Journal of Dynamics and Differential Equations, Vol. 4, No. 2, 1992

On Null Controllability of Linear Systems with Recurrent Coefficients and Constrained Controls

Russell Johnson[1] **and Mahesh Nerurkar**[2]

Received June 9, 1988

Given a family of time-dependent linear control processes, we study conditions under which local null controllability implies global null controllability. This is done by employing methods of dynamical systems and the Sacker-Sell spectral theory. We show that the above implication holds "almost surely" for recurrent families provided the spectrum of the associated linear system is contained in $(-\infty, 0]$.

KEY WORDS: Uniform controllability; recurrence; dynamical systems; Sacker-Sell spectrum.

1. INTRODUCTION

In this paper we study the time-dependent linear control problem

$$x'(t) = A(t)\, x(t) + B(t)\, u(t) \qquad (x \in \mathbf{R}^n,\, u \in \mathbf{R}^m) \tag{1.1}$$

where $u = u(t)$ is an appropriate control function. We are interested in the following question: When does local null controllability of (1.1) imply global null controllability? We study this question under the assumption that u is admissible, that is, $u(t)$ lies in a fixed compact convex subset $\Omega \subseteq \mathbf{R}^m$ which contains the origin.

We recall the concept of local (respectively, global) null controllability. If $(x_0, t_0) \in \mathbf{R}^n \times \mathbf{R}$, we say that (x_0, t_0) can be *steered to* y in time $T > 0$ if

[1] Department of Mathematics, University of Southern California, Los Angeles, California 90089.

[2] Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08102.

1040-7294/92/0400-0259$06.50/0

there is an admissible control u_0: $[t_0, t_0+T] \to \Omega$ such that the solution of the initial value problem

$$x'(t) = A(t)\,x(t) + B(t)\,u_0(t)$$
$$x(t_0) = x_0$$

also satisfies $x(t_0+T) = y$. Most of the time y will be the origin in $\mathbf{R}^n$ and t_0 will equal zero; in this case we say that x_0 can be steered to $y=0$ in time T.

Define $D(T) = \{x \in \mathbf{R}^n \mid x \text{ can be steered to } y=0 \text{ in time } T\}$ and set

$$D = \bigcup \{D(T) \mid T > 0\} \tag{1.2}$$

One says that (1.1) is *locally null controllable* if there exists $T>0$ and a neighborhood V of $0 \in \mathbf{R}^n$ such that $V \subseteq D(T)$. Note that one can define a weaker notion of local null controllability by letting T depend on $x \in V$. However, these notions turn out to be the same under mild assumptions on Ω (see Lemma 2.8, below). The control process (1.1) is called *globally null controllable* if $D = \mathbf{R}^n$.

The relation between local and global null controllability is well understood when A and B are constant matrices and Ω is a compact convex set containing zero. Then it is known that global null controllability is equivalent to local null controllability together with the condition that the real parts of the eigenvalues of A are all nonpositive. It is easy to see that generalized eigenvectors of A corresponding to eigenvalues with a negative real part decay to zero exponentially under the action of e^{tA} (the fundamental matrix solution of $x' = Ax$), and thus they can be steered to zero by first bringing them into a neighborhood of the origin (note that $0 \in \Omega$) and then using local null controllability. However, steering generalized eigenvectors with purely imaginary eigenvalues to zero is not quite as simple. In this case, one can use arguments based either on the Jordan normal form (see Ref. 10) or on the convexity of the reachable set (see Ref. 11).

The above discussion points out two general principles which carry over to the study of nonautonomous control processes. First, once one is given local null controllability, global null controllability depends only on the qualitative behavior of the solution of the associated linear system

$$x' = A(t)x \tag{1.3}$$

and does not have much to do with the matrix $B(t)$. Second, it may be difficult in general to steer vectors with "zero exponents" to zero because they may exhibit a sort of "transient" behavior. In fact solutions of the nonautonomous system (1.3) may wander away to infinity even when the exponent is zero. Thus the additional condition one needs to impose is that these solutions of (1.3) should be recurrent in some sense. This in turn suggests that ergodic theory can play a role in studying our controllability question.

We study the control process (1.1) under the assumption that $A(t)$, $B(t)$ are *uniformly recurrent* matrix-valued functions. We define this concept below; it seems a natural assumption to make in studying the relation between local and global null controllability for (1.1). We remark that if $A(t)$, $B(t)$ are almost periodic in the sense of Bohr, then they are recurrent. So our theory applies when (1.1) has almost periodic coefficients.

To analyze the qualitative behavior of solutions of the linear system (1.3), we use the results of Sacker and Sell [14] and Johnson *et al.* [9]. In the language of these papers, the additional condition one needs to impose is that the zero exponent is the right end point of the "spectral interval" to which it belongs (see Section 3). We translate this condition into the language of ergodic theory, and then, using an ergodic-theoretic recurrence result on an appropriate probability space, we show that the norm of a vector with a zero exponent returns close to its initial value after a sufficiently long time. Using this fact, we are able to steer such vectors closer to zero than before and, finally, to zero in finitely many steps (because of local null controllability).

In the case when (1.3) has negative exponents (thus avoiding the troublesome case of vectors with zero exponents), global null controllability of (1.1) was discussed by several authors (e.g., Ref. 13). On the other hand, in Ref. 16, Tonkov states a result about global null controllability with nonpositive exponents but under a very strong assumption of reducibility of the associated linear system. Our assumption is far weaker and also illustrates exactly where the problem lies.

