Journal of Dynamics and Differential Equations, Vol. 3, No. 4, 1991

Existence and Stability of Traveling Waves in Periodic Media Governed by a Bistable Nonlinearity

Xue Xin[1]

Received April 27, 1990

We prove the existence of multidimensional traveling wave solutions of the bistable reaction-diffusion equation with periodic coefficients under the condition that these coefficients are close to constants. In the case of one space dimension, we prove their asymptotic stability.

KEY WORDS: Bistable reaction-diffusion equation; periodic media; traveling waves; spectral theory; existence and stability.

1. INTRODUCTION

We consider the initial value problem of the bistable reaction-diffusion equation with periodic coefficients:

$$u_t = \sum_{i,j} (a_{ij}(x)\, u_{x_i})_{x_j} + \sum_i b_i(x)\, u_{x_i} + f(x, u) \tag{1.1}$$

The coefficients are assumed to be smooth, and 2π-periodic in each component of x, $x \in R^n$; $a(x) = (a_{ij}(x))$ is an $n \times n$ positive definite matrix uniformly in x, and $f(x, u)$ is a cubic bistable nonlinearity. Typically, $f(x, u) = u(1-u)(u-\mu)$, $\mu \in (0, 1/2)$. We are interested in the large time behavior of solutions of (1.1), in particular, the convergence of these solutions to a traveling wave solution as time tends to infinity.

An equation like (1.1) with constant coefficients was first studied in the classic paper by Kolmogorov, Petrovskii, and Piskunov (1937), where they consider

$$u_t = u_{xx} + f(u) \tag{1.2}$$

[1] Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, New York 10012.

1040-7294/91/1000-0541$06.50/0

with $f(u)=u(1-u)$, called KPP nonlinearity. In the constant coefficient problem, a traveling wave is a solution of the form $u=\varphi(x-ct)$, where φ satisfies

$$\begin{aligned} &\varphi''+c\varphi'+f(\varphi)=0 \\ &\varphi(-\infty)=0, \qquad \varphi(+\infty)=1 \end{aligned} \tag{1.3}$$

and c is a constant. Kolmogorov *et al.* (1937) established the existence of traveling waves and the convergence of solutions to a traveling wave solution with $c=2\sqrt{f'(0)}$, when the initial condition is the indicator function of the set $(0, +\infty)$. Since then, Kanel (1964), Fife and McLeod (1977), Aronson and Weinberger (1975), and others studied the large time behavior of solutions of (1.2) and its multidimensional analog for various nonlinearities $f(u)$ and various initial conditions. More recently, equations like (1.1) with periodic or random coefficients and KPP nonlinearity were investigated by Freidlin and Gartner (1979). Their results for the periodic case are as follows. For any $y \in R^n$, let

$$L_z=\sum_{i,j}(\partial_{x_i}-z_i)(a_{ij}(x)(\partial_{x_j}-z_j))+\sum_i b_i(x)(\partial_{x_i}-z_i)+f_u(x,0) \tag{1.4}$$

L_z is obtained from the right-hand side of Eq. (1.1) by substituting $\partial_{x_i}-z_i$ for ∂_{x_i} and $f_u(x,0)$ for $f(x,u)$. L_z is a linear strongly elliptic operator on T^n, the n-dimensional torus of size 2π, so L_z has a unique principal eigenvalue $\lambda=\lambda(z)$, differentiable in z. It is easy to show that λ is also convex in z. Thus it has a convex dual function $H=H(y)$ defined by

$$H(y)=\sup_{z\in R^n}((y,z)-\lambda(z))$$

where $y\in R^n$. Freidlin and Gartner (1979) showed that if the initial condition is nonnegative, continuous, compactly supported, but not identically zero, then for any given $y\in R^n$, the asymptotic behavior of u is described by $H=H(y)$ as follows.

$$\lim_{t\to\infty} u(t,ty)=\begin{cases}0 & \text{if} \quad H(y)>0\\ 1 & \text{if} \quad H(y)<0\end{cases}$$

In the above sense, the set $t\cdot\{y\in R^n \mid H(y)=0\}$ can be regarded as the wave front and the wave speed v in the unit direction $e\in R^n$ is obtained by solving the equation $H(ve)=0$. Their approach is based on applying the Feynman–Kac formula and limit theorems for large deviation probabilities. In the KPP case, it turns out that the wave speed can be determined independently of the wave shape, and the large deviation method is well suited to capturing the wave speed and neglecting the more delicate problem of the wave profile. In fact, the linearization of $f(x, u)$ at $u=0$ is enough to determine the wave speed as seen from the definition of L_z where only $f_u(x, 0)$ appears.

In the bistable case, however, the wave speed and the wave profile are usually coupled and have to be determined at the same time. On the other hand, the traveling waves, if they exist, are more stable than those in the KPP case. This is evident in the work of Gartner (1983) and Sattinger (1976) on related front propagation and stability problems. Their results imply that traveling waves are in general not stable in the KPP case unless stability is considered in a suitably weighted Banach space, whereas in the bistable case, stability holds without weight. We therefore look for traveling wave solutions of (1.1) which not only carry the usual features of traveling wave solutions of constant coefficient semilinear parabolic equations, but also take into account the effect of the nonuniform but periodic medium. Without loss of generality, we put $b_i(x)$ to zero and assume that $f(x, u)$ does not depend on x. We find that solutions of the form $u = U(k \cdot x - ct, x)$ serve our purpose, where $k \in R^n$, $x \in R^n$, c is a constant, and $U = U(s, y)$ is 2π-periodic in y, $U(-\infty, y) = 0$, $U(+\infty, y) = 1$. Indeed, substituting it into Eq. (1.1) gives

$$\begin{gathered}(\nabla_y + k\partial_s)(a(y)(\nabla_y + k\partial_s) U) + cU_s + f(U) = 0 \\ U(-\infty, y) = 0, \qquad U(+\infty, y) = 1, \qquad U(s, \cdot)\, 2\pi - \text{periodic}\end{gathered} \tag{1.5}$$

This is a degenerate elliptic equation on an infinite cylinder.

In Section 2, we prove the existence of solutions under the condition that the coefficients are close to constants. The idea is to write (U, c) as a perturbation of (φ, c_0), which is the solution of (1.3), i.e., $U = \varphi + \delta V$, $c = c_0 + \delta c_1$, and analyzing the equation satisfied by (V, c_1). It can be recast into the form $L_0 V = R(V, c_1, \delta)$, where L_0 is a linear degenerate elliptic operator with coefficients homogeneous in y, and R contains $(\nabla_y + k\partial_s)^2 V$, $\partial_s V$, $\nabla_y V$, c_1, and is nonlinear in V. A priori estimates for L_0 are obtained by using Fourier series in y and spectral theory of second-order ODEs on R^1. L_0 is shown to have a one-dimensional null space and a solvability condition must be satisfied to solve the inhomogeneous problem $L_0 u = f$. To remove the uncertainty in u due to the one-dimensional null space, a normalization condition is introduced. When the solvability condition is satisfied and normalization is done, L_0^{-1} gains one derivative in s, y, and two derivatives of the form $(\nabla_y + k\partial_s)^2$. Because of the degeneracy of the second derivative terms of L_0, the usual elliptic $W^{2,p}$ estimates are not available. Instead we have weaker parabolic-type estimates. This property is then used to set up the iteration scheme $L_0 V^{n+1} = R(V^n, c_1^n, \delta)$, with

$c_1^n = c_1^n(V^n)$ given by the solvability condition. The iteration scheme is shown to converge in a suitable Sobolev space by the contraction mapping principle. The normalization condition implies that the solution to equation (1.5) is unique up to a constant shift in the s variable.

In Section 3, we prove the asymptotic stability of traveling waves we constructed in Section 2, when space dimension is equal to 1. Nonlinear stability is proved in the sense that if the initial condition is a sufficiently small H^1 perturbation of the traveling wave profile, then the solution of (1.1) converges in H^1 to a shifted traveling wave as t goes to infinity, and the decay rate is exponential in time. The idea of the proof is that after changing to the moving front coordinate $\xi = x - ct$, the above traveling wave solution becomes a time periodic solution of (1.1), and the perturbation satisfies $v_t = L(t)v + N(v)$, where $L(t)$ is linear and periodic in t, and $N(v)$ is nonlinear. We analyze $L(t)$ using the Poincaré map and its spectrum. By L^2 integration methods and perturbation theory of the spectrum of bounded linear operators, we show that 1 is a simple eigenvalue of the Poincaré map, and the rest of the spectrum stays strictly inside the unit circle. This spectral property is then used to establish the linearization principle which implies the nonlinear asymptotic stability.

2. MULTI-DIMENSIONAL EXISTENCE OF TRAVELING WAVES

2.1. Introduction and Statement of Main Theorem

In this section, we consider existence of traveling waves, i.e., solutions of the following equation:

$$(\nabla_y + k\partial_s)(a(y)(\nabla_y + k\partial_s)U) + c\partial_s U + f(U) = 0 \tag{2.1}$$

satisfying the boundary condition: $U(-\infty, y) = 0$, $U(+\infty, y) = 1$, and $U(s, \cdot)$ 2π-periodic in y. Here $f(U) = U(1-U)(U-\mu)$, $\mu \in (0, 1/2)$, the typical bistable nonlinearity; $a(y)$ is a positive definite matrix, 2π-periodic in y, $y \in R^n$; $s \in R^1$ and k is any unit vector in R^n.

We are interested in the case when $a(y)$ is not far from a constant positive definite matrix. We assume that $a(y) = I + \delta a_1(y)$, where $a_1(y)$ is 2π-periodic in y and smooth, I is the identity matrix in R^n, and δ is taken to be small.

Let $\varphi = \varphi(s)$ be the classical traveling wave solution of

$$\varphi''_{ss} + c_0\varphi'_s + f(\varphi) = 0 \tag{2.2}$$

where $s \in (-\infty, +\infty)$, $\varphi(-\infty) = 0$, $\varphi(+\infty) = 1$. It is known that $\varphi_s > 0$, and $c_0 < 0$. Solution of (2.2) is unique up to a constant shift in s. To remove this translation invariance, we impose a normalization condition: $\varphi(0)=\theta$, $\theta \in (0, 1)$. A similar normalization condition is proposed for (2.1):

$$\frac{1}{(2\pi)^n}\int_{T_y^n} U(0, y)\,dy=\theta, \qquad \theta\in(0,1) \tag{2.3}$$

where T_y^n denotes the n-dimensional torus with size 2π.

