$$

$$(w \cdot (D\tau^{-1}))\, w = \begin{pmatrix} [\omega_1(D_1 \tau_1^{-1}) + w_2(D_2 \tau_2^{-1})]\, w_1 \\ [w_1(D_1 \tau_1^{-1}) + w_2(D_2 \tau_2^{-1})]\, w_2 \end{pmatrix} \in \mathbf{M} \times \mathbf{M} \tag{3.9}$$

In all of the above definitions, indexes are always treated periodically as in (2.5). The shifted model for (3.1) is defined by

$$\begin{cases} \dfrac{du}{dt} + \dfrac{1}{2}\,((\tau u) \cdot D + u \cdot (D\tau^{-1}))\, u + vAu + (D\tau^{-1})\, p = \Gamma \\ \mathrm{DIV} u = 0 \\ \tilde{u} = 0 \end{cases} \tag{3.10}$$

where $u \in \mathbf{M} \times \mathbf{M}$ and $p \in \mathbf{M}$ are unknowns, and $\Gamma \in \mathbf{M} \times \mathbf{M}$ is fixed and satisfies (H).

Remark 3.1. If we recall approximation (2.2), then modification (3.10) is the same as (3.1) in an infinitesimal sense. ▯

One of the advantages of such a modification is stated in Lemma 2.4.

Lemma 3.1. *For any $v, w \in \mathbf{M}$, we have*

$$D_k(vw) = (D_k v)\, w + (\tau_k v)(Dw) \qquad (k = 1, 2) \tag{3.11}$$

Proof.

$$\begin{aligned} (D_1(vw))_{ij} &= m(v_{ij} w_{ij} - v_{i-1,j} w_{i-1,j}) \\ &= m(v_{ij} - v_{i-1,j})\, w_{ij} + m v_{i-1,j}(w_{ij} - w_{i-1,j}) \\ &= [(D_1 v)\, w + (\tau_1 v)(D_1 w)]_{ij} \end{aligned}$$

This proves (3.11) for $k = 1$. The case $k = 2$ can be proved similarly. ▯

Lemma 3.2. *For any $v \in \mathbf{M}$, we have*

$$\sum_{i,j=1}^{m} D_k v_{ij} = 0 \qquad (k = 1, 2) \tag{3.12}$$

Proof. This is true because the indexes are treated periodically as in (2.5). ▯

Lemma 3.3. *For any* $v \in \mathbf{M} \times \mathbf{M}$ *with* $\mathrm{DIV} v = 0$ *and any* $w \in \mathbf{M}$, *we have*

$$\langle D\tau^{-1} w, v \rangle = \frac{1}{m^2} (D\tau^{-1} w, v) = 0 \tag{3.13}$$

Proof.

$$\langle D\tau^{-1} w, v \rangle = \frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} ((D_k \tau_k^{-1} w)\, v_k)_{ij}$$

By (3.11),

$$(D_k \tau_k^{-1} w)\, v_k = D_k(\tau_k^{-1} w v_k) - \tau_k(\tau_k^{-1} w)\, D_k v_k = D_k(\tau_k^{-1} w v_k) - w D_k v_k$$

Hence

$$\langle D\tau^{-1} w, v \rangle = \frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} (D_k(\tau_k^{-1} w v_k))_{ij} - \frac{1}{m^2} \sum_{i,j=1}^{m} \left(w \left(\sum_{k=1}^{2} D_k v_k \right) \right)_{ij}$$

By (3.12),

$$\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} D_k(\tau_k^{-1} w v_k) = 0$$

Moreover, since $\mathrm{DIV} v = 0$, then

$$\frac{1}{m^2} \sum_{i,j=1}^{m} w \left(\sum_{k=1}^{2} D_k v_k \right) = \frac{1}{m^2} \sum_{i,j=1}^{m} (w \mathrm{DIV} v)_{ij} = 0$$

Therefore,

$$\langle D\tau^{-1} w, v \rangle = 0$$

This completes the proof. □

Now we turn to the nonlinear term of (3.10).

Denote, for every $u, v, w \in \mathbf{M} \times \mathbf{M}$, by

$$B(u, v) = ((\tau u) \cdot D + u \cdot (D\tau^{-1}))\, v \tag{3.14}$$

and

$$b(u, v, w) = \langle B(u, v), w \rangle \tag{3.15}$$

Lemma 3.4. *For any u, $v \in \mathbf{M}$, if u satisfies* DIV $=0$, *then*

$$b(u, v, v) = 0$$

Proof. By (3.12),

$$\frac{1}{m^2} \sum_{i,j=1}^{m} \sum_{k,l=1}^{2} (D_k(u_k v_l v_l))_{ij} = 0$$

But by (3.11),

$$\sum_{k,l=1}^{2} D_k(u_k v_l v_l) = \left(\sum_{k=1}^{2} D_k u_k\right)\left(\sum_{l=1}^{2} v_l v_l\right) + \sum_{k,l=1}^{2} (\tau_k u_k)\, D_k(v_l v_l)$$

Since DIV$u = \sum_{k=1}^{2} D_k u_k = 0$, we have

$$\frac{1}{m^2} \sum_{i,j=1}^{m} \sum_{k,l=1}^{2} ((\tau_k u_k)\, D_k(v_l v_l))_{ij} = 0$$

Together with (3.11), we have

$$\begin{aligned}
&\frac{1}{m^2} \sum_{i,j=1}^{m} \sum_{k,l=1}^{2} ((\tau_k u_k)(D_k v_l)\, v_l)_{ij} \\
&\quad = -\frac{1}{m^2} \sum_{i,j=1}^{m} \sum_{k,l=1}^{2} ((\tau_k u_k)(\tau_k v_l)(D_k v_l))_{ij} \\
&\quad = -\frac{1}{m^2} \sum_{i,j=1}^{m} \sum_{l=1}^{2} u_{1,i-1,j} v_{l,i-1,j} (D_1 v_l)_{ij} \\
&\qquad -\frac{1}{m^2} \sum_{i,j=1}^{m} \sum_{l=1}^{2} u_{2,i,j-1} v_{l,i,j-1} (D_2 v_l)_{ij} \\
&\quad = -\frac{1}{m^2} \sum_{i,j=1}^{m} \sum_{l=1}^{2} u_{1,ij} v_{l,ij} (D_1 v_l)_{i+1,j} \\
&\qquad -\frac{1}{m^2} \sum_{i,j=1}^{m} \sum_{l=1}^{2} u_{2,ij} v_{l,ij} (D_2 v_l)_{i,j+1} \\
&\quad = -\frac{1}{m^2} \sum_{i,j=1}^{m} \sum_{k,l=1}^{2} (u_k v_l (D_k \tau_k^{-1} v_l))_{ij}
\end{aligned}$$

or

$$\langle ((\tau u) \cdot D + u \cdot (D\tau^{-1}))\, v, v \rangle = 0$$

This proves the Lemma. □

3.2. Global Existence and Global $\|\cdot\|_0$-Attractors

Recall (3.10),

$$\begin{cases} \dfrac{du}{dt}+\dfrac{1}{2}\left((\tau u)\cdot D+u\cdot(D\tau^{-1})\right)u+\nu Au+(D\tau^{-1})\,p=\Gamma \\ \mathrm{DIV}u=0 \\ \bar{u}=0 \end{cases}$$

where $u\in\mathbf{M}\times\mathbf{M}$ and $p\in\mathbf{M}$ are unknowns, and $\Gamma\in\mathbf{M}\times\mathbf{M}$ is fixed and satisfies (H).

The last two equations of (3.10) can be absorbed in the space expression of $\mathbf{M}\times\mathbf{M}$. To do so, we define two subspaces $\mathbf{V}$ and $\mathbf{W}$ of $\mathbf{M}\times\mathbf{M}$:

$$\mathbf{V}=\{v\in\mathbf{M}\times\mathbf{M}\mid \mathrm{DIV}v=0\} \tag{3.16}$$

and

$$\mathbf{W}=\{v\in\mathbf{V}\mid \bar{v}=0\} \tag{3.17}$$

where $\bar{v}$ is the average of v defined as in (2.22).

Note that we have the inclusion relation $\mathbf{W}\subset\mathbf{V}\subset\mathbf{M}\times\mathbf{M}$. Note also that we have the dimensions $\dim(\mathbf{M}\times\mathbf{M})=2m^2$, $\dim(\mathbf{V})=m^2$, and $\dim(\mathbf{W})=m^2-2$.

Lemma 3.5. *$\|\cdot\|_\infty$, $\|\cdot\|_0$, $|\cdot|_1$, and $|\cdot|_2$ are norms on* $\mathbf{W}$.

Proof. This can be verified from the definitions (2.34), (2.23), (2.32), (2.33) together with remark 2.6, and the nondegeneracy property of A shown in (2.46). ▯

Lemma 3.6. *The operators A and D defined by (2.9) and (2.11) commute,* i.e.,

$$D_iA=AD_i \qquad (i=1,2) \tag{3.18}$$

Proof. One can check (3.18) by considering expressions (2.9) and (2.11) or their matrix forms (2.15) and (2.18). ▯

Corollary 3.1. $\mathbf{V}$ *and* $\mathbf{W}$ *are invariant subspaces of A.* ▯

Let

$$P:\mathbf{M}\times\mathbf{M}\to\mathbf{W} \tag{3.19}$$

be the orthogonal projection with respect to the inner product $\langle\cdot,\cdot\rangle$ defined in (2.36) and remark 2.6.

By applying P to the first equation in (3.10), we obtain

$$\frac{dPu}{dt}+\frac{1}{2}PB(u, u)+vPAu+P(D\tau^{-1})\,p=P\Gamma \tag{3.20}$$

where $B(u, u)$ is as in (3.14).

By Corollary 3.1, we have $PA=AP=A$ on $\mathbf{W}$. By Lemma 3.3, we have $P(D\tau^{-1})\,p=0$. Since u satisfies the last two equations in (3.10), which is equivalent to the fact that $u\in\mathbf{W}$, we have $Pu=u$. Hence (3.20) yields

$$\frac{du}{dt}+\frac{1}{2}PB(u, u)+vAu=P\Gamma \qquad (u=u(t)\in\mathbf{W}) \tag{3.21}$$

If we take into account the initial data, we obtain

$$\begin{cases}\dfrac{du}{dt}+\dfrac{1}{2}PB(u, u)+vAu=P\Gamma \qquad (u=u(t)\in\mathbf{W})\\ u(0)=u^{(0)}\in\mathbf{W}\end{cases} \tag{3.22}$$

Remark 3.2. In solving for u, (3.22) is equivalent to the initial value problem of (3.10). In the former case, we ignore the term p. □

Note that $\mathbf{W}$ is an (m^2-2)-dimensional subspace of $\mathbf{M}\times\mathbf{M}$ and the first equation of (3.22) is also of dimension (m^2-2). Furthermore, since $PB(u, u)$ and Au can be expressed as (quadratic and linear) polynomials of u, it is certainly Lipschitz continuous with respect to u. So we can solve for $u=u(t)\in\mathbf{W}$ on some interval $t\in[0, T)$ with $u(0)=u^{(0)}$. This is summarized as

Theorem 3.1. *For any $u^{(0)}\in\mathbf{W}$, there is a unique $u=u(t)\in\mathbf{W}$ satisfying (3.10) and/or (3.22) on $t\in[0, T)$ with $u(0)=u^{(0)}$.* □

The main purpose of this section is to show global existence for solutions of (3.10). We consider only for $u=u(t)\in\mathbf{W}$.

To show the global existence, we prove an a priori estimate of the weighted norm $\|\cdot\|_0$ of $u=u(t)$ on the existence interval.

Theorem 3.2. *There exists a constant*

$$\rho_0=\frac{C}{16v}$$

which is independent of m, where C is as in (H) *and $v>0$ is as in (3.22), such that for any constants $\rho_0'>\rho_0$ and $R_0>0$ independent of m, there exists* *a constant* $T_0 > 0$ *independent of* m, *such that as long as the initial data* $u^{(0)} \in \mathbf{W}$ *satisfies*

$$\|u^{(0)}\|_0 \leqslant R_0$$

then

$$\|u(t)\|_0 \leqslant \rho_0' \qquad (\forall t \geqslant T_0) \tag{3.23}$$

Proof. By taking the inner product $\langle \cdot, \cdot \rangle$ of u and (3.21), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_0^2 + \frac{1}{2}\langle PB(u, u), u\rangle + \nu\langle Au, u\rangle = \langle P\Gamma, u\rangle \tag{3.24}$$

Since $u \in \mathbf{W}$, then $u = Pu$, so

$$\langle PB(u, u), u\rangle = \langle B(u, u), u\rangle = b(u, u, u)$$

By Lemma 3.4,

$$\langle PB(u, u), u\rangle = 0$$

so Lemma 3.4 yields

$$\frac{1}{2}\frac{d}{dt}\|u\|_0^2 + \nu\langle Au, u\rangle = \langle P\Gamma, u\rangle \tag{3.25}$$

Since $u \in \mathbf{W}$, by (2.46),

$$\langle Au, u\rangle \geqslant 16\,\|u\|_0^2 \tag{3.26}$$

Then (3.25) yields

$$\frac{1}{2}\frac{d}{dt}\|u\|_0^2 + 16\nu\,\|u\|_0^2 \leqslant \langle P\Gamma, u\rangle \leqslant \|P\Gamma\|_0\,\|u\|_0 \leqslant \frac{1}{32\nu}\,\|\Gamma\|_0^2 + 8\nu\,\|u\|_0^2$$

By combining this with (H), we have

$$\frac{d}{dt}(\|u\|_0^2\, e^{16\nu t}) \leqslant \frac{C^2}{16\nu}\, e^{16\nu t}$$

Integrating from 0 to t, we obtain

$$\|u(t)\|_0^2 \leqslant R_0^2 e^{-16\nu t} + \frac{C^2}{(16\nu)^2}(1 - e^{-16\nu t})$$

Choosing $\rho_0 = C/16\nu$ and $T_0 = 1/8\nu \log R_0/(\rho_0'^2 - \rho_0^2)$, we complete the proof of the Theorem. □

Remark 3.3. The absorbing constant ρ_0 (of $\|\cdot\|_0$-norm) tends to 0 as C [the uniform bound of $\|\Gamma\|_0$ as in (H)] tends to 0. □

Remark 3.4. Theorem 3.2 shows that if $\rho_0' > \rho_0$ is arbitrarily chosen, then for any initial data $u^{(0)} \in \mathbf{W}$, the solution $u = u(t)$ to (3.22) [or equivalently, (3.10)] has the property $\|u(t)\|_0 \leqslant \rho_0'$ for t large enough. This is the same as to say that

$$\limsup_{t \to \infty} \|u(t)\|_0 \leqslant \rho_0 \qquad (\forall u^{(0)} \in \mathbf{W}) \quad \square \tag{3.27}$$

Since

$$|u_{k,ij}(t)| \leqslant m^2 \, \|u(t)\|_0 \qquad (k = 1, 2,\ 1 \leqslant i, j \leqslant m)$$

we know that

$$\limsup_{t \to \infty} |u_{k,ij}(t)| \leqslant m^2 \rho_0 < \infty \qquad (k = 1, 2,\ 1 \leqslant i, j \leqslant m)$$

That is to say, $u = u(t)$, the solution to (3.22) [or equivalently, (3.10)] does not blow up in finite time. So we have proved the following global existence.

Theorem 3.3. *For any initial data $u^{(0)} \in \mathbf{W}$, the solution, $u = u(t) \in \mathbf{W}$, to (3.22) [or equivalently, (3.10)] exists for $t \in [0, +\infty)$.* □

As an immediate consequence of Theorems 3.2 and 3.3, we have the following theorem.

Theorem 3.4. *The solution, $u = u(t) \in \mathbf{W}$, to (3.22) [or equivalently, (3.10)] has a global attractor $\mathscr{A} = \mathscr{A}_m$ in $\mathbf{W}$ in the norm $\|\cdot\|_0$.* □

4. DISCRETIZED INEQUALITIES AND THE DIMENSIONS OF THE ATTRACTORS

In this section, we first prove a discretized Sobolev embedding inequality. Then we prove the discretized Lieb-Thirring inequality, which plays an important role in proving the exponential decay of the wedge product of the solutions of the linearized equation of (3.10). Finally, from the exponential decay property, we infer an estimate of the Hausdorff dimension of the attractor obtained in Theorem A.3.

4.1. Discretized Sobolev Embedding Inequalities

It is well-known that if $kp = n$ and $\Omega \subset \mathbf{R}^n$ is bounded and satisfies the cone property, then the continuous embedding

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad (\forall q \in [1, \infty)) \tag{4.1}$$

is true (see, e.g., [A]).

In this section we prove a special case of (4.1) ($k = 1$, $p = 2$, $n = 2$, and $\Omega = (0, 1)^2$) in the discretized version.

Recall that functions in $\mathbf{M}$ are our "discretized" functions. We temporarily define the following norms in $\mathbf{M}$ for any p, $\rho \in [1, \infty)$:

$$|v|_q = \left(\frac{1}{m^2} \sum_{i,j=1}^{m} |v_{ij}|^q\right)^{1/q} \tag{4.2}$$

$$\begin{aligned}|v|_{1,p} &= \left(\frac{1}{m^2} \sum_{i,j=1}^{m} (|v_{ij}|^p + |(Dv)_{ij}|^p)\right)^{1/p} \\ &= \left(\frac{1}{m^2} \sum_{i,j=1}^{m} (|v_{ij}|^p + |(D_1 v)_{ij}|^p + |(D_2 v)_{ij}|^p)\right)^{1/p}\end{aligned} \tag{4.3}$$

The following lemma is the key point of the discretized Sobolev embedding.