The paper is organized as follows. In Section 2, we show that a weak notion of local null controllability implies a stronger one, namely, uniform local null controllability (see Theorem 2.10). This result is a generalization of a result of Artstein [1] and is false in general if $A(t)$, $B(t)$ are not uniformly recurrent (see Ref. 1 for counterexamples). In Section 3, we state and prove our main theorem (Theorem 3.2) relating local and global null controllability.

We finish the introduction with basic definitions and terminology. Let $M(n, m)$ be the set of $n \times m$ real matrices with the Euclidean norm $\| \ \|$ and let $L^p_{\text{loc}}(n, m) = \{f: \mathbf{R} \to M(n, m) \mid f$ is locally L^p-integrable$\}$. Here $p \geq 1$. This space is given the distribution topology. Thus $f_j \to f$ in $L^p_{\text{loc}}(n, m)$ if and only if $\int_{\mathbf{R}} f_j(t)\, \varphi(t)\, dt \to \int_{\mathbf{R}} f(t)\, \varphi(t)\, dt$, where φ ranges over the set of real-valued, C^∞ functions on R with compact support. If $f \in L^p_{\text{loc}}(n, m)$, defines the translations

$$\tau_t(f)(s) = f(t+s), \qquad t, s \in \mathbf{R}$$

We also write $\tau_t(f) = \tau(f, t)$. The translations are continuous and satisfy the group property:

$$\tau_t \circ \tau_s = \tau_{t+s}, \qquad t, s \in \mathbf{R}$$

Now suppose that $\int_t^{t+1} \|A(s)\|^p \, ds$, $\int_t^{t+1} \|B(s)\|^p \, ds$ are bounded independently of $t \in \mathbf{R}$. If $p = 1$, we suppose, in addition, that

$$\operatorname*{Lim}_{\varepsilon \to 0} \int_t^{t+\varepsilon} \|A(s)\| \, ds = 0 = \operatorname*{Lim}_{\varepsilon \to 0} \int_t^{t+\varepsilon} \|B(s)\| \, ds$$

uniformly in t. Under these conditions, the sets $C_A = cls\{\tau_t(A) \mid t \in \mathbf{R}\} \subseteq L^p_{\text{loc}}(n, n)$ and $C_B = cls\{\tau_t(B) \mid t \in \mathbf{R}\} \subseteq L^p_{\text{loc}}(n, m)$ are compact and translation invariant. Consider the product space $C_A \times C_B$. The restriction of the set of mappings $\{\tau_t \mid t \in \mathbf{R}\}$ defines a *flow* on $C_A \times C_B$, that is, a one-parameter group of homeomorphisms of $C_A \times C_B$ which is jointly continuous in $\xi \in C_A \times C_B$ and in $t \in \mathbf{R}$. Define

$$E = cls\{\tau_t(A, B) \mid t \in \mathbf{R}\} \subseteq C_A \times C_B$$

Definitions 1.4. (i) Let d be a metric on $L^p_{\text{loc}}(n, n) \times L^p_{\text{loc}}(n, m)$ which is compatible with the distribution topology. The point $\xi_0 = (A, B)$ is called *uniformly recurrent* if, given $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ with the property that every interval $[t, t+T] \subseteq \mathbf{R}$ contains a point t_0 such that $d(\xi_0, \tau(\xi_0, t_0)) < \varepsilon$.

(ii) The flow $(E, \{\tau_t \mid t \in \mathbf{R}\})$ is called *minimal* if, for each $\xi \in E$, the orbit $\{\tau_t(\xi) \mid t \in \mathbf{R}\}$ is dense in E.

Theorem 1.5 (e.g., Ref. 5). *If $\xi_0 = (A, B)$ is recurrent, then $(E, \{\tau_t \mid t \in \mathbf{R}\})$ is minimal. Conversely, if $(E, \{\tau_t \mid t \in \mathbf{R}\})$ is minimal, then every point $\xi \in E$ is uniformly recurrent in the sense of Definition 1.4.*

The condition of minimality is a strong restriction on the flow $(E, \{\tau_t\})$. On the other hand, for certain purposes the minimal subset of an arbitrary compact metric flow may be regarded as its "building blocks," and results valid on minimal sets may be used to obtain information valid for the entire flow. An instance of this point of view is presented in Ref. 8.

Instead of concentrating on the single control system (1.1), we consider the family of systems

$$x'(t) = A(\tau_t(\xi))x + B(\tau_t(\xi))u \qquad (\xi \in E) \tag{1.1$_\xi$}$$

Here we have abused notation and written $A(\tau_t(\xi))$ for that matrix function $\tilde{A}(t)$ which is the projection onto C_A of the point $\xi \in C_A \times C_B$; $B(\tau_t(\xi))$ is to be interpreted similarly. We say that $(x_0, t_0) \in \mathbf{R}^n \times \mathbf{R}$ can be *ξ-steered* to $y \in \mathbf{R}^n$ in time $T > 0$ if there is an admissible control u_0: $[t_0, t_0 + T] \to \Omega$ such that the solution of $(1.1)_\xi$ satisfying $x(t_0) = x_0$ also satisfies $x(t_0 + T) = y$. The sets defined in (1.2) are generalized in the natural way. Set $D(\xi, T) = \{x \in \mathbf{R}^n \mid x \text{ can be } \xi\text{-steered to } y = 0 \text{ in time } T\}$ and

$$D(\xi) = \bigcup \{D(\xi, T) \mid T > 0\} \qquad \text{for each} \quad \xi \in E \tag{1.6}$$

We also need the concept of an ergodic measure in Section 3.

Definition 1.7. Let $(E, \{\tau_t \mid t \in \mathbf{R}\})$ be a flow where E is a compact metric space. A (Radon) probability measure m on E is *invariant* if $m(\tau_t(B)) = m(B)$ for any Borel subset $B \subseteq E$. An invariant measure is *ergodic* if, in addition, for each Borel subset $B \subseteq E$, $m(\tau_t(B) \Delta B) = 0$ implies $m(B) = 0$ or 1, where Δ denotes the symmetric difference of sets.

