Potential Analysis **4**: 1–45, 1995.

Stochastic Partial Differential Equations in M-Type 2 Banach Spaces

ZDZISŁAW BRZEŹNIAK
*Fakultät für Mathematik, Ruhr-Universität Bochum, 4630 Bochum, Germany**.

(Received: 16 February 1992; accepted: 10 March 1993)

Abstract. We study abstract stochastic evolution equations in M-type 2 Banach spaces. Applications to stochastic partial differential equations in L^p spaces with $p \geqslant 2$ are given. For example, solutions of such equations are Hölder continuous in the space variables.

Mathematics Subject Classifications (1991). 35R60, 60H15, 46E35, 47F05, 47D06.

Key words. Stochastic parabolic equations, M-type 2 and UMD Banach spaces, real and complex interpolation, Sobolev and Besov spaces.

1. Introduction

In this paper we will be concerned with the theory of Stochastic Partial Differential Equations with multiplicative noise in Banach spaces.

The theory of such equations in Hilbert spaces originated in the seventies, see for example [11]. The fundamental works of Pardoux and Krylov–Rosovskii (see [33]–[34], [28]) have given it a mature shape. These authors considered nonlinear monotone stochastic evolution equations and for such equations, roughly speaking, they proved existence and uniqueness of solutions. Although in [28] the authors consider their equations in Banach spaces, the basic space for their treatment is a Hilbert space. The main assumption of all these papers is monotonicity of the operators involved and the authors make use of the methods developed earlier for monotone deterministic partial differential equations, see [30].

Another approach, based on theory of semigroups of linear operators was initiated by Dawson, see [11] and later developed by Da Prato and his collaborators, see [9], [10], [18] and references therein.

In this paper we follow mainly the latter method. Our main object is to develop a theory of stochastic evolution equations in Banach spaces. However, not all Banach spaces fit into such a theory, only a class of the so-called M-type 2 Banach spaces

* *Permanent address*: Instytut Matematyki, Uniwersytet Jagielloński, Reymonta 4, Kraków, Poland.
The author is an Alexander von Humboldt Stiftung fellow.

seems to be well suited. Let us remark that L^p space with $2 \leqslant p \leqslant \infty$ is a good example of an M-type 2 Banach space. Comparing with [40] we have different assumptions and our results are of different nature.

We describe briefly the content of the first part of this paper. In Section 2 we present the basic definitions used throughout the paper, in particular those of M-type 2 and UMD Banach spaces. In Section 3 we consider stochastic Itô integrals of processes with values in M-type 2 Banach spaces. We recall some basic properties of this integral (including an inequality of Doob type and a simple version of Itô's formula). Let us mention the works of Dettweiler, [12]–[14], where a theory of integration in M-type 2 Banach spaces is built and where one can find the proofs of all the results we state in Section 3. In Section 4 we treat linear stochastic differential equations in M-type 2 Banach spaces of the following type

$$\begin{aligned} du(t) + Au(t)dt &= \sum B^j u(t)dw^j(t) + f(t), \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

where $-A$ is a generator of an analytic semigroup $\{e^{-tA}\}_{t \geqslant 0}$ on X, an M-type 2 Banach space, $B^1, \ldots, B^d$ are linear (generally unbounded) operators in X and $w(t) = (w^1(t), \ldots, w^d(t))$ is a d-dimensional Wiener process (we take it finite dimensional only for simplicity of exposition).

We prove existence and uniqueness of solutions for such equations, where the basic space (i.e. the space of initial conditions) is some real interpolation space between $D(A)$, the domain of A, and X.

In Section 5 we show that under some additional assumptions on A, the initial condition u_0 can as well be taken from X (however we lose some regularity of the solutions). In Section 6 we show what happens with our theory in the Hilbert space framework. In fact our results generalize those from [9]. In Section 7 we give some simple examples of equations satisfying our assumptions.

In the second part of our work (consisting of Sections 8 to 11) we will mainly be concerned with applying the results from Section 4 to stochastic parabolic equations.

The model equation for our study is the following special type of equation (1.1),

$$\begin{aligned} du - \Delta u \, dt + \sum b_{j,k}(x) \frac{\partial u}{\partial x_k} dw^j(t) &= 0, \quad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{1.2}$$

with $w(t) = (w^j(t))$ as before. The results of Section 4 allow us to study the problem (1.2) in the real interpolation space $D_A(\frac{1}{2}, 2) = (L^p, D(A))_{\frac{1}{2},2}$ with $D(A)$ being the domain of the operator A, the appropriate realization of $-\Delta$ in the L^p space.

There, one of the necessary conditions for studying (1.2) is that the linear operators B_j, $B_j u = b_j(\partial/\partial x_j)u$, map the space $D(A)$ into $D_A(\frac{1}{2}, 2)$. But here one encounters two difficulties. One is that the elements of $D_A(\frac{1}{2}, 2)$ usually satisfy some boundary conditions and so even if u satisfies them, $(\partial/\partial x_j)u$ does not have to. This difficulty can be overcome (by means of some trick and complex interpolation method) by changing the pair of spaces $(D(A), D_A(\frac{1}{2}, 2))$ so that the new space corresponding to $D_A(\frac{1}{2}, 2)$ do not contain boundary conditions, see Section 5. The second difficulty is that the space $D_A(\frac{1}{2}, 2)$ is too big (for $p > 2$), in the sense that B_j is not a bounded map from $D(A)$ into $D_A(\frac{1}{2}, 2)$ even if we forget in the latter space about the boundary conditions. And this difficulty does not disappear by means of the methods used in Sections 5 and 6.

We show how to overcome this difficulty in Sections 8–11. Roughly speaking, the real interpolation method should be used instead of the complex one. To explain this point more clearly, let us consider the case of equation (1.2) in the whole domain $\mathbb{R}^n$. Then the real interpolation spaces are the Besov spaces. So $D(A) = W^{2,p}(\mathbb{R}^n)$, $D_A(\frac{1}{2}, 2) = B^1_{p,2}(\mathbb{R}^n)$ and thus $\partial/\partial x_j, j = 1, \dots, n$, are not bounded operators from $D(A)$ to $D_A(\frac{1}{2}, 2)$. However $\partial/\partial x_j$ is a bounded operator from $B^2_{p,2}(\mathbb{R}^n)$ to $B^1_{p,2}(\mathbb{R}^n)$ and from $B^1_{p,2}(\mathbb{R}^n)$ to $B^0_{p,2}(\mathbb{R}^n)$. This example (in a general framework of this paper) will be our guideline in Sections 8–11.