Lemma 4.1. *If $1 \leqslant p < 2$ and $q = 2p/(2-p)$, then there is a constant*

$$K_1 = K_1(p) = 2^{(p+1)/p} \cdot 3^{(p-1)/p} \cdot 3^{(p-1)/p} \cdot \frac{p}{2-p}$$

independent of m such that

$$|v|_q \leqslant K_1 \, |v|_{1,p} \qquad (\forall v \in \mathbf{M})$$

Proof. Let $\gamma = p/(2-p)$. Then $\gamma - 1 = 2 \cdot [(p-1)/(2-p)] \geqslant 0$, or, $\gamma \geqslant 1$.

For $i \geqslant m/2$,

$$|v_{ij}|^\gamma = |v_{1j}|^\gamma + \sum_{k=1}^{i-1} (|v_{k+1,j}|^\gamma - |v_{kj}|^\gamma)$$

hence

$$\frac{1}{m} \sum_{i=\lceil m/2 \rceil}^{m} |v_{ij}|^\gamma = \frac{1}{m} \left\lceil \frac{m}{2} \right\rceil |v_{1j}|^\gamma + \frac{1}{m} \sum_{i=\lceil m/2 \rceil}^{m} \sum_{k=1}^{i-1} (|v_{k+1,j}|^\gamma - |v_{kj}|^\gamma)$$

So

$$\frac{1}{2}|v_{1j}|^\gamma \leqslant \frac{1}{m}\left\lceil \frac{m}{2} \right\rceil |v_{ij}|^\gamma$$

$$= \frac{1}{m} \sum_{i=\lceil m/2 \rceil}^{m} |v_{ij}|^\gamma - \frac{1}{m} \sum_{i=\lceil m/2 \rceil}^{m} \sum_{k=1}^{i-1} (|v_{k+1,j}|^\gamma - |v_{kj}|^\gamma)$$

$$\leqslant \frac{1}{m} \sum_{i=1}^{m} |v_{ij}|^\gamma + \frac{1}{m} \sum_{i=1}^{m-1} (m-i)\,||v_{i+1,j}|^\gamma - |v_{ij}|^\gamma|$$

By the mean value theorem, for any a, $b \geqslant 0$, there is a $\theta \in (0, 1)$ such that

$$|a^\gamma - b^\gamma| = \gamma(\theta a + (1-\theta)\, b)^{\gamma-1}\, |a-b| \leqslant \gamma(a^{\gamma-1} + b^{\gamma-1})\, |a-b|$$

so

$$\frac{1}{2}|v_{1j}|^\gamma \leqslant \frac{1}{m} \sum_{i=1}^{m} |v_{ij}|^\gamma + \frac{\gamma}{m} \sum_{i=1}^{m-1} (|v_{i+1,j}|^{\gamma-1} + |v_{ij}|^{\gamma-1})\, |(D_1 v)_{i+1,j}| \tag{4.4}$$

Note that the right-hand side of (4.4) is independent of the first index of v, and the indexes of v are arranged periodically as in (2.5). Thus we can replace the index "1" on the left-hand side of (4.4) by any index "i." Hence we have

$$\max_{1 \leqslant i \leqslant m} |v_{ij}|^\gamma \leqslant \frac{2}{m} \sum_{i=1}^{m} |v_{ij}|^\gamma + \frac{2\gamma}{m} \sum_{i=1}^{m} (|v_{i+1,j}|^{\gamma-1} + |v_{ij}|^{\gamma-1})\, |(Dv)_{i+1,j}|$$

By taking the sum over the index j, we have

$$\frac{1}{m} \sum_{j=1}^{m} (\max_{1 \leqslant i \leqslant m} |v_{ij}|^\gamma) \leqslant \frac{2}{m^2} \sum_{i,j=1}^{m} |v_{ij}|^\gamma + \frac{2\gamma}{m^2} \sum_{i,j=1}^{m} (|v_{i+1,j}|^{\gamma-1} + |v_{ij}|^{\gamma-1})\, |(Dv)_{i+1,j}|$$

By applying Hölder's inequality, we have

$$\frac{1}{m} \sum_{j=1}^{m} (\max_{1 \leqslant i \leqslant m} |v_{ij}|^\gamma)$$

$$\leqslant 2\gamma \left(\frac{1}{m^2} \sum_{i,j=1}^{m} (|v_{ij}|^p + 2\,|(D_1 v)_{i+1,j}|^p) \right)^{1/p}$$

$$\cdot \left(\frac{1}{m^2} \sum_{i,j=1}^{m} (|v_{ij}|^{(\gamma-1)p'} + |v_{i+1,j}|^{(\gamma-1)p'} + |v_{ij}|^{(\gamma-1)p'}) \right)^{1/p'}$$

$$\leqslant 2 \cdot 2^{1/p'} \cdot 3^{1/p'} \cdot \gamma \left(\frac{1}{m^2} \sum_{i,j=1}^{m} (|v_{ij}|^p + |(Dv)_{i+1,j}|^p) \right)^{1/p}$$

$$\cdot \left(\frac{1}{m^2} \sum_{i,j=1}^{m} |v_{ij}|^{(\gamma-1)p'} \right)^{1/p'}$$

Note that $p' = p/(p-1)$ and $\gamma - 1 = [2(p-1)]/(2-p)$, so $(\gamma-1)\,p' = [2p/(2-p)] = q$. Hence

$$\frac{1}{m}\sum_{j=1}^{m}\big(\max_{1\leqslant i\leqslant m}|v_{ij}|^{\gamma}\big) \leqslant 2^{(p+1)/p}\cdot 3^{(p-1)/p}\cdot\gamma\cdot|v|_{1,p}\cdot|v|_q^{\gamma-1}$$

By the same procedure, we can prove

$$\frac{1}{m}\sum_{i=1}^{m}\big(\max_{1\leqslant j\leqslant m}|v_{ij}|^{\gamma}\big) \leqslant 2^{(p+1)/p}\cdot 3^{(p-1)/p}\cdot\gamma\cdot|v|_{1,p}\cdot|v|_q^{\gamma-1}$$

Hence, since $q = [2p/(2-p)] = 2\gamma$, we have

$$\begin{aligned}
|v|_q^q &= \frac{1}{m^2}\sum_{i,j=1}^{m}(|v_{ij}|^{\gamma})^2\\
&\leqslant \frac{1}{m^2}\sum_{i,j=1}^{m}\big(\max_{1\leqslant i\leqslant m}|v_{ij}|^{\gamma}\big)\cdot\big(\max_{1\leqslant j\leqslant m}|v_{ij}|^{\gamma}\big)\\
&= \left(\frac{1}{m}\sum_{i=1}^{m}\big(\max_{1\leqslant j\leqslant m}|v_{ij}|^{\gamma}\big)\right)\cdot\left(\frac{1}{m}\sum_{j=1}^{m}\big(\max_{1\leqslant i\leqslant m}|v_{ij}|^{\gamma}\big)\right)\\
&\leqslant 2^{2(p+1)/p}\cdot 3^{2(p-1)/p}\cdot\gamma^2\cdot|v|_{1,p}^2\cdot|v|_q^{2(\gamma-1)}.
\end{aligned}$$

If $|v|_q = 0$ then $v = 0$, so the lemma is trivial in this case. Otherwise we have

$$|v|_q^{q-2(\gamma-1)} \leqslant 2^{2(p+1)/p}\cdot 3^{2(p-1)/p}\gamma^2\,|v|_{1,p}^2$$

But $q - 2(\gamma-1) = q - 2q/p' = 2p/2 - p\cdot(1 - 2(p-1)/p) = 2$. Therefore

$$|v|_q \leqslant 2^{(p+1)/p}\cdot 3^{(p-1)/p}\cdot\frac{p}{2-p}\cdot|v|_{1,p}$$

This proves the lemma with $K_1 = 2^{(p+1)/p}\cdot 3^{(p-1)/p}\cdot[p/(2-p)]$. $\square$

With the lemma above, we can prove the following discretized embedding Theorem.

Theorem 4.1. *For any $q \in [1, \infty)$, the following embedding*

$$(\mathbf{M}, |\cdot|_{1,2}) \hookrightarrow (\mathbf{M}, |\cdot|_q)$$

is true. In other words, there exists a constant K_2 independent of m such that

$$|v|_q \leqslant K_2\,|v|_{1,2} \qquad (\forall v \in \mathbf{M})$$

Proof. If $q \leqslant 2$, by Hölder's inequality, $|v|_q \leqslant |v|_2$. Hence we need only to prove the Lemma for $q \geqslant 2$.

Assume $q \geqslant 2$, and take $s = 2q/(2+q)$, then we have $1 \leqslant s < 2$. Apply Hölder's inequality again,

$$
\begin{aligned}
|v|_{1,s}^{s} &= \frac{1}{m^2} \sum_{i,j=1}^{m} (|v_{ij}|^s + |(Dv)_{ij}|^s) \\
&\leqslant \left(\frac{1}{m^2} \sum_{i,j=1}^{m} (|v_{ij}|^2 + |(Dv)_{ij}|^2) \right)^{s/2} \\
&\quad \cdot \left(\frac{1}{m^2} \sum_{i,j=1}^{m} (1^{2/(2-s)} + 1^{2/(2-s)}) \right)^{(2-s)/s} \\
&= 2^{(2-s)/s} |v|_{1,2}^{s}
\end{aligned}
$$

Since $q = 2s/(2-s)$, by Lemma 4.1, $|v|_q \leqslant K_1 |v|_{1,s}$. Hence

$$|v|_q \leqslant 2^{(2-s)/s^2} K_1 |v|_{1,2}$$

Now take

$$K_2 = \begin{cases} 2^{(2q^2+3q+2)/q^2} \cdot 3^{(q-2)/2q} \cdot q & \text{if } q \geqslant 2 \\ 32 & \text{if } 1 \leqslant q < 2 \end{cases}$$

which completes the proof. □

Corollary 4.1. *For any $q \in [1, \infty)$, there is a constant $K_3 = \sqrt{(17/4)}\, K_2$ independent of m such that*

$$|v|_q \leqslant K_3 \|v\|_1 \qquad (\forall v \in \mathbf{W})$$

Proof. By Corollary 2.2,

$$|v|_{1,2}^2 = \frac{1}{m^2} \sum_{i,j=1}^{m} |v_{ij}|^2 + \frac{1}{m^2} \sum_{i,j=1}^{m} |(Dv)_{ij}|^2 \leqslant \frac{17}{16} \|v\|_1^2$$

Hence by Theorem 4.1,

$$|v|_q \leqslant K_2 |v|_{1,2} \leqslant \frac{\sqrt{17}}{4} K_2 \|v\|_1 \quad \square$$

We are particularly interested in the norm $|\cdot|_4$ in $\mathbf{M}$ or $\mathbf{W}$. By Corollary 4.1, we have the embedding inequality $|v|_4 \leqslant K_3 \|v\|_1$ $(\forall v \in \mathbf{W})$. We prove the following discretized interpolation inequality, which is a special case of the discretized Lieb-Thirring inequality, and which will be used to simplify the proof of the discretized H^1-absorbing property of the discretized Navier-Stokes equations.

Lemma 4.2.

$$|u|_4 \leqslant K_4 \, \|u\|_0^{1/2} \, \|u\|_1^{1/2} \qquad (\forall v \in \mathbf{W}) \tag{4.5}$$

where $K_4 = 4$.

Proof. By Lemma 4.1, with $p=1$ and $q=2$, we have, for any $v \in \mathbf{M} \times \mathbf{M}$,

$$|v|_2 \leqslant K_1 \, |v|_{1,1} \tag{4.6}$$

where $K_1 = 4$. For any $u \in \mathbf{W}$, we define $u^2 \in \mathbf{W}$ by $(u^2)_{k,ij} = (u_{k,ij})^2$. Then (4.6) yields $|u^2|_2 \leqslant K_1 \, |u^2|_{1,1}$, or

$$\left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} u_{k,ij}^4\right)^{1/2} \leqslant K_1 \cdot |u|_2^2 + K_1 \cdot \left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} (|D_1 u_{k,ij}^2| + |D_2 u_{k,ij}^2|)\right) \tag{4.7}$$

Since

$$\begin{aligned}\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} (|D_1 u_{k,ij}^2| + |D_2 u_{k,ij}^2|) &= \frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} (|(D_1 u)_{k,ij}| \cdot |u_{k,ij} + u_{k,i-1,j}| \\ &\quad + |(D_2 u)_{k,ij}| \cdot |u_{k,ij} + u_{k,i,j-1}|)\end{aligned}$$

by the Cauchy-Schwarz inequality,

$$\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} (|D_1 u_{k,ij}^2| + |D_2 u_{k,ij}^2|) \leqslant 2\sqrt{2} \cdot \|u\|_0 \cdot \|u\|_1$$

Hence, by (4.7),

$$\left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} u_{k,ij}^4\right)^{1/2} \leqslant K_1 \cdot |u|_2^2 + 2\sqrt{2}\, K_1 \cdot \|u\|_0 \cdot \|u\|_1 \tag{4.8}$$

But $u \in \mathbf{W}$, so (2.46) gives

$$|u|_2^2 = \|u\|_0^2 \leqslant \tfrac{1}{4} \cdot \|u\|_0 \cdot \|u\|_1$$

Thus, (4.8) gives

$$\begin{aligned}\left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} u_{k,ij}^4\right)^{1/2} &\leqslant \frac{1+8\sqrt{2}}{4} K_1 \cdot \|u\|_0 \cdot \|u\|_1 \\ &\leqslant (1+8\sqrt{2}) \cdot \|u\|_0 \cdot \|u\|_1\end{aligned}$$

or

$$|u|_4 \leqslant \sqrt{1+8\sqrt{2}} \cdot \|u\|_0^{1/2} \cdot \|u\|_1^{1/2} \leqslant 4 \cdot \|u\|_0^{1/2} \cdot \|u\|_1^{1/2} \quad \square$$

In the continuous case, this is equivalent to

$$|f|_{L^4} \leqslant C\,|f|_{L^2}^{1/2}\,|f|_{H^1}^{1/2} \qquad (\forall f \in H^1(\Omega))$$

where $\Omega \subset \mathbf{R}^2$. This is a special case of Nirenberg's inequality, which gives flexible indexes of order of differentiation and interpolation (see [N], [F], and [L1]). It is also a special case of the Lieb-Thirring inequality, which applies to a family of orthonormal functions (see [L-T], [T3]).

4.2. Discretized Lieb-Thirring Inequality

Define a norm $\|\cdot\|$ in $\mathbf{M} \times \mathbf{M}$ by

$$\|u\| = (\|u\|_1^2 + \|u\|_0^2)^{1/2} = (\|u_1\|_1^2 + \|u_2\|_1^2 + \|u_1\|_0^2 + \|u_2\|_0^2)^{1/2}$$
$$(\forall u \in \mathbf{M} \times \mathbf{M}) \tag{4.9}$$

For a fixed $g \in \mathbf{M} \times \mathbf{M}$, consider the discretized Schrödinger-type linear operator $A+1+g\colon \mathbf{M} \times \mathbf{M} \to \mathbf{M} \times \mathbf{M}$ defined by

$$((A+1+g)\,u)_{k,ij} = (Au)_{k,ij} + u_{k,ij} + g_{k,ij}u_{k,ij} \qquad (\forall u \in \mathbf{M} \times \mathbf{M}) \tag{4.10}$$

This operator induces a bilinear form on $\mathbf{M} \times \mathbf{M}$

$$\begin{aligned} L_g(u, v) &= \langle u, (A+1+g)\,v\rangle \\ &= \langle u, (A+1)\,v\rangle + \frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} g_{k,ij}u_{k,ij}v_{k,ij} \end{aligned} \tag{4.11}$$

which in turn induces a quadratic form on $\mathbf{M} \times \mathbf{M}$

$$Q_g(u) = L_g(u, u) \tag{4.12}$$

From (4.11), one sees that $A+1+g$ is a symmetric operator on $\mathbf{M} \times \mathbf{M}$, so it has $2m$ real eigenvalues. The eigenvalues of $A+1+g$ can be obtained by the min-max method

$$\mu_j(g) = \min_{u^{(1)},\ldots,u^{(j-1)} \in \mathbf{M} \times \mathbf{M}} \ \max_{\substack{v \in \mathbf{M} \times \mathbf{M} \\ \langle v, u^{(i)}\rangle = 0 \\ i=1,\ldots,j-1}} Q_g(v) \tag{4.13}$$

with

$$\mu_1(g) \leqslant \mu_2(g) \leqslant \cdots \leqslant \mu_{2m}(g) \tag{4.14}$$

In this section, we give an estimate (for any $\gamma > 0$) of

$$\sum_{\mu_j(g)<0} |\mu_j(g)|^\gamma \tag{4.15}$$

in terms of g and γ.

If g, $h \in \mathbf{M} \times \mathbf{M}$ satisfy $g \leqslant h$, i.e.,

$$g_{k,ij} \leqslant h_{k,ij} \qquad (k=1, 2,\ 1 \leqslant i, j \leqslant m) \tag{4.16}$$

then, by (4.13),

$$\mu_j(g) \leqslant \mu_j(h) \qquad (1 \leqslant j \leqslant 2m) \tag{4.17}$$

For $g \in \mathbf{M} \times \mathbf{M}$, define g_+, $g_- \in \mathbf{M} \times \mathbf{M}$ by

$$(g_+)_{ij} = \max(g_{ij}, 0), \qquad (g_-)_{ij} = \max(-g_{ij}, 0) \tag{4.18}$$

Then $-g_- \leqslant g$ yields

$$\mu_j(-g_-) \leqslant \mu_j(g) \qquad (1 \leqslant j \leqslant 2m) \tag{4.19}$$

For every $r \in \mathbf{R}$, define $N_r(g)$ to be the number of $\mu_j(g) \leqslant r$ (counting the multiplicity).