It is a standard fact that every compact metric flow admits at least one ergodic measure. See, e.g., Ref. 12 for a proof and for a detailed discussion of the concept of invariance and ergodicity.

2. LOCAL NULL CONTROLLABILITY

In this section we study the local null controllability of the family of control systems

$$x'(t) = A(\tau_t(\xi))x + B(\tau_t(\xi))u \qquad (\xi \in E) \tag*{$(1.1)_\xi$}$$

where $(E, \{\tau_t\})$ is a minimal flow as described in Section 1. Our goal is to show that if, for some $\xi_0 \in E$, the system $(1.1)_{\xi_0}$ is locally null controllable, then every Eq. $(1.1)_\xi$ is uniformly locally null controllable in the sense of the following.

Definition 2.1. The family of control systems $(1.1)_\xi$ is said to be *uniformly locally null controllable* if there exists a neighborhood of the origin $V \subseteq \mathbf{R}^n$ and a number $T > 0$ such that $V \subseteq D(\xi, T)$ for all $\xi \in E$.

We first need to recall a necessary and sufficient condition for local null controllability proved in Ref. 3 and some terminology that goes with it.

Definition 2.2. Let Ω be a subset of $\mathbf{R}^m$. Define H_Ω: $\mathbf{R}^m \to \mathbf{R}$ by setting

$$H_\Omega(\alpha) = \operatorname{Sup}\{\langle \alpha, \omega \rangle \mid \omega \in \Omega\}$$

where $\langle\ ,\ \rangle$ is the usual inner product on $\mathbf{R}^m$. The map H_Ω is called the *support function* of Ω.

Remarks 2.3. (a) If $0 \in \Omega$, then H_Ω is nonnegative. (b) If Ω is compact, then H_Ω is continuous.

We quote a lemma and a proposition from Ref. 3, then prove a corollary of these results. The statements are true for linear control system (1.1) solely under the condition that $A(t)$, $B(t)$ are locally integrable. The assumption of uniform recurrence is not needed.

Lemma 2.4. *Let $x_0 \in \mathbf{R}^n$, $t_0 \in \mathbf{R}$ and suppose Ω is compact and contains zero. Consider the control process*

$$x'(t) = A(t)\,x(t) + B(t)\,u(t) \tag{1.1}$$

Then (x_0, t_0) can be steered to zero in time T if and only if

$$\langle x_0, z^*(t_0)\rangle + \int_{t_0}^{t_0+T} H_\Omega(B^*(s)\,z^*(s))\,ds \geqslant 0$$

for all solutions $z^*(t)$ of the associated adjoint linear system

$$z' = -A^*(t)z \tag{2.5}$$

Here the asterisk denotes transpose.

Proposition 2.6. *Let the control process (1.1) and the set Ω be as in Lemma 2.4. The process is locally null controllable if and only if there exists $\varepsilon > 0$ such that $\int_0^\infty H_\Omega(B^*(s)\,z^*(s))\,ds \geqslant \varepsilon$ for each solution $z^*(t)$ of the adjoint system (2.5) which satisfies $\|z^*(0)\| = 1$, $\|\ \|$ being the Euclidean norm.*

Corollary 2.7. *Let the control process (1.1) be as in Lemma 2.4. Suppose there is a neighborhood V of the origin in $\mathbf{R}^n$ such that, for each $x \in V$, there exists $T = T(x) > 0$ with the property that x can be steered to zero in time T. Then the control process (1.1) is locally null controllable if either (a) Ω is compact, convex and contains zero or (b) Ω is compact and* $0 \in \operatorname{int}(\Omega)$.

Proof. (a) Suppose that Ω is compact, convex and that $0 \in \Omega$. Choose $x_1, x_2, \ldots, x_r \in V$ such that the convex hull of the x_i's contains a ball B around zero. Let $u_i = u_i(t)$ $(1 \leqslant i \leqslant r)$ be admissible controls which steer x_i to zero in time T_i. Set $T = \max\{T_i \mid 1 \leqslant i \leqslant r\}$. If $x \in B$, then x is a convex combination of the x_i's, say $x = \sum_{i=1}^r \varepsilon_i x_i$, where $0 \leqslant \varepsilon_i \leqslant 1$. Define

$u = \sum_{i=1}^{r} \varepsilon_i \tilde{u}_i$ where $\tilde{u}_i = u_i$ on $[0, T_i]$ and zero otherwise. Then u steers x to zero in time T. Hence the same T can be used for all points in B and part (a) is proved.

(b) Suppose now that Ω is compact and $0 \in \text{int}(\Omega)$. Let $\tilde{\Omega}$ be the convex hull of Ω. Let δ and R be positive numbers chosen so that (i) the ball $B(\delta)$ of radius δ centered at 0 is contained in Ω and (ii) the ball $B(R)$ centered at 0 of radius R contains $\tilde{\Omega}$. By part (a) and Proposition 2.6 applied to the constrained set $B(R)$, we obtain an $\varepsilon > 0$ such that, for each solution $z^*(t)$ of the adjoint system (2.5) which satisfies $\|z^*(0)\| = 1$, one has the following.

$$\int_0^\infty H_\Omega(B^*(s)\, z^*(s))\, ds \geqslant \varepsilon$$

Now we reason as follows:

$$\begin{aligned}
\int_0^\infty H_\Omega(B^*(s)\, z^*(s))\, ds &\geqslant \int_0^\infty H_{B(\delta)}(B^*(s)\, z^*(s))\, ds \\
&\geqslant \int_0^\infty \text{Sup}\{\langle \alpha, B^*(s)\, z^*(s)\rangle \mid \|\alpha\| < \delta\}\, ds \\
&\geqslant \frac{\delta}{R} \int_0^\infty \text{Sup}\{\langle \alpha, B^*(s)\, z^*(s)\rangle \mid \|\alpha\| < R\}\, ds \\
&\geqslant \frac{\delta}{R} \int_0^\infty H_{B(R)}(B^*(s)\, z^*(s))\, ds \geqslant \frac{\delta\varepsilon}{R}
\end{aligned}$$

for each solution $z^*(t)$ of (2.5) satisfying $\|z^*(0)\| = 1$. Another application of Proposition 2.6 completes the proof of (b).