A natural question one should ask himself is the following. What new information concerning the equations (1.2) can be obtained comparing with the Hilbert (i.e. L^2) space methods? One of the possible answers can be easily seen in the case of a boundary value problem, see Section 10, written formally as an equation (1.2). For, by taking $X = L^p$ with $p > n$, the value $u(t)$ of the solution u to (1.2) at time $t > 0$ belongs to $B^1_{p,2}(\mathbb{R}^n)$ a.s. in $\omega \in \Omega$ for all $t > 0$. Hence, in view of the Sobolev imbedding theorem $u(t, \cdot)$ belongs to $\mathscr{C}^\alpha(\mathbb{R}^n)$ for $\alpha < 1 - n/p$. And this property is the main motivation of this paper. It is especially important that it is also valid for boundary value problems in bounded domains without imposing a long series of compatibility conditions as it has been done in [17] by using only Hilbert space methods.

Let us now describe briefly the content of the second part of this paper.

Section 8 can be considered as the main part of it. Loosely speaking, we construct an extension Z of the Banach space X and a semigroup $\{T(t)\}$ on Z (with a generator denoted by $-A_Z$), an extension of the semigroup $\{e^{-tA}\}$ such that $D(A_Z^{\frac{1}{2}}) = (Z, D(A_Z))_{\frac{1}{2},2}$. This result is used in following sections to study equations of type (1.2).

In a short Section 9 we identify some of the spaces constructed in Section 8 by means of duality.

Section 10 provides two simple (but systematically worked out) examples. The first one concerns the equation (1.2) in the full domain $\mathbb{R}^n$, while in the second one we treat a similar equation but in bounded domains with Dirichlet boundary conditions.

In Section 11 we extend the results from the preceding one, so that we are able to treat general systems (of any order) of stochastic parabolic equations in bounded or unbounded domains, with general boundary conditions.

As already said, the case of Hilbert space is discussed in Sections 5 and 6 by means of complex interpolation method. Therefore let us only add a remark that under quite general assumptions on the operator A, the real and the complex interpolation methods coincide one with another (of course in a Hilbert space setting). However, there are some examples of generators of analytic semigroups $-A$ for which these two methods are different (even in Hilbert spaces). Related problems and questions will be treated elsewhere.

NOTATION. Throughout this paper X will always denote a Banach space, $-A$ a generator of a uniformly bounded analytic semigroup $\{e^{-tA}\}_{t\geqslant 0}$ on X and $\{w(t)\}_{t\geqslant 0} = \{(w^j(t))_{j=1}^d\}_{t\geqslant 0}$ a d-dimensional Wiener process with respect to a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ on a complete probability space $(\Omega, \mathscr{F}, P)$. By writing *a.s.* we mean almost surely on $(\Omega, \mathscr{F}, P)$. $M^2(0,T;X)$ is a space of all progressively measurable X-valued processes such that

$$\int_0^T \mathbb{E}|\xi(t)|^2\, dt < \infty. \tag{1.3}$$

If D is a domain in $\mathbb{R}^n$, $k\in\mathbb{N}$, $1\leqslant p<\infty$, then $W^{k,p}(D)$ denotes the Sobolev space of functions u belonging to $L^p(D)$ such that all derivatives of u up to order k belong to $L^p(D)$. $B^s_{p,q}(D)$, for $s\in\mathbb{R}$, $1\leqslant p,q\leqslant\infty$ is the Besov space of functions defined on D.

2. Notation and General Information

In this paper $(\Omega, \mathscr{F}, P)$ will be a complete probability space, $I\subset\mathbb{R}$, $\{\mathscr{F}_t\}_{t\in I}$ an increasing family of sub-σ-algebras of $\mathscr{F}$ and X a Banach space. An X-valued process $\{M_t\}_{t\in I}$ is an X-valued martingale, see [15] or [32], if and only if $M_t\in L^1(\Omega, \mathscr{F}_t, P;\ X)$ for $t\in I$ and

$$\mathbb{E}(M_t|\mathscr{F}_s) = M_s \text{ a.s., for all } s\leqslant t\in I.$$

For a nice and precise definition of conditional expectation of a Banach valued random function the reader is referred to Chapter 5 in [15].

Following [37] we take

DEFINITION 2.1. A Banach space X is of M-type p, for $p\in[1,\infty)$, iff for any X-valued martingale $\{M_n\}_{n\in\mathbb{N}}$ the following inequality holds

$$\sup_n \mathbb{E}|M_n|^p \leqslant L_p(X)\sum_n \mathbb{E}|M_n - M_{n-1}|^p, \tag{2.1}$$

where the constant $L_p(X)$ (depending only on X (and so on its norm)) is the smallest possible. As usual $M_{-1}=0$.

X is of type p iff for any finite sequence $\varepsilon_1, \ldots, \varepsilon_n : \Omega \to \{-1, 1\}$ of symmetric i.i.d. random variables and for any finite sequence $x_1, \ldots, x_n$ of elements of X, the following inequality holds

$$\mathbb{E}\left|\sum_{i=1}^{n} \varepsilon_i x_i\right|^p \leqslant K_p(X) \sum_{i=1}^{n} |x_i|^p, \tag{2.2}$$

where the constant $K_p(X) = K_p$ is the smallest possible.

If X and Y are isomorphic Banach spaces then X is of M-type p iff Y is and X is of type p iff Y is so. Moreover

$$L_p(Y) \leqslant \{\operatorname{dist}(X, Y)\}^p L_p(X), \quad K_p(Y) \leqslant \{\operatorname{dist}(X, Y)\}^p K_p(X),$$

where $\operatorname{dist}(X, Y) = \inf\{|T||T^{-1}| : T \text{ is an isomorphism between } X \text{ and } Y\}$.

If X is of M-type p then X of type p and $K_p(X) \leqslant L_p(X)$, see [37] but the converse is not true. It is important to determine the constants $K_p(X)$ and $L_p(X)$ for various spaces X and different p. In particular, if H is a Hilbert space then $K_2(H) = L_2(H) = 1$.

In the definition of type p Banach spaces, one could instead of the Bernoulli sequence ε_i take as well a sequence ξ_i of i.i.d. Gausian random variables with $\mathbb{E}\xi_i = 0$ and $\mathbb{E}|\xi_i| = \sigma^2 > 0$. Indeed, X is of type p iff for any sequence ξ_i as above the following inequality holds

$$\mathbb{E}\left|\sum_{i=1}^{n} \xi_i x_i\right|^p \leqslant \tilde{K}_p(X)\mathbb{E}|\xi_i|^p \sum_{i=1}^{p} |x_i|^p. \tag{2.3}$$

Here, as usual, $\tilde{K}_p(X)$ is the smallest possible constant. Writing $\xi_i = \varepsilon_i|\xi_i|$ and using a standard conditional expectation trick we have

$$\mathbb{E}\left|\sum_{i=1}^{n} \xi_i x_i\right|^p = \mathbb{E}\left|\sum_{i=1}^{n} \varepsilon_i|\xi_i|x_i\right|^p \leqslant K_p(X)\mathbb{E}\sum_{i=1}^{n} ||\xi_i|x_i|^p = K_p(X)\mathbb{E}|\xi_1|^p \sum_{i=1}^{n} |x_i|^p,$$

so that $\tilde{K}_p(X) \leqslant K_p(X)$.