Lemma 4.3. *For any* $\gamma \geqslant 0$,

$$\sum_{\mu_j(g)<0} |\mu_j(g)|^\gamma = \gamma \int_0^\infty r^{\gamma-1} N_{-r}(g)\, dr \tag{4.20}$$

Proof. Let $\beta = N_0(g)$. Then

$$\begin{aligned}
&\gamma \int_0^\infty r^{\gamma-1} N_{-r}(g)\, dr \\
&= \gamma \left(\int_{-\mu_1(g)}^\infty + \int_{-\mu_2(g)}^{-\mu_1(g)} + \cdots + \int_{-\mu_\beta(g)}^{-\mu_{\beta-1}(g)} + \int_0^{-\mu_\beta(g)} \right) r^{\gamma-1} N_{-r}(g)\, dr \\
&= \gamma \left(\int_{-\mu_1(g)}^\infty r^{\gamma-1} \cdot 0 dr + \int_{-\mu_2(g)}^{-\mu_1(g)} r^{\gamma-1} \cdot 1 dr + \cdots \right. \\
&\qquad \left. + \int_{-\mu_\beta(g)}^{-\mu_{\beta-1}(g)} r^{\gamma-1} \cdot (\beta-1)\, dr + \int_0^{-\mu_\beta(g)} r^{\gamma-1} \cdot \beta dr \right) \\
&= \sum_{\mu_j(g)<0} |\mu_j(g)|^\gamma. \quad \square
\end{aligned}$$

Hence, in order to estimate (4.15), we need to estimate $N_{-r}(g)$.

Lemma 4.4. *If $g \in \mathbf{M} \times \mathbf{M}$ satisfies $g \leqslant 0$ (i.e., $g_{k,ij} \leqslant 0$ for all k, i, and j) but not identically equal to zero (i.e., there are k, i, and j such that $g_{k,ij} < 0$) and $r \leqslant 0$, then for any fixed j with $1 \leqslant j \leqslant 2m$, $\mu_j(\kappa g)$ is continuous and strictly decreasing with respect to κ.*

Proof. [K] proved the same result for the continuous version. In the discretized version, since the global space is of finite dimension, we can prove this Lemma quickly.

Since $\mu_j(\kappa g)$ is the jth eigenvalue of the operator $A + 1 + g$, i.e., $\mu_j(\kappa g)$ is the jth root of the characteristic polynomial $\lambda(A + 1 + \kappa g)$ of the operator $A + 1 + \kappa g$. But $\lambda(A + 1 + \kappa g)$ is continuous with respect to κ, so $\mu_j(\kappa g)$ is continuous with respect to κ.

By the min-max method (4.13),

$$\begin{aligned}\mu_j(g) &= \min_{u^{(1)},\ldots,u^{(j-1)} \in \mathbf{M} \times \mathbf{M}} \ \max_{\substack{v \in \mathbf{M} \times \mathbf{M} \\ \langle v, u^{(i)} \rangle = 0 \\ i = 1,\ldots,j-1}} Q_{\kappa g}(v) \\ &= \min_{u^{(1)},\ldots,u^{(j-1)} \in \mathbf{M} \times \mathbf{M}} \ \max_{\substack{v \in \mathbf{M} \times \mathbf{M} \\ \langle v, u^{(i)} \rangle = 0 \\ i = 1,\ldots,j-1}} \langle v, (A+1)\, v \rangle + \kappa \langle v, gv \rangle\end{aligned}$$

Since $\langle \cdot, (A+1) \cdot \rangle$ is positive and $\langle \cdot, g \cdot \rangle$ is nonpositive and not identically 0, so $\mu_j(\kappa g)$ is strictly decreasing when κ is strictly increasing.

This proves the lemma. □

Lemma 4.5. *With the same assumptions as in Lemma 4.4, $N_r(g)$ is equal to the number of κ's in $(0, 1]$ such that $\mu_j(\kappa g) = r$ for some j.*

Proof. Consider $\mu_j(\kappa g)$.

If $\kappa = 0$, then $\mu_j(\kappa g)$ are the eigenvalues of $A + 1 = A + \mathrm{id}_{\mathbf{M} \times \mathbf{M}}$, which are positive since the eigenvalues of A are nonnegative.

If $\kappa = 1$, then $\mu_j(\kappa g)$ are the eigenvalues of $A + 1 + g$.

Hence by Lemma 4.4, for any $\mu_j(g) \leqslant r (\leqslant 0)$, we can find one and only one $\kappa_j \in (0, 1]$ such that $\mu_j(\kappa_j g) = r$. On the other hand, for any $\mu_j(g) > r$ there is no $\kappa \in (0, 1]$ such that $\mu_j(\kappa g) = r$. This proves the lemma. □

For any $r \leqslant 0$, $A + 1 - r$ is a symmetric linear operator, and its eigenvalues are all positive. Hence the inverse

$$(A + 1 - r)^{-1}$$

exists, and its eigenvalues are also positive.

For any $h \in \mathbf{M} \times \mathbf{M}$, define the multiplicative operator

$$h \colon \mathbf{M} \times \mathbf{M} \to \mathbf{M} \times \mathbf{M}$$

by

$$(hv)_{k,ij} = h_{k,ij} v_{k,ij} \qquad (\forall v \in \mathbf{M} \times \mathbf{M}) \tag{4.21}$$

Let us also define for every $r \leqslant 0$

$$G_r = |g|^{1/2} (A + 1 - r)^{-1} |g|^{1/2} \tag{4.22}$$

where $|g|^{1/2}$ is an element of $\mathbf{M} \times \mathbf{M}$ defined by

$$(|g|^{1/2})_{k,ij} = |g_{k,ij}|^{1/2} \tag{4.23}$$

One sees that G_r is a positive operator since $r \leqslant 0$ and $A + 1 - r$ is positive.

Lemma 4.6. *We make the same assumptions as in Lemma 4.4. We also assume that $\kappa \in (0, 1]$. Then $A + 1 + \kappa g$ has r as an eigenvalue of multiplicity l if and only if $1/\kappa$ is an eigenvalue of G_r of multiplicity l.*

Proof. If $\kappa \in (0, 1]$ and

$$(A + 1 + \kappa g)\, v = rv$$

for some $v \in \mathbf{M} \times \mathbf{M}$ with $v \neq 0$. Let $u = -|g|^{1/2} v$, then $u \neq 0$ and

$$G_r u = |g|^{1/2} (A + 1 - r)^{-1} |g|^{1/2} (-|g|^{1/2} v) = |g|^{1/2} \left(-\frac{1}{\kappa} v \right) = \frac{1}{\kappa} u$$

Assume that $v^{(1)}, v^{(2)}, \ldots, v^{(l)}$ are linearly independent eigenvectors of $A + 1 + \kappa g$, then the corresponding eigenvectors of G_r are $\{u^{(i)} = -|g|^{1/2} v^{(i)}\}_{i=1}^{l}$. Let $\sum_{i=1}^{l} k_i u^{(i)} = 0$. Then $|g|^{1/2} \sum_{i=1}^{l} k_i v^{(i)} = 0$, or $g \sum_{i=1}^{l} k_i v^{(i)} = 0$. But $(A + 1 + \kappa g) \sum_{i=1}^{l} k_i v^{(i)} = r \sum_{i=1}^{l} k_i v^{(i)}$, or $(A + 1) \sum_{i=1}^{l} k_i v^{(i)} = r \sum_{i=1}^{l} k_i v^{(i)}$. Since $A + 1$ is a positive operator and $r \leqslant 0$, we have $\sum_{i=1}^{l} k_i v^{(i)} = 0$, or $k_1 = k_2 = \cdots = k_l = 0$. Hence $\{u^{(i)}\}_{i=1}^{l}$ are linearly independent.

Conversely, if $u \in \mathbf{M} \times \mathbf{M}$ with $u \neq 0$ and $k \in (0, 1]$ satisfy

$$G_r u = \frac{1}{\kappa} u$$

Then

$$\begin{aligned} &\kappa g (A + 1 - r)^{-1} |g|^{1/2} u \\ &\qquad = -|g|^{1/2} u = (A + 1 - r)(A + 1 - r)^{-1} (-|g|^{1/2} u) \end{aligned}$$

If we take $v=(A+1-r)^{-1}(-|g|^{1/2}u)$, we have $v \neq 0$ and

$$(A+1+\kappa g)\,v=rv$$

Assume that $u^{(1)}, u^{(2)},..., u^{(l)}$ are linearly independent eigenvectors of G_r, then the corresponding eigenvectors of $(A+1+\kappa g)$ are $\{v^{(i)}=(A+1-r)^{-1}(-|g|^{1/2}u^{(i)})\}_{i=1}^{l}$. Let $\sum_{i=1}^{l} k_i v^{(i)}=0$, then $|g|^{1/2}\sum_{i=1}^{l} k_i u^{(i)}=0$. But $G_r \sum_{i=1}^{l} k_i u^{(i)} = (1/\kappa)\sum_{i=1}^{l} k_i u^{(i)}$ or $(1/\kappa)\sum_{i=1}^{l} k_i u^{(i)} = 0$. Hence $\sum_{i=1}^{l} k_i u^{(i)} = 0$ or $k_1 = k_2 = \cdots = k_l = 0$. So $\{v^{(i)}\}_{i=1}^{l}$ are linearly independent.

This proves the lemma. ∎

Lemma 4.7. *Assumptions are as in Lemma 4.4. We also assume that $t \in [0, 1]$ and $k \geqslant 1$. Then*

$$N_r(g) \leqslant Tr((g-(1-t)\,r)^{1/2}_- (A+1-rt)^{-1}(g-(1-t)\,r)^{1/2}_-)^k$$

where "Tr" is the trace of a linear operator.

Proof. By Lemmas 4.5 and 4.6, $N_r(g)$ is equal to the number of eigenvalues of G_r which are $\geqslant 1$. Moreover, G_r is positive. Then, denoting by σ_j the eigenvalues of G_r, we have

$$N_r(g)=\sum_{\sigma_j \geqslant 1} 1 \leqslant \sum_{\sigma_j \geqslant 1} \sigma_j^k \leqslant \mathrm{Tr}(G_r^k)$$

This proves the Lemma for $t=1$.

For $t \in [0, 1)$, note that $(A+1+g)\,u=ru$ if and only if $(A+1+(g-(1-t)\,r))\,u=rtu$. Hence by Lemmas 4.5 and 4.6 and the fact that $N_s(h)$ is decreasing with respect to h [which is implied by that fact that $\mu_j(h)$ is increasing with respect to h], we have

$$N_r(g)=N_{rt}(g-(1-t)\,r) \leqslant N_{rt}(-(g-(1-t)\,r)_-)$$

Using the result of this Lemma for the case $t=1$ completes the proof. ∎

To estimate (4.15) by means of the estimate for $N_r(g)$, we need to consider the eigenvectors of the operator $A\colon \mathbf{M} \to \mathbf{M}$. The eigenvectors $e(k, l)$ of A are given by Theorem 2.1.

Lemma 4.8. *Let $e(k, l) \in \mathbf{M}(1 \leqslant k, l \leqslant m)$ be given by Theorem 2.1. Then the only possible values that $\|e(k, l)\|_0$ can take are 1 or $(1/\sqrt{2})$ or $(1/2)$.*

Proof. Since by Theorem 2.1, $e(k, l) = r(k)_i \cdot s(l)_j$ $(0 \leqslant k, l \leqslant m-1)$, it follows that

$$\|e(k, l)\|_0^2 = \frac{1}{m^2} \sum_{i,j=1}^{m} e(k, l)_{ij}^2$$
$$= \left(\frac{1}{m} \sum_{i=1}^{m} r(k)_i^2\right) \cdot \left(\frac{1}{m} \sum_{j=1}^{m} s(l)_j^2\right) \tag{4.24}$$

But by Theorem 2.1, for a fixed k, $r(k)_i$ is of the form

$$r(k)_j = \begin{cases} 1 & (k=0) \\ \sin\left(j \cdot \dfrac{2k\pi}{m}\right) & \left(1 \leqslant k \leqslant \left\lfloor \dfrac{m}{2} \right\rfloor\right) \\ \cos\left(j \cdot \dfrac{2(m-k)\pi}{m}\right) & \left(\left\lfloor \dfrac{m}{2} \right\rfloor < k \leqslant m-1\right) \end{cases}$$

$(1/m) \sum_{j=1}^{m} r(0)_j^2 = 1$; in the second case, $(1/m) \sum_{j=1}^{m} r(k)_j^2 = \frac{1}{2} - 1/2m \sum_{j=1}^{m} \cos(j \cdot (4n\pi/m)) = \frac{1}{2}$; in the third case, $(1/m) \sum_{j=1}^{m} r(k)_j^2 = \frac{1}{2} + (1/2m) \sum_{j=1}^{m} \cos[j \cdot (4n\pi/m)] = \frac{1}{2}$. Hence we have

$$\frac{1}{m} \sum_{j=1}^{m} r(k)_j^2 = \begin{cases} 1 \\ \dfrac{1}{2} \end{cases} \tag{4.25}$$

Similarly,

$$\frac{1}{m} \sum_{j=1}^{m} s(l)_j^2 = \begin{cases} 1 \\ \dfrac{1}{2} \end{cases} \tag{4.26}$$

Thus, (4.24), together with (4.25) and (4.26), shows that the lemma holds. □

We reorder $\{e(k, l)\}_{k,l=0}^{m-1}$ as $\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(m^2)}$ so that the corresponding eigenvalues of A satisfy $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_{m^2}$. By Lemma 4.8, $\|\eta^{(n)}\|_0 \geqslant \frac{1}{2}$. Since $|e(k, l)_{ij}| = |r(k)_i| \cdot |s(l)_j| \leqslant 1$, so

$$|\eta_{ij}^{(n)}| \leqslant 1 \qquad (1 \leqslant i, j \leqslant m,\ 1 \leqslant n \leqslant m^2) \tag{4.27}$$

Hence the normalized family

$$\left\{\varphi^{(n)} = \frac{\eta^{(n)}}{\|\eta^{(n)}\|_0}\right\}_{n=1}^{m^2} \tag{4.28}$$

satisfies [by Lemma 4.8 and (4.27)]

$$|\varphi_{ij}^{(n)}| \leqslant 2 \tag{4.29}$$

Proposition 4.1. *The assumptions are as in Lemma 4.4. We also assume that $\gamma > 0$ and $1 < k < \gamma + 1$. Then there is a constant $K > 0$ independent of m such that*

$$\sum_{\mu_j(g)<0} |\mu_j(g)|^\gamma \leqslant K \cdot \left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} (g_{k,ij})_-^{\gamma+1} \right) \tag{4.30}$$

Proof. By Lemma 4.7,

$$N_r(g) \leqslant \operatorname{Tr}((g-(1-t)\,r)_-^{1/2}\,(A+1-rt)^{-1}\,(g-(1-t)\,r)_-^{1/2})^l$$

But $(A+1-rt)$ and the multiplicative operator $(g-(1-t)\,r)_-^{1/2}$ are both positive and symmetric, and the result of [L-T] (Appendix B) implies that for any $\beta \in [1, +\infty)$,

$$\begin{aligned} &\operatorname{Tr}((g-(1-t)\,r)_-^{1/2}\,(A+1-rt)^{-1}\,(g-(1-t)\,r)_-^{1/2})^\beta \\ &\quad \leqslant \operatorname{Tr}((g-(1-t)\,r)_-^{\beta/2}\,(\mathrm{A}+1-rt)^{-\beta}\,(g-(1-t)\,r)_-^{\beta/2}) \end{aligned}$$

By taking $t = \frac{1}{2}$, so for any $r \geqslant 0$,

$$N_{-r}(g) \leqslant \operatorname{Tr}\left(\left(g+\frac{r}{2}\right)_-^{\beta/2} \left(A+1+\frac{r}{2}\right)^{-\beta} \left(g+\frac{r}{2}\right)_-^{\beta/2} \right) \tag{4.31}$$

Consider the orthonormal and complete family $\{\varphi^{(n)}\}_{n=1}^{2m^2}$ of $\mathbf{M} \times \mathbf{M}$, as in (4.28), consisting of the eigenfunctions of A. We have, for any $v \in \mathbf{M} \times \mathbf{M}$,

$$\begin{aligned} &\left(g+\frac{r}{2}\right)_-^{\beta/2} \left(A+1+\frac{r}{2}\right)^{-\beta} \left(g+\frac{r}{2}\right)_-^{\beta/2} v \\ &\quad = \sum_{n=1}^{2m^2} \left(\lambda_n+1+\frac{r}{2}\right)^{-\beta} \left\langle \left(g+\frac{r}{2}\right)_-^{\beta/2} v, \varphi^{(n)} \right\rangle \left(g+\frac{r}{2}\right)_-^{\beta/2} \varphi^{(n)} \end{aligned}$$