We now return to our family of control systems $(1.1)_\xi$, where ξ runs over a minimal set E. Consider the associated linear systems for each $\xi \in E$.

$$x' = A(\tau_t(\xi))x \tag{2.8}$$

$$z' = -A^*(\tau_t(\xi))z \tag{2.9}$$

Let the corresponding fundamental matrix solutions be $X(\xi, t)$ and $X^*(\xi, t)^{-1}$, respectively. Each of these maps satisfies the following *cocycle identity*.

$$X(\xi, t+s) = X(\tau(\xi, t), s)\, X(\xi, t), \qquad \xi \in E, \qquad t, s \in \mathbf{R}$$

and similarly for $Z(\xi, t) = X^*(\xi, t)^{-1}$. Now we can prove the main result of this section.

Theorem 2.10. *Let $\Omega \subseteq \mathbf{R}^m$ be compact and suppose $0 \in \Omega$. Suppose that there exists a $\xi_0 \in E$ such that the process $(1.1)_{\xi_0}$ is locally null controllable. Then the family of Eqs. $(1.1)_{\xi}$, $\xi \in E$ is uniformly locally null controllable.*

Proof. For convenience we set

$$f_\xi^p(t) = B^*(\tau_t(\xi))\, X^*(\xi, t)^{-1} p, \qquad p \in \mathbf{R}^n, \qquad \xi \in E$$

By local null controllability of $(1.1)_{\xi_0}$ and Proposition 2.6, there exists $\varepsilon > 0$ such that

$$\int_0^\infty H_\Omega(f_{\xi_0}^p(s))\, ds \geqslant 8\varepsilon \qquad (p \in \mathbf{R}^n,\ \|p\| = 1)$$

Since the map $p \to H_\Omega(f_{\xi_0}^p(s))$ is continuous and nonnegative for each s, the compactness of the unit sphere in $\mathbf{R}^n$ implies that there exists $J > 0$ such that

$$\int_0^J H_\Omega(f_{\xi_0}^p(s))\, ds \geqslant 4\varepsilon \qquad (p \in \mathbf{R}^n,\ \|p\| = 1)$$

Let d be a metric on E. By continuity of the map $\xi \to \int_0^J H_\Omega(f_\xi^p(s))\, ds$, we can find $\delta > 0$ such that, if $d(\xi, \xi_0) \leqslant \delta$, then the value of this map changes by no more than ε for all $\|p\| = 1$.

Now the point ξ_0 is uniformly recurrent. Therefore, corresponding to the number δ just defined, there exists $L > 0$ and a sequence $T_n \to -\infty$ with the following properties: (i) $T_n < T_{n-1}$; (ii) $|T_n - T_{n-1}| < L$; and (iii) $\tau(\xi_0, T_n)$ lies in the δ-ball centered at ξ_0, for all $n \geqslant 1$ (see Theorem 1.5).

These conditions along with our choice of δ imply that

$$\int_0^J H_\Omega(f_{\tau(\xi_0, T_n)}^p(s))\, ds \geqslant 2\varepsilon \qquad (p \in \mathbf{R}^n,\ \|p\| = 1, n \geqslant 1) \tag{2.11}$$

Next let $M = \operatorname{Sup}\{\max(\|X^*(\xi, t)^{-1}\|, 1) \mid t \in [-L, 0],\ d(\xi, \xi_0) \leqslant \delta\}$. We claim that

$$\int_0^{J+L} H_\Omega(f_{\tau(\xi_0, T)}^p(s))\, ds \geqslant \frac{2\varepsilon}{M} \quad \text{if} \quad \|p\| = 1 \quad \text{and} \quad T < 0 \tag{2.12}$$

To see this, note that for each $\xi \in E$ and $t \in [-L, 0]$ we have

$$\begin{aligned} H_\Omega(f_{\tau(\xi, t)}^p(s)) &= H_\Omega(B^*(\tau_s \tau_t(\xi))\, X^*(\tau_t(\xi), s)^{-1} p) \\ &= H_\Omega(B^*(\tau_{s+t}(\xi))\, X^*(\xi, t+s)^{-1}\, X^*(\xi, t)\, p) \end{aligned}$$

where we used the cocycle identity for X^{*-1}. Thus letting $\tilde{p} = X^*(\xi, t)\, p$, we get

$$\inf\left\{\int_0^{J+L} H_\Omega(f^p_{\tau_t(\xi)}(s))\, ds \,\middle|\, \|p\| = 1\right\}$$

$$= \inf\left\{\int_0^{J+L} H_\Omega(f^{\tilde{p}}_\xi(t+s))\, ds \,\middle|\, \|p\| = 1\right\}$$

$$= \inf\left\{\int_t^{t+J+L} H_\Omega(f^{\tilde{p}}_\xi(s))\, ds \,\middle|\, \|p\| = 1\right\}$$

$$\geqslant \inf\left\{\int_0^{t+J+L} H_\Omega(f^{\tilde{p}}_\xi(s))\, ds \,\middle|\, \|p\| = 1\right\} \qquad \text{(since } t<0\text{)}$$

$$= \inf\left\{\|X^*(\xi, t)\, p\| \int_0^{t+J+L} H_\Omega(f^{(p/\|\tilde{p}\|)}_\xi(s)) \,\middle|\, \|p\| = 1\right\}$$