We proceed to show existence as follows. Write $U=\varphi+\delta v$, $c=c_0+\delta c_1$, and substitute these expressions into (2.1):

$$\begin{aligned}(\nabla_y+k\partial_s)((I+\delta a_1)(\nabla_y+k\partial_s)(\varphi+\delta v))\\ +(c_0+\delta c_1)(\varphi+\delta v)_s+f(\varphi+\delta v)=0\end{aligned} \tag{2.4}$$

which is the same as

$$\begin{aligned}&\delta(\nabla_y+k\partial_s)(I+\delta a_1)(\nabla_y+k\partial_s)v+(\nabla_y+k\partial_s)(I+\delta a_1)(k\varphi_s)\\ &\quad+f(\varphi)+(c_0+\delta c_1)\,\varphi_s+\delta(c_0+\delta c_1)\,v_s\\ &\quad+\delta f'(\varphi)v+\tfrac{1}{2}\delta^2 f''(\varphi)\,v^2-\delta^3v^3=0\end{aligned} \tag{2.5}$$

Simplifying the above equation using (2.2), we get

$$\begin{aligned}&\delta(\nabla_y+k\partial_s)(I+\delta a_1)(\nabla_y+k\partial_s)v+\delta(\nabla_y+k\partial_s)\,a_1(k\varphi_s)+\delta c_1\varphi_s\\ &\quad+\delta(c_0+\delta c_1)\,v_s+\delta f'(\varphi)v+\tfrac{1}{2}\delta^2f''(\varphi)\,v^2-\delta^3v^3=0\end{aligned} \tag{2.6}$$

Canceling δ and letting $L_0=(\nabla_y+k\partial_s)^2+c_0\partial_s+f'(\varphi)$, we arrive at

$$\begin{aligned}L_0v=&-\delta(\nabla_y+k\partial_s)\,a_1(\nabla_y+k\partial_s)v-(\nabla_y+k\partial_s)\,a_1(k\varphi_s)\\ &+\delta^2v^3-c_1(\varphi_s+\delta v_s)-\tfrac{1}{2}\delta f''(\varphi)\,v^2\end{aligned} \tag{2.7}$$

Notice that the normalization condition is now reduced to

$$\int_{T_y^n} v(0, y)\,dy=0 \tag{2.8}$$

Our existence result is the following:

Theorem 2.1. *Consider problems (2.1)–(2.3) and (2.7)–(2.8) with $\theta=1/2$. Let*

$$X_t=\{v\in H^{t+1}(R_s^1\times T_y^n)\mid(\nabla_y+k\partial_s)^2\,v\in H^t(R_s^1\times T_y^n)\}$$

where k is a unit vector in R^n, $t\in Z^+$, and $t-[t/2]>(n+1)/2$. Then $\exists\delta_0=\delta_0(c_0, n, t)$, such that if δ is less than δ_0, there exist unique $v\in X_t$ and $c_1\in R$, *solutions of the problem (2.7)–(2.8), so that (U, c) given by $U = \varphi + \delta v$, and $c = c_0 + \delta c_1$, solves the problem (2.1) along with its boundary conditions. Moreover, if $(V(s, y), C)$ is another solution of (2.1), then $C = c$ and $V(s, y) = U(s - s_0, y)$ for some $s_0 \in R$.*

The proof of consists of four steps.

(1) Fourier decompose L_0 into ODEs on R^1 (beginning of Section 2.2).

(2) Use spectral theorem of second-order ODEs (Henry, 1981, pp. 136–142) and Fourier transform to study their invertibility in L^2 and high-order Sobelev spaces (Lemmas 2.1 to 2.3).

(3) Piece together the results on the ODEs in step 2 to get the invertibility of L_0 (Proposition 2.1).

(4) Set up the iteration scheme based on step 3 and show its convergence by the contraction mapping principle (Section 2.3).

2.2. Invertibility of L_0 and Related Estimates

Let $V = L^2(R_s \times T_y^n)$, $V_1 = \{h \in V \mid (h, \varphi_s) = 0\}$, where $(\cdot, \cdot)$ is the usual L^2 inner product. We are going to analyze

$$L_0 \psi = g, \qquad g \in V \tag{2.9}$$

where ψ satisfies the normalization condition:

$$\int_{T_y^n} \psi(0, y)\, dy = 0 \tag{2.10}$$

To do this, we expand ψ and g into Fourier series in y:

$$\begin{aligned} \psi(s, y) &= \sum_{m \in Z^n} \alpha_m(s)\, e^{im \cdot y} \\ \alpha_m(s) \in L^2(R), &\qquad \sum_{m \in Z^n} \|\alpha_m\|_2^2 < +\infty \\ g(s, y) &= \sum_{m \in Z^n} g_m(s)\, e^{im \cdot y} \\ g_m(s) \in L^2(R), &\qquad \sum_{m \in Z^n} \|g_m\|_2^2 < +\infty \end{aligned} \tag{2.11}$$

then $L_0 \psi = g$ is equivalent to the following ODEs indexed by m:

$$\alpha_m'' + (c_0 + 2(k \cdot m) i)\, \alpha_m' + (f'(\varphi) - |m|^2)\, \alpha_m = g_m \tag{2.12}$$

here prime means d/ds, and $|m| = \sqrt{(m \cdot m)}$.

When $m=0$, we have

$$\alpha_0'' + c_0\alpha_0' + f'(\varphi)\,\alpha_0 = g_0 \tag{2.13}$$

Let $N_0 = d_{ss} + c_0 d_s + f'(\varphi)$, $N_m = d_{ss} + (c_0 + 2(k\cdot m)i)\,d_s + (f'(\varphi) - |m|^2)$ for $m \neq 0$, where $d_s = d/ds$ and $d_{ss} = d^2/ds^2$, and consider N_0, N_m $(m \neq 0)$ on $L^2(R)$.

Differentiating (2.2), we see at once $\varphi_s \in \mathrm{Ker}(N_0)$. Notice that $f'(\varphi) \to -1+\mu$, as $s \to +\infty$; $f'(\varphi) \to -\mu$, as $s \to -\infty$. So $N_0\alpha_0 = 0$ is asymptotic to

$$\alpha_0'' + c_0\alpha_0' + (-1+\mu)\,\alpha_0 = 0, \qquad s \to +\infty \tag{2.14}$$

$$\alpha_0'' + c_0\alpha_0' - \mu\alpha_0 = 0, \qquad s \to -\infty \tag{2.15}$$

In both cases, there are two linearly independent solutions; one is exponentially decaying, and the other is exponentially growing. So $N_0\alpha_0 = 0$ has at most one nontrivial L^2 solution, and $\dim \mathrm{Ker}(N_0) = 1$, $\mathrm{Ker}(N_0) = \mathrm{span}\{\varphi_s\}$. It is easy to check that $e^{c_0 s}\varphi_s$ is in $\mathrm{Ker}(N_0^*)$, and thus $\dim \mathrm{Ker}(N_0^*) = 1$, $\mathrm{Ker}(N_0^*) = \mathrm{span}\{e^{c_0 s}\varphi_s\}$. By Fredholm alternative, (2.13) has L^2 solution if and only if

$$(g_0, e^{c_0 s}\varphi_s) = 0 \tag{2.16}$$

moreover, if we restrict ourselves to V_1, the solution is unique. Let α_{00} be such a solution, then

$$\|\alpha_{00}\|_{H^2} \leqslant M_0\, \|g_0\|_{L^2} \tag{2.17}$$

where M_0 depends only on φ.

The normalization condition now becomes

$$\alpha_0(0) = 0 \tag{2.18}$$

Since $\alpha_0(s) = \alpha_{00}(s) + \gamma\varphi_s(s)$, where γ is a constant, the normalization condition gives:

$$\gamma = -\alpha_{00}(0)/\varphi_s(0) \tag{2.19}$$

By Sobolev inequality, $|\alpha_{00}(0)| \leqslant M_1\, \|\alpha_{00}\|_{H^2}$, which implies, together with (2.17),

$$|\gamma| \leqslant M_2\, \|g_0\|_{L^2} \tag{2.20}$$

or:

$$\|\alpha_0\|_{H^2} \leqslant M_3\, \|g_0\|_{L^2} \tag{2.21}$$

where M_3 depends only on φ.

Now we turn to the invertibility of N_m. We use spectral theory of second-order ordinary differential operators. We state a definition of essential spectrum and a related theorem, proof of which is given by Henry (1981, pp. 136–142).

Definition 2.1. If L is a linear operator in a Banach space, a normal point of L is any complex number which is either in the resolvent set or an isolated eigenvalue of L of finite multiplicity. Any other number is in the essential spectrum.

Theorem 2.2. *Suppose $M(x)$, $N(x)$ are bounded real matrix functions, and D is a constant positive definite matrix; $M(x) \to M_\pm$, as $x \to \pm\infty$, and $N(x) \to N_\pm$, as $x \to \pm\infty$. In any spaces $L^p(R)$, $1 \leqslant p \leqslant \infty$, $C_0(R)$, $C_{\text{unif}}(R)$, define*

$$Lu(x) = -Du_{xx} + M(x)\,u_x + N(x)u, \qquad -\infty < x < +\infty$$

Let $S_\pm = \{\lambda \mid \det(\tau^2 D + i\tau M_\pm + N_\pm - \lambda I) = 0$, for some real $\tau \in (-\infty, +\infty)\}$, then the essential spectrum of L is contained in P, which is the union of the regions inside or on the curves $S_\pm$.

Remark 2.1. $S_\pm$ consist of finitely many algebraic curves, and they are asymptotically parabolas as τ becomes large.