So the kernel of $[g+(r/2)]_-^{\beta/2}\,[A+1+(r/2)]^{-\beta}\,[g+(r/2)]_-^{\beta/2}$ is

$$\begin{aligned} &G(k, i, j; k', i', j') \\ &\quad = \sum_{n=1}^{2m^2} \left(\lambda_n+1+\frac{r}{2}\right)^{-\beta} \left(g_{k,ij}+\frac{r}{2}\right)_-^{\beta/2} - \varphi_{k,ij}^{(n)} \left(g_{k',i'j'}+\frac{r}{2}\right)_-^{\beta/2} \varphi_{k',i'j'}^{(n)} \end{aligned}$$

and its trace is

$$\frac{1}{m^2}\sum_{k=1}^{2}\sum_{i,j=1}^{m} G(k,i,j;k,i,j)$$
$$=\frac{1}{m^2}\sum_{k=1}^{2}\sum_{i,j=1}^{m}\sum_{n=1}^{2m^2}\left(\lambda_n+1+\frac{r}{2}\right)^{-\beta}\left(g_{k,ij}+\frac{r}{2}\right)_-^{\beta}(\varphi_{k,ij}^{(n)})^2$$

Hence

$$N_{-r}(g)\leqslant \operatorname{Tr}\left(\left(g+\frac{r}{2}\right)_-^{\beta/2}\left(A+1+\frac{r}{2}\right)^{-\beta}\left(g+\frac{r}{2}\right)_-^{\beta/2}\right)$$
$$=\frac{1}{m^2}\sum_{k=1}^{2}\sum_{i,j=1}^{m}\sum_{n=1}^{2m^2}\left(\lambda_n+1+\frac{r}{2}\right)^{-\beta}\left(g_{k,ij}+\frac{r}{2}\right)_-^{\beta}(\varphi_{k,ij}^{(n)})^2$$

By (4.29),

$$N_{-r}(g)\leqslant 4\cdot\left(\frac{1}{m^2}\sum_{k=1}^{2}\sum_{i,j=1}^{m}\left(g_{k,ij}+\frac{r}{2}\right)_-^{\beta}\right)\cdot\left(\sum_{n=1}^{2m^2}\left(\lambda_n+1+\frac{r}{2}\right)^{-\beta}\right)$$

By Theorem 2.1, $\lambda(p,q)=4m^2[\sin^2(p\pi/m)+\sin^2(q\pi/m)]$ $(1\leqslant p,q\leqslant m)$. Hence

$$\sum_{n=1}^{2m^2}\left(\lambda_n+1+\frac{r}{2}\right)^{-\beta}$$
$$=2\cdot\sum_{p,q=1}^{m}\left[1\Big/\left(4m^2\left(\sin^2\frac{p\pi}{m}+\sin^2\frac{q\pi}{m}\right)+1+\frac{r}{2}\right)^{\beta}\right]$$
$$=2\cdot\sum_{p,q=1}^{m}\left[1\Big/\left(4\left(\frac{\sin^2(p\pi/m)}{(p\pi/m)^2}\cdot(p\pi)^2+\frac{\sin^2(q\pi/m)}{(q\pi/m)^2}\cdot(q\pi)^2\right)+1+\frac{r}{2}\right)^{\beta}\right]$$

Since $(\sin x)/x\geqslant 2/\pi$ $(\forall x\in[0,(\pi/2)])$, we have

$$\sum_{n=1}^{2m^2}\left(\lambda_n+1+\frac{r}{2}\right)^{-\beta}\leqslant 2\cdot\sum_{p,q=1}^{m}\frac{1}{(16(p^2+q^2)+1+(r/2))^{\beta}}$$
$$\leqslant 2\cdot\int_0^{\infty}\int_0^{\infty}\frac{dx\,dy}{(16(x^2+y^2)+(r/2))^{\beta}}$$
$$=2^{\beta-5}\pi\cdot r^{1-\beta}\int_0^{\infty}\frac{t\,dt}{(t^2+1)^{\beta}}$$

If we choose $\beta>1$, then

$$S=2^{\beta-5}\pi\int_0^{\infty}\frac{t\,dt}{(t^2+1)^{\beta}}=\frac{2^{\beta-6}\pi}{\beta}<\infty \tag{4.32}$$

Hence

$$N_{-r}(g) \leqslant 4Sr^{1-\beta} \cdot \left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} \left(g_{k,ij} + \frac{r}{2} \right)_-^{\beta} \right) \tag{4.33}$$

Now (4.20) yields

$$\begin{aligned} \sum_{\mu_j(g)<0} |\mu_j(g)|^{\gamma} &\leqslant \gamma \int_0^{\infty} r^{\gamma-1} N_{-r}(g)\, dr \\ &= 4S\gamma \cdot \left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} \left(\int_0^{\infty} r^{\gamma-\beta} \left(g_{k,ij} + \frac{r}{2} \right)_-^{\beta} dr \right) \right) \end{aligned}$$

For fixed k, i, and j, consider $\int_0^{\infty} r^{\gamma-\beta}[g_{k,ij} + (r/2)]_-^{\beta}\, dr$. We need to consider only those (k, i, j) such that $g_{k,ij} + (r/2) < 0$ or $g_{k,ij} < -(r/2) < 0$. The change of variables $r = 2(g_{k,ij}) - \rho$ gives

$$\int_0^{\infty} r^{\gamma-\beta} \left(g_{k,ij} + \frac{r}{2} \right)_-^{\beta} dr = 2^{\gamma-\beta+1} \cdot \left(\int_0^1 \rho^{\gamma-\beta} (1-\rho)^{\beta}\, d\rho \right) \cdot (g_{k,ij})_-^{\gamma+1}$$

Since $1 < \beta < \gamma + 1$, we know that

$$S_1 = 2^{\gamma-\beta+1} \cdot \left(\int_0^1 \rho^{\gamma-\beta} (1-\rho)^{\beta}\, d\rho \right) < \infty$$

Hence

$$\sum_{\mu_j(g)<0} |\mu_j(g)|^{\gamma} \leqslant 4SS_1\gamma \cdot \left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} (g_{k,ij})_-^{\gamma+1} \right)$$

This completes the proof of the proposition for $K = 4SS_1\gamma$. □

Corollary 4.2. *Assumptions are as in Lemma 4.4. Then for* $0 < \gamma \leqslant 1$, *we have*

$$\sum_{\mu_j(g)<0} |\mu_j(g)| \leqslant K^{1/\gamma} \cdot \left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} (g_{k,ij})_-^{\gamma+1} \right)^{1/\gamma} \tag{4.34}$$

Proof. Let $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_N \leqslant 0 < \mu_{N+1} \leqslant \cdots$. Then

$$\sum_{\mu_j<0} |\mu_j| \leqslant \left(\sum_{\mu_j<0} |\mu_j|^{\gamma} \right) |\mu_1|^{1-\gamma} \leqslant \left(\sum_{\mu_j<0} |\mu_j|^{\gamma} \right)^{1/\gamma}$$

By Proposition 4.1, we obtain (4.34). □

With the estimate shown in Proposition 4.1 (especially in Corollary 4.2), we can prove the following theorem, which is a generalization of Lemma 4.2.

Theorem 4.2. (Discretized Lieb-Thirring Inequality). *Let $\varphi^{(l)}$, $1 \leqslant l \leqslant N(<2m^2)$ be a family in $\mathbf{M}\times\mathbf{M}$ which is orthonormal with respect to the inner product $\langle\cdot,\cdot\rangle$. Let $\rho\in\mathbf{M}\times\mathbf{M}$ defined by*

$$\rho_{k,ij}=\sum_{l=1}^{N}|\varphi_{k,ij}^{(l)}|^2 \qquad (k=1,2,\ 1\leqslant i,j\leqslant m) \tag{4.35}$$

Then for every p with

$$1<p\leqslant 2$$

there exists a constant $\kappa>0$ independent of m such that

$$\left(\frac{1}{m^2}\sum_{k=1}^{2}\sum_{i,j=1}^{m}\rho_{k,ij}^{p/(p-1)}\right)^{p-1}\leqslant\kappa\left(\sum_{l=1}^{N}(\|\varphi^{(l)}\|_1^2+1)\right) \tag{4.36}$$

Proof. Define the operator $B\colon\mathbf{M}\times\mathbf{M}\to\mathbf{M}\times\mathbf{M}$ by $B=A+1-\alpha\rho^{1/(p-1)}$. Then

$$\sum_{l=1}^{N}\langle\varphi^{(l)},B\varphi^{(l)}\rangle=\sum_{l=1}^{N}(\|\varphi^{(l)}\|_1^2+\|\varphi^{(l)}\|_0^2)-\alpha\sum_{l=1}^{N}\langle\varphi^{(l)},\rho^{1/(p-1)}\varphi^{(l)}\rangle$$

Since

$$\begin{aligned}\sum_{l=1}^{N}\langle\varphi^{(l)},\rho^{1/(p-1)}\varphi^{(l)}\rangle&=\frac{1}{m^2}\sum_{k=1}^{2}\sum_{i,j=1}^{m}\sum_{l}^{N}\rho_{k,ij}^{1/(p-1))^2}\\&=\frac{1}{m^2}\sum_{k=1}^{2}\sum_{i,j=1}^{m}\rho_{k,ij}^{p/(p-1)}\end{aligned}$$

we have

$$\begin{aligned}&-\alpha\cdot\frac{1}{m^2}\sum_{k=1}^{2}\sum_{i,j=1}^{m}\rho_{k,ij}^{p/(p-1)}+\sum_{l=1}^{N}(\|\varphi^{(l)}\|_1^2+\|\varphi^{(l)}\|_0^2)\\&=\sum_{l=1}^{N}\langle\varphi^{(l)},B\varphi^{(l)}\rangle\end{aligned}$$

Since $\varphi^{(1)},\ldots,\varphi^{(N)}$ are orthonormal in $\mathbf{M}\times\mathbf{M}$, a well-known wedge-product argument shows that

$$\begin{aligned}\sum_{l=1}^{N}\langle\varphi^{(l)},B\varphi^{(l)}\rangle=&\,\mu_{i_1}(-\alpha\rho^{1/(p-1)})+\mu_{i_2}(-\alpha\rho^{1/(p-1)})\\&+\cdots+\mu_{i_N}(-\alpha\rho^{1/(p-1)})\end{aligned}$$

Hence

$$\sum_{l=1}^{N} \langle \varphi^{(l)}, B\varphi^{(l)} \rangle \geqslant \mu_1(-\alpha\rho^{1/(p-1)}) + \mu_2(-\alpha\rho^{1/(p-1)}) + \cdots$$
$$+ \mu_N(-\alpha\rho^{1/(p-1)}) \geqslant \sum_{\mu_j(-\alpha\rho^{1/(p-1)})<0} \mu_j(-\alpha\rho^{1/(p1-)})$$

By Corollary 4.2 (take $\gamma = p-1$), we have

$$\sum_{\mu_j(-\alpha\rho^{1/(p-1)})<0} \mu_j(-\alpha\rho^{1/(p-1)})$$
$$\geqslant -K^{1/(p-1)} \cdot \left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} (\alpha\rho_{k,ij}^{1/(p-1)})^p \right)^{1/(p-1)}$$

Thus

$$-\left(K\alpha^p \cdot \frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} \rho_{k,ij}^{p/(p-1)} \right)^{1/(p-1)} + \alpha \cdot \left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} \rho_{k,ij}^{p/(p-1)} \right)$$
$$\leqslant \sum_{l=1}^{N} (\|\varphi^{(l)}\|_1^2 + \|\varphi^{(l)}\|_0^2)$$

Let

$$\alpha = 2^{-(p-1)} K^{-1} \cdot \left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} \rho_{k,ij}^{p/(p-1)} \right)^{p-2}$$

we have

$$\sum_{l=1}^{N} (\|\varphi^{(l)}\|_1^2 + \|\varphi^{(l)}\|_0^2) \geqslant 2^{-(p-1)} K^{-1} \cdot \left(\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} \rho_{k,ij}^{p/(p-1)} \right)^{p-1}$$

This completes the proof for

$$K = \frac{1}{2^{p-1} \cdot 4SS_1\gamma} = \frac{\beta}{2^{2p-5}\pi \int_0^1 s^{p-1-\beta}(1-s)^\beta\, ds}$$

where $\beta \in (1, p)$. ▯

Corollary 4.3. *Let $\varphi^{(1)}, \ldots, \varphi^{(N)}$ be as in Theorem 4.2. Then*

$$\frac{1}{m^2} \sum_{k=1}^{2} \sum_{i,j=1}^{m} \rho_{k,ij}^2 \leqslant \kappa \left(\sum_{l=1}^{N} (\|\varphi^{(l)}\|_1^2 + 1) \right) \tag{4.37}$$

where κ can be chosen as

$$\kappa = \lim_{\beta \to 1^+} \frac{2\beta}{\pi \int_0^1 s^{1-\beta}(1-s)^\beta \, ds} = \frac{4}{\pi}$$

Proof. By taking $p=2$ in Theorem 4.2, we obtain (4.37). ▯

4.3. The Hausdorff and Fractal Dimensions of the Global Attractor

In this section, we consider the shifted version of the discretized Navier-Stokes equation (3.22) or, equivalently, (3.10),

$$\begin{cases} \dfrac{du}{dt} + \dfrac{1}{2}\left((\tau u)\cdot D + u\cdot(D\tau^{-1})\right) u + \nu A u + (D\tau^{-1})\, p = \Gamma \\ \mathrm{DIV} u = 0 \\ \bar{u} = 0 \end{cases} \tag{4.38}$$

with the initial condition

$$u(0) = u^{(0)} \in \mathbf{W} = \{ w \in \mathbf{M} \times \mathbf{M} \mid \mathrm{DIV} w = 0, \ \bar{w} = 0 \} \tag{4.39}$$

where $u \in \mathbf{M} \times \mathbf{M}$ and $p \in \mathbf{M}$ are unknowns, and $\Gamma \in \mathbf{M} \times \mathbf{M}$ is fixed and satisfies (H).

By taking the formal derivative of the solution u to (4.38), for u starting from the attractor $\mathcal{A}_m$ as in Theorem 3.4, i.e., $u(0) = u^{(0)} \in \mathcal{A}_m$, with respect to the time variable t, we obtain the following equation

$$\begin{cases} \dfrac{dv}{dt} + \nu A v + \dfrac{1}{2}\left((\tau v)\cdot D + v\cdot(D\tau^{-1})\right) u + \dfrac{1}{2}\left((\tau u)\cdot D + u \right. \\ \qquad \left. \cdot\, (D\tau^{-1})\right) v + (D\tau^{-1})\, q = 0 \\ \mathrm{DIV} v = 0 \\ \bar{v} = 0 \end{cases} \tag{4.40}$$

where $v \in \mathbf{M} \times \mathbf{M}$ and $u = u(t)$ is the solution of (4.38).

Equation (4.40) is equivalent to

$$\frac{dv}{dt} + \nu A v + \frac{1}{2} P\left((\tau v)\cdot D + v\cdot(D\tau^{-1})\right) u + \frac{1}{2} P\left((\tau u) \cdot D + u\cdot(D\tau^{-1})\right) v = 0 \tag{4.41}$$

where $P\colon \mathbf{M} \times \mathbf{M} \to \mathbf{W}$ is the orthogonal projection with respect to the inner product $\langle \cdot, \cdot \rangle$.

Equation (4.41) is associated, as usual, with the initial condition

$$v(0)=\xi\in\mathbf{W} \tag{4.42}$$

Since by (4.41), dv/dt is equal to a (quadratic) polynomial in v with smooth coefficients (depending on t), the local existence of the solutions to (4.41) and (4.42) is evident.

In this section, we estimate the Hausdorff dimension of the attractor $\mathscr{A}=\mathscr{A}_m$ obtained in Theorem 3.4, i.e., the attractor $\mathscr{A}=\mathscr{A}_m$ under the norm $\|\cdot\|_0$.

We first prove the following result.

Lemma 4.9. *For any bounded set $B\subset\mathbf{W}$, we have*

$$\limsup_{t\to+\infty}\sup_{u^{(0)}\in B}\frac{1}{t}\int_0^t\|u(s)\|_1^2\,dt\leqslant\frac{C^2}{16\nu^2} \tag{4.43}$$

where C is as in (H).