Now, given any $T<0$, we can write $T = T_n + t$ for some n, with $t \in [-L, 0]$. Applying the above inequality with $\xi = \tau(\xi_0, T_n)$, we obtain

$$\inf\left\{\int_0^{J+L} H_\Omega(f^p_{\tau(\xi_0, T)}(s))\, ds \,\middle|\, \|p\| = 1\right\}$$

$$= \inf\left\{\|X^*(\tau(\xi_0, T_n), t)\, p\| \int_0^{t+J+L} H_\Omega(f^{(p/\|\tilde{p}\|)}_{\tau(\xi_0, T_n)}(s))\, ds \,\middle|\, \|p\| = 1\right\}$$

$$\geqslant (2\varepsilon) \inf\{\|X^*(\tau(\xi_0, T_n), t)\, p\| \mid \|p\| = 1\} \qquad \text{by (2.11)}$$

$$\geqslant (2\varepsilon) \inf\{\|X^*(\tau(\xi_0, T_n), t)\, p\| \mid \|p\| = 1,\ t \in [-L, 0]\}$$

Since $\|p\| \leqslant \|A^{-1}\|\ \|Ap\|$ for any nonsingular matrix A, we finally obtain

$$\inf\left\{\int_0^{J+L} H_\Omega(f^p_{\tau(\xi_0, T)}(s))\, ds \,\middle|\, \|p\| = 1\right\} \geqslant \frac{2\varepsilon}{M}$$

which implies (2.12).

Now note that minimality of E implies that $\{\tau_t(\xi_0) \mid t<0\}$ is dense in E [5]. Using continuity of the map $\xi \to \int_0^{J+L} H_\Omega(f^p_\xi(s))\, ds$, we obtain

$$\int_0^{J+L} H_\Omega(f^p_\xi(s))\, ds \geqslant \frac{2\varepsilon}{M} \qquad (\xi \in E,\ \|p\| = 1) \tag{2.13}$$

Now define $V = \{x \in \mathbf{R}^n \mid \|x\| < (\varepsilon/M)\}$. We show that $V \subseteq D(\xi, T)$ for all $\xi \in E$, where T is independent of ξ. This completes the proof of Theorem 2.10. We do so by essentially repeating the steps of the proof of Proposition 2.6 above (see Ref. 3).

By Lemma 2.4, we must show that there exists $T>0$ such that, for each $x\in V$ and $\xi\in E$, one has the following:

$$\langle x, p\rangle+\int_0^T H_\Omega(f_\xi^p(s))\,ds\geqslant 0 \qquad (p\in\mathbf{R}^n,\ \|p\|=1)$$

If this statement is not true, then there exists $x_0\in V$ and a sequence $p_n\in\{p\in\mathbf{R}^n,\ \|p\|=1\}$, $\xi_n\in E$, and $\tilde{T}_n\to\infty$ such that

$$\int_0^{\tilde{T}_n} H_\Omega(f_{\xi_n}^{p_n}(s))\,ds< -\langle x_0, p_n\rangle\leqslant\|x_0\|<\frac{\varepsilon}{M}, \qquad \text{for} \quad n\geqslant 1$$

By compactness of the unit sphere in $\mathbf{R}^n$ and of E, we can assume that $p_n\to p$ and $\xi_n\to\xi$. Then

$$\lim_{n\to\infty} H_\Omega(f_{\xi_n}^{p_n}(s))\,\chi_{[0,\tilde{T}_n]}(s)=H_\Omega(f_\xi^p(s))$$

where the limit holds pointwise on $\mathbf{R}$. Here $\chi_{[0,\tilde{T}_n]}$ denotes the characteristic function of $[0, \tilde{T}_n]$. We thus have

$$\int_0^\infty H_\Omega(f_\xi^p(s))\,ds\leqslant\liminf_{n\to\infty}\int_0^{\tilde{T}_n} H_\Omega(f_{\xi_n}^{p_n}(s))\,ds$$
$$\leqslant\limsup_{n\to\infty}\int_0^{\tilde{T}_n} H_\Omega(f_{\xi_n}^{p_n}(s))\,ds\leqslant\|x_0\|<\frac{\varepsilon}{M}$$

where the first inequality uses Fatou's lemma. However, this violates (2.13). The proof of Theorem 2.10 is thus complete.

Remark 2.14. In Ref. 16 Tonkov states a result where uniform null controllability at ξ_0 is obtained from null controllability at ξ_0, under the additional assumptions that there exists a uniformly recurrent ξ_1, such that (i) the system $(1.1)_{\xi_1}$ is globally null controllable and (ii) $d(\xi_0(t), \xi_1(t))\to 0$ as $t\to\infty$. Condition (ii) appears to weaken the assumption of uniform recurrence of ξ_0; however, condition (i) is much too strong.