PROPOSITION 2.1. (1) *If a Banach space X is of M-type p and $r \in [0, \infty)$ then for any X-valued martingale $\{M_n\}_{n\in\mathbb{N}}$ the following inequality holds*

$$\mathbb{E}\sup_n |M_n|^r \leqslant C_{p,r}(X)\mathbb{E}\left\{\sum_n |M_n - M_{n-1}|^p\right\}^{\frac{r}{p}}. \tag{2.4}$$

(2) (Doob Inequality) *If X is a Banach space X and $r \in (1, \infty)$ then for any X-valued martingale $\{M_n\}_{n\in\mathbb{N}}$*

$$\mathbb{E}\sup_{j\leqslant n} |M_j|^r \leqslant \left(\frac{r}{r-1}\right)^r \mathbb{E}|M_n|^r. \tag{2.5}$$

Proof. For a proof of (2.4) see [38]. (2.5) can be proved by observing that in view of 1.18, ch I in [32], $|M_j|$ is a nonnegative submartingale and then applying the real version of Doob inequality, see [26]. See also Theorem 8.2, ch. IV in [32]. □

It is now well known, see [37], [38] that X is of M-type p iff it is p-smooth, i.e. X can be equivalently renormed in such a way that its modulus of smoothness

$$\rho_X(t) = \sup\{\tfrac{1}{2}(|x + ty| + |x - ty|) - 1 : |x|, |y| = 1\}$$

satisfies: $\rho_X(t) \leqslant Kt^p$ for all $t > 0$ and some $K > 0$.

Therefore all spaces L^p for $p \in [2, \infty)$ are of M-type 2.

DEFINITION 2.2. A Banach space X is a UMD space (i.e. X has unconditional margtingale difference property) iff for any $p \in (1, \infty)$, for any X-valued martingale difference $\{\xi_j\}$ (i.e.: $\Sigma_{j=1}^n \xi_j$ is a martingale), for any $\epsilon: \mathbb{N} \to \{-1, 1\}$ and for any $n \in \mathbb{N}$

$$\mathbb{E}\left|\sum_{j=1}^n \epsilon_j \xi_j\right|^p \leqslant \beta_p(X) \mathbb{E}\left|\sum_{j=1}^n \xi_j\right|^p, \tag{2.6}$$

where $\beta_p(X) > 0$ is a smallest possible constant.

It is known, see [8] and references therein, that for a Banach space X the following conditions are equivalent.

(i) X is a UMD space;

(ii) X is ζ convex, i.e., there is a biconvex function $\zeta: X \times X \to \mathbb{R}$ with the properties: $\zeta(0, 0) > 0$, $\zeta(x, y) \leqslant |x + y|$ for $|x|, |y| = 1$;

(iii) Hilbert transform for X-valued functions is a bounded linear operator in $L^p(\mathbb{R}, X)$ for any (or some) $p \in (1, \infty)$.

EXAMPLE 2.1. Every L^p space with $p \in (1, \infty)$ is a UMD space. Also, for a given Hilbert space H and $p \in (1, \infty)$, the space

$$\mathscr{C}_p = \{A: H \to H : A \text{ is compact and } \|A\| := (\mathrm{tr}(AA^*)^{p/2})^{1/p} < \infty\}$$

is a UMD space, see [22] and [8].

3. Itô Type Integrals

In this section we assume X to be of M-type 2 Banach space, see Definition 2.1. Below we shall show how to define Itô integral for X-valued processes. Although the definition, construction and properties of this integral highly resemble the ones in Hilbert space setting, for convenience of the reader we present them. All details (and proofs, which we mostly omit) one can find in the work of Dettweiler, especially [13].

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, $\{\mathscr{F}_t\}_{t\geqslant 0}$ a right-continuous, increasing filtration of sub-σ-algebras of $\mathscr{F}$. Finally let $\{w_t\}_{t\geqslant 0}$ be a d-dimensional Wiener process on $(\Omega, \mathscr{F}, P)$ with respect to the filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$. Let $\mathscr{B}$ denote the σ-algebra of Borel sets in $\mathbb{R}$. On the product space X^d we consider a norm defined by

$$|x| = \left\{\sum_i |x^i|^2\right\}^{\frac{1}{2}}, \quad x = (x^1, \ldots, x^d) \in X^d.$$

DEFINITION 3.1. Let $f: [0, T) \times \Omega \longrightarrow \mathscr{L}(\mathbb{R}^d, X) \cong X^d$ be an adapted random step function, i.e. $f(s) = f_i$ a.e. for $s \in [t_i, t_{i+1})$ for some sequences $\{t_i\}_{i=0}^n : 0 = t_0 < \cdots < t_n = T$ and $\{f_i\}_{i=0}^{n-1} : f_i \in L^2(\Omega, \mathscr{F}_{t_i}, P; X)$. Let $\Delta_i w = w(t_{i+1}) - w(t_i)$, $\Delta_i t = t_{i+1} - t_i$ and for $f \in X^d$, $\alpha \in \mathbb{R}^d \langle f, \alpha \rangle = \Sigma \alpha_i f_i$. Then we put

$$I(f) := \sum_{i=0}^{n-1} \langle f_i, \Delta_i w \rangle. \tag{3.1}$$

Let us observe that $I(f): \Omega \longrightarrow X$ is $\mathscr{F}_T$ measurable and $I(f) \in L^2(\Omega, \mathscr{F}_T, P; X)$. Moreover, in view of M-type 2 and type 2 properties of X we have

$$\begin{aligned} \mathbb{E}|I(f)|^2 &\leqslant L_2(X) \sum_i \mathbb{E}|\langle f_i, \Delta_i w \rangle|^2 \\ &\leqslant L_2(X) K_2(X) \sum_i \sum_j |f_i^j|^2 \Delta_i t = C_2(X) \sum_i |f_i|^2 (t_{i+1} - t_i). \end{aligned}$$

This gives (with $C_2(X) = K_2(X) L_2(X)$)

PROPOSITION 3.1. *For any adapted random step function* $f: [0, T) \times \Omega \longrightarrow \mathscr{L}(\mathbb{R}^d, X) \cong X^d$ *the following inequality holds*

$$\mathbb{E}|I(f)|^2 \leqslant C_2(X) \int_0^T \mathbb{E}|f(s)|^2 \, ds. \tag{3.2}$$

Let us notice that the constant $C_2(X)$ is independent of d, the dimension of the Wiener process. The above result allows one, in a standard way, to extend the operator I to a larger class of random functions.