To apply the above theorem, we write N_m into the equivalent real second-order system. Let $\alpha_m = \beta_m + i\gamma_m$, then the operator in matrix form is

$$\begin{pmatrix} \beta_m \\ \gamma_m \end{pmatrix}_{ss} + \begin{pmatrix} c_0 & -2(k\cdot m) \\ 2(k\cdot m) & c_0 \end{pmatrix} \cdot \begin{pmatrix} \beta_m \\ \gamma_m \end{pmatrix}_s + \begin{pmatrix} -|m|^2 + f'(\varphi) & 0 \\ 0 & -|m|^2 + f'(\varphi) \end{pmatrix} \cdot \begin{pmatrix} \beta_m \\ \gamma_m \end{pmatrix}$$

So

$$S_\pm = \{\lambda \mid \det(-\tau^2 I + i\tau M + N_\pm - \lambda I) = 0 \text{ for some real } \tau\}$$

where

$$M = \begin{pmatrix} c_0 & -2(k\cdot m) \\ 2(k\cdot m) & c_0 \end{pmatrix}$$

$$N_\pm = \begin{pmatrix} -|m|^2 + f'_\pm & 0 \\ 0 & -|m|^2 + f'_\pm \end{pmatrix}$$

and

$$f'_{\pm} = -1+\mu, \qquad f'_{-} = -\mu$$

It follows that

$$\det\begin{pmatrix} -\tau^2-\lambda-|m|^2+f'_{\pm}+i\tau c_0 & -2i(k\cdot m)\tau \\ 2i(k\cdot m)\tau & -\tau^2-\lambda-|m|^2+f'_{\pm}+i\tau c_0 \end{pmatrix}=0$$

which is

$$(\tau^2+\lambda+|m|^2-f'_{\pm}-i\tau c_0)^2-4(k\cdot m)^2\tau^2=0$$

or

$$\tau^2+\lambda_1+|m|^2-f'_{\pm}-i\tau c_0-2(k\cdot m)\tau=0$$
$$\tau^2+\lambda_2+|m|^2-f'_{\pm}-i\tau c_0+2(k\cdot m)\tau=0$$

Thus we get

$$\begin{aligned}\lambda_1 &= -|m|^2+f'_{\pm}-\tau^2+2(k\cdot m)\tau+ic_0\tau\\ \lambda_2 &= -|m|^2+f'_{\pm}-\tau^2-2(k\cdot m)\tau+ic_0\tau\end{aligned} \tag{2.22}$$

It is easy to see that both λ_1 and λ_2 live on parabolas. In case of λ_1, set $x=-\tau^2-|m|^2+2(k\cdot m)\tau+f'_{\pm}$, $y=c_0\tau$, then

$$\begin{aligned}x &= -y^2/c_0^2-|m|^2+2(k\cdot m)y/c_0+f'_{\pm}\\ &= -(y-c_0(k\cdot m))^2/c_0^2-|m|^2+(k\cdot m)^2+f'_{\pm}\end{aligned}$$

This is an equation of a parabola in the left half (x, y) plane, with the vertex being

$$x_0 = -|m|^2+(k\cdot m)^2+f'_{\pm}$$
$$y_0 = c_0(k\cdot m)$$

Notice $x_0 \leqslant f'_{\pm} \leqslant \max(\mu-1, -\mu) = -\mu < 0$, so λ_1 is strictly inside the left half plane. Similarly, λ_2 stays strictly inside the left half plane. Therefore, 0 is not in the essential spectrum of N_m. It is either in the resolvent set or an isolated eigenvalue of finite multiplicity.

Suppose 0 is an isolated eigenvalue, then there exists $u \in L^2$ such that

$$u''+(c_0+2(k\cdot m)i)u'+(f'(\varphi)-|m|^2)u=0$$

which is asymptotic to

$$u'' + (c_0 + 2(k \cdot m)i)u' + (f'_{\pm} - |m|^2)u = 0$$

as $s \to \pm\infty$. At s near $-\infty$, $u = O(e^{\eta s})$ where η is

$$\eta = -c_0/2 - (k \cdot m)i \pm 1/2 \sqrt{c_0^2 - 4f'_{\pm} + 4(k \cdot m)\, c_0 i + 4\,|m|^2 - 4(k \cdot m)^2}$$

Therefore,

$$\operatorname{Re} \eta = -c_0/2 \pm 1/2 \operatorname{Re} \sqrt{c_0^2 - 4f'_{-} + 4(k \cdot m)\, c_0 i + 4\,|m|^2 - 4(k \cdot m)^2}$$

By the inequality

$$|\operatorname{Re} \sqrt{z}| \geqslant \sqrt{\operatorname{Re} z}, \qquad \text{if} \quad \operatorname{Re} z \geqslant 0$$

and the fact that

$$c_0^2 - 4f'_{-} + 4\,|m|^2 - 4(k \cdot m)^2 \geqslant c_0^2 - 4f'_{-} > c_0^2$$

we see that $\operatorname{Re} \eta$ is dominated by the second term in the sum. So one characteristic root has negative real part; the other has positive real part. Then at $-\infty$, u behaves like $O(e^{s \operatorname{Re} \eta_+})$, where η_+ is the characteristic root with positive real part. From the above formula for $\operatorname{Re} \eta$, we have $\operatorname{Re} \eta_+ + c_0/2 > 0$. Hence, if $u \in L^2$, then $v = e^{1/2(c_0 + 2(k \cdot m)i)s} u \in L^2$, in view of c_0 being negative. However, v satisfies

$$v'' + (f'(\varphi) - c_0^2/4 + (k \cdot m)\, c_0 i - /m/^2 + (k \cdot m)^2)v = 0 \tag{2.23}$$

or

$$v'' + (f'(\varphi) - c_0^2/4 - |m|^2 + (k \cdot m)^2)v = -i(k \cdot m)\, c_0 v \tag{2.24}$$

Since the operator in the left-hand side is self-adjoint on $L^2(R)$, its spectrum is real, so $(k \cdot m) = 0$. This implies that

$$v'' + (f'(\varphi) - c_0^2/4 - |m|^2)v = 0$$

or equivalently for u

$$u'' + c_0 u' + f'(\varphi)u = |m|^2 u$$

from the known fact that operator $d_{ss} + c_0 d_s + f'(\varphi)$ does not have eigenvalue in the right half-plane, we get a contradiction. Thus 0 is in the resolvent set of N_m, and N_m's are all invertible on $L^2(R)$.

Lemma 2.1. *For any $m \neq 0$, N_m has a bounded inverse on $L^2(R)$.*

In order to derive more properties of N_m's, we study their associated constant coefficient differential operators.

Let $N_{m,0} = (d^2/ds^2) + (c_0 + 2(k \cdot m)i)(d/ds) - |m|^2$, $m \neq 0$, and consider the problem

$$N_{m,0} u = g \qquad \text{on } L^2(R) \tag{2.25}$$

After transform the above equation becomes

$$(-\xi^2 + (c_0 + 2i(k \cdot m))\, i\xi - |m|^2)\hat{u} = \hat{g}, \qquad \xi \in R$$

which is

$$(-\xi^2 - 2(k \cdot m)\xi - |m|^2 + ic_0\xi)\hat{u} = \hat{g}$$

Letting $S(\xi) = -\xi^2 - 2(k \cdot m)\xi - |m|^2 + ic_0\xi$, then we have the following.

Lemma 2.2. *There exists constant $M_4 = M_4(c_0)$, such that*

$$|S(\xi)|^2 \geqslant M_4(\xi^2 + |m|^2)$$

Proof. Since $|S(\xi)|^4 = |\xi k + m|^4 + c_0^2\xi^2$, we only need to show that $|S(\xi)|^2 \geqslant \text{const} \cdot |m|^2$, where the constant is independent of m.

If $|\xi| \leqslant |m|/2$, then

$$|S(\xi)|^2 \geqslant (|m| - |k\xi|)^4 + c_0^2\xi^2 \geqslant (|m|/2)^4 + c_0^2\xi^2 \geqslant |m|^4/16$$

If $|\xi| > |m|/2$, then $|S(\xi)|^2 \geqslant c_0^2\xi^2 \geqslant c_0^2\,|m|^2/4$. Combining the above inequalities, we prove the lemma. ▮

Corollary 2.1. *If $N_{m,0}u = g$, then there exists $M_5 = M_5(c_0)$ such that*

$$\|u\|_{L^2} \leqslant \frac{M_5}{|m|}\,\|g\|_{L^2}$$

$$\|u\|_{H^1} \leqslant M_5\,\|g\|_{L^2}$$

Proof. It follows from Lemma 2.2 that

$$\hat{u} = \hat{g}/S(\xi)$$

and

$$\|\hat{u}\|_{L^2} \leqslant \frac{1}{|m|}\sqrt{\frac{1}{M_4}}$$

which implies the first inequality. Also,

$$\|u\|_{H^1}^2 = \|(1+\xi^2)\,\hat{u}^2\|_{L^2} \leqslant \frac{1}{M_4}\,\|\hat{g}\|_{L^2}^2$$

from which the second inequality follows. ∎

Lemma 2.3. *There exists constant M_7 depending only on c_0 such that*

$$\|N_m^{-1}g\|_{L^2} \leqslant \frac{M_7}{|m|}\,\|g\|_{L^2}$$

$$\|N_m^{-1}g\|_{H^2} \leqslant M_7\,\|g\|_{L^2}$$

Proof. Let us consider

$$N_m u = g, \qquad \text{i.e.,}$$

$$N_{m,0}u + f'(\varphi)u = g$$

or

$$N_{m,0}u = g - f'(\varphi)u$$

By Corollary 2.1, we have

$$\|u\|_{L^2} \leqslant \frac{M_5}{|m|}\,(\|g\|_{L^2} + \|f'\|_{L^\infty}\,\|u\|_{L^2})$$

If $|m| \geqslant 2M_5\,\|f'\|_{L^\infty}$, then

$$1/2\,\|u\|_{L^2} \leqslant \left(1 - \frac{M_5\,\|f'\|_{L^\infty}}{|m|}\right)\|u\|_{L^2} \leqslant \frac{M_5}{|m|}\,\|g\|_{L^2}$$

or

$$\|u\|_{L^2} \leqslant \frac{2M_5}{|m|}\,\|g\|_{L^2}$$

However, there are only finitely many m's that satisfy $|m| \leqslant 2M_5\,\|f'\|_{L^\infty}$, for each of them we have from Lemma 2.1 that

$$\|u\|_{L^2} \leqslant \tilde{M}_m\,\|g\|_{L^2}$$

where $\tilde{M}_m$ is the bound of N_m^{-1} as given in Lemma 2.1. Let $n_0 = 2M_5\,\|f'\|_{L^\infty}$, and $M_6 = \max(\max_{1 \leqslant |m| \leqslant n_0}(\tilde{M}_m\,|m|), 2M_5)$, then

$$\|u\|_{L^2} \leqslant \frac{M_6}{|m|}\,\|g\|_{L^2}$$

where $M_6 = M_6(c_0)$. In other words,

$$\|N_m^{-1}\|_{L^2} \leqslant \frac{M_6}{|m|} \tag{2.26}$$

From $N_{m,0}u = g - f'(\varphi)u$, Corollary 2.1, and the above inequality, we obtain

$$\begin{aligned}\|u\|_{H^1} &\leqslant M_5(\|g\|_{L^2} + \|f'\|_{L^\infty} \|u\|_{L^2}) \\ &\leqslant M_5\left(\|g\|_{L^2} + \frac{1}{|m|}\|f'\|_{L^\infty} M_6 \|g\|_{L^2}\right) \\ &\leqslant M_5(1 + \|f'\|_{L^\infty} M_6)\|g\|_{L^2}\end{aligned}$$

therefore, $\|N_m^{-1}g\|_{H^2} \leqslant M_7 \|g\|_{L^2}$, where $M_7 = \max(M_6, M_5(1 + \|f'\|_{L^\infty} M_6))$.