Proof. By (3.25),

$$\frac{1}{2}\frac{d}{dt}\|u\|_0^2+\nu\,\|u\|_1^2=\langle P\Gamma,u\rangle\leqslant\|u\|_1\cdot\|\Gamma\|_0\cdot\frac{1}{4}\leqslant\frac{\nu}{2}\,\|u\|_1^2+\frac{\|\Gamma\|_0^2}{32\nu}$$

Hence

$$\frac{1}{\nu t}\,\|u(t)\|_0^2+\frac{1}{t}\int_0^t\|u(s)\|_1^2\,ds\leqslant\frac{C^2}{16\nu^2}+\frac{1}{\nu t}\,\|u^{(0)}\|_0^2$$

This proves the lemma. □

Proposition 4.2. *There exist constants $d>0$ and $\alpha>0$ independent of m such that if $l\geqslant d$, then there is an $\alpha_l>0$ such that*

$$\begin{aligned}|v^{(1)}(t)\wedge v^{(2)}(t)\wedge\cdots\wedge v^{(l)}(t)|_{\wedge^l\mathbf{W}}&\\ \leqslant|\xi^{(1)}\wedge\xi^{(2)}\wedge\cdots\wedge\xi^{(l)}|_{\wedge^l\mathbf{W}}\,e^{-\alpha_l t}&\end{aligned} \tag{4.44}$$

Proof. Fix a number l with $1\leqslant l\leqslant m^2-2$. Let $v^{(j)}(t)$ be the solution to (4.41) with $v^{(j)}(0)=\xi^{(j)}\in\mathbf{W}$. Then a standard procedure gives (see [T3])

$$\begin{aligned}&|v^{(1)}(t)\wedge v^{(2)}(t)\wedge\cdots\wedge v^{(l)}(t)|_{\wedge^l\mathbf{W}}\\ &\quad=|\xi^{(1)}\wedge\xi^{(2)}\wedge\cdots\wedge\xi^{(l)}|_{\wedge^l\mathbf{W}}\exp\left(\int_0^t\operatorname{Tr}F'(S(s)\,u^{(0)})\circ Q_l(s)\,ds\right)\end{aligned} \tag{4.45}$$

where $S(s)\,u^{(0)} = u(s)$ and F' is the operator given by

$$F'(S(s)\,u^{(0)})\,w = -vAw - \tfrac{1}{2}((\tau w)\cdot D + w\cdot(D\tau^{-1}))\,u$$
$$- \tfrac{1}{2}((\tau u)\cdot D + u\cdot(D\tau^{-1}))\,w \tag{4.46}$$

and $Q_l(s)$ is the orthogonal projection of $\mathbf{W}$ to the space spanned by $v^{(1)}(s)$, $v^{(2)}(s),\ldots, v^{(l)}(s)$. Equation (4.45) implies that if $\xi^{(1)}$, $\xi^{(2)},\ldots, \xi^{(l)}$ are linearly dependent, then $v^{(1)}(t), v^{(2)}(t),\ldots, v^{(l)}(t)$ are always linearly dependent [in this case (4.44) is trivial]; otherwise, if $\xi^{(1)}$, $\xi^{(2)},\ldots, \xi^{(l)}$ are linearly independent, then $v^{(1)}(t)$, $v^{(2)}(t),\ldots, v^{(l)}(t)$ are never linearly dependent. We shall always assume that $\xi^{(1)}$, $\xi^{(2)},\ldots, \xi^{(l)}$ are linearly independent. Therefore

$$\dim(Q_l(s)\,\mathbf{W}) = l$$

for any $s \geqslant 0$. Hence we can find an orthonormal basis $\{\varphi^{(i)}\}_{i=1}^{l}$ of $Q_l(s)\,\mathbf{W}$, where $\varphi^{(i)} = \varphi^{(i)}(s)$ depends on the time s. We have

$$\begin{aligned}\operatorname{Tr} F'(S(s)\,u^{(0)})\circ Q_l(s) &= \sum_{n=1}^{l} \langle F'(u(s))\circ Q_l(s)\,\varphi^{(n)}(s), \varphi^{(n)}(s)\rangle \\ &= \sum_{n=1}^{l} \langle F'(u(s))\,\varphi^{(n)}(s), \varphi^{(n)}(s)\rangle \\ &= -v\sum_{n=1}^{l} \|\varphi^{(n)}\|_1^2 - \frac{1}{2}\sum_{n=1}^{l} b(u(s), \varphi^{(n)})\end{aligned} \tag{4.47}$$

where

$$\begin{aligned}b(u(s), \varphi^{(n)}) &= \langle \tau\varphi^{(n)} Du(s), \varphi^{(n)}\rangle + \langle \varphi^{(n)}(D\tau^{-1})\,u(s), \varphi^{(n)}\rangle \\ &\quad + \langle((\tau u(s))\cdot D + u(s)\cdot(D\tau^{-1}))\,\varphi^{(n)}, \varphi^{(n)}\rangle \\ &= b_1(n) + b_2(n) + b_3(n)\end{aligned} \tag{4.48}$$

By Lemma 3.4, $b_3(n) = 0$. Hence we need to estimate

$$b(u(s), \varphi^{(n)}) = b_1(n) + b_2(n) \tag{4.49}$$

Define $\rho \in \mathbf{M}\times\mathbf{M}$ by

$$\rho_{k,ij} = \sum_{n=1}^{l} (\varphi_{k,ij}^{(n)})^2 \tag{4.50}$$

Since

$$\begin{aligned}\left|\sum_{n=1}^{l} b_1(n)\right| &= \left|\sum_{n=1}^{l} \langle \tau\varphi^{(n)} Du(s), \varphi^{(n)}\rangle\right| \\ &= \left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{n=1}^{l}\sum_{k=1}^{2} \tau_k \varphi_{k,ij}^{(n)} (D_k u(s))_{h,ij}\varphi_{h,ij}^{(n)}\right| \\ &\leqslant \left|\frac{1}{2}\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{h,k=1}^{2}\left((D_k u(s))_{h,ij}\sum_{n=1}^{l}(\tau_k\varphi_{k,ij}^{(n)})^2\right)\right. \\ &\quad \left. +\frac{1}{2}\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{h,k=1}^{2}\left((D_k u(s))_{h,ij}\sum_{n=1}^{l}(\varphi_{h,ij}^{(n)})^2\right)\right|\end{aligned}$$

by the Cauchy-Schwarz inequality,

$$\begin{aligned}\left|\sum_{n=1}^{l} b_1(n)\right| &\leqslant \frac{1}{2}\left(\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{h,k=1}^{2}(D_k u(s))_{h,ij}^2\right)^{1/2} \\ &\quad \cdot\left(\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{h,k=1}^{2}\left(\sum_{n=1}^{l}(\tau_k\varphi_{k,ij}^{(n)})^2\right)^2\right)^{1/2} \\ &\quad +\frac{1}{2}\left(\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{h,k=1}^{2}(D_k u(s))_{h,ij}^2\right)^{1/2} \\ &\quad \cdot\left(\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{h,k=1}^{2}\left(\sum_{n=1}^{l}(\varphi_{h,ij}^{(n)})^2\right)^2\right)^{1/2} \\ &= \sqrt{2}\,\|u(s)\|_1\cdot\|\rho\|_0\end{aligned}$$

Similarly,

$$\left|\sum_{n=1}^{l} b_2(n)\right| \leqslant \sqrt{2}\,\|u(s)\|_1\cdot\|\rho\|_0$$

Thus (4.49) yields

$$\left|\sum_{n=1}^{l} b(u,\varphi^{(n)})\right| \leqslant 2\sqrt{2}\,\|u(s)\|_1\,\|\rho\|_0$$

By Corollary 4.3,

$$\|\rho\|_0 \leqslant \left(\kappa\sum_{n=1}^{l}\|\varphi^{(n)}\|_1^2 + \kappa l\right)^{1/2}$$

where κ is a constant which can be taken by $\kappa = 4/\pi$. Hence

$$\left| \sum_{n=1}^{l} b(u, \varphi^{(n)}) \right| \leqslant 2\sqrt{2}\, \kappa^{1/2} \, \|u(s)\|_1 \cdot \left(\sum_{n=1}^{l} \left\| \varphi^{(n)} \right\|_1 + l \right)^{1/2}$$

Thus, (4.47) yields

$$\begin{aligned} &\operatorname{Tr} F'(S(s)\, u^{(0)}) \circ Q_l(s) \\ &\quad \leqslant -\nu \sum_{n=1}^{l} \|\varphi^{(n)}\|_1^2 + \sqrt{2}\, \kappa^{1/2} \, \|u(s)\|_1 \cdot \left(\sum_{n=1}^{l} \|\varphi^{(n)}\|_1^2 + l \right)^{1/2} \end{aligned}$$

Since $\{\varphi^{(i)}\}_{i=1}^{l}$ are orthonormal, by using (3.26), we have

$$l = \sum_{n=1}^{l} \|\varphi^{(n)}\|_0^2 \leqslant \frac{1}{16} \sum_{n=1}^{l} \|\varphi^{(n)}\|_1^2$$

Hence

$$\begin{aligned} &\operatorname{Tr} F'(S(s)\, u^{(0)}) \circ Q_l(s) \\ &\quad \leqslant -\nu \sum_{n=1}^{l} \|\varphi^{(n)}\|_1^2 + \sqrt{2}\, \kappa^{1/2} \, \|u(s)\|_1 \cdot \frac{\sqrt{17}}{4} \left(\sum_{n=1}^{l} \|\varphi^{(n)}\|_1^2 \right)^{1/2} \end{aligned}$$

By Young's inequality, we have

$$\begin{aligned} \operatorname{Tr} F'(S(s)\, u^{(0)}) \circ Q_l(s) &\leqslant -\nu \sum_{n=1}^{l} \|\varphi^{(n)}\|_1^2 + \frac{\nu}{2} \sum_{n=1}^{l} \|\varphi^{(n)}\|_1^2 + \frac{17\kappa}{16\nu} \|u(s)\|_1^2 \\ &= -\frac{\nu}{2} \sum_{n=1}^{l} \|\varphi^{(n)}\|_1^2 + \frac{17}{4\pi\nu} \|u(s)\|_1^2 \end{aligned} \tag{4.51}$$

But a standard wedge product argument shows that

$$\sum_{n=1}^{l} \|\varphi^{(n)}\|_1^2 \geqslant \lambda_1 + \lambda_2 + \cdots + \lambda_l \tag{4.52}$$

where $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_l$ are the l smallest eigenvalues of $A: \mathbf{W} \to \mathbf{W}$. We need a lower bound for λ_l in terms of l.

By Theorem 2.1, the eigenvalues of $A: \mathbf{M} \to \mathbf{M}$ are

$$\begin{aligned} \lambda(k, j) &= 4\mathrm{m}^2 \left(\sin^2 \frac{k\pi}{m} + \sin^2 \frac{j\pi}{m} \right) \\ &= 4 \left(\frac{\sin^2(k\pi/m)}{(\kappa\pi/m)^2} \cdot (k\pi)^2 + \frac{\sin^2(j\pi/m)}{(j\pi/m)^2} \cdot (j\pi)^2 \right) \\ &\geqslant 16(k^2 + j^2) \end{aligned} \tag{4.53}$$

with $0 \leqslant k,\ j \leqslant m-1$, where we used the fact that $(\sin x)/x \geqslant 2/\pi$ $(\forall x \in (0, \pi/2])$.

All eigenvalues $0 = 0 < \lambda_1' \leqslant \lambda_2' \leqslant \cdots \leqslant \lambda_{2M^2-2}'$ of A: $\mathbf{M} \times \mathbf{M} \to \mathbf{M} \times \mathbf{M}$ are obtained by doubling the $\lambda(k, j)$'s. Hence

$$\lambda_k \geqslant k_k' \qquad (1 \leqslant k \leqslant m^2 - 2) \tag{4.54}$$

Let $\tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_{m^2-1}$ be an ordering of

$$\{16(k^2 + j^2) \mid 0 \leqslant k, j \leqslant m-1, k+j \neq 0\} \tag{4.55}$$

and let $\theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_{2m^2-2}$ be defined by

$$\theta_{2l-1} = \theta_{2l} = \tau_l \qquad (1 \leqslant l \leqslant m^2 - 1) \tag{4.56}$$

Then

$$\lambda_k \geqslant \lambda_k' \geqslant \theta_k \qquad (1 \leqslant k \leqslant m^2 - 2) \tag{4.57}$$

Let

$$\# B(N) = \# \{(k, j) \mid 0 \leqslant k, j \leqslant m-1, k+j \neq 0, k^2 + j^2 \leqslant N^2\}$$

Then

$$\# B(N) \leqslant 2(N+1)^2 - 2$$

Hence

$$\theta_{\# B(N)+1} \geqslant 16N^2$$

or

$$\lambda_{2N^2+4N+1} \geqslant 16N^2 \qquad (N \geqslant 1)$$

For any $l \geqslant 7$, there exists an $N \geqslant 1$ such that

$$2N^2 + 4N + 1 \leqslant l < 2(N+1)^2 + 4(N+1) + 1$$

Then

$$\begin{aligned}\theta_l &\geqslant \theta_{2N^2+4N+1} \geqslant 16N^2 \\ &= \tfrac{16}{17}(2(N+1)^2 + 4(N+1) + 1) + \tfrac{1}{17}(240N^2 - 128N - 112)\end{aligned}$$

By elementary algebra,

$$240N^2 - 128N - 112 \geqslant 0 \qquad (\forall N \geqslant 1)$$

So for any $l \geqslant 7$,

$$\theta_l \geqslant \tfrac{16}{17}(2(N+1)^2+4(N+1)+1) > \tfrac{16}{17} l \tag{4.58}$$

Since $\theta_1=\theta_2=\theta_3=\theta_4=16$, $\theta_5=\theta_6=32$, we know that (4.58) is still true for $1 \leqslant l \leqslant 6$. Thus $\theta_l > (16/17)\, l$ $(\forall l \geqslant 1)$. By (4.57), we have $\lambda_l > (16/17)\, l$ $(\forall l \geqslant 1)$. Hence

$$\lambda_1+\lambda_2+\cdots+\lambda_l > \frac{16}{17}\cdot\frac{l(l+1)}{2} > \frac{8}{17} l^2 \qquad (\forall l \geqslant 1)$$

Then (4.52) gives

$$\sum_{n=1}^{l} \|\varphi^{(n)}\|_1^2 \geqslant \frac{8}{17} l^2 \qquad (\forall l \geqslant 1)$$

Combining this with (4.51), we have

$$\operatorname{Tr} F'(S(s)\, u^{(0)}) \circ Q_l(s) \leqslant -\frac{4\nu}{17} l^2 + \frac{17}{4\pi\nu} \|u(s)\|_1^2$$

Thus

$$\frac{1}{t}\int_0^t \operatorname{Tr} F'(S(s)\, u^{(0)}) \circ Q_l(s)\, ds \leqslant -\frac{4\nu}{17} l^2 + \frac{17}{4\pi\nu}\cdot\frac{1}{t}\int_0^t \|u(s)\|_1^2\, ds \tag{4.59}$$

By Lemma 4.9, for any G_1 such that

$$G_1 > \frac{C^2}{16\nu^2} \tag{4.60}$$

we have

$$\frac{1}{t}\int_0^t \|u(s)\|_1^2\, ds \leqslant G_1 \tag{4.61}$$

for t large enough. Hence (4.59) yields

$$\frac{1}{t}\int_0^t \operatorname{Tr} F'(S(s)\, u^{(0)}) \circ Q_l(s)\, ds \leqslant -\frac{4\nu}{17}\left(l^2 - \frac{17^2}{2^4\pi\nu^2}\cdot G_1\right) \tag{4.62}$$

for t large enough. Let

$$d = \frac{17}{2^4\pi^{1/2}}\cdot G \tag{4.63}$$

where

$$G = \frac{C}{\nu^2} \tag{4.64}$$

then if $l > d$ and t is large enough, (4.62) yields

$$\frac{1}{t}\int_0^t \operatorname{Tr} F'(S(s)\, u^{(0)}) \circ Q_l(s)\, ds \leqslant -\alpha_l < 0 \tag{4.65}$$

with

$$\alpha_l = \frac{4\nu}{17}(l^2 - d^2) > 0 \tag{4.66}$$

Therefore, by (4.45), the Proposition holds. □

Corollary 4.4. *There exist global solutions to (4.40) for any initial value $\xi \in \mathbf{W}$.*

Proof. Take $l = 1$. Then (4.45) and (4.65) guarantee that $v(t) = v(t; \xi)$ does not blow up in finite time. □

Let $V(t; u^{(0)})$ be the operator with

$$V(t; u^{(0)})\, \xi = v(t)$$

where $v(t)$ is the solution to (4.41) and (4.42). Then, by Proposition 4.2, the global Lyapunov exponents $\mu_l = \mu_l(t)$, for $u^{(0)} \in \mathscr{A} = \mathscr{A}_m$ and t large enough, satisfy

$$\mu_1 + \mu_2 + \cdots + \mu_l \leqslant \alpha_l \tag{4.67}$$

By applying Theorem V.3.3 of [T3], we obtain

Theorem 4.3. *Let $\mathscr{A} = \mathscr{A}_m$ be the attractor given by Theorem 3.4, i.e., the attractor under the norm $\|\cdot\|_0$. Then there is a constant d, which is independent of m, such that the Hausdorff dimension of $\mathscr{A} = \mathscr{A}_m$ satisfies*

$$d_{\mathrm{H}}(\mathscr{A}) = d_{\mathrm{H}}(\mathscr{A}_m) \leqslant d = \frac{17}{2^4 \pi^{1/2}} \cdot G = \frac{17C}{2^4 \pi^{1/2} \nu^2} \tag{4.68}$$

In other words, the Hausdorff dimension of the attractor $\mathscr{A} = \mathscr{A}_m$ of the semiflow of (4.38) is bounded by a constant which is independent of m. □

The fractal dimension $d_{\mathrm{F}}(B)$ (or capacity) gives another measure of the complexity of a geometric set B. It is always true that

$$d_{\mathrm{F}}(B) \geqslant d_{\mathrm{H}}(B)$$

for any set B. Hence the fractal dimension gives an upper bound for the dimension information.

To estimate the fractal dimension of $\mathscr{A} = \mathscr{A}_m$, we need Lemma VI.2.2 of [T3].

Lemma VI.2.2. *We assume that the sequence of numbers $\mu_j, j \geqslant 1$ satisfies the following inequalities*:

$$\mu_1 + \mu_2 + \cdots + \mu_j \leqslant -\alpha j^\theta + \beta, \qquad \forall j \geqslant 1$$

where α, β, $\theta > 0$. Let $J \in \mathbf{N}$ be defined as

$$J - 1 < \left(\frac{2\beta}{\alpha}\right)^{1/\theta} \leqslant J$$

Then $\mu_1 + \mu_2 + \cdots + \mu_J < 0$ and

$$\frac{(\mu_1 + \mu_2 + \cdots + \mu_j)_+}{|\mu_1 + \mu_2 + \cdots + \mu_J|} \leqslant 1 \qquad (j = 1, 2, \ldots, J) \quad \square$$

By this lemma, together with Theorem V.3.3 of [T3] and (4.67), we have the following.