DEFINITION 3.2. If Y is a Banach space and $p \in (1, \infty)$, by $M^p(0, T; Y)$ we denote the space of all functions $\xi: [0, T) \times \Omega \longrightarrow Y$ with the properties:

(i) ξ is progresively measurable,

(ii) $\xi \in L^p([0, T) \times \Omega, \mathscr{B} \times \mathscr{F}_T, P; Y)$ and

$$\int_0^T \mathbb{E}|\xi(s)|^p \, ds < \infty.$$

By $M^p_{loc}(0,\infty;Y)$ we denote the space of functions $\xi:[0,\infty)\times\Omega\to Y$ such that $\xi\in M^p(0,T;Y)$ for any $T>0$. The most common case will be $p=2$.

The following facts are well known.

(a) If $\xi\in M^p(0,T;Y)$ then the function $[0,T)\ni t\to\xi(t)\in L^p(\Omega,\mathscr{F}_T;Y)$ is strongly measurable.

(b) $M^p(0,T;Y)$ is a closed subspace of $L^p([0,T)\times\Omega,\mathscr{B}\times\mathscr{F}_T;Y)$.

(c) The family of random adapted step functions belongs to $M^p(0,T;Y)$ is dense in $M^p(0,T;Y)$.

COROLLARY 3.2. *There exists a unique linear and bounded operator*

$$I:M^2(0,T;X^d)\to L^2(\Omega,\mathscr{F}_T,P;X), \tag{3.3}$$

which is an extension of the operator introduced in Definition 3.1. *Moreover the inequality* (3.2) *holds for any* $f\in M^2(0,T;X^d)$.

If $\xi\in M^2_{loc}(0,\infty;X^d)$ and $t>0$ then we put

$$\int_0^t\langle\xi(s),\mathrm{d}w(s)\rangle:=I(1_{[0,t)}\xi) \tag{3.4}$$

and we call $\int_0^t\langle\xi(s),\mathrm{d}w(s)\rangle$ the Itô integral of the process ξ up to time t.

PROPOSITION 3.3 (i) *If* $\xi^1,\xi^2\in M^2_{loc}(0,\infty;Y^d)$ *and* $\alpha_1,\alpha_2\in\mathbb{R}$ *then for* $t\geqslant 0$, *a.s.*

$$\int_0^t\langle\alpha_1\xi^1(r)+\alpha_2\xi^2(r),\mathrm{d}w(r)\rangle=\alpha_1\int_0^t\langle\xi^1(r),\mathrm{d}w(r)\rangle+\alpha_2\int_0^t\langle\xi^2(r),\mathrm{d}w(r)\rangle.$$

(ii) *If* $0\leqslant t_1<t_2\leqslant t$, $\eta\in L^2(\Omega,\mathscr{F}_{t_1},P;Y^d)$ *and* $\xi=1_{[t_1t_2)}\eta$ *then a.s.*

$$\int_0^t\langle\xi(r),\mathrm{d}w(r)\rangle=\langle\eta,w(t_2)-w(t_1)\rangle.$$

(iii) *If* $\xi\in M^p_{loc}(0,T;Y^d)$ *then* $\int_0^t\langle\xi(r),\mathrm{d}w(r)\rangle$ *is* $\mathscr{F}_t$ *measurable and for* $r\leqslant s\leqslant t$,

$$\mathbb{E}\left(\int_0^t\langle\xi(r),\mathrm{d}w(r)\rangle\mid\mathscr{F}_s\right)=\int_0^s\langle\xi(r),\mathrm{d}w(r)\rangle.$$

(iv) *If X and Y are Banach spaces of M-type* 2, *Z is any Banach space and* $\beta:X\times Y\to Z$ *is a continuous bilinear function, then for all* $\xi\in M^2_{loc}(0,\infty;X^d)$, $\eta\in M^2_{loc}(0,\infty:Y^d)$ *and all* $t\geqslant 0$

$$\mathbb{E}\beta\left(\int_0^t\langle\xi(r),\mathrm{d}w(r)\rangle,\int_0^t\langle\eta(r),\mathrm{d}w(r)\rangle\right)=\int_0^t\mathbb{E}\tilde{\beta}(\xi(r),\eta(r))\,\mathrm{d}r,$$

where for $\xi\in X^d$, $\eta\in X^d$ *we put* $\tilde{\beta}(\xi,\eta)=\Sigma\,\beta(\xi^j,\eta^j)\in Z$.

(v) (Stochastic Fubini Theorem) *If* $g\in L^2([0,T)\times[0,T)\times\Omega,\ \mathscr{F}_T;X^d)$ *is such that* $g(t,\cdot)\in M^2(0,T;X^d)$ *for almost all* $t\in[0,T)$, *then for all* $t\geqslant 0$, *a.s.*,

$$\int_0^t\int_0^t\langle g(s,r),\mathrm{d}w(r)\rangle\,\mathrm{d}s=\int_0^t\left\langle\int_0^t g(s,r)\,\mathrm{d}s,\mathrm{d}w(r)\right\rangle. \tag{3.5}$$

PROPOSITION 3.4. *If* $\xi\in M^2_{loc}(0,\infty;X^d)$ *then*
(1) $t\longrightarrow x(t)=\int_0^t\langle\xi(s),\mathrm{d}w(s)\rangle$ *is an* X*-valued martingale, with almost all paths continuous; moreover* $x\in M^2_{loc}(0,\infty;X)$.
(2) *for any* $r\in(1,\infty)$

$$\mathbb{E}\sup_{t\leqslant T}\left|\int_0^t\langle\xi(s),\mathrm{d}w(s)\rangle\right|^r\leqslant\left(\frac{r}{r-1}\right)^r C_r(X)\mathbb{E}\left\{\int_0^T|\xi(s)|^2\,\mathrm{d}s\right\}^{r/2}. \tag{3.6}$$

In particular $x\in L^2(\Omega,\mathbb{C}(0,T;X))$.

COROLLARY 3.5. (A simple case of Itô's formula) *Assume that* X, Y, Z *are M-type 2 Banach spaces and* $\beta\colon X\times Y\longrightarrow Z$ *is a continuous bilinear map. Let* $\xi\in M^p_{loc}(0,\infty;X^d)$ *and* $\eta\in M^q_{loc}(0,\infty;Y^d)$ *for some* $p,q\in(2,\infty)$ *and let*

$$x(t)=\int_0^t\langle\xi(s),\mathrm{d}w(s)\rangle\quad\textit{and}\quad y(t)=\int_0^t\langle\eta(s),\mathrm{d}w(s)\rangle.$$

Then the process $\beta(x(\cdot),y(\cdot))$ *belongs to* $M^2_{loc}(0,\infty;Z)$ *and for all* $t\geqslant 0$ *a.s.*,

$$\beta(x(t),y(t))=\sum_{j=1}^d\int_0^t\{\beta(\xi^j(s),y(s))+\beta(x(s),\eta^j(s))\}\,\mathrm{d}w^j(s)+\int_0^t\beta(\bar{\xi}(s),\bar{\eta}(s))\,\mathrm{d}s. \tag{3.7}$$

where $\bar{\xi}(s)=\Sigma_{j=1}^d\xi^j(s)$ *and* $\bar{\eta}(s)=\Sigma_{j=1}^d\eta^j(s)$.