Corollary 2.2. *If $N_m u = g$, then there exists constant $M_8 = M_8(c_0)$ such that*

$$\|u\|_{H^1} \leqslant M_8 \|g\|_{L^2} \tag{2.27}$$

$$\|(kd_s + im)u\|_{L^2} \leqslant M_8 \|g\|_{L^2} \tag{2.28}$$

$$\|(kd_s + im)^2 u\|_{L^2} \leqslant M_8 \|g\|_{L^2} \tag{2.29}$$

Proof. The first two inequalities are direct consequences of Lemma 2.3, while the last one can be seen by writing the equation $N_m u = g$ as

$$(kd_s + im)^2 u = g - c_0 u_s - f'(\varphi)u$$

and taking the L^2 norm of both sides. ∎

Corollary 2.3. *If $N_m u = g$, and $g \in H^t$, then $\exists M_9 = M_9(c_0, t)$ such that*

$$\|u\|_{H^{t+1}} \leqslant M_9 \|g\|_{H^t} \tag{2.30}$$

$$\|(kd_s + im)u\|_{H^t} \leqslant M_9 \|g\|_{H^t} \tag{2.31}$$

$$\|(kd_s + im)^2 u\|_{H^t} \leqslant M_9 \|g\|_{H^t} \tag{2.32}$$

where $t \in Z^+$.

Proof. Notice $N_m u = N_{m,0}u + f'(\varphi)u = g$, and that $D^j = (d_s)^j$ commute with $N_{m,0}$, so if we apply D^j to the above equation, we get

$$N_{m,0}D^j u + \sum_{0 \leqslant l \leqslant j-1} C_j^l D^{j-l}(f'(\varphi))\, D^l u + f'(\varphi)\, D^j u = D^j g$$

or

$$N_m D^j u + \sum_{0 \leqslant l \leqslant j-1} C_j^l D^{j-l}(f'(\varphi))\, D^l u = D^j g$$

By induction on j, Lemma 2.3 and Corollary 2.2, we see that the expected inequalities hold. ▌

Proposition 2.1. *Consider problem*

$$L_0 u = g \qquad \text{on } L^2(R_s^1 \times T_y^n)$$

where $L_0 = (\nabla_y + k\partial_s)^2 + c_0\partial_s + f'(\varphi)$, *and* g *satisfies*

$$(g, e^{c_0 s}\varphi_s) = 0$$

moreover, u *satisfies the normalization condition:*

$$\int_{T_y^n} u(0, y)\, dy = 0$$

then there exists constant $M_{10} = M_{10}(c_0, t)$ *such that*

$$\|u\|_{H^{t+1}(R_s^1 \times T_y^n)} \leqslant M_{10}\, \|g\|_{H^t(R_s^1 \times T_y^n)} \tag{2.33}$$

$$\|(\nabla_y + k\partial_s)^2\, u\|_{H^t(R_s^1 \times T_y^n)} \leqslant M_{10}\, \|g\|_{H^t(R_s^1 \times T_y^n)} \tag{2.34}$$

where $t \in Z^+$.

Proof. Write u and g in terms of Fourier series of y

$$u = \sum_{m \in Z^n} u_m e^{im \cdot y}$$

$$g = \sum_{m \in Z^n} g_m e^{im \cdot y}$$

where u_m, $g_m \in L^2(R)$, and $\sum_{m \in Z^n} (u_m^2 + g_m^2) < +\infty$.
By analysis in (2.13)–(2.21), we have

$$\|u_0\|_{H^{t+2}(R)} \leqslant M_{10}(c_0, t)\, \|g_0\|_{H^t(R)}$$

From Corollary 2.3, it follows

$$\|u_m\|_{H^{t+1}(R)} \leqslant M_{10}(c_0, t)\, \|g_m\|_{H^t(R)}$$

$$\|mu_m\|_{H^t(R)} \leqslant M_{10}(c_0, t)\, \|g_m\|_{H^t(R)}$$

$$\|(kd_s + im(^2\, u_m\|_{H^t(R)} \leqslant M_{10}(c_0, t)\, \|g_m\|_{H^t(R)}$$

for $m \neq 0$.

Combining the above inequalities, we get

$$\begin{aligned}
\|u\|_{H^{t+1}(R_s^1\times T_y^n)} &\leqslant \sum_{m\in Z^n}\ \sum_{0\leqslant l\leqslant t+1} |m|^l\, \|u_m^{(t+1-l)}\|_{L^2(R)}\\
&\leqslant M_{10}\sum_{m\in Z^n}\|g_m\|_{H^t(R)}+\sum_{m\in Z^n}\ \sum_{1\leqslant l\leqslant t+1}|m|^l\,\|u_m^{(t+1-l)}\|_{L^2(R)}\\
&\leqslant M_{10}\sum_{m\in Z^n}\left(\|g_m\|_{H^t(R)}+\sum_{0\leqslant l\leqslant t}|m|^{l+1}\,\|u_m^{(t-l)}\|_{L^2(R)}\right)\\
&\leqslant M_{10}\sum_{m\in Z^n}\left(\|g_m\|_{H^t(R)}+\sum_{0\leqslant l\leqslant t}|m|^{l}\,\|g_m^{(t-l)}\|_{L^2(R)}\right)\\
&\leqslant M_{10}\,\|g\|_{H^t(R_s^1\times T_y^n)}
\end{aligned}$$

Similarly, we establish the second inequality. ∎

We end this section with a Sobolev imbedding lemma.

Lemma 2.4. *Let $u\in H^t(R_s^1\times T_y^n)$, and $t-[t/2]>(n+1)/2$, then there exists constant $M=M(t)$ such that*

$$\|u^2\|_{H^t}\leqslant M\,\|u\|_{H^t}^2$$
$$\|u^3\|_{H^t}\leqslant M\,\|u\|_{H^t}^3$$

Assume that $t\in Z^+$.

Proof. It suffices to show that $\|D^tu^2\|_{L^2}\leqslant M\,\|u\|_{H^t}^2$, where D is ∂_s or ∂_{y_j}. The mixed derivatives can be treated similarly. Since

$$D^t(u^2)=\sum_{0\leqslant l\leqslant t} C_t^l D^l u D^{(t-l)}u$$

If $t-[t/2]>(n+1)/2$, then $H^t(R_s^1\times T_y^n)$ can be continuously imbedded into $C^{[t/2]}(R_s^1\times T_y^n)$, which is

$$\|D^{(l)}u\|_{L^\infty}\leqslant M\,\|u\|_{H^t}$$

for $0\leqslant l\leqslant [t/2]$. Now taking the L^2 norm of both sides of the identity of differentiation, and using the imbedding inequality, we arrive at our conclusion. In the same manner, one can show the inequality for u^3. ∎

2.3. Iteration Scheme and Its Convergence

Let us consider equation (2.7), that is

$$\begin{aligned}
L_0v=&-\delta(\nabla_y+k\partial_s)\,a_1(\nabla_y+k\partial_s)v-(\nabla_y+k\partial_s)\,a_1(k\varphi_s)\\
&-\delta^2v^3-c_1(\varphi_s+\delta v_s)-\tfrac{1}{2}\delta f''(\varphi)\,v^2
\end{aligned}$$

along with the normalization condition:

$$\int_{T_y^n} v(0, y)\, dy = 0$$

The related iteration scheme is

$$L_0 v_{n+1} = -\delta(\nabla_y + k\partial_s)\, a_1(\nabla_y + k\partial_s)\, v_n - (\nabla_y + k\partial_s)\, a_1(k\varphi_s) - c_1^{n+1}(\varphi_s + \delta v_{n,s}) - \tfrac{1}{2}\delta f''(\varphi)\, v_n^2 + \delta^2 v_n^3 \tag{2.35}$$

and

$$c_1^{n+1} = \frac{1}{(\varphi_s + \delta v_{n,s}, \psi)} (F(v_n), \psi) \tag{2.36}$$

where

$$F(v) = -\delta(\nabla_y + k\partial_s)\, a_1(\nabla_y + k\partial_s) v - (\nabla_y + k\partial_s)\, a_1(k\varphi_s) - \tfrac{1}{2}\delta f''(\varphi)\, v^2 + \delta^2 v^3 \tag{2.37}$$

and

$$\psi = e^{c_0 s}\varphi_s \tag{2.38}$$

Moreover, v_n's satisfy the normalization condition.

Let

$$X_t = \{v \in H^{t+1}(R_s^1 \times T_y^n) \mid (\nabla_y + k\partial_s)^2\, v \in H^t(R_s^1 \times T_y^n)\}$$

equipped with norm:

$$\|v\|_{X_t} = (\|v\|_{H^{t+1}}^2 + \|(\nabla_y + k\partial_s)^2\, v\|_{H^t}^2)^{1/2}$$

X_t is a Hilbert space with above norm. Notice that in the iteration, $c_1^{n+1} = c_1^{n+1}(v_n)$, and $v_{n+1} = v_{n+1}(c_1^{n+1}, v_n) = v_{n+1}(v_n)$. By Proposition 2.1 in Section 2.2, we see that the mapping $T: v_n \to v_{n+1}$ is a mapping from X_t to X_t if $t - [t/2] > (n+1)/2$, since each term in the right-hand side of (2.35) is in H^t and thus v_{n+1} belongs to X_t.