Theorem 4.4. *Let $\mathscr{A} = \mathscr{A}_m$ be the attractor given by Theorem 3.4, i.e., the attractor under the norm $\|\cdot\|_0$. Then the fractal dimension of $\mathscr{A} = \mathscr{A}_m$ satisfies*

$$d_F(\mathscr{A}) = d_F(\mathscr{A}_m) \leqslant 2\sqrt{2}\, d = \frac{17G}{4\sqrt{2\pi}} = \frac{17C}{4\nu^2\sqrt{2\pi}} \quad \square$$

Remark 4.1. Theorems 4.3 and 4.4 show that improving the mesh size $h = 1/m$ (or increasing m) does not change the estimate for the dimension of the attractor $\mathscr{A} = \mathscr{A}_m$. In other words, there is a limit for the mesh size beyond which the discretized system does not give more valuable information of the dynamical behavior. $\square$

APPENDIX: EXISTENCE OF $\|\cdot\|_1$-Attractors

In the continuous version of the Navier-Stokes equations, to prove the existence of the global L^2-attractor, it is necessary to prove both L^2- and H^1-absorbing properties. In our discretized model, the phase space is finite dimensional, so all bounded sets are compact. This leads to the direct proof of the existence of the global $\|\cdot\|_.$-attractors once the $\|\cdot\|_0$-absorbing property is proved. In Section 4, the estimates of Hausdorff and fractal dimensions of the attractors are established.

We shall prove the existence of the global $\|\cdot\|_1$-attractors. We do this by proving the discretized $\|\cdot\|_1$-absorbing property. In the proof, a discretized interpolation inequality plays an important role. This inequality can be proved by using discrete Fourier transform.

At the end of this section, we prove a discrete interpolated inequality directly rather than using the discrete Fourier transform, which considerably shortens the proof of the existence of the $\|\cdot\|_1$-attractors.

A.1. Discrete Fourier Transform and an Interpolated Inequality

In this section, we first discuss the two-dimensional discrete Fourier transform; we then apply this theory to our model **M** set in Section 2.2 to obtain an interpolated inequality, which plays an important role in studying the dynamical behaviors of (3.22).

Fourier analysis is a well-developed theory which is a powerful tool in the research of partial differential equations. [S-W] gives an introduction to this beautiful theory.

In (3.22), the "space" variable (i, j) is discrete. So the Fourier transform of the continuous version is not applicable to our model. Discrete Fourier analysis is developed in numerical analysis. In [V-B], there is a discussion of discrete Fourier analysis of one-dimension.

In the first part of this section we list some properties of the two-dimensional Fourier transform.

Let $h>0$ be fixed. Temporarily we use symbols $n=(n_1, n_2)$, $m=(m_1, m_2)$, $k=(k_1, k_2)\in\mathbf{Z}^2$, $w=(w_1, w_2)$, $t=(t_1, t_2)\in\mathbf{R}^2$. We denote by

$$Q_h=\left\{w=(w_1, w_2)\in\mathbf{R}^2 \,\middle|\, -\frac{\pi}{h}\leqslant w_1, w_2\leqslant\frac{\pi}{h}\right\} \tag{A.1}$$

And we assume that $v\colon \mathbf{Z}^2\to\mathbf{R}$, i.e.,

$$v=v(n)=v(n_1, n_2)\in\mathbf{R} \qquad (\forall n\in\mathbf{Z}^2) \tag{A.2}$$

We say that $v\in L^\infty(\mathbf{Z}^2)$ or $v\in L^1(\mathbf{Z}^2)$ or $v\in L^2(\mathbf{Z}^2)$ or $v\in H^2(\mathbf{Z}^2)$ if the following corresponding condition is true:

$$\sup_{n\in\mathbf{Z}^2}|v(n)|<\infty \qquad (v\in L^\infty(\mathbf{Z}^2)) \tag{A.3}$$

or

$$h^2\sum_{n\in\mathbf{Z}^2}|v(n)|<\infty \qquad (v\in L^1(\mathbf{Z}^2)) \tag{A.4}$$

or

$$h^2 \sum_{n \in \mathbf{Z}^2} |v(n)|^2 < \infty \qquad (v \in L^2(\mathbf{Z}^2)) \tag{A.5}$$

or

$$\frac{1}{h^2} \sum_{n \in \mathbf{Z}^2} |4v(n_1, n_2) - v(n_1+1, n_2) - v(n_1-1, n_2) - v(n_1, n_2+1) - v(n_1, n_2-1)|^2 < \infty \qquad (v \in H^2(\mathbf{Z}^2)) \tag{A.6}$$

If $v \in L^1(\mathbf{Z}^2)$, we define its *discrete Fourier transform* as a function $\hat{v}\colon \mathbf{R}^2 \to \mathbf{C}$, which is defined for every $w \in \mathbf{R}^2$ by

$$\begin{aligned} \hat{v} = \hat{v}(w) = \hat{v}(w_1, w_2) &= h^2 \sum_{n \in \mathbf{Z}^2} v(n)\, e^{-ihw \cdot n} \\ &= h^2 \sum_{n_1, n_2 = -\infty}^{+\infty} v(n_1, n_2)\, e^{-ih(w_1 n_1 + w_2 n_2)} \end{aligned} \tag{A.7}$$

One can check that the series is convergent in this case.

In the continuous version, the existence of the inverse Fourier transform is a difficult problem, especially the pointwise existence. But in discrete version, it is surprisingly simple to prove the pointwise existence of the inverse Fourier transform.

Lemma A.1. *If $v \in L^1(\mathbf{Z}^2)$, then the inverse Fourier transform is given by*

$$v(n) = \frac{1}{(2\pi)^2} \int_{Q_h} \hat{v}(w)\, e^{ihn \cdot w}\, dw \tag{A.8}$$

Proof.

$$\begin{aligned} &\frac{1}{(2\pi)^2} \int_{Q_h} \hat{v}(w)\, e^{ihn \cdot w}\, dw \\ &= \frac{h^2}{(2\pi)^2} \int_{Q_h} \left(\sum_{k \in \mathbf{Z}^2} v(k)\, e^{-ihk \cdot w} \right) e^{ihn \cdot w}\, dw \\ &= \frac{h^2}{(2\pi)^2} \sum_{k \in \mathbf{Z}^2} \left(\int_{Q_h} e^{ih(n-k) \cdot w} \right) v(k) = v(n) \qquad \square \end{aligned}$$

For $v \in L^2(\mathbf{Z}^2)$, we define its L^2-*norm* by

$$\|v\|_0^2 = h^2 \sum_{n \in \mathbf{Z}^2} |v(n)|^2 \tag{A.9}$$

Note that this is an extension of (2.24).

For the discrete Fourier transform $\hat{v}$ of v, we define the L^2-*norm* by

$$\|\hat{v}\|_{L^2}^2 = \frac{1}{(2\pi)^2} \int_{Q_h} |\hat{v}(w)|^2 \, dw \tag{A.10}$$

It is well-known that the continuous Fourier transform is a unitary operator in L^2. We claim that the discrete Fourier transform also preserves the L^2-norm. We prove this by using the results of the continuous version.

We want to extend u: $\mathbf{Z}^2 \to \mathbf{R}$ to u^*: $\mathbf{R}^2 \to \mathbf{R}$.

We construct a function

$$\Psi_n(x) = \frac{\sin[\pi(x_1 - hn_1)/h]}{[\pi(x_1 - hn_1)/h]} \cdot \frac{\sin[\pi(x_2 - hn_2)/h]}{[\pi(x_2 - hn_2)/h]}$$
$$(\forall x \in \mathbf{R}^2, \ \forall n \in \mathbf{Z}^2) \tag{A.11}$$

Note that $(0, x_2)$ and $(x_1, 0)$ are removable singular points of Ψ_n, Also, we have the following shifting formula

$$\Psi_n(x) = \Psi_0(x - hn) \tag{A.12}$$

Now we define for $x \in \mathbf{R}^2$,

$$v^*(x) = \sum_{n \in \mathbf{Z}^2} \Psi_n(x) \, v(n) \tag{A.13}$$

v^* is not a direct extension of v, but it is an extension in the following sense:

$$v^*(hn) = v(n) \qquad (\forall n \in \mathbf{Z}^2) \tag{A.14}$$

Since in (A.11) and (A.13), the spatial variable x of Ψ_n and v^* is continuous, we can apply the continuous Fourier transform to Ψ_n and v^*.

Let $\hat{\Psi}_n$ be the continuous Fourier transform of Ψ_n.

Lemma A.2. *For $t \in \mathbf{R}^2$, we have*

$$\hat{\Psi}_0(t) = \begin{cases} h^2 & t \in Q_h \\ 0 & \textit{elsewhere} \end{cases} \tag{A.15}$$

and

$$\hat{\Psi}_n(t) = e^{-ih \cdot n} \hat{\Psi}_0(t) \qquad (A.16)$$

Proof. By (1.68) of [V-B] (actually it is from [P]), for any $s \in \mathbf{R}$, we have

$$\int_{-\infty}^{\infty} \frac{\sin(\pi x/h)}{(\pi x/h)} e^{-is \cdot x} \, dx = \begin{cases} h & s \in [-(\pi/h), (\pi/h)] \\ 0 & \text{elsewhere} \end{cases}$$

then

$$\begin{aligned} \hat{\Psi}_0(t) &= \int_{\mathbf{R}^2} \Psi_0(x) \, e^{-it \cdot x} \, dx \\ &= \left(\int_{-\infty}^{\infty} \frac{\sin(\pi x_1/h)}{(\pi x_1/h)} e^{-it_1 x_1} \, dx_1 \right) \cdot \left(\int_{-\infty}^{\infty} \frac{\sin(\pi x_2/h)}{(\pi x_2/h)} e^{-it_2 x_2} \, dx_2 \right) \\ &= \begin{cases} h^2 & t \in Q_h \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

This proves (A.15).

To prove (A.16), by using (A.12), we have

$$\begin{aligned} \hat{\Psi}_n(t) &= \int_{\mathbf{R}^2} \Psi_n(x) \, e^{-it \cdot x} \, dx = \int_{\mathbf{R}^2} \Psi_0(x - hn) \, e^{-it \cdot x} \, dx \\ &= e^{-iht \cdot n} \int_{\mathbf{R}^2} \Psi_0(x) \, e^{-it \cdot x} \, dx = e^{-iht \cdot n} \hat{\Psi}_0(t) \end{aligned}$$

This proves (A.16). □

Lemma A.3. *If $v \in L^1(\mathbf{Z}^2)$, then*

$$\hat{v}^*(t) = \frac{1}{h^2} \Psi_0(t) \, \hat{v}(t) \qquad (A.17)$$

and

$$\|\hat{v}\|_{L^2} = \|v^*\|_{L^2} \qquad (A.18)$$

Proof. Again, by (A.12),

$$\begin{aligned} \hat{v}^*(t) &= \int_{\mathbf{R}^2} v^*(x) \, e^{-it \cdot x} \, dx = \int_{\mathbf{R}^2} \left(\sum_{n \in \mathbf{Z}^2} \Psi_n(x) \, v(n) \right) \cdot e^{-it \cdot x} \, dx \\ &= \sum_{n \in \mathbf{Z}^2} \left(\int_{\mathbf{Z}^2} \Psi_0(x - hn) \, e^{-it \cdot (x - hn)} \, dx \right) \cdot v(n) \, e^{-iht \cdot n} \\ &= \frac{1}{h^2} \hat{\Psi}_0(t) \, \hat{v}(t) \end{aligned}$$

This proves (A.17).

Hence by (A.15),

$$\hat{v}^*(t) = \begin{cases} \hat{v}(t) & t \in Q_h \\ 0 & \text{elsewhere} \end{cases} \tag{A.19}$$

But by Parseval's equation,

$$\|v^*\|_{L^2}^2 = \int_{\mathbf{R}^2} |v^*(x)|^2 \, dx = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} |\hat{v}^*(w)|^2 \, dw$$

So (A.19) yields

$$\|v^*\|_{L^2}^2 = \frac{1}{(2\pi)^2} \int_{Q_h} |\hat{v}(w)|^2 \, dw = \|\hat{v}\|_{L^2}^2$$

This proves (A.18). □

Lemma A.4. *If $v \in L^1(\mathbf{Z}^2)$, then*

$$\|v\|_0 = \|v^*\|_{L^2} \tag{A.20}$$

Proof. Since the continuous Fourier transform is a unitary operator on $L^2(\mathbf{R}^2)$, we have

$$(\Psi_m, \Psi_n)_{L^2} = \int_{\mathbf{R}^2} \Psi_m(x) \, \Psi_n(x) \, dx = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} \hat{\Psi}_m(t) \overline{\hat{\Psi}_n(t)} \, dt$$

$$= \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} \hat{\Psi}_m(t) \, \hat{\Psi}_n(-t) \, dt$$

By (A.16) and (A.15), we have

$$(\Psi_m, \Psi_n)_{L^2} = \frac{h^4}{(2\pi)^2} \left(\int_{-\pi/h}^{\pi/h} e^{iht_1(n_1 - m_1)} \, dt_1 \right) \cdot \left(\int_{-\pi/h}^{\pi/h} e^{iht_2(n_2 - m_2)} \, dt_2 \right)$$

$$= h^2 \frac{\sin[(n_1 - m_1)\pi]}{[(n_1 - m_1)\pi]} \cdot \frac{\sin[(n_1 - m_1)\pi]}{[(n_1 - m_1)\pi]} = \begin{cases} h^2 & n = m \\ 0 & n \neq m \end{cases}$$

Hence

$$\|v^*\|_{L^2}^2 = \sum_{m,n \in \mathbf{Z}^2} v(n) \, v(m) (\Psi_n, \Psi_m)_{L^2}$$

$$= \sum_{n \in \mathbf{Z}^2} |v(n)|^2 \, (\Psi_n, \Psi_n)_{L^2} + \sum_{\substack{n,m \in \mathbf{Z} \\ n \neq m}} v(n) \, v(m) (\Psi_n, \Psi_m)_{L^2}$$

$$= h^2 \sum_{n \in \mathbf{Z}^2} |v(n)|^2 = \|v\|_0^2$$

This proves (A.20). □

From Lemmas A.3 and A.4, we know that the discrete Fourier transform preserves the L^2-norm.

Theorem A.1. *If $v \in L^1(\mathbf{Z}^2)$, then*

$$\|\hat{v}\|_{L^2} = \|v\|_0 \quad \square \tag{A.21}$$

* * * * *

We now extend the definition of the operator A in (2.9). For any $v: \mathbf{Z}^2 \to \mathbf{R}$, define $Av: \mathbf{Z}^2 \to \mathbf{R}$ by

$$\begin{aligned}(Av)(n) = \frac{1}{h^2}\,&(4v(n_1, n_2) - v(n_1+1, n_2) - v(n_1-1, n_2)\\ &- v(n_1, n_2+1) - v(n_1, n_2-1)) \qquad (\forall n = (n_1, n_2) \in \mathbf{Z}^2)\end{aligned} \tag{A.22}$$

In the rest of this section, we prove an interpolated formula by using the discrete Fourier transform discussed above.

Theorem A.2. *If $v \in L^\infty(\mathbf{Z}^2) \cap L^1(\mathbf{Z}^2) \cap L^2(\mathbf{Z}^2) \cap H^2(\mathbf{Z}^2)$ as defined in (A.3)–(A.6), then*

$$\begin{aligned}\|v\|_\infty &\leqslant \frac{9}{4\pi^{3/2}} \|v\|_0^{1/2} (\|v\|_0^2 + \|Av\|_0^2)^{1/4}\\ &\leqslant \frac{9}{4\pi^{3/2}} \|v\|_0^{1/2} (\|v\|_0 + \|Av\|_0)^{1/2}\end{aligned} \tag{A.23}$$

where $\|v\|_\infty$ is defined by

$$\|v\|_\infty = \sup_{n \in \mathbf{Z}^2} |v(n)| \tag{A.24}$$

Proof. For v as in the statement of the Theorem, we can define its discrete Fourier transform as above, we have

$$\begin{aligned}\widehat{Av}(w) &= h^2 \sum_{n \in \mathbf{Z}^2} (Av)(n)\, e^{-ihn\cdot w}\\ &= \left(\sum_{n \in \mathbf{Z}^2} v(n_1, n_2)\, e^{-ihn\cdot w}\right) \cdot (4 - e^{ihw_1} - e^{-ihw_1} - e^{ihw_2} - e^{-ihw_2})\\ &= \left(\frac{4 - e^{ihw_1} - e^{-ihw_1} - e^{ihw_2} - e^{-ihw_2}}{h^2}\right) \cdot \hat{v}(w)\end{aligned}$$

Define

$$K(w;h)=\frac{4-e^{ihw_1}-e^{-ihw_1}-e^{ihw_2}-e^{-ihw_2}}{h^2} \tag{A.25}$$

Then we have

$$\widehat{Av}(w)=K(w;h)\,\hat{v}(w) \tag{A.26}$$

If $Av\in L^1(\mathbf{Z}^2)\cap L^2(\mathbf{Z}^2)$ [see (A.4) and (A.5) for the definitions], then

$$\begin{aligned}\|\widehat{Av}\|_{L^2}^2&=\frac{1}{(2\pi)^2}\int_{Q_h}|\widehat{Av}(w)|^2\,dw\\&=\frac{1}{(2\pi)^2}\int_{Q_h}|K(w;h)|^2\,|\hat{v}(w)|^2\,dw\end{aligned} \tag{A.27}$$

Now by (A.8), for any $n\in\mathbf{Z}^2$, we have

$$\begin{aligned}|v(n)|&=\left|\frac{1}{(2\pi)^2}\int_{Q_h}\hat{v}(w)\,e^{ihn\cdot w}\,dw\right|\leqslant\frac{1}{(2\pi)^2}\int_{Q_h}|\hat{v}(w)|\,dw\\&=\frac{1}{(2\pi)^2}\int_{Q_h\cap\{|w|<\alpha\}}|\hat{v}(w)|\,dw+\frac{1}{(2\pi)^2}\int_{Q_h\cap\{|w|\geqslant\alpha\}}|\hat{v}(w)|\,dw\\&=I_1+I_2\end{aligned} \tag{A.28}$$

where α is a positive parameter to be determined.