4. The Main Results

In all this section, but in Proposition 4.1 where X is assumed to be only UMD Banach space, X denotes an M-type 2 and UMD Banach space. $\mathscr{L}(X)$ denotes the space of all linear bounded operators in X. Finally, $-A$ is a generator of an analytic semigroup $\{e^{-tA}\}_{t\geqslant 0}$ on X and satisfies the assumptions of [16], i.e.:

(H1) $\exists M>0$ such that for $\lambda\geqslant 0$, $(A+\lambda)^{-1}$ exists and

$$|(A+\lambda)^{-1}|\leqslant\frac{M}{1+\lambda};$$

(H2) $\forall s\in\mathbb{R}\,A^{is}$ exists and belongs to $\mathscr{L}(X)$, $\{A^{is}\}_{s\in\mathbb{R}}$ is a strongly continuous group on X and for some $K>0$ and $\vartheta_A<\pi/2$ the following inequality holds

$$|A^{is}|\leqslant Ke^{\vartheta_A|s|},\quad s\in\mathbb{R}.$$

REMARK 4.1. (i) What in fact we shall assume is the following condition

(H3) $A^z \in \mathscr{L}(X)$ for all $z \in \mathbb{C}$ with $\operatorname{Im} z \leqslant 0$ and for some $K > 0$, $\vartheta_A < \pi/2$ the following inequality holds

$$|A^z| \leqslant K e^{\vartheta_A |\operatorname{Im} z|}, \quad \text{for } \operatorname{Im} z \leqslant 0,$$

where $\operatorname{Im} z$ denotes the imaginary part of the complex number z.

Indeed, it is the stronger condition that is proved in all examples known to the author, see [41], [42], [20] and others. Although **(H2)** may be satisfied even if **(H3)** is not, no such an example is known to the author.

(ii) It may happen that the operator A does not satisfy the conditions **(H1)**–**(H2)** but for some $\omega > 0$, $A + \omega I$ does. However all the statements below remain true in that case.

By $D(A)$ we denote the domain of the operator A, which endowed with a natural norm is a Banach space. Moreover, as A^{-1} is bounded, $D(A)$ is isomorphic to X, and so is an M-type 2 and UMD Banach space.

We introduce now the real interpolation spaces $D_{A^m}(\vartheta, p) = (X, D(A^m))_{\vartheta,p}$ between X and $D(A^m)$, the domain of the m-th power of A, with parameters $\vartheta \in (0,1)$ and $p \in [1, \infty)$, where m is a nonnegative natural number, see [9] or [45],

$$D_{A^m}(\vartheta, p) = \left\{ x \in X : \int_0^\infty |t^{m(1-\vartheta)} A^m e^{-tA} x|^p \frac{dt}{t} < \infty \right\}. \tag{4.1}$$

In particular, the norm in $(X, D(A^m))_{\vartheta,p}$ is given by

$$|x|_{D_{A^m}(\vartheta,p)} = \int_0^\infty |t^{m(1-\vartheta)} A^m e^{-tA} x|^p \frac{dt}{t}.$$

In the case $m = 1$ the norm $|\cdot|_{D_A(\vartheta,p)}$ will be also denoted by $|\cdot|_{\vartheta,p}$.

REMARK 4.2. (i) If $\vartheta_1 \leqslant \vartheta_2$ then $D_A(\vartheta_2, p) \subseteq D_A(\vartheta_1, p)$ and if $\alpha > \vartheta$ then $D(A^\alpha) \subseteq D_A(\vartheta, p)$; if $p > q \geqslant 1$ then $D_A(\vartheta, q) \subseteq D_A(\vartheta, p)$.

(ii) The spaces $D_A(\vartheta, p)$ are invariant with respect to the semigroup $\{e^{-tA}\}$. In addition, $\{e^{-tA}\}$ is an analytic semigroup on $D_A(\vartheta, p)$ and if $\vartheta \leqslant 1 - 1/p$ it is a contraction semigroup; this is the case when $\vartheta = \frac{1}{2}$ and $p = 2$.

(iii) As observed above, if X is an M-type p Banach space then so is $D(A)$. Moreover the interpolation spaces $D_A(\vartheta, p)$ are of M-type p (with the same p). The same concerns the UMD property. See also Appendix A.

We will be particularly interested in the case $m = 1$, $\vartheta = \frac{1}{2}$ and $p = 2$. Then we put

$$\mathbb{V} = D_A(\tfrac{1}{2}, 2) = \left\{ x \in X : |x|^2_{D_A(\frac{1}{2},2)} = \int_0^\infty |A e^{-tA} x|^2 \, dt < \infty \right\}.$$

If X is a Hilbert space and A is a positive self-adjoint operator in X then $D_A(\frac{1}{2},2) = D(A^{\frac{1}{2}})$ and $|x|^2_{D_A(\frac{1}{2},2)} = \frac{1}{2}|A^{\frac{1}{2}}x|^2$. By $\|\cdot\|$ we shall denote any norm on $\mathbb{V} = D_A(\frac{1}{2},2)$ which is equivalent to $|\cdot|_{D_A(\frac{1}{2},2)}$.

First we recall a result which is a direct consequence of Theorem 3.2 from [16] and Theorems 1.8.2, 1.14.5 and 1.15.3 from [45].

PROPOSITION 4.1. *Let X be a UMD Banach space and suppose that a closed operator A satisfies conditions* **(H1)–(H2)**, *and let $T \in (0,\infty]$, $p \in (1,\infty)$.*

Then for every $u_0 \in D_A(1-1/p,p)$ and $f \in L^p(0,T;X)$ there exists a unique $u \in W^{1,p}(0,T)$ which is a solution to the following Cauchy problem

$$\begin{aligned} u'(t) + Au(t) &= f(t), \quad t \in (0,T), \\ u(0) &= u_0. \end{aligned} \tag{4.2}$$

Here $W^{1,p}(0,T) = \{u \in L^p(0,T;D(A))\}: u' \in L^p(0,T;X)\}$. Moreover, for some constant $C_p(X,A) > 0$

$$\left(\int_0^T |u(t)|^p_{D(A)}\,\mathrm{d}t + \int_0^T |u'(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} \leqslant \left(C_p(X,A)\int_0^T |f(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} + |u_0|_{D_A(1-1/p,p)}. \tag{4.3}$$

Let $\{w_t\}$ be a d-dimensional Wiener process as in Section 3. Let B^j, for $j = 1,\dots,d$, be linear operators in X, which are bounded as operators from $D(A)$ into $D_A(\frac{1}{2},2)$, and moreover satisfy

$$\sum_{j=1}^d |B^j x|^2_{D_A(\frac{1}{2},2)} \leqslant C_1|x|^2_{D(A)} + C_2|x|^2_{D_A(\frac{1}{2},2)}, \quad x \in D(A). \tag{4.4}$$

for some constants $C_1 > 0$ and $C_2 > 0$. By $|B|^2$ we denote the smallest constant C_1 for which (4.4) holds for some $C_2 > 0$.