Let M be the maximum of all constants in all our previous estimates ($M = M(c_0, n, t)$), then Proposition 2.1 yields

$$\|v_{n+1}\|_{X_t} \leqslant M(\delta\, \|v_n\|_{X_t} + \delta^2\, \|v_n\|_{X_t}^3 + 1 + |c_1^{n+1}| + \delta\, |c_1^{n+1}|\, \|v_n\|_{X_t} + \delta\, \|v_n\|_{X_t}^2) \tag{2.39}$$

$$|c_1^{n+1}| \leqslant \frac{M(1 + \delta\, \|v_n\|_{X_t} + \delta\, \|v_n\|_{X_t}^2 + \delta^2\, \|v_n\|_{X_t}^3)}{(\varphi_s, \psi) - \delta\, \|v_n\|_{X_t} \cdot \|\psi\|_{L^2}} \tag{2.40}$$

Suppose $\|v_n\|_{X_t} \leqslant R$, then (2.39) and (2.40) give

$$|c_1^{n+1}| \leqslant \frac{M(1+\delta R+\delta R^2+\delta^2 R^3)}{1-\delta MR} \tag{2.41}$$

$$\begin{aligned}\|v_{n+1}\|_{X_t} \leqslant{} & \delta MR+\delta MR^2+\delta^2 MR^3+M \\ & + (M+\delta R)\frac{M(1+\delta R+\delta R^2+\delta^2 R^3)}{1-\delta MR}\end{aligned} \tag{2.42}$$

Choose $R=2(M^2+M)$, then there exists a $\delta_0=\delta_0(M)$, suitably small such that when $\delta<\delta_0$, the right-hand side of (2.42) is less than $R=2(M^2+M)$; and $\delta_0 M(M^2+M)<1/4$. As a result, the mapping $T\colon v_n \to v_{n+1}$ is from $B_{X_t}(0, R)$ to itself, where $B_{X_t}(0, R)$ denotes a ball of radius R with center at 0 in our iteration space X_t. Moreover, (2.41) says that $\{c_1^n\}$ is a bounded sequence. From now on, we use M for all possible positive constants appearing in our analysis which depend only on c_0, n, and t.

Consider $v_{n+1}=Tv_n$, and $v_{n+2}=Tv_{n+1}$, then

$$\begin{aligned}L_0(v_{n+2}-v_{n+1}) ={} & -\delta(\nabla_y+k\partial_s)\, a_1(\nabla_y+k\partial_s)(v_{n+1}-v_n) \\ & -\delta c_1^{n+2} v_{n+1,s}+\delta c_1^{n+1} v_{n,s}-(c_1^{n+2}-c_1^{n+1})\,\varphi_s \\ & -\tfrac{1}{2}\delta f''(\varphi)(v_{n+1}^2-v_n^2)+\delta^2(v_{n+1}^3-v_n^3)\end{aligned}$$

Taking the H^t-norm of both sides and using Schwarz inequalities:

$$\begin{aligned}\|v_{n+1}^2-v_n^2\|_{H^t} &\leqslant C_1(t)\, \|v_{n+1}-v_n\|_{H^t}\cdot \|v_{n+1}+v_n\|_{H^t} \\ &\leqslant 2RC_1(t)\, \|v_{n+1}-v_n\|_{H^t} \\ &\leqslant M\, \|v_{n+1}-v_n\|_{H^t}\end{aligned} \tag{2.43}$$

and

$$\|v_{n+1}^3-v_n^3\|_{H^t} \leqslant M\, \|v_{n+1}-v_n\|_{H^t}$$

we have

$$\begin{aligned}\|L_0(v_{n+2}-v_{n+1})\|_{H^t} \leqslant{} & \|-\delta(\nabla_y+k\partial_s)\, a_1(\nabla_y+k\partial_s)(v_{n+1}-v_n)\|_{H^t} \\ & + \|-\tfrac{1}{2}\delta f''(\varphi)(v_{n+1}^2-v_n^2)+\delta^2(v_{n+1}^3-v_n^3)\|_{H^t} \\ & + \|-\delta c_1^{n+2} v_{n+1,s}+\delta c_1^{n+1} v_{n,s}-(c_1^{n+2}-c_1^{n+1})\,\varphi_s\|_{H^t} \\ \leqslant{} & \delta M\, \|v_{n+1}-v_n\|_{X^t}+M\, |c_1^{n+2}-c_1^{n+1}| \\ & + \|\delta c_1^{n+1}(v_{n+1,s}-v_{n,s})+\delta(-c_1^{n+1}+c_1^{n+2})\, v_{n+1,s}\|_{H^t}\end{aligned} \tag{2.44}$$

Notice

$$c_1^{n+2}-c_1^{n+1}=\frac{1}{(\varphi_s,\psi)+\delta(v_{n+1},\psi)}F(v_{n+1})-\frac{1}{(\varphi_s,\psi)+\delta(v_n,\psi)}F(v_n)$$
$$=\left(\frac{1}{(\varphi_s,\psi)+\delta(v_{n+1},\psi)}-\frac{1}{(\varphi_s,\psi)+\delta(v_n,\psi)}\right)F(v_{n+1})$$
$$+\frac{1}{(\varphi_s,\psi)+\delta(v_n,\psi)}(F(v_{n+1})-F(v_n))$$

It is easy to see that

$$|F(v_{n+1})|\leqslant M$$
$$\left|\frac{1}{(\varphi_s,\psi)+\delta(v_{n+1},\psi)}\right|\leqslant M$$
$$\left|\frac{1}{(\varphi_s,\psi)+\delta(v_n,\psi)}\right|\leqslant M$$

and that

$$|F(v_{n+1})-F(v_n)|\leqslant \delta M\,\|v_{n+1}-v_n\|_{X_t}$$

Therefore by the boundedness of $\{c_1^n\}$ sequence and (2.44), we have

$$|c_1^{n+1}-c_1^n|\leqslant \delta M\,\|v_{n+1}-v_n\|_{X_t} \tag{2.45}$$

and thus

$$\|L_0(v_{n+2}-v_{n+1})\|_{H^t}\leqslant \delta M\,\|v_{n+1}-v_n\|_{X_t}$$

or

$$\|v_{n+2}-v_{n+1}\|_{X_t}\leqslant \delta M\,\|v_{n+1}-v_n\|_{X_t} \tag{2.46}$$

So if $\delta<\delta_0=1/M$, then T is a contraction mapping, it has a unique fixed point in X_t. When the v_n sequence converges, (2.36) shows the corresponding convergence of c_1^n sequence. As a result, we obtain the unique solution (v, c), being limits of the sequence (v_n, c_1^n), to the problem (2.7)–(2.8).

Without loss of generality, we can take θ in (2.3) to be 1/2. Suppose $V=V(s, y)$ is any other solution of problem (2.1), we can always shift its s variable by s_0 such that the normalization condition is satisfied by $V(s+s_0, y)$, which solves the same (2.1) that $V(s, y)$ satisfies. By the uniqueness of contraction mapping, $V(s+s_0, y)$ is just the solution we constructed by iteration, therefore $V(s, y)=U(s-s_0, y)$. In other words, if δ is less than δ_0, the problem (2.1) has a unique solution up to a constant shift in the s variable; especially the traveling speed c is unique. This completes the proof of the theorem.

3. ONE-DIMENSIONAL STABILITY OF TRAVELING WAVES

3.1. Introduction and Statement of Main Theorem

In this section, we study the stability of the traveling wave solutions that we constructed in the last section when space dimension is equal to 1. Consider the following reaction-diffusion equation:

$$\begin{aligned} u_t &= (a(x)\, u_x)_x + f(u) \\ u\,|_{t=0} &= U(x, x) + u_0(x) \end{aligned} \tag{3.1}$$

where $a(x)=1+\delta a_1(x)$, $a_1(x)$ is a smooth 2π-periodic function, $f(u)=u(1-u)(u-\mu)$, with $\mu \in (0, 1/2)$, and $U=U(s, y)$ is the solution of the following equation:

$$\begin{gathered} (\partial_s+\partial_y)(a(y)(\partial_s+\partial_y)\, U)+cU_s+f(U)=0 \\ U(+\infty, y)=1, \qquad U(-\infty, y)=0, \qquad U(s, \cdot)\ 2\pi\text{-periodic} \end{gathered} \tag{3.2}$$

Writing $u=u(t, x)$ as a perturbation of $U=U(x-ct, x)$:

$$u=u(x, t)=U(x-ct, x)+v(x, t) \tag{3.3}$$

and substituting into (3.1) gives

$$\begin{aligned} v_t &= (a(x)\, v_x)_x + f(U+v) - f(U) \\ v\,|_{t=0} &= u_0(x) \end{aligned} \tag{3.4}$$

Changing to moving coordinate $(\xi, t)=(x-ct, t)$, one gets from equation (3.1):

$$\begin{aligned} u_t &= (a(\xi+ct)\, u_\xi)_\xi + cu_\xi + f(u) \\ u\,|_{t=0} &= U(\xi, \xi) + u_0(\xi) \end{aligned} \tag{3.5}$$

Equation (3.4) becomes under (ξ, t):

$$\begin{aligned} v_t &= (a(\xi+ct)\, v_\xi)_\xi + cv_\xi + f(U+v) - f(U) \\ v\,|_{t=0} &= u_0(\xi) \end{aligned} \tag{3.6}$$

Let

$$L(t)v=(a(\xi+ct)\,v_\xi)_\xi+cv_\xi+f'(U)v$$
$$R(t,v)=f(U+v)-f(U)-f'(U)v$$

Equation (3.6) can then be put into

$$v_t=L(t)v+R(t,v) \tag{3.7}$$

Notice that L as an operator on $v=v(\xi,t)$ is time dependent. However, due to the fact that $a=a(\xi+ct)$ and $U=U(x-ct,x)=U(\xi,\xi+ct)$ are all periodic functions in t with period $2\pi/c$, $L=L(t)$ is periodic in t with period being $2\pi/c$.

Applying operator $d=c\partial_\xi-\partial_t$ to Eq. (3.5), in view of the property that ∂_ξ and ∂_t commute with d, and $d(a(\xi+ct))=0$, we have

$$(du)_t=(a(\xi+ct)(du)_\xi)_\xi+c(du)_\xi+f'(u)(du) \tag{3.8}$$

Especially if we make $u=U(\xi,\xi+ct)$, then

$$d(U)=d(U(\xi,\xi+ct))=cU_s+cU_y-cU_y=cU_s$$

or $U_s=U_s(\xi,\xi+ct)$ is a time periodic solution of

$$v_t=Lv=L(t)v \tag{3.9}$$

where $L(t+2\pi/c)=L(t)$.

We see that the time evolution of v in (3.9) is governed by the spectrum of Poincaré return map and 1 is its eigenvalue. We show that 1 is a simple eigenvalue and all the other points in the spectrum are bounded in absolute value by 1. This spectral property is shown to imply the asymptotic stability of the traveling waves.