3. GLOBAL NULL CONTROLLABILITY

In this section we state and prove our main result. Theorem 3.2, relating local and global null controllability for recurrent linear control processes. We use results of Sacker and Sell [14] and Johnson *et al.* [9] concerning the family of linear equations,

$$x'=A(\tau_t(\xi))x, \qquad (\xi\in E) \tag{2.8}$$

Define $\Sigma = \{\lambda \in \mathbf{R} \mid x' = [A(\tau_t(\xi)) - \lambda I]x$ admits a nontrivial bounded solution for some $\xi \in E\}$. The set Σ is called the (Sacker–Sell) *spectrum* of Eqs. (2.8). Analogously one defines the spectrum Σ^* of the adjoint system (2.9). It is easily seen that $\Sigma = -\Sigma^*$; that is, $\lambda \in \Sigma$ if and only if $-\lambda \in \Sigma^*$.

Theorem 3.1 [14]. *The spectrum Σ of the family (2.8) is a disjoint union of $k \leqslant n$ compact intervals $[a_i, b_i] \subseteq \mathbf{R}$, $(1 \leqslant i \leqslant k)$. Furthermore, there exists k continuous subbundles $V_1, V_2, ..., V_k$ of the trivial bundle $\mathbf{R}^n \times E$ such that*

(i) $\mathbf{R}^n \times E = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ *(Whitney sum);*

(ii) each V_i is invariant under the flow $\{\hat{\tau}_t \mid t \in \mathbf{R}\}$ on $\mathbf{R}^n \times E$ defined by

$$\hat{\tau}_t(x, \xi) = (X(\xi, t)x, \tau_t(\xi))$$

(iii) for each $(x, \xi) \in V_i$, one has

$$\operatorname*{Lim\,sup}_{t \to \pm\infty} \frac{1}{t} \ln \|X(\xi, t)x\| \in [a_i, b_i]$$

and

$$\operatorname*{Lim\,inf}_{t \to \pm\infty} \frac{1}{t} \ln \|X(\xi, t)x\| \in [a_i, b_i]$$

and

(iv) $\Sigma|_{V_i} = [a_i, b_i]$.

We can now state our main result.

Theorem 3.2. *Let $\Omega \subseteq \mathbf{R}^n$ be compact with $0 \in \Omega$. Suppose the uniform recurrent family of control processes $(1.1)_\xi$ $(\xi \in E)$ is uniformly locally null controllable [see (2.1)]. Suppose, further, that the spectrum Σ of Eq. (2.8) is contained in $(-\infty, 0]$. Let m be an invariant measure on E. Then the process $(1.1)_\xi$ is globally null controllable for m-almost all $\xi \in E$.*

It is worth noting that the hypothesis that $\Sigma \subseteq (-\infty, 0]$ is equivalent to the following statement: to each $\varepsilon > 0$ there corresponds a constant K_ε such that, for all $\xi \in E$, all $x \in \mathbf{R}^n - \{0\}$ and all $t \geqslant 0$, one has

$$\|X(\xi, t)x\| \leqslant K_\varepsilon e^{\varepsilon t} \|x\|$$

We begin the proof of Theorem 3.2 with an elementary observation.

Lemma 3.3. *Consider the process $(1.1)_\xi$ for fixed $\xi \in E$. If x_0 can be ξ-steered to x_1 in time T_1 and if x_1 can be $\tau(\xi, T_1)$-steered to x_2 in time T_2, then x_0 can be ξ-steered to x_2 in time $T_1 + T_2$.*

Proof. Let u_1 and u_2 be admissible controls which steer x_0 to x_1 and x_1 to x_2, respectively. Define

$$u(t)=\begin{cases} u_1(t) & \text{if } 0\leqslant t\leqslant T_1 \\ u_2(t-T_1) & \text{if } T_1\leqslant t\leqslant T_1+T_2 \\ 0 & \text{otherwise} \end{cases}$$

It is straightforward to check that $u(t)$ ξ-steers x_0 to x_2.

We now continue with the proof of Theorem 3.2. First, assuming m to be ergodic, we prove that there is a set $E^*\subseteq E$ such that $m(E^*)=1$ and such that, for any $\xi\in E^*$ and $x\in \mathbf{R}^n$, there exists T (which may depend on ξ and x) such that x can be ξ-steered to 0 in time T.

We begin by showing that it is sufficient to prove this statement when A is lower triangular, i.e., has zeros above the main diagonal. This is achieved by making a uniformly recurrent change of variable $x=P(t)\,y$, which triangularizes the matrix function A (see Ref. 9). Briefly, there is a minimal flow $(\hat{E}, \{\hat{\tau}_t \mid t\in\mathbf{R}\})$, a surjective continuous map $\pi\colon \hat{E}\to E$ which commutes with the flow, and a continuous mapping $P\colon \hat{E}\to GL(n,\mathbf{R})$ such that $\hat{\xi}\colon\to (d/dt)\,P(\hat{\tau}_t(\hat{\xi}))\,|_{t=0}$ is continuous, with the following properties. First, make the change of variables $x=P(\hat{\tau}_t(\hat{\xi}))\,y$ in the linear system $x'=A(\tau_t(\pi(\hat{\xi}))x$; then the resulting coefficient matrix,

$$C(\hat{\tau}_t(\hat{\xi}))=P^{-1}(\hat{\tau}_t(\hat{\xi}))\,A(\tau_t(\pi(\hat{\xi}))\,P(\hat{\tau}_t(\hat{\xi}))-P^{-1}(\hat{\tau}_t(\hat{\xi}))\,\frac{d}{dt}\,P(\hat{\tau}_t(\hat{\xi}))$$

is lower triangular. Second, there is an invariant measure $\hat{m}$ on $\hat{E}$ which projects to m under π, i.e., $\pi_*\hat{m}=m$. Thus if $\hat{E}^*\subseteq\hat{E}$ has $\hat{m}$ measure 1, then the m measure of $E^*=\pi(\hat{E}^*)\subseteq E$ is also 1.