Our goal now is to extend Proposition 4.1 to Itô equations. We consider the linear equation (1.1) and assume that

$$f \in M^2(0,T;X), u_0 \in L^2(\Omega,\mathscr{F}_0,P;D_A(\tfrac{1}{2},2)).$$

By Z_T we denote the space

$$Z_T = Z_T(A) = M^2(0,T;D(A)) \cap \mathscr{C}(0,T;L^2(\Omega,\mathscr{F},P;D_A(\tfrac{1}{2},2))). \tag{4.5}$$

DEFINITION 4.1. A strict solution to problem (1.1) is a function $u \in Z_T(A)$ that satisfies equation (1.1) in the following integral form

$$u(t) + \int_0^\infty Au(s)\,\mathrm{d}s = u_0 + \int_0^t \sum_{j=1}^d B^j u(s)\,\mathrm{d}w^j(s) + \int_0^t f(s)\,\mathrm{d}s, \quad \text{a.s., } t \geqslant 0. \tag{4.6}$$

A function $u \in Z_T(A)$ is a mild solution to problem (1.1) if it satisfies

$$u(t) = e^{-tA}u_0 + \sum_{j=1}^{d} \int_0^t e^{-(t-r)A} B^j u(r)\, \mathrm{d}w^j(r) + \int_0^t e^{-(t-r)A} f(r)\, \mathrm{d}r \tag{4.7}$$

a.s., for any $t \geqslant T$.

Following some ideas of [9] we shall prove

PROPOSITION 4.2. *Under the assumptions mentioned above a function $u \in Z_T(A)$ is a strict solution to problem* (1.1) *if and only if it is a mild solution.*

The proof of Proposition 4.12 will be concluded before stating Lemma 4.5. We will need the following

LEMMA 4.3. *Let $g \in M^2(0, T; (D_A(\frac{1}{2}, 2))^d)$, $f \in M^2(0, T; X)$ and let*

$$y(t) = \sum_{j=1}^{d} \int_0^t e^{-(t-r)A} g^j(r)\, \mathrm{d}w^j(r) = \int_0^t e^{-(t-r)A} \langle g(r), \mathrm{d}w(r)\rangle, \tag{4.8}$$

$$z(t) = \int_0^t e^{-(t-r)A} f(r)\, \mathrm{d}r. \tag{4.9}$$

Then both y and z belong to $M^2(0, T; D(A))$ and

$$\mathbb{E}\int_0^T |y(s)|^2_{D(A)}\, \mathrm{d}s \leqslant C_2(X)\mathbb{E}\int_0^T |g(s)|^2_{D_A(\frac{1}{2},2)^d}\, \mathrm{d}s, \tag{4.10}$$

$$\mathbb{E}\int_0^T |z(s)|^2_{D(A)}\, \mathrm{d}s \leqslant C_2(X, A)\mathbb{E}\int_0^T |f(s)|^2\, \mathrm{d}s, \tag{4.11}$$

where $C_2(X)$ (resp. $C_2(X, A)$) is as in Proposition 3.1 *(resp.* 4.1).

Moreover $y(t)$ and $z(t)$ satisfy respectively

$$y(t) + \int_0^t Ay(s)\, \mathrm{d}s = \int_0^t \langle g(r), \mathrm{d}w(r)\rangle, \quad t \geqslant 0, \tag{4.12}$$

$$z'(t) + Az(t) = f(t), t > 0; \quad z(0) = 0. \tag{4.13}$$

Proof. By Remark 4.2(iii) $D(A)$ is an M-type 2 and UMD Banach space. L_2 $(D(A))$, the M-type 2 constant of $D(A)$ is equal to $L_2(X)$ (due to the definition of the norm in $D(A)$). Similarly $K_2(D(A)) = K_2(X)$ and thus $C_2(D(A)) = C_2(X)$. Therefore, in view of Corollary 3.2 we have

$$\mathbb{E}|y(t)|^2_{D(A)} \leqslant C_2(X)\mathbb{E}\int_0^t |e^{-(t-s)A} g(s)|^2_{D(A)^d}\, \mathrm{d}s,$$

hence by Fubini Theorem

$$\mathbb{E}\int_0^T |y(t)|^2_{D(A)}\,dt \leqslant C_2(X)\mathbb{E}\int_0^T\int_0^t |e^{-(t-s)A}g(s)|^2_{D(A)^d}\,ds\,dt$$
$$\leqslant C_2(X)\mathbb{E}\int_0^T\int_r^T |e^{-(t-r)A}g(r)|^2_{D(A)^d}\,dt\,dr.$$

However, for fixed $r\in[0,T]$, the definition of the norm in $D_A(\frac{1}{2},2)$ yields

$$\int_r^T |e^{-(t-r)A}g(r)|^2_{D(A)^d}\,dt \leqslant |g(r)|^2_{D_A(\frac{1}{2},2)^d}$$

and so (4.10) is proven.

To prove (4.12), in view of (4.10) we may employ stochastic Fubini formula (3.5). We have

$$(-A)\int_0^t y(s)\,ds = \int_0^t\int_0^s (-A)e^{-(s-r)A}\langle g(r), dw(r)\rangle\,ds$$
$$= \int_0^t \left\langle \int_r^t (-A)^{e-(s-r)A}g(r)\,ds\,dw(r)\right\rangle = \int_0^t \langle e^{-(t-r)A}g(r) - g(r), dw(r)\rangle$$
$$= \int_0^t e^{-(t-r)A}\langle g(r), dw(r)\rangle - \int_0^t \langle g(r), dw(r)\rangle = y(t) - \int_0^t \langle g(r), dw(r)\rangle$$

and hence (4.12) holds. To prove (4.11) we observe that z is a pathwise solution to (4.13). Therefore by Proposition 4.1, $z(t)$ is progressively measurable and

$$\int_0^T |z(t)|^2_{D(A)}\,dt \leqslant C_2(X,A)\int_0^T |f(r)|^2\,dr, \text{ a.s.}$$

that concludes the proof of (4.11). □

The following extension of Lemma 4.3 will also be useful in the sequel.