Our stability result is

Theorem 3.1. *Consider Eq. (3.1) with initial data* $u(x,0)=U(x,x)+\varepsilon u_0(x)$, *or equivalently Eq. (3.5) with initial data* $U(\xi,\xi)+\varepsilon u_0(\xi)$. *Assume that* $u_0(\xi)\in H^1(R)$, $U=U(s,y)\in C^3(R_s\times T_y)$, $U_s\in H^3(R_s\times T_y)$, *then if* ε *is sufficiently small, there exist a function* $\gamma=\gamma(\varepsilon)$, *and a constant* $K=K(\rho_0)$, $\rho_0\in(0,1)$, *such that*

$$\|u(\xi,t)-U(\xi+\gamma(\varepsilon),\xi+ct)\|_{H^1}\leqslant K\rho_0^t,\qquad \forall t\geqslant 0 \tag{3.10}$$

where $\gamma(\varepsilon)=\varepsilon h(\varepsilon)$, h *is* C^1 *in* ε, *and has a finite limit as* $\varepsilon\to 0$, *i.e.,* $h(0)=(u_0,e_0^*)$, *here* e_0^* *is the initial value of the unique time periodic solu-*

tion of $-v_t = L^* v$, *with* $(v, U_s(\xi, \xi + ct)) = 1$, *and* L^* *is the adjoint of* L *in* $L^2(R)$.

Remark 3.1. The assumptions on U and U_s in Theorem 3.1 are satisfied when t in Theorem 2.1 is chosen large enough.

The proof consists of two steps.

(1) Spectral analysis of the Poincaré map of operator $L(t)$ (Section 3.2). Using L^2 integration methods and perturbation theory of the spectrum of bounded linear operators, we show that the Poincaré map associated with L has its spectrum strictly inside the unit circle except 1, which is a simple eigenvalue (Lemma 3.1 and Corollary 3.1).

(2) Following Sattinger (1976), we use the spectral properties of the Poincaré map to establish the nonlinear asymptotic stability (Section 3.3).

3.2. Spectral Properties of Poincaré Return Map

Let $p = 2\pi/c$, and consider problem (3.9).

Definition 3.1. The Poincaré map is

$$U: v(\xi, 0) \to v(\xi, p)$$

where $v(\xi, t)$ is the solution of (3.9) with initial condition $v(\xi, 0)$. It is easy to see that U is a bounded map from $L^2(R)$ to itself by the standard parabolic estimates. In our problem, $U = U(\delta)$. We are interested in analyzing its spectrum when δ is small. To do this, let us write (3.9) in perturbation form adopting the notation of Section 2:

$$\begin{aligned} v_t &= L_0 v + \delta L_1 v \\ v\,|_{t=0} &= v_0 \in L^2(R) \end{aligned} \tag{3.11}$$

where

$$L_0 v = v_{\xi\xi} + c_0 v_\xi + f'(\varphi) v \tag{3.12}$$

$$L_1 v = (a_1(\xi + ct)\, v_\xi)_\xi + c_1 v_\xi + (f''(\varphi)\, \varphi_1 - 3\delta\varphi_1^2) v \tag{3.13}$$

recalling that $U = \varphi + \delta\varphi_1$, $c = c_0 + \delta c_1$. Here to avoid abuse of notation, we replace $v = v(s, y)$ in Theorem 2.1 by φ_1. So in the above equation, $\varphi_1 = \varphi_1(\xi, \xi + ct)$.

Define $Bv = (f''(\varphi)\, \varphi_1 - 3\delta\varphi_1^2) v$, then

$$L_1 v = (a_1(\xi + ct)\, v_\xi)_\xi + c_1 v_\xi + Bv \tag{3.14}$$

It is clear that B is a bounded operator from $L^2(R)$ to itself.

The spectrum of L_0 lies in $\{\lambda \mid \operatorname{Re}\lambda \leqslant -\mu\}$ except 0, which is a simple eigenvalue. This picture of L_0's spectrum can be obtained by applying the spectral theory of second-order ordinary differential operators, in particular, Theorem 2.2 in Section 2.2. As a result, the Poincaré map U_0 for the problem

$$v_t = L_0 v \tag{3.15}$$

has spectrum strictly inside the unit circle except 1, which is a simple eigenvalue with the corresponding eigenfunction being φ_s.

Let us consider $v_\lambda = e^{-\lambda t}v$, where v is the solution of (3.15) and $\lambda > 0$. Then v_λ satisfies (3.15) with L_0 replaced by $L_{0,\lambda} \equiv L_0 - \lambda$. Similarly, if v solves (3.9), then v_λ solves (3.9) with $L_{0,\lambda}$ in the place of L_0. Let U_λ and $U_{0,\lambda}$ be the return maps for $L_\lambda \equiv L_{0,\lambda} + \delta L_1$ and $L_{0,\lambda}$, respectively, then it is clear that $U_\lambda = e^{-p\lambda}U$, and $U_{0,\lambda} = e^{-p\lambda}U_0$.

Define $R(\delta) = U(\delta) - U_0$. We have the following.

Lemma 3.1. *Consider $R(\delta)$ as an operator from $L^2(R)$ to itself, then*

$$\|R(\delta)\|_{L^2} = \|U(\delta) - U_0\|_{L^2} = O(\sqrt{\delta})$$

as $\delta \to 0$.

Proof. Let v^0 be the solution of

$$\begin{aligned} v_t^0 &= L_0 v^0 \\ v^0 \mid_{t=0} &= v_0 \in L^2 \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} v_t &= L_0 v + \delta L_1 v \\ v \mid_{t=0} &= v_0 \in L^2 \end{aligned} \tag{3.17}$$

Define $v_\lambda^0 = e^{-\lambda t}v^0$, $v_\lambda = e^{-\lambda t}v^0$, then v_λ^0 and v_λ satisfy

$$\begin{aligned} v_{\lambda,t}^0 &= L_{0,\lambda} v_\lambda^0 \\ v_\lambda^0 \mid_{t=0} &= v_0 \end{aligned} \tag{3.18}$$

$$\begin{aligned} v_{\lambda,t} &= L_{0,\lambda} v_\lambda + \delta L_1 v_\lambda \\ v_\lambda \mid_{t=0} &= v_0 \end{aligned} \tag{3.19}$$

Choose $\lambda = |f'(\varphi)|_{L^\infty} + 1$, then multiply (3.18) by v_λ^0, and integrate over ξ. Integration by parts gives

$$\tfrac{1}{2}\partial_t \int (v_\lambda^0)^2 = -\int (v_{\lambda,\xi}^0)^2 + \int (f'(\varphi) - \lambda)(v_\lambda^0)^2$$

By our choice of λ, we have

$$\tfrac{1}{2}\partial_t \int (v_\lambda^0)^2 \leqslant -\int (v_{\lambda,\xi}^0)^2 - \int (v_\lambda^0)^2 \tag{3.20}$$

So

$$\begin{gathered} \partial_t \int (v_\lambda^0)^2 \leqslant 0 \\ \int (v_\lambda^0)^2 \leqslant \int v_0^2 \end{gathered} \tag{3.21}$$

for $t \in [0, p]$.

Integrating (3.20) from 0 to p over t gives

$$\tfrac{1}{2}\int (v_\lambda^0)^2 \bigg|_{t=p} - \tfrac{1}{2}\int v_0^2 \leqslant -\int_0^p \int (v_{\lambda,\xi}^0)^2 - \int_0^p \int (v_\lambda^0)^2$$

therefore

$$\int_0^p \int (v_{\lambda,\xi}^0)^2 \leqslant \tfrac{1}{2}\int v_0^2 - \tfrac{1}{2}\int (v_\lambda^0)^2 \bigg|_{t=p} - \int_0^p \int (v_\lambda^0)^2$$

It follows

$$\int_0^p \int (v_{\lambda,\xi}^0)^2 \leqslant \tfrac{1}{2}\int v_0^2$$

which is

$$\int_0^p \int (v_{\lambda,\xi}^0)^2 \leqslant c_2 \, \|v_0\|_2^2 \tag{3.22}$$

where $c_2 \equiv 1/2$, the subscript 2 means L^2 norm.

Let $w = v_\lambda - v_\lambda^0$, then w satisfies

$$w_t = L_{0,\lambda} w + \delta L_1 v_\lambda = L_{0,\lambda} w + \delta L_1 w + \delta L_1 v_\lambda^0 \tag{3.23}$$

Multiply w to (3.23) and integrate over ξ. Integrating by parts, we obtain

$$1/2\ \partial_t \int w^2 \leqslant -\int w_\xi^2 - \int w^2 - \delta \int (a_1 w_\xi^2 - B w^2) + \delta \int w \cdot L_1 v_\lambda^0 \tag{3.24}$$

Suppose δ is small, and let $LHS = 1/2\, \partial_t \int w^2$, we have

$$LHS \leqslant -1/2 \int w_\varepsilon^2 - 1/2 \int w^2 + \delta \int w L_1 v_\lambda^0$$

$$= -1/2 \int w_\varepsilon^2 - 1/2 \int w^2 - \delta \int (w_\xi a_1 v_{\lambda,\xi}^0 + c_1 \omega_\xi v_\lambda^0 - Bwv_\lambda^0)$$

$$\leqslant -1/2 \int w_\varepsilon^2 - 1/2 \int w^2 + \delta\, |a_1|_{L^\infty} \int |w_\xi v_{\lambda,\xi}^0|$$

$$+ \frac{\delta\, |c_1|}{2} \int (w_\xi^2 + (v_\lambda^0)^2) + \frac{|B|_{L^\infty}\, \delta}{2} \int (w^2 + (v_\lambda^0)^2)$$

Letting $c_3 = \max(|B|_{L^\infty}, |a_1|_{L^\infty}, |c_1|)$, we have

$$LHS \leqslant -1/3 \int w_\varepsilon^2 - 1/3 \int w^2 + \delta c_3\, \|v_\lambda^0\|_2^2 + \delta c_3 \int |w_\xi v_{\lambda,\xi}^0|$$

$$\leqslant -1/3 \int w_\varepsilon^2 - 1/3 \int w^2 + \delta c_3\, \|v_\lambda^0\|_2^2 + \frac{\delta c_3}{2} \int (w_\xi^2 + (v_{\lambda,\xi}^0)^2)$$

$$\leqslant \delta c_3\, \|v_\lambda^0\|_2^2 + \frac{\delta c_3}{2} \int (v_{\lambda,\xi}^0)^2 \tag{3.25}$$