Since the above change of variables clearly preserves local and global null controllability (with the same control), we can assume that A is lower triangular, as asserted. With this assumption let a_{ii} denote the ith diagonal element of A. The assumption that all spectral intervals are contained in $(-\infty, 0]$ implies that (see Ref. 9)

$$\overline{\lim_{t\to\infty}}\,\frac{1}{t}\int_0^t a_{ii}(\tau_s(\xi))\,ds\leqslant 0 \qquad (\xi\in E)$$

We show that this implies, for m-a.a. $\xi\in E$, that the process $(1.1)_\xi$ is globally null controllable. We fix numbers $r>0$, $T_0>0$ such that the ball $B(r)$ of radius r around $x=0$ can be ξ-steered to zero in time T_0 for all $\xi\in E$.

We use the following ergodic theoretic result from Ref. 15 or, rather, the corollary which follows from it.

***Proposition* 3.4.** *Let* $(Y, \{\tau_t\}, \mu)$ *be a flow where* Y *is a compact metric space and* μ *is an ergodic probability measure on* Y. *Let* $h\colon Y \to \mathbf{R}$ *be a* μ*-integrable function with* $\int_Y h\,d\mu = 0$. *Let* $\tilde{Y} = \{y \in Y \mid$ *given* $\varepsilon > 0$ *and* $N > 0$, *there exists* $t > N$ *such that* $|\int_0^t h(\tau_s(y))\,ds| \leqslant \varepsilon\}$. *Then* $\mu(\tilde{Y}) = 1$.

***Corollary* 3.5.** *Let* $(Y, \{\tau_t\}, \mu)$ *be as above and let* $h\colon Y \to \mathbf{R}$ *be a* μ*-integrable function satisfying* $\int_Y h\,d\mu \leqslant 0$. *Let* $\tilde{Y} = \{y \in Y \mid$ *given* $\varepsilon > 0$, $T_0 > 0$, *and* $k \in \mathbf{N}$, *there are numbers* $Q_j > T_0$ $(1 \leqslant j \leqslant k)$ *such that, if* $S_0 = 0$ *and* $S_p = \sum_{i=1}^p Q_i$, *then* $\int_0^{Q_j} h(\tau(y, S_{j-1} + s))\,ds < \varepsilon$ $(1 \leqslant j \leqslant k)\}$. *Then* $\mu(\tilde{Y}) = 1$.

Proof. The statement (of the corollary) follows from the Birkhoff's ergodic theorem if $\int_Y h\,d\mu < 0$. If $\int_Y h\,d\mu = 0$, we fix ε, T_0, and k and use Proposition 3.4 to choose $Q_j > T_0$ such that $|\int_0^{S_j} h(\tau_s(y))\,ds| \leqslant (\varepsilon/2)$ for all $1 \leqslant j \leqslant k$; here $S_j = Q_1 + \cdots + Q_j$ and $y \in \tilde{Y}$. This implies the statement of the corollary for each $y \in \tilde{Y}$ and completes the proof.

Now let $\tilde{E} = \{\xi \in E \mid$ the conclusion of Corollary 3.5 applies to each function $a_{ii}(\xi)$ $(1 \leqslant i \leqslant n)\}$. Then $m(\tilde{E}) = 1$. Define $E^* = \bigcap \{\tau(\tilde{E}, N) \mid N \in \mathbf{Z}\}$. We have $m(E^*) = 1$. Also, $\xi \in E^*$ implies $\tau(\xi, N) \in E^*$ for each integer N. We show that $(1.1)_\xi$ is globally null controllable for each $\xi \in E^*$.

Fix a vector $x = (x_1, x_2,..., x_n) \in \mathbf{R}^n$ and let $\xi \in E^*$. We claim that x can be ξ-steered to a vector whose first component is zero. To see this, let k be an integer such that $k^{-1}|x_1| < r$. Write $x = (\alpha x_1, 0,..., 0) + ((1-\alpha)x_1, x_2,..., x_n)$ where $\alpha|x_1| < r$ and $(1/k) < \alpha < 1$. Then using the definition of E^*, we can find $Q_1 > T_0$ and a control $V_1(t)$ such that x is ξ-steered in time Q_1 to a vector $(y_1, y_2,..., y_n)$ with $y_1 = \beta x_1$, $0 \leqslant \beta \leqslant (k-1)/k$ by $V_1(t)$. This procedure can clearly be repeated. After j $(j \leqslant k)$ steps, we can use Lemma 3.3 to obtain a time $T_1 = Q_1 + Q_2 + \cdots + Q_j > T_0$ and a control $U_1(t)$ such that x is ξ-steered in time T_1 to a vector whose first component is zero. This proves our claim.

The next step is, of course, to apply the preceding argument to the vector $(0, z_2,..., z_n)$, which we have just obtained. It is clear that there exists $Q \geqslant T_1$ such that $\tau(\xi, Q) \in E^*$. Note that, setting $U_1(t) = 0$ for $T_1 \leqslant t \leqslant Q$, without loss of generality we can assume $\tau(\xi, T_1) \in E^*$. Having done so, we choose a new k such that $k^{-1}|z_2| < r$. Replacing a_{11} by a_{22} and proceeding as above and, in particular, making use of the triangular form of Eq. $(2.8)_\xi$, we obtain a number $T_2 > T_0$ and a control $U_2(t)$ such that $(0, z_2,..., z_n)$ is $\tau(\xi, T_1)$-steered to $(0, 0, w_3,..., w_n)$ in time T_2 by $U_2(t)$.

It is now clear that, after n repetitions of our argument, we obtain (using Lemma 3.3) a time $T = T_1 + T_2 + \cdots + T_n$ and a control U which ξ-steers x to zero in time T. This proves Theorem 3.2 if m is ergodic.