LEMMA 4.4 *Let* $g\in M^2(0,T;(D_A(\frac{1}{2},2))^d)$, $f\in M^2(0,T;X)$ *and* y *and* z *be defined by* (4.8) *and* (4.9) *respectively. Then* y *and* z *belong to* $Z_T(A)$ *and*

$$\sup_{0\leqslant s\leqslant T}\mathbb{E}|y(s)|^2_{D_A(\frac{1}{2},2)} \leqslant C_2(X)\mathbb{E}\int_0^T |g(s)|^2_{D_A(\frac{1}{2},2)^d}\,ds, \tag{4.14}$$

$$\mathbb{E}\sup_{0\leqslant s\leqslant T}|z(s)|^2_{D_A(\frac{1}{2},2)} \leqslant C_2(X,A)\mathbb{E}\int_0^T |f(s)|^2_{D_A(\frac{1}{2},2)^d}\,ds, \tag{4.15}$$

with $C_2(X)$ *and* $C_2(X,A)$ *as before.*

Proof. The first part of this Lemma (i.e. concerning $M^2(0,T;D(A))$) is proven in Lemma 4.3. In the second part the claim about z is obvious (see the proof of Lemma

4.3). The inequality (4.14) follows from Corollary 3.2 and Remark 4.2. In order to prove that $y \in \mathscr{C}(0, T; L^2(\Omega, \mathscr{F}, P; D_A(\frac{1}{2}, 2)))$ we first assume that $g \in M^2(0, T; (D(A))^d)$. Then with $v(t)$ being a pathwise solution to $v'(t) + Av(t) = -AM(t)$, $t \in (0, T)$, $v(0) = 0$ and $M(t) = \int_0^t \langle g(r), \mathrm{d}w(r) \rangle$ we have $y(t) = v(t) + M(t)$ for all $t \geqslant 0$. Indeed, if $\tilde{y}(t) = v(t) + M(t)$ then $\tilde{y}(0) = 0$ and

$$\begin{aligned} \mathrm{d}\tilde{y}(t) &= \mathrm{d}v(t) + \mathrm{d}M(t) = -Av(t)\,\mathrm{d}t - AM(t)\,\mathrm{d}t + \langle g(r), \mathrm{d}w(r) \rangle \\ &= -A\tilde{y}(t) + \langle g(r), \mathrm{d}w(r) \rangle. \end{aligned}$$

Hence by Lemma 4.5 below (the implication (ii) $\Rightarrow$ (i)),

$$\tilde{y}(t) = \int_0^t e^{-(t-r)A} \langle g(r), \mathrm{d}w(r) \rangle = y(t).$$

Since $AM \in M^2(0, T; X)$, by Proposition 4.1 we get

$$v \in M^2(0, T; D(A)) \cap L^2(\Omega, \mathscr{F}, P;\ \mathscr{C}(0, T; D_A(\tfrac{1}{2}, 2))) \subset \mathscr{C}(0, T; L^2(\Omega, \mathscr{F}, P; D_A(\tfrac{1}{2}, 2))).$$

Moreover, in view of Proposition 3.4, $M \in \mathscr{C}(0, T; L^2(\Omega, \mathscr{F}, P; D_A(\frac{1}{2}, 2)))$. Density of $M^2(0, T; (D(A))^d)$ in $M^2(0, T; (D_A(\frac{1}{2}, 2))^d)$ and the inequality (4.14) conclude the proof. □

Since $B^j \in \mathscr{L}(D(A), D_A(\frac{1}{2}, 2))$, if $u \in M^2(0, T; (D(A)))$ then $g^j(t) := B^j u(t) \in M^2(0, T; (D_A(\frac{1}{2}, 2))^d)$ and hence Proposition 4.2 is a direct consequence of the following

LEMMA 4.5. *Assume that* $u_0 \in L^2(\Omega, \mathscr{F}_0, P; D_A(\frac{1}{2}, 2))$, $g \in M^2(0, T; (D_A(\frac{1}{2}, 2))^d)$, $f \in M^2(0, T; X)$ *and* $u \in M^2(0, T; D(A))$. *Then the following two conditions are equivalent:*

(*i*) $u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-r)A} \langle g(r), \mathrm{d}w(r) \rangle + \int_0^t e^{-(t-r)A} f(r)\,\mathrm{d}r$, *a.s., for* $t \leqslant T$.

(*ii*) $u(t) + \int_0^t Au(s)\,\mathrm{d}s = u_0 + \int_0^t \langle g(s), \mathrm{d}w(s) \rangle + \int_0^t f(s)\,\mathrm{d}s$, *a.s., for* $t \leqslant T$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from Lemma 4.3. In order to prove the converse one let us consider a process $v(t)$ given by

$$v(t) = e^{-tA} u_0 + \int_0^t e^{-(t-r)A} \langle g(r), \mathrm{d}w(r) \rangle + \int_0^t e^{-(t-r)A} f(r)\,\mathrm{d}r.$$

In view of Lemma 4.3, $v \in M^2(0, T; D(A))$ and

$$v(t) + \int_0^t Av(s)\,\mathrm{d}s = u_0 + \int_0^t \langle g(s), \mathrm{d}w(s) \rangle + \int_0^t f(s)\,\mathrm{d}s, \text{ a.s., } t \leqslant T.$$

Therefore it remains to consider the case $f = 0$, $g = 0$ and $u_0 = 0$. In other words, we need to show that if $u \in M^2(0, T; D(A))$ satisfies $u(t) + \int_0^t Au(s)\,ds = 0$ a.s. for all $t \in [0, T]$ then $u(t) = 0$ a.s., $t \in [0, T]$. However this is an easy application of Itô's formula (see Corollary 3.5). Indeed, for fixed $t \leqslant T$ we have

$$\begin{aligned} de^{-(t-s)A}u(s) &= Ae^{-(t-s)A}u(s) + e^{-(t-s)A}\,du(s) \\ &= Ae^{-(t-s)A}u(s) + e^{-(t-s)A}(-A)u(s) = 0,\ s \leqslant t \end{aligned}$$

and hence $e^{-(t-0)A}u(0) = e^{-(t-t)A}u(t)$ a.s., which concludes the proof. □

REMARK 4.3. The last part of the proof can be derived in another way, see Chapter 4 in [36]. For $k \in \mathbb{N}$ let $x_k(t) := (A + kI)^{-1}u(t)$. Obviously $x_k(0) = 0$ and

$$x_k \in M^2(0, T; X) \text{ and } dx_k(t) = \{-u(t) + kx_k(t)\}\,dt.$$

Taking $\langle x_k(t), \phi\rangle$ for $\phi \in X^*$ we easily have

$$x_k(t) = -\int_0^t e^{k(t-s)}u(s)\,ds$$

and, as

$$\|x_k(t)\|_{L^2(\Omega, \mathscr{F}_0, X)} \leqslant \|(k + A)^{-1}\|\,\|u(t)\|_{L^2(\Omega, \mathscr{F}_0, X)} \to 0 \text{ in } L^2(\Omega, \mathscr{F}_0, X),$$

we get

$$\int_0^t e^{k(t-s)}u(s)\,ds \to 0 \text{ in } L^2(\Omega, \mathscr{F}_t, X) \text{ for any } k \in \mathbb{N} \text{ and } t \in [0, T].$$

Applying Lemma 1.1 from Chapter 4 in [36] (which in fact is valid not only for continuous functions but for any L^1 function), we infer that $u = 0$ as an element of $L^2(0, T; L^2(\Omega, \mathscr{F}_0, X))$. In particular we get $u(t) = 0$ for all $t \geqslant 0$.