Integrating the above inequality from 0 to p over t, we obtain

$$1/2 \int w^2 \bigg|_{t=p} \leqslant \delta c_3 \int_0^p \|v_\lambda^0\|_2^2 + \frac{\delta c_3}{2} \int_0^p \int (v_{\lambda,\xi}^0)^2 \tag{3.26}$$

Using (3.21)–(3.22), we get

$$1/2 \int w^2 \bigg|_{t=p} \leqslant \delta c_3\, p\, \|v_0\|_2^2 + \frac{\delta c_3}{2} c_2\, \|v_0\|_2^2$$

$$= O(\delta)\, \|v_0\|_2^2 \tag{3.27}$$

that is,

$$\int (v_\lambda - v_\lambda^0)^2 \bigg|_{t=p} = \|(U_\lambda - U_{0,\lambda})\, v_0\|_2^2 \leqslant O((\delta)\, \|v_0\|_2^2 \tag{3.28}$$

or

$$\|U_\lambda - U_{0,\lambda}\|_2 \leqslant O(\sqrt{\delta}) \tag{3.29}$$

which implies

$$\|U(\delta)-U_0\|_2 \leqslant O(\sqrt{\delta}) \tag{3.30}$$

this completes the proof of the lemma. ∎

Corollary 3.1. *If δ is small, then $\exists \rho=\rho(\delta)<1$ such that the spectrum of $U(\delta)$ lies inside $B_\rho(0)$ (the ball of radius ρ with center at 0) except* 1, *which is a simple eigenvalue. In other words, $\sigma(U(\delta))\backslash\{1\}\subset B_\rho(0)$.*

Proof. We know that $\sigma(U_0)\backslash\{1\}$ is separated from 1 by a circle of radius less than 1, and 1 is a simple eigenvalue. Lemma 3.1 says $U(\delta)$ is a $O(\sqrt{\delta})$ perturbation of U_0. By perturbation theory of bounded linear operators, we see that $\sigma(U(\delta))$ has a simple eigenvalue in the neighborhood of 1 and the rest of the spectrum still lies inside a ball of radius less than 1, which depends on the size of δ. From our preliminary analysis, we observe that 1 is still an eigenvalue of $U(\delta)$, so it must be the simple eigenvalue predicted by the spectral perturbation theory. ∎

Remark 3.2. In general, the Poincaré return map is defined as

$$U(s)\colon v(\xi, s)\to v(\xi, s+p)$$

where $v(s, t)$ is the solution of (3.9). It is well known that $U(t+p)=U(t)$ for all t, and the nonzero spectrum of $U(t)$ is independent of t.

3.3. Stability of Traveling Wave Solutions

Let $X=L^2(R_\xi)$, and decompose X according to the spectral point $\{1\}$, i.e., $X(t)=X_1(t)\oplus X_2(t)$, where $X_1(t)=\operatorname{span}\{\psi(\xi, t)\}$ for all $t\geqslant 0$, here $\psi(\xi, t)=U_s(\xi, \xi+ct)$, and $U=U(s, y)$ is the traveling wave solution constructed in Section 2. In order to define the projection operator, we consider the following backward parabolic equation:

$$\begin{aligned} -v_t &= L^*v, \qquad t\in[0, p] \\ v\,|_{t=p} &= v_1 \end{aligned} \tag{3.31}$$

where L^* is the adjoint of L in terms of L^2 inner product $(\cdot,\cdot)$ in ξ, and L is the time periodic linear operator in (3.9).

Define operator $V\colon v(\xi, p)\to v(\xi, 0)$. V is well defined from L^2 to itself. Let u be the solution of the problem:

$$\begin{aligned} u_t &= Lu, \qquad t\in[0, p] \\ u\,|_{t=0} &= u_0 \end{aligned} \tag{3.32}$$

with which is associated the Poincaré map $U: u(\xi, 0) \to u(\xi, p)$. Direct computation shows that

$$(u, v)_t = (u_t, v) + (u, v_t) = (Lu, v) + (u, -L^*v) = (Lu, v) - (Lu, v) = 0$$

so

$$(u(\xi, p), v(\xi, p)) = (u(\xi, 0), v(\xi, 0))$$

or

$$(U(u(\xi, 0)), v(\xi, p)) = (u(\xi, 0), V(v(\xi, p)))$$

Since $u(\xi, 0)$ and $v(\xi, p)$ are arbitrary, we see that

$$V = U^*$$

therefore $\sigma(V) = \sigma(U^*) = \sigma(U)$.

Let $e^* = e^*$ be the time periodic solution of (3.31) such that $e^*(\xi, 0)$ is the eigenfunction of U^* corresponding to $\{1\}$. Since $\{1\}$ is a simple eigenvalue of U, hence of U^*, $(e^*(\xi, 0), \psi(\xi, 0)) \neq 0$. It is clear that the forward problem:

$$\begin{aligned} -v_t &= L^*v, \qquad t \geqslant 0 \\ v\,|_{t=0} &= v_0 \end{aligned} \tag{3.33}$$

has bounded L^2 solution for all $t \geqslant 0$ if and only if $v_0 = ce^*(\xi, 0)$, where c is a constant. After normalization, we assume that $c = 1$ and $(e^*(\xi, 0), \psi(\xi, 0)) = 1$, therefore $(e^*(\xi, t), \psi(\xi, t)) = 1$, for all $t \geqslant 0$.

Define the projection operator $P: X \to X_1(t)$ by

$$Pu = (u, e^*)\psi$$

where $u = u(\xi, t)$, $u \in L^2(R_\xi)$, for all $t \geqslant 0$, and $(\cdot, \cdot)$ is L^2 inner product in ξ. From P we have another projection $Q: X \to X_2(t)$ given by

$$Qu = u - Pu$$

It follows that

$$\begin{aligned} (Qu, e^*) &= (u - Pu, e^*) = (u, e^*) - (Pu, e^*) \\ &= (u, e^*) - (u, e^*)(\psi, e^*) = (u, e^*) - (u, e^*) = 0, \qquad \forall t \geqslant 0 \end{aligned}$$

Definition 3.2. For any $\rho \in (0, 1)$,

$$X_\rho = \left\{ u(t, \cdot) \in H^1, \forall t \geqslant 0 \;\middle|\; \|u\|_\rho = \sup_{t \geqslant 0} \frac{1}{\rho^t} \|u(t, \cdot)\|_{H^1} < +\infty \right\}$$

$$QX_\rho = \{u \in X_\rho \mid (u, e^*) = 0, \forall t \geqslant 0\}$$

Lemma 3.2. *Consider the following initial value problem:*

$$\begin{aligned} u_t &= Lu + Qh \\ u\,|_{t=0} &= 0 \end{aligned} \tag{3.34}$$

where $h = h(\cdot, t) \in H^1(R)$, $\forall t \geqslant 0$.

Let K be the linear map from h to u, then $\exists \rho_0 \in (0, 1)$ *such that K is a bounded map from* X_{ρ_0} *to* QX_{ρ_0}.

Proof. Let $T(t, s)$ be the evolution operator of equation:

$$u_t = Lu, \qquad 0 \leqslant s \leqslant t \tag{3.35}$$

and

$$U(s) = T(s + p, s)$$

By the periodicity of $L(t)$, it is well known that $\sigma(U(s)) \backslash \{0\}$ is independent of s. Because of the spectrum of $U(s)$ as described in Corollary 3.1 and the gradient estimates of parabolic equations, there exist constant M, $\rho \in (0, 1)$, such that

$$\begin{aligned} &\|U^n(s)\, Qx(s)\|_2 \leqslant \rho^n \, \|Qx(s)\|_2 \\ &\|U^n(s)\, Qx(s)\|_{H^1} \leqslant M\rho^n \, \|Qx(s)\|_{H^1} \\ &\|T(s + r, s)\|_{H^1} \leqslant M \end{aligned}$$

where $r \in [0, p)$, $s \geqslant 0$, $n \in Z^+$, $x \in H^1$ and M and ρ are independent of r, s, n, x.

Therefore, for any s, $0 \leqslant s \leqslant t$, such that $t = s + np + r$, $r \in [0, p)$, we have

$$T(t, s) = T(s + np + r, s + np)\, T(s + np, s) \tag{3.36}$$

thus,

$$\begin{aligned} \|T(t, s)\, Qx(s)\|_{H^1} &\leqslant \|T(s + r, s)\|_{H^1} \cdot \|U^n(s)\, Qx(s)\|_{H^1} \\ &\leqslant M\, \|U^n(s)\, Qx(s)\|_{H^1} \leqslant M^2 \rho^n \, \|Qx(s)\|_{H^1} \end{aligned}$$

since

$$\rho^n = (\rho^{1/p})^{np} = (\rho^{1/p})^{t - s - r} \leqslant \rho^{(t - s)/p - 1}$$

let $\eta = \rho^{1/p}$, we obtain

$$\|T(t, s)\, Qx(s)\|_{H^1} \leqslant \frac{M^2}{\rho} \eta^{t - s} \, \|Qx(s)\|_{H^1} \overset{\text{def}}{=} M_1 \eta^{t - s} \, \|Qx(s)\|_{H^1}$$

By variation of constant formula, we have for u in (3.34)

$$u(t)=\int_0^t T(t,s)\,Qh(s)\,ds \tag{3.37}$$

$$\begin{aligned}\|u(t)\|_{H^1} &\leqslant \int_0^t \|T(t,s)\,Qh(s)\|_{H^1}\,ds \leqslant M_1 \int_0^t \eta^{t-s}\,\|h(s)\|_{H^1}\,ds\\
&\leqslant M_1 \int_0^t \eta^{t-s}\,\|h(s)\|_{H^1}\frac{1}{\rho_0^s}\rho_0^s\,ds, \qquad \rho_0\in(\eta,1)\\
&\leqslant M_1\,\|h\|_{\rho_0}\int_0^t \eta^{t-s}\rho_0^s\,ds\\
&\leqslant M_1\,\|h\|_{\rho_0}\,\rho_0^t\int_0^t\left(\frac{\eta}{\rho_0}\right)^{t-s}ds\\
&\leqslant M_1\,\|h\|_{\rho_0}\,\rho_0^t\frac{1}{\ln(\rho_0/\eta)}\end{aligned}$$