By the Cauchy-Schwarz inequality,

$$\begin{aligned}I_1&=\frac{1}{(2\pi)^2}\int_{Q_h\cap\{|w|<\alpha\}}|\hat{v}(w)|\,dw\\&\leqslant\left(\frac{1}{(2\pi)^2}\int_{Q_h\cap\{|w|<\alpha\}}1^2dw\right)^{1/2}\cdot\left(\frac{1}{(2\pi)^2}\int_{Q_h\cap\{|w|<\alpha\}}|\hat{v}(w)|^2\,dw\right)^{1/2}\\&\leqslant\frac{1}{(2\pi)^2}(\pi\alpha^2)^{1/2}\cdot\|\hat{v}\|_{L^2}\end{aligned}$$

and combining this with (A.21), we have

$$I_1\leqslant\frac{\alpha}{4\pi^{3/2}}\|v\|_0 \tag{A.29}$$

Again, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
I_2 &= \frac{1}{(2\pi)^2} \int_{Q_h \cap \{|w| \geqslant \alpha\}} |\hat{v}(w)| \, dw \\
&= \frac{1}{(2\pi)^2} \int_{Q_h \cap \{|w| \geqslant \alpha\}} \frac{1}{\sqrt{1+K(w;h)^2}} \cdot \sqrt{1+K(w;h)^2}\, |\hat{v}(w)| \, dw \\
&\leqslant \frac{1}{(2\pi)^2} \cdot \left(\int_{Q_h \cap \{|w| \geqslant \alpha\}} \frac{dw}{1+K(w;h)^2} \right)^{1/2} \\
&\quad \cdot \left(\int_{Q_h \cap \{|w| \geqslant \alpha\}} (1+K(w;h)^2)\, |\hat{v}(w)|^2 \, dw \right)^{1/2} \\
&= \frac{1}{(2\pi)^2} \cdot I_3^{1/2} \cdot I_4^{1/2}
\end{aligned}
\tag{A.30}
$$

Since

$$
\begin{aligned}
I_4 &= \int_{Q_h \cap \{|w| \geqslant \alpha\}} (1+K(w;h)^2)\, |\hat{v}(w)|^2 \, dw \\
&\leqslant \int_{Q_h} |\hat{v}(w)|^2 \, dw + \int_{Q_h} K(w;h)^2\, |\hat{v}(w)|^2 \, dw
\end{aligned}
$$

by (A.10), (A.27), and (A.21),

$$
I_4 \leqslant (2\pi)^2 \left(\|\hat{v}\|_{L^2}^2 + \|\widehat{Av}\|_{L^2}^2 \right) = (2\pi)^2 \left(\|v\|_0^2 + \|Av\|_0^2 \right) \tag{A.31}
$$

We need to estimate

$$
I_3 = \int_{Q_h \cap \{|w| \geqslant \alpha\}} \frac{dw}{1+K(w;h)^2}
$$

By (A.25),

$$
h^2 K(w;h) = 4 \left(\sin^2\left(\frac{hw_1}{2}\right) + \sin^2\left(\frac{hw_2}{2}\right) \right)
$$

hence

$$
K(w;h) = \left(\frac{\sin^2\left(\frac{hw_1}{2}\right)}{\left(\frac{hw_1}{2}\right)^2} \cdot w_1^2 + \frac{\sin^2\left(\frac{hw_2}{2}\right)}{\left(\frac{hw_2}{2}\right)^2} \cdot w_2^2 \right)
$$

Since $w \in Q_h$, then $|w_i| \leqslant \pi/h$, or $|hw_i/2| \leqslant \pi/2$, we have

$$\frac{\sin^2 (hw_2/2)}{(hw_2/2)^2} \geqslant \frac{4}{\pi^2} \qquad (i=1, 2)$$

Therefore,

$$K(w; h) \geqslant \frac{4}{\pi^2} (w_1^2 + w_2^2)$$

Now

$$I_3 \leqslant \int_{Q_h \cap \{|w| \geqslant \alpha\}} \frac{dw}{1 + K(w; h)^2} \leqslant \frac{4}{\pi^2} \int_0^{2\pi} d\theta \int_\alpha^{+\infty} \frac{r}{r^4} \, dr = \frac{16}{\pi\alpha^2} \tag{A.32}$$

By substituting (A.31) and (A.32) back in (A.30), we obtain

$$\begin{aligned} I_2 &\leqslant \frac{1}{(2\pi)^2} \cdot \left(\frac{16}{\pi\alpha^2}\right)^{1/2} \cdot ((2\pi)^2 (\|v\|_0^2 + \|Av\|_0^2))^{1/2} \\ &\leqslant \frac{2}{\pi^{3/2}\alpha} \cdot (\|v\|_0^2 + \|Av\|_0^2)^{1/2} \end{aligned}$$

By using this, together with (A.29) and (A.28), we have

$$|v(n)| \leqslant \frac{\alpha}{4\pi^{3/2}} \|v\|_0 + \frac{2}{\alpha\pi^{3/2}} (\|v\|_0^2 + \|Av\|_0^2)^{1/2} \qquad (\forall n \in \mathbf{Z}^2)$$

If $\|v\|_0 = 0$, then (A.23) is trivial. Otherwise, we take

$$\alpha = \frac{(\|v\|_0^2 + \|Av\|_0^2)^{1/4}}{\|v\|_0^{1/2}}$$

so that

$$\begin{aligned} |v(n)| &\leqslant \frac{9}{4\pi^{3/2}} \|v\|_0^{1/2} (\|v\|_0^2 + \|Av\|_0^2)^{1/4} \\ &\leqslant \frac{9}{4\pi^{3/2}} \|v\|_0^{1/2} (\|v\|_0 + \|Av\|_0)^{1/2} \qquad (\forall n \in \mathbf{Z}^2) \end{aligned}$$

This proves (A.23). □

A.2. The Interpolation Inequality in M

In the last section, we proved the interpolation inequality (A.23). This inequality holds for $v: \mathbf{Z}^2 \to \mathbf{R}$. But in our discretized model (2.28), elements in $\mathbf{M}$ are not defined on whole $\mathbf{Z}^2$. So we need to modify (A.23).

First, we need to extend a $v \in \mathbf{M}$ to a $\tilde{v}: \mathbf{Z}^2 \to \mathbf{R}$. Since $\mathbf{M}$ is the discretization for periodic functions, one may try to use (2.5) as a natural extension of any $v \in \mathbf{M}$. Unfortunately, such an extension causes the blowing up of the discretized H^1-norm defined in (A.6). In other words, using (2.5) will ruin the well-definedness of the operator A defined in (A.22). To get around this, we multiply the periodic extension of v by a cutoff function.

Consider the following polynomial

$$\rho(s) = -10(s-2)^3 - 15(s-2)^4 - 6(s-2)^5 \tag{A.33}$$

defined on the interval $s \in [1, 2]$. One may check that

$$\rho(1) = 1, \qquad \rho'(1) = \rho''(1) = \rho(2) = \rho'(2) = \rho''(2) = 0 \tag{A.34}$$

and

$$0 \leqslant \rho(s) \leqslant 1, \qquad -\frac{10}{\sqrt{3}} \leqslant \rho''(s) \leqslant \frac{10}{\sqrt{3}} \qquad (\forall s \in [1, 2]) \tag{A.35}$$

We define a 2-D cutoff function θ by

$$\theta(x) = \theta(x_1, x_2) = \theta_1(x_1) \cdot \theta_2(x_2) \qquad (\forall x = (x_1, x_2) \in \mathbf{R}^2) \tag{A.36}$$

where

$$\theta_i(s) = \begin{cases} 0 & -\infty < s \leqslant -1 \\ \rho(1-s) & -1 < s \leqslant 0 \\ 1 & 0 < s \leqslant 1 \\ \rho(s) & 1 < s \leqslant 2 \\ 0 & s > 2 \end{cases} \qquad (i = 1, 2) \tag{A.37}$$

Elementary calculus shows that the following is true.

Lemma A.5. *$\theta \in C^2(\mathbf{R}^2)$ and*

$$0 \leqslant |\theta(x)| \leqslant 1 \qquad (\forall x \in \mathbf{R}^2)$$

$$\theta(x) = \theta(x_1, x_2) = 1 \qquad (0 \leqslant x_1, x_2 \leqslant 1)$$

$$\theta(x) = \theta(x_1, x_2) = 0 \qquad (x_1 \leqslant -1 \text{ or } x_2 \leqslant -1 \text{ or } x_1 \geqslant 2 \text{ or } x_2 \geqslant 2)$$

$$\left| \frac{\begin{array}{c} 4\theta(x_1, x_2) - \theta(x_1+h, x_2) - \theta(x_1-h, x_2) \\ -\ \theta(x_1, x_2+h) - \theta(x_1, x_2-h) \end{array}}{h^2} \right| \leqslant \frac{40}{\sqrt{3}} \qquad (Zx = (x_1, x_2) \in \mathbf{R}^2)$$

Proof. The first three statements are trivial.

For the last inequality, according to the definition of θ in (A.36) and (A.37), we can rearrange the left-hand side as

$$\left|\frac{\begin{array}{c}4\theta(x_1, x_2)-\theta(x_1+h, x_2)-\theta(x_1-h, x_2)\\ -\ \theta(x_1, x_2+h)-\theta(x_1, x_2-h)\end{array}}{h^2}\right|$$

$$\leqslant \left|\frac{2\rho(x_1)-\rho(x_1+h)-\rho(x_1-h)}{h^2}\right| \cdot |\rho(x_2)|$$

$$+\ |\rho(x_1)| \cdot \left|\frac{2\rho(x_2)-\rho(x_2+h)-\rho(x_2-h)}{h^2}\right|$$

By the mean value theorem, there exist c_i, ξ_i, $\zeta_i \in (x_i-h, x_i+h)$ such that

$$\frac{2\rho(x_i)-\rho(x_i+h)-\rho(x_i-h)}{h^2} = -\rho''(c_i)\cdot\frac{\xi_i-\zeta_i}{h} \qquad (i=1, 2)$$

But by (A.35) and the fact that $|\zeta_i-\xi_i| \leqslant 2h$ $(i=1, 2)$, we have

$$\left|\frac{2\rho(x_i)-\rho(x_i+h)-\rho(x_i-h)}{h^2}\right| \leqslant \frac{20}{\sqrt{3}} \qquad (i=1, 2)$$

Hence the lemma is proved. □

Define the extension of any $v \in \mathbf{M}$ by

$$\tilde{v}(n) = v_{(n_1 \bmod_+ m),\ (n_2 \bmod_+ m)} \cdot \theta\left(\frac{n_1}{m}, \frac{n_2}{m}\right)$$

$$(\forall n = (n_1, n_2) \in \mathbf{Z}^2) \tag{A.38}$$

We can prove that this extension preserves the various norms of **M**.

Lemma A.6. *For any $v \in \mathbf{M}$, $\tilde{v} \in L^\infty(\mathbf{Z}^2) \cap L^1(\mathbf{Z}^2) \cap L^2(\mathbf{Z}^2) \cap H^2(\mathbf{Z}^2)$ and*

$$\|v\|_\infty = \|\tilde{v}\|_\infty \tag{A.39}$$

$$\|v\|_0 \leqslant \|\tilde{v}\|_0 \leqslant 3\ \|v\|_0 \tag{A.40}$$

$$\|Av\|_0 \leqslant \|A\tilde{v}\|_0 \leqslant (4800\ \|v\|_0^2 + 9\ \|Av\|_0^2)^{1/2} \tag{A.41}$$

where $\|\tilde{v}\|_0$ and $\|A\tilde{v}\|_0$ are as in (A.9) and $h = 1/m$

Proof. Let $v \in \mathbf{M}$. By Lemma A.5 and the definition of $\tilde{v}$,

$$\tilde{v}(n) = \tilde{v}(n_1, n_2) = 0$$

if $n_1 \leqslant -m$ or $n_2 \leqslant -m$ or $n_1 \geqslant 2m$ or $n_2 \geqslant 2m$. So $\tilde{v} \in L^\infty(\mathbf{Z}^2) \cap L^1(\mathbf{Z}^2) \cap L^2(\mathbf{Z}^2) \cap H^2(\mathbf{Z}^2)$. Moreover, since

$$\tilde{v}(n) = \tilde{v}(n_1, n_2) = v_{n_1, n_2}$$

if $1 \leqslant n_1, n_2 \leqslant m$, and

$$|\tilde{v}(n)| \leqslant |v_{(n_1 \bmod_+ m), (n_2 \bmod_+ m)}|$$

so

$$\|v\|_\infty = \|\tilde{v}\|_\infty$$

and

$$\|v\|_0 \leqslant \|\tilde{v}\|_0, \qquad \|Av\|_0 \leqslant \|A\tilde{v}\|_0$$

For the other inequalities, note that by Lemma A.5,

$$\begin{aligned}
\|\tilde{v}\|_0^2 &= h^2 \sum_{n \in \mathbf{Z}^2} |\tilde{v}(n)|^2 \\
&= h^2 \sum_{n \in \mathbf{Z}^2} \left| v_{(n_1 \bmod_+ m), (n_2 \bmod_+ m)} \theta\left(\frac{n_1}{m}, \frac{n_2}{m}\right)\right|^2 \\
&\leqslant h^2 \sum_{-m \leqslant n_1, n_2 \leqslant 2m} |v_{(n_1 \bmod_+ m), (n_2 \bmod_+ m)}|^2 \\
&= \frac{9}{m^2} \sum_{i,j=1}^{m} |v_{ij}|^2 = 9\ \|v\|_0^2
\end{aligned}$$

Furthermore, by using Lemma A.5, we have

$$\begin{aligned}
\|A\tilde{v}\|_0^2 &= h^2 \sum_{n \in \mathbf{Z}^2} |(A\tilde{v})(n)|^2 = \frac{1}{m^2} \sum_{-m \leqslant i,j \leqslant 2m} |(A\tilde{v})(i,j)|^2 \\
&\leqslant \frac{1}{m^2} \sum_{-m \leqslant i,j \leqslant 2m} \left| (Av)_{(i \bmod_+ m), (j \bmod_+ m)}\ \theta\left(\frac{i}{m}, \frac{j}{m}\right)\right|^2 \\
&\quad + \frac{1}{m^2} \sum_{-m \leqslant i,j \leqslant 2m} |v_{(i \bmod_+ m), (j \bmod_+ m)}|^2 \\
&\quad \cdot \left| m^4 \left(4\theta\left(\frac{i}{m}, \frac{j}{m}\right) - \theta\left(\frac{i+1}{m}, \frac{j}{m}\right) - \theta\left(\frac{i-1}{m}, \frac{j}{m}\right)\right.\right. \\
&\quad \left.\left. - \theta\left(\frac{i}{m}, \frac{j+1}{m}\right) - \theta\left(\frac{i}{m}, \frac{j-1}{m}\right)\right)\right|^2 \\
&\leqslant 9\ \|Av\|_0^2 + 9\ \|v\|_0^2 \left(\frac{40}{\sqrt{3}}\right)^2
\end{aligned}$$

This completes the proof of Lemma A.6. □

Now we can state the modified version of the interpolation inequality (Theorem A.1) in $\mathbf{M}$.

Proposition A.1. *For any* $v \in \mathbf{M}$,

$$\|v\|_\infty \leqslant \frac{9}{\pi^{3/2}} \|v\|_0^{1/2} \|Av\|_0^{1/2} \tag{A.42}$$

Proof. Extend $v \in \mathbf{M}$ to a $\tilde{v}: \mathbf{Z}^2 \to \mathbf{R}$ as in (A.38), then by Lemma A.6,

$$\|v\|_\infty = \|\tilde{v}\|_\infty$$

Theorem A.2 yields

$$\|v\|_\infty = \|\tilde{v}\|_\infty \leqslant \frac{9}{4\pi^{3/2}} \|\tilde{v}\|_0^{1/2} (\|\tilde{v}\|_0^2 + \|A\tilde{v}\|_0^2)^{1/4}$$

Using Lemma A.6 and Corollary 2.2,

$$\begin{aligned}\|v\|_\infty &= \frac{9}{4\pi^{3/2}} \cdot \sqrt{3} \cdot \|v\|_0^{1/2} \cdot (9\,\|v\|_0^2 + 4800\,\|v\|_0^2 + 9\,\|Av\|_0^2)^{1/4} \\ &\leqslant \frac{9}{\pi^{3/2}} \cdot \|v\|_0^{1/2} \cdot \|Av\|_0^{1/2}. \quad \square\end{aligned}$$

A.3. Global $\|\cdot\|_1$-Attractors

In this section, we prove the existence of global attractors of the shifted Navier-Stokes equation (3.22) under the norm $\|\cdot\|_1$. We actually prove the $\|\cdot\|_1$-norm absorbing property, which implies the existence of global attractors in the sense of the $\|\cdot\|_1$-norm.