Finally, in the general case set $E^* = \{\xi \in E \mid$ the process $(1.1)_\xi$ is globally null controllable). Then we have shown that $m(E^*)=1$ for every ergodic measure m on E and, hence, for every invariant measure m [12]. This completes the proof of Theorem 3.2.

Remark 3.6. (a) Even if the spectral interval $[a, b]$ containing 0 satisfies $b>0$, using generic results developed in Ref. 6, under some additional conditions (e.g., for two-dimensional systems) we can show that there is a residual subset $E^* \subseteq E$ (which, however, has measure zero) such that, if $\xi \in E^*$, then $(1.1)_\xi$ is globally null controllable.

(b) Suppose that all solutions in the jth spectral bundle V_j are bounded for $-\infty < t < \infty$. Then it can be proved that $(1.1)_\xi$ is globally null controllable for all $\xi \in E$.

We close the paper with a partial converse to Theorem 3.2.

Theorem 3.7. *Suppose the minimal flow $(E, \{\tau_t\})$ supports exactly one invariant measure m, which is then necessarily ergodic. This is the case if, for example, $A(t)$ and $B(t)$ are Bohr almost periodic. Suppose that $m\{\xi \in E \mid$ the process $(1.1)_\xi$ is globally null controllable$\}$ is positive. Then the spectrum Σ of $x' = A(\tau_t(\xi))x$ is contained in $(-\infty, 0]$.*

Proof. Assume that the spectrum Σ intersects the positive real axis. Then the spectrum Σ^* of the adjoint system (2.9) meets the negative real axis. Let $b<0$ be the left end point of one of the intervals in Σ^*. Then by Ref. 9 there is an ergodic probability measure μ on *projective bundle* $\mathbf{P}^{n-1}(\mathbf{R}) \times E$ such that $\mathrm{Lim}_{t \to +\infty}(1/t) \ln \|X^*(\xi, t)^{-1}v\| = b < 0$ for μ-almost all $(v, \xi) \in \mathbf{P}^{n-1}(\mathbf{R}) \times E$. Here we confuse $v \neq 0$ with the line in $\mathbf{R}^n$ on which it lies. Since μ projects onto m, we have

$$m\left\{\xi \in E \,\middle|\, \text{for some } v \neq 0 \text{ in } \mathbf{R}^n,\ \lim_{t \to +\infty} \frac{1}{t} \ln \|X^*(\xi, t)^{-1}v\| < 0\right\} = 1$$

Now Proposition 2.1 of Ref. 13 shows that, if $(1.1)_\xi$ is globally null controllable, then $\mathrm{Lim}_{t \to +\infty}(1/t) \ln \|X^*(\xi, t)^{-1}v\| \geq 0$ for all $v \in \mathbf{R}^n$. Thus

$$m\left\{\xi \in E \,\middle|\, \lim_{t \to +\infty} \frac{1}{t} \ln \|X^*(\xi, t)^{-1}v\| \geq 0 \text{ for all } v \in \mathbf{R}^n,\ v \neq 0\right\} > 0$$

This contradicts the previous paragraph and completes the proof.

ACKNOWLEDGMENTS

The research of R.J. is supported by NSF Grant DMS 86-00199, and that of M.N. is supported by a Research Council Grant from Rutgers University.

The first author would like to thank Dr. R. M. Bianchini for valuable advice and conversations during the preparation of this paper.

REFERENCES

1. Z. Artstein, Uniform controllability via the limiting systems. *Appl. Math. Optimiz.* **9**, 111–131 (1982).
2. B. Barmish and W. Schimittendorf, Null controllability of linear systems with constrained controls. *SIAM J. Control Optimiz.* **18**(4), 327–345 (1980).
3. B. Barmish and W. Schimittendorf, A necessary and sufficient condition for local constrained controllability of linear systems. *IEEE Trans. Auto. Control* **25**(1), 97–100 (1980).
4. I. Blinov and E. Tonkov, Global controllability of conditionally periodic linear systems. *Trans. Mat. Zametki* **32** (1982).
5. R. Ellis, *Lectures on Topological Dynamics*, Benjamin, New York, 1969.
6. R. Johnson, Minimal functions with unbounded integral. *Israel J. Math.* **31**, 133–141 (1978).
7. R. Johnson, Ergodic theory and linear differential equations. *J. Diff. Eq.* **28**, 23–34 (1978).
8. R. Johnson and M. Nerurkar, Stabilization and linear regulator problem for linear nonautonomous control processes. Preprint.
9. R. Johnson, K. Palmer, and G. Sell, Ergodic theory of linear dynamical systems. *SIAM J. Math. Anal.* **18**, 1–33 (1987).
10. E. Lee and L. Marcus, *Foundations of Optimal Control Theory*, John Wiley and Sons, New York, 1967.
11. J. Maki and A. Strauss, *Introduction to Optimal Control Theory*, Springer-Verlag, Heidelberg, 1982.
12. V. Nemytski and V. Stepanov, *Qualitative Theory of Ordinary Differential Equations*, Princeton University Press, Princeton, N.J., 1960.
13. L. Pandolfi, Linear control systems. Controllability with constrained control. *J. Optimiz. Theory Appl.* **19** (1976).
14. R. Sacker and G. Sell, A spectral theory of linear differential equations. *J. Diff. Eq.* **27**, 320–358 (1978).
15. I. Ya. Schneiberg, Zeroes of integrals along trajectories of ergodic systems. *Trans. Funk. Anal. Priloyhenia* **19** (1985).
16. E. Tonkov, Dynamic system of translations and questions of uniform controllability of linear systems. *Sov. Math. Doklady* **23** (1981).