Let $M_2=M_1[1/(\ln\rho_0/\eta)]$, we see that

$$\begin{aligned}\|u(t)\|_{H^1}\cdot\frac{1}{\rho_0^t}&\leqslant M_2\,\|h\|_{\rho_0}\\
\|Kh\|_{\rho_0}=\|u\|_{\rho_0}&\leqslant M_2\,\|h\|_{\rho_0}\end{aligned} \tag{3.38}$$

Moreover,

$$\begin{aligned}(Kh,e^*)_t&=(u_t,e^*)+(u,e_t^*)\\
&=(Lu+Qh,e^*)+(u,-L^*e^*)\\
&=(Lu,e^*)+(u,-L^*e^*)=0\\
(Kh,e^*)&=(Kh,e^*)\,|_{t=0}=(u,e^*)\,|_{t=0}=0\end{aligned}$$

therefore $Kh\in QX_{\rho_0}$ if $h\in X_\rho$. ▌

Consider $u(x,t)$ solution of (3.1) with initial data of the form $u(x,0)=U(x,x)+\varepsilon u_0(x)$, or equivalently the following equation in the moving coordinate (ξ,t):

$$\begin{aligned}u_t&=(a(\xi+ct)\,u_\xi)_\xi+cu_\xi+f(u)\\
u\,|_{t=0}&=U(\xi,\xi)+\varepsilon u_0(\xi)\end{aligned} \tag{3.39}$$

Proof of Theorem 3.1. Write

$$u=u(\xi,t)=U(\xi+\gamma(\varepsilon),\xi+ct)+\varepsilon v(\xi,t,\varepsilon)$$

and

$$U^\gamma = U(\xi + \gamma(\varepsilon), \xi + ct)$$

then v satisfies

$$v_t = (a(\xi + ct)\, v_\xi)_\xi + cv_\xi + \frac{f(U^\gamma + \varepsilon v) - f(U^\gamma)}{\varepsilon}$$

or

$$v_t = (a(\xi + ct)\, v_\xi)_\xi + cv_\xi + f(U^\gamma)v + \varepsilon R$$

where

$$R = R(\xi, t, v, \varepsilon) = \varepsilon^2\{f(U^\gamma + \varepsilon v) - f(U^\gamma) - \varepsilon f(U^\gamma)v\}$$

or

$$v_t = (a(\xi + ct)\, v_\xi)_\xi + cv_\xi + f(U)v + \varepsilon B + \varepsilon R$$

where

$$\begin{aligned} B = B(v, h) &= \frac{(f(U^\gamma) - f(U))v}{\varepsilon} \\ &= (1/\varepsilon)(f(U(\xi + \gamma(\varepsilon), \xi + ct)) - f(U(\xi, \xi + ct)))v) \\ &= (1/\varepsilon)\{f(U(\xi, \xi + ct) + \gamma(\varepsilon)\, U_s(\xi, \xi + ct) + O(\gamma^2)) \\ &\quad - f(U(\xi, \xi + ct))\}\, v \\ &= \frac{1}{\varepsilon}\, \{f'(U)(\gamma U_s + O(\gamma^2))\}\, v \\ &= h(\varepsilon)\{f'(U)\, U_s + O(\gamma)\}\, v \end{aligned}$$

B is a bounded map from $X_{\rho 0} \times R \to X_{\rho 0}$, and Fréchet differentiable. While R,

$$R = \varepsilon^{-2}\{1/2\varepsilon^2 v^2 f''(U^\gamma) - \varepsilon^3 v^3\} = \frac{v^2}{2} f''(U^\gamma) - \varepsilon v^3$$

Since $\|v^2\|_{H^1} \leqslant c\, \|v\|^2_{H^1}$, $\|v^3\|_{H^1} \leqslant c\, \|v\|^3_{H^1}$, we see that $R = R(v, h)$ is a bounded map from $X_{\rho 0} \times R \to X_{\rho 0}$, and Fréchet differentiable. At $t = 0$,

$$U(\xi + \gamma, \xi) + \varepsilon v(\xi, 0, \varepsilon) = U(\xi, \xi) + \varepsilon u_0(\xi)$$

$$\begin{aligned} v(\xi, 0, \varepsilon) &= u_0(\xi) + \frac{1}{\varepsilon}\, (U(\xi, \xi) - U(\xi + \gamma, \xi)) \\ &= u_0(\xi) - h(\varepsilon)\, U_s(\xi, \xi) + \varepsilon g(\xi, \gamma, \varepsilon) \end{aligned}$$

where

$$g(\xi, h, \varepsilon) = -h^2 \int_0^1 U_{ss}(\xi + \varepsilon h\tau, \xi)\tau \, d\tau$$

g is uniformly bounded as $\varepsilon \to 0$, and differentiable in h. For fixed ε, h, $g \in H^1(R_\xi)$. By assumption, $u_0 \in H^1$, so $v(\xi, 0, \varepsilon) \in H^1$, then $v(\xi, t) \in H^1$ for all $t \geqslant 0$, from classical parabolic theory.

Decompose $v(\xi, t)$ as $v = Pv + Qv = p(t)\, U_s + \zeta$, where $p(t) = (v, e^*)$, $(\zeta, e^*) = 0$. Substitute the above equality into the equation of v:

$$\begin{aligned} p_t U_s + p(U_s)_t + \zeta_t &= L(pU_s) + L\zeta + \varepsilon B + \varepsilon R \\ &= pL(U_s) + L\zeta + \varepsilon B + \varepsilon R \end{aligned}$$

So

$$p_t U_s + \zeta_t = L\zeta + \varepsilon B + \varepsilon R \tag{3.40}$$

Form the inner product of both sides of (3.40) with e^*,

$$p_t + (\zeta_t, e^*) = (L\zeta, e^*) + \varepsilon(B + R, e^*) \tag{3.41}$$

which is

$$p_t + (\zeta_t, e^*) = (\zeta, L^*e^*) + \varepsilon(B + R, e^*)$$

$$p_t + (\zeta_t, e^*) = -(\zeta, e_t^*) + \varepsilon(B + R, e^*)$$

since

$$(\zeta_t, e^*) + (\zeta, e_t^*) = (\zeta, e^*)_t = 0$$

we have

$$p_t = \varepsilon(B + R, e^*)$$

Plug this equality into (3.40)

$$\begin{aligned} \zeta_t &= L\zeta + \varepsilon(B + R) - \varepsilon(B + R, e^*)\, U_s \\ &= L\zeta + \varepsilon Q(B + R) \end{aligned}$$

Summarizing, we have

$$\begin{aligned} \zeta_t &= L\zeta + \varepsilon Q(B + R) \\ p_t &= \varepsilon(B + R, e^*) \end{aligned}$$

Decomposing initial condition yields

$$p(0) = (u_0, e_0^*) - h(\varepsilon) + \varepsilon(g, e_0^*)$$
$$\zeta(0) = Q(u_0 + \varepsilon g)$$

where $e_0^* = e^*(\xi, 0)$.

By Lemma 3.2, the above equations are equivalent to

$$\zeta(t) = \varepsilon KQR + \varepsilon KQB + T(t, 0)\, Q(u_0 + \varepsilon g)$$

$$p(t) = \varepsilon \int_0^t (B + R, e^*)\, ds + (u_0, e_0^*) - h(\varepsilon) + \varepsilon(g, e_0^*)$$

where the last equation can be rewritten into

$$p(t) = -\varepsilon \int_t^\infty (B + R, e^*)\, ds + \varepsilon \int_0^\infty (B + R, e^*)\, ds$$
$$+ (u_0, e_0^*) - h(\varepsilon) + \varepsilon(g, e_0^*)$$

which can be split into the following two equations, recalling that h is to be determined:

$$p(t) = -\varepsilon \int_t^\infty (B + R, e^*)\, ds \tag{3.42}$$

$$h = (u_0, e_0^*) + \varepsilon(g, e_0^*) + \varepsilon \int_0^\infty (B + R, e^*)\, ds \tag{3.43}$$

These two equations along with the equation for ζ can be recast into

$$F_1(\zeta, p, h, \varepsilon) \stackrel{\text{def}}{=} \zeta - \varepsilon KQR - \varepsilon KQB - T(t, 0)\, Q(u_0 + \varepsilon g) = 0 \tag{3.44}$$

$$F_2(\zeta, p, h, \varepsilon) \stackrel{\text{def}}{=} p(t) + \varepsilon \int_t^\infty (B + R, e^*)\, ds = 0 \tag{3.45}$$

$$F_3(\zeta, p, h, \varepsilon) \stackrel{\text{def}}{=} h - (u_0, e_0^*) - \varepsilon(g, e_0^*) - \varepsilon \int_0^\infty (B + R, e^*)\, ds = 0 \tag{3.46}$$

Define mapping $F = (F_1, F_2, F_3)$, the above equations are the same as

$$F(\zeta, p, h, \varepsilon) = 0 \tag{3.47}$$

Define function space

$$R_\rho = \left\{ p(t) \in C(R) \,\middle|\, \sup_{t \geq 0} \frac{|p(t)|}{\rho^t} < \infty, \rho \in (0, 1) \right\}$$

Then from Lemma 3.2, and properties of R, B, g, it is straightforward to check that F is a Fréchet differentiable mapping from $QX_{\rho_0} \times R_{\rho_0} \times R$ to itself.

When $\varepsilon = 0$, Eq. (3.47) has solution in $QX_{\rho_0} \times R_{\rho_0} \times R$:

$$h = h_0 = (u_0, e_0^*) \tag{3.48}$$

$$p = 0 \tag{3.49}$$

$$\zeta = T(t, 0)\, Qu_0 \in QX_{\rho_0} \tag{3.50}$$

moreover,

$$F' \,|_{(\zeta_0, p_0, h_0, 0)} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is invertible.

Implicit Function Theorem in Banach Space says that if ε is sufficiently small, then there exists $(\zeta(\varepsilon), p(\varepsilon), h(\varepsilon))$ in $QX_{\rho_0} \times R_{\rho_0} \times R$, differentiable in ε and satisfying

$$F(\zeta(\varepsilon), p(\varepsilon), h(\varepsilon)) = 0$$

which then implies (3.10), and thus the asymptotic stability of traveling wave solution. ▮

ACKNOWLEDGMENTS

This paper is a part of my dissertation. I would like to express my deepest gratitude to my thesis advisor Prof. George Papanicolaou for his continued guidance, support, and encouragement during the preparation of this work. I would like to thank Professors L. Nirenberg and H. Berestycki for their interest in this work and many helpful suggestions. I would also like to thank Professors M. Avellaneda, J. Goodman, and Z. Xin for many interesting and motivating discussions.

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