First, by Lemma 3.5, $\|\cdot\|_\infty$, $\|\cdot\|_0$, $|\cdot|_1$, and $|\cdot|_2$ are norms in $\mathbf{W}$. We change the symbols in (2.32) and (2.33). Denote by

$$\|v\|_1 = \|Dv\|_0 = \left(\sum_{k,l=1}^{2} \|D_k v_l\|_0^2\right)^{1/2} \qquad (\forall v \in \mathbf{W}) \tag{A.43}$$

and

$$\|v\|_2 = \|Av\|_0 = \left(\sum_{k=1}^{2} \|Av_k\|_0^2\right)^{1/2} \qquad (\forall v \in \mathbf{W}) \tag{A.44}$$

We want to prove an a priori estimate for $\|\cdot\|_1$ as in Theorem 32. We begin with the following lemma.

Lemma A.7. *For any fixed $r>0$, there exists a constant*

$$\rho_{1/2}=\left(\frac{rC^2}{16\nu^2}+\frac{\rho_0'^2}{\nu}\right)^{1/2}$$

with arbitrarily fixed $\rho_0'>\rho_0$ such that for R_0, T_0, and the initial data $u^{(0)}\in \mathbf{W}$ described in Theorem 3.2, we have

$$\int_t^{t+r} \|u(s)\|_1^2\, ds \leqslant \rho_{1/2}^2 \qquad (\forall t\geqslant T_0)$$

Proof. (3.25) yields

$$\frac{1}{2}\frac{d}{dt}\|u\|_0^2+\nu\|u\|_1^2\leqslant \langle P\Gamma, u\rangle \leqslant \|P\Gamma\|_0\|u\|_0\leqslant \frac{1}{32\nu}\|P\Gamma\|_0^2+8\nu\|u\|_0^2$$

By (3.26) and (H), we have

$$\nu\int_t^{t+r}\|u(s)\|_1^2\,ds\leqslant \frac{rC^2}{16\nu}+\|u(t)\|_0^2-\|u(t+r)\|_0^2\leqslant \frac{rC^2}{16\nu}+\|u(t)\|_0^2$$

Now for R_0, T_0, and $u^{(0)}$ as in Theorem 3.2, by the conclusion of Theorem 3.2, we have

$$\|u(s)\|_0\leqslant \rho_0' \qquad (\forall s\geqslant T_0)$$

Hence

$$\int_t^{t+r}\|u(s)\|_1^2\,ds\leqslant \frac{rC^2}{16\nu^2}+\frac{\rho_0'^2}{\nu} \qquad (\forall t\geqslant T_0)$$

This completes the proof. □

Remark A.1. $\rho_{1/2}\to 0$ as $C\to 0$. □

To prove the absorbing property for $\|\cdot\|_1$, we need a lemma stated in [T3] (Lemma III.1.1).

Lemma [The Uniform Gronwall Lemma]. *Let g, h, y be three positive locally integrable functions on $t\in(t_0, +\infty)$ such that y' is locally integrable on $(t_0, +\infty)$, and which satisfy*

$$\frac{dy}{dt}\leqslant gy+h \qquad (\forall t\geqslant t_0)$$

$$\int_t^{t+r} g(s)\,ds\leqslant a_1, \qquad \int_t^{t+r} h(s)\,ds\leqslant a_2, \qquad \int_t^{t+r} y(s)\,ds\leqslant a_3 \qquad (\forall t\geqslant t_0)$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t+r)\leqslant\left(\frac{a_3}{r}+a_2\right)e^{a_1}\qquad(\forall t\geqslant t_0)\quad\square$$

With this lemma, we can prove the following H^1-absorbing property.

Theorem A.3. *There exists a constant*

$$\rho_1=\frac{C}{16\nu^{3/2}}\exp\left(\frac{3^{11}C^4}{2^{19}\pi^6\nu^7}\right)$$

which is independent of m, such that for any constant $\rho_1'>\rho_1$ arbitrarily fixed and $R_1>0$ independent of m, there exists a constant $T_1>0$ independent of m, such that as long as the initial data $u^{(0)}\in\mathbf{W}$ satisfy

$$\|u^{(0)}\|_0\leqslant R_1$$

then

$$\|u(t)\|_1\leqslant\rho_1'\qquad(\forall t\geqslant T_1)\tag{A.45}$$

Proof. By taking the inner product $\langle\cdot,\cdot\rangle$ of Au and (3.21), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_1^2+\frac{1}{2}\langle PB(u,u),Au\rangle+\nu\|Au\|_0^2=\langle P\Gamma,Au\rangle$$

Since

$$\langle P\Gamma,Au\rangle\leqslant\|P\Gamma\|_0\,\|Au\|_0\leqslant\frac{1}{2\nu}\|P\Gamma\|_0^2+\frac{\nu}{2}\|Au\|_0^2$$

so

$$\frac{d}{dt}\|u\|_1^2+\langle PB(u,u),Au\rangle+\nu\|Au\|_0^2\leqslant\frac{1}{\nu}\|\Gamma\|_0^2\tag{A.46}$$

By Corollary 3.1, $Au\in\mathbf{W}$. Thus $Au=PAu$, therefore

$$\begin{aligned}\langle PB(u,u),Au\rangle&=\langle B(u,u),Au\rangle\\&=\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(\tau_k u_k(D_k u_l)(Au_l))_{ij}\\&\quad+\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(u_k(D_k\tau_k^{-1}u_l)(Au_l))_{ij}\end{aligned}\tag{A.47}$$

Hence

$$|\langle PB(u,u), Au\rangle| \leqslant \|u\|_\infty \cdot \left(\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2} |((D_k u_l)(Au_l))_{ij}| + \frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2} |((D_k \tau_k^{-1} u_l)(Au_l))_{ij}|\right)$$
$$\leqslant 2\sqrt{2}\,\|u\|_\infty\,\|Du\|_0\,\|Au\|_0$$

by (A.42), we have

$$|\langle PB(u,u), Au\rangle| = 18\sqrt{2}\,\|u\|_0^{1/2}\,\|u\|_1 \cdot \left(\frac{\|Au\|_0}{\pi}\right)^{3/2} \tag{A.48}$$

By using Young's inequality

$$ab \leqslant \frac{a^4}{4} + \frac{b^{4/3}}{4/3} \qquad (\forall a, b \geqslant 0)$$

we obtain

$$|\langle PB(u,u), Au\rangle| \leqslant \frac{(18\sqrt{2})^4\,\|u\|_0^2\,\|u\|_1^4}{4\varepsilon^4}$$
$$\leqslant \frac{648^2\,\|u\|_0^2\,\|u\|_1^4}{4\varepsilon^4} + \frac{3\varepsilon^{4/3}}{4\pi^2}\cdot\|Au\|_0^2$$

Set $3\varepsilon^{4/3}/4\pi^2 = \nu$, then $\varepsilon^4 = (4\nu\pi^2/3)^3$. Hence

$$|\langle PB(u,u), Au\rangle| \leqslant \frac{3^{11}}{4\nu^3\pi^6}\,\|u\|_0^2\,\|u\|_1^4 + \nu\,\|Au\|_0^2 \tag{A.49}$$

Together with (A.46), we obtain

$$\frac{d}{dt}\|u\|_1^2 + \nu\,\|Au\|_0^2 \leqslant \frac{3^{11}}{4\nu^3\pi^6}\,\|u\|_0^2\,\|u\|_1^2\cdot\|u\|_1^2 + \frac{C^2}{\nu} \tag{A.50}$$

Set $y(s) = \|u(s)\|_1^2$, $h(s) = (1/\nu)\,\|\Gamma\|_0^2$ and $g(s) = (3^{11}/4\nu^3\pi^6)\|u(s)\|_0^2\,\|u(s)\|_1^2$. Let $R_1 > 0$ be as in the assumption. Then, by Theorem 3.2 and Lemma A.7, there is a $t_0 > 0$ such that $\|u(s)\|_0 \leqslant \rho_0'$ and $\int_t^{t+r}\|u(s)\|_1^2\,ds \leqslant \rho_{1/2}$ if $t \geqslant t_0$. Then there are positive constants a_1, a_2, and a_3 such that

$$\int_t^{t+r} g(s)\,ds \leqslant a_1, \qquad \int_t^{t+r} h(s)\,ds \leqslant a_2, \qquad \int_t^{t+r} y(s)\,ds \leqslant a_3 \qquad (\forall t \geqslant t_0)$$

where

$$a_1 = \frac{3^{11}}{4\nu^3\pi^6}(\rho_0')^2\rho_{1/2}^2, \qquad a_2 = \frac{r}{\nu}C^2, \qquad a_3 = \rho_{1/2}^2 \tag{A.51}$$

Choose $(\rho_1')^2 = (a_3 + a_2)\,e^{a_1}$ and $T_1 = t_0 + r$. Using the uniform Gronwell lemma, we obtain

$$\|u(t)\|_1 \leqslant \rho_1' \qquad (\forall t \geqslant T_1)$$

with

$$(\rho_1')^2 = \left(\frac{rC^2}{16\nu^2} + \frac{rC^2}{\nu} + \frac{(\rho_0')^2}{\nu}\right)\exp\left(3^{11}(\rho_0')^2\left(\frac{rC^2}{16\nu^2} + \frac{(\rho_0')^2}{\nu}\right)\Big/4\nu^3\pi^6\right)$$

for arbitrarily chosen $r > 0$ and $\rho_0' > \rho_0 = C/16\nu$.

Let $r \to 0$, $\rho_0' \to \rho_0$, we have, for any $u^{(0)} \in \mathbf{W}$,

$$\limsup_{t \to +\infty} \leqslant \frac{C}{16\nu^{3/2}}\exp\left(\frac{3^{11}C^4}{2^{19}\pi^6\nu^7}\right)$$

This completes the proof. □

Remark A.2. The absorbing radius ρ_1 (of $\|\cdot\|_1$-norm) tends to 0 as C, the uniform bound of $\|\Gamma\|_0$ as in (H), tends to 0. □

As an immediate consequence of Theorem A.3, we have the following.

Theorem A.4. *The solution $u = u(t)$ to (3.22) [or equivalently, (3.10)] has a global attractor $\mathscr{A} = \mathscr{A}_m$ in $\mathbf{W}$ under the norm $\|\cdot\|_1$.* □

Remark A.3. Since $\mathbf{W}$ is finite dimensional, by the fact that all norms in a finite dimensional space are equivalent to each other, we know that $\mathscr{A} = \mathscr{A}_m$ in Theorems 3.4 and A.4 are geometrically the same. Moreover, by the proofs of Theorems 3.4 and A.4, the rates of convergence T_0 and T_1 differ only by a fixed constant r. □

A.4. Another Proof of the Existence of the Global $\|\cdot\|_1$-Attractors

One sees that in the proof of the existence of the $\|\cdot\|_1$-attractors in Section A.3, the following inequality shown as in (A.48) plays a key role:

$$|\langle PB(u, u), Au\rangle| \leqslant c\,\|u\|_0^{1/2} \cdot \|u\|_1 \cdot \|Au\|_0^{3/2} \tag{A.52}$$

where c is a constant independent of u and the mesh number m.

In this additional section, we give a very short proof of (A.52) by using the inequality (4.5) proved in Lemma 4.2. The new proof avoids the long argument by the discrete Fourier transform. By the proof of Theorem A.3, we know that as long as (A.52) is holds, all other arguments of the proof of Theorem A.3 can be applied without any change.

By (A.47),

$$\begin{aligned}|\langle PB(u,u), Au\rangle| &= |\langle B(u,u), Au\rangle| \\ &\leqslant \left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(\tau_k u_k(D_k u_l)(Au_l))_{ij}\right| \\ &\quad + \left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(u_k(D_k(D_k\tau_k^{-1}u_l)(Au_l))_{ij}\right| \qquad \text{(A.53)}\end{aligned}$$

Using the generalized Hölder's inequality

$$\sum_i a_i b_i c_i \leqslant \left(\sum_i a_i^4\right)^{1/4}\cdot\left(\sum_i b_i^4\right)^{1/4}\cdot\left(\sum_i c_i^2\right)^{1/2}$$

we have

$$\begin{aligned}&|\langle PB(u,u), Au\rangle| \\ &\leqslant \left(\left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(\tau_k u_k)_{ij}^4\right|\right)^{1/4}\cdot\left(\left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(D_k u_l)_{ij}^4\right|\right)^{1/4} \\ &\quad\cdot\left(\left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(Au_l)_{ij}^2\right|\right)^{1/2}+\left(\left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(u_k)_{ij}^4\right|\right)^{1/4} \\ &\quad\cdot\left(\left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(D_k\tau_k^{-1}u_l)_{ij}^4\right|\right)^{1/4}\cdot\left(\left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(Au_l)_{ij}^2\right|\right)^{1/2} \\ &= 2^{7/4}\cdot\left(\left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(u_k)_{ij}^4\right|\right)^{1/4}\cdot\left(\left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(D_k u_l)_{ij}^4\right|\right)^{1/4} \\ &\quad\cdot\left(\left|\frac{1}{m^2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{2}(Au_l)_{ij}^2\right|\right)^{1/2}\end{aligned}$$

We need the following lemmas.

Lemma A.8. *Let D_1, D_2, and A be as in (2.12), (2.14),* and *(2.15). Then they commute with each other, and $D_1^{tr}D_1$, $D_2^{tr}D_2$, and A are all positive in BW, and*

$$D_1^{\rm tr}D_1 + D_2^{\rm tr}D_2 = A \qquad (A.54)$$

Furthermore, define

$$\begin{cases} \tilde{D}_1 = aD_1 + bD_2 \\ \tilde{D}_2 = cD_1 + dD_2 \end{cases} \tag{A.55}$$

where a, b, c, $d \in \mathbf{R}$ *and* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is an orthonormal matrix. Then* $\tilde{D}_1$ *and* $\tilde{D}_2$ *commute, and* $\tilde{D}_1^{\mathrm{tr}}\tilde{D}_1$, $\tilde{D}_2^{\mathrm{tr}}\tilde{D}_2$, *and* $\tilde{D}_1^{\mathrm{tr}}\tilde{D}_1 + \tilde{D}_2^{\mathrm{tr}}\tilde{D}_2$ *are all positive over* $\mathbf{R}$. *Moreover,*

$$\tilde{D}_1^{\mathrm{tr}}\tilde{D}_1 + \tilde{D}_2^{\mathrm{tr}}\tilde{D}_2 = D_1^{\mathrm{tr}}D_1 + D_2^{\mathrm{tr}}D_2 \tag{A.56}$$

Proof. The first part of this lemma follows from the definitions and Theorem 2.1. The second part can be checked directly by the basic properties of orthogonal matrixes. □

Lemma A.9. *For any* $u \in \mathbf{W}$, *we have*

$$\|D_iD_ju\|_0 \leqslant \|Au\|_0 \qquad (i, j = 1, 2) \tag{A.57}$$

Proof. If $i = j = 1$ or 2, then for any $u \in \mathbf{W}$, by Lemma A.8,

$$\begin{aligned} \|D_iD_iu\|_0^2 &= \langle D_iD_iu, D_iD_iu \rangle = \langle u, D_i^{\mathrm{tr}}D_iD_i^{\mathrm{tr}}D_iu \rangle \\ &\leqslant \langle u, AAu \rangle = \langle Au, Au \rangle = \|Au\|_0^2 \end{aligned}$$

Otherwise let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -(1/\sqrt{2}) \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$, and let $\tilde{D}_1$ and $\tilde{D}_2$ be as in (A.55). Then for any $u \in \mathbf{W}$,

$$\begin{aligned} \|D_1D_2u\|_0 = \|D_2D_1u\|_0 &= \|\tilde{D}_1\tilde{D}_1u - \tilde{D}_2\tilde{D}_2u\|_0 \\ &\leqslant \|\tilde{D}_1\tilde{D}_1u\|_0 + \|\tilde{D}_2\tilde{D}_2u\|_0 \end{aligned}$$

In a similar way, we can prove that

$$\|\tilde{D}_k\tilde{D}_ku\|_0 \leqslant \|(\tilde{D}_1^{\mathrm{tr}}\tilde{D}_1u + \tilde{D}_2^{\mathrm{tr}}\tilde{D}_2)\,u\|_0 \qquad (k = 1, 2)$$

By Lemma A.8,

$$\|D_1D_2u\|_0 = \|D_2D_1u\|_0 \leqslant \|Au\|_0 \quad □$$

Now we can prove the following.

Proposition A.2.

$$|\langle PB(u, u), Au \rangle| \leqslant 2^8 \cdot \|u\|_0^{1/2} \cdot \|u\|_1 \cdot \|Au\|_0^{3/2} \tag{A.58}$$

Proof. By Lemmas 4.2 and A.9, we have

$$
\begin{aligned}
|\langle PB(u,u), Au\rangle| &\leqslant 2^{7/4}\cdot 4\, \|u\|_0^{1/2}\, \|u\|_1^{1/2}\cdot 4\cdot 4\, \|u\|_1^{1/2}\, \|u\|_2^{1/2}\cdot \|u\|_2 \\
&\leqslant 2^8\cdot \|u\|_0^{1/2}\cdot \|u\|_1\cdot \|u\|_2^{3/2}
\end{aligned}
$$

This proves the proposition. ∎

Remark. The proof of (A.52) depends heavily on the fact that $PA = A$. It is true only in the periodic boundary condition case. ∎

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