THEOREM 4.6. *Assume that X is both an M-type 2 and UMD Banach space and that $-A$ a generator of an analytic semigroup $\{e^{-sA}\}_{s \geqslant 0}$ on X. Assume also $-A$ satisfies the conditions* **(H1)–(H2)**.

Then there exists a positive number ϵ_0 with the following property. For any linear operators $B^1, \ldots, B^d$ satisfying (4.4) *with $|B| < \epsilon_0$ and for any*

$$u_0 \in L^2(\Omega, \mathscr{F}_0, P; D_A(\tfrac{1}{2}, 2)),\ f \in M^2(0, T; X), \tag{4.16}$$

the problem (1.1) *has a unique strict solution u in the class $Z_T(A)$ (defined in* (4.5)*).*

Proof. For a fixed $T > 0$ let us consider the following norm on the Banach space $Z_T = Z_T(A)$,

$$\|u\|_T = \max\{|u|_{M^2(0,T;D(A))}, |u|_{\mathscr{C}(0,T;L^2(\Omega,\mathscr{F},P;D_A(\frac{1}{2},2)))}\}.$$

We shall show that for T small enough, the equation (4.7) has a unique solution in Z_T. To do this let us consider an affine mapping $\Phi_T: Z_T \to Z_T$ defined by

$$v = \Phi_T(u) \text{ iff } v(t) = e^{-tA}u_0 + \sum_{j=1}^{d} \int_0^t e^{-(t-r)A}B^j u(r)\,\mathrm{d}u^j(r) + \int_0^t e^{-(t-r)A}f(r)\,\mathrm{d}r.$$

Due to Lemmata 4.3 and 4.4 Φ_T is a bounded affine operator. We take $u^1, u^2 \in Z_T$, put $u = u^1 - u^2$ and compute

$$v = v^1 - v^2 = \Phi(u^1) - \Phi(u^2) = \sum_{j=1}^{d} \int_0^t e^{-(t-r)A}B^j u(r)\,\mathrm{d}w^j(r).$$

Since $B^j u \in M^2(0, T; D_A(\frac{1}{2}, 2))$, in view of Lemma 4.3 and inequality (4.4) we have

$$|v|^2_{M^2(0,T;D(A))} \leqslant C_1 C_2(X) |u|^2_{M^2(0,T;D(A))} + C_2 C_2(X) \int_0^T \mathbb{E}|u(s)|^2_{D_A(\frac{1}{2},2)}\,\mathrm{d}s.$$

But the last integral is not greater than

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}|u(t)|_{D_A(\frac{1}{2},2)} \int_0^T \mathbb{E}|u(s)|_{D_A(\frac{1}{2},2)}\,\mathrm{d}s \leqslant \gamma T^{1/2} |u|_{M^2(0,T;D(A))} \times |u|_{\mathscr{C}(0,T;L^2(\Omega,\mathscr{F},P;D_A(\frac{1}{2},2)))},$$

where γ is norm of the inclusion $D(A) \hookrightarrow D_A(\frac{1}{2}, 2)$.

Similarly, applying Lemma 4.4 we obtain

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}|v(t)|^2_{D_A(\frac{1}{2},2)} \leqslant C_2(X) \int_0^T \mathbb{E}|Bu(s)|^2_{D_A(\frac{1}{2},2)}\,\mathrm{d}s.$$

Therefore

$$\|v\|_T^2 \leqslant C_1 C_2(X) |u|^2_{M^2(0,T;D(A))} + C_2 C_2(X) \gamma T^{1/2} |u|_{M^2(0,T;D(A))} |u|_{L^2(\Omega,\mathscr{F},P;\mathscr{C}(0,T;D_A(\frac{1}{2},2)))},$$

which allows us to infer that $\|\Phi_T\|^2 \leqslant C_2(X)\{C_1 + C_2 \gamma T^{1/2}\}$. Therefore by putting

$$\epsilon_0 = C_2(X)^{-\frac{1}{2}} \tag{4.17}$$

we find that for T sufficiently small, Φ_T is a strict contraction and hence possesses a unique fix point u. This u is obviously a solution to (4.7) on the time interval $[0, T)$. Since $u(T) \in L^2(\Omega, \mathscr{F}_T, P; D_A(\frac{1}{2}, 2))$ and T depends only on γ, C_1, C_2 and $C_2(X)$, we may proceed step by step to obtain a solution on any given *a priori* time interval.

□

COROLLARY 4.7. *Under the assumptions of Theorem* 4.6, *if for some* $\alpha \in (\frac{1}{2}, 1)$ *the linear operators* B^j *are bounded from* $D(A^\alpha)$ *to* $D_A(\frac{1}{2}, 2)$, *then the conclusions of Theorem* 4.6 *hold true, i.e. for any* $u_0 \in L^2(\Omega, \mathscr{F}_0, P; D_A(\frac{1}{2}, 2))$, $f \in M^2(0, T; X)$ *the problem* (1.1) *has a unique strict solution* u *that belongs to* $Z_T(A)$.

Proof. It is enough to observe that for any $\delta > 0$ there is C_δ such that

$$|x|^2_{D(A^\alpha)} \leqslant \delta |x|^2_{D(A)} + C_\delta |x|^2_{D_A(\frac{1}{2},2)}, \quad x \in D(A).$$

Indeed, we see that then the condition (4.4) is satisfied with the constant C_1 as small as we want. □

REMARK 4.4. If we had assumed that $B^j \in \mathscr{L}(D(A), D(A^\alpha))$ for some $\alpha \in (\frac{1}{2}, 1)$ then the conclusion would not have changed. Indeed, if $\gamma \in (\frac{1}{2}, \alpha)$ then from the intermediacy property of $D(A^\nu)$, $0 < \nu < 1$, by applying Young inequality we get that for any $\delta > 0$ there is $C_\delta > 0$ such that

$$|x|^2_{D_A(\frac{1}{2},2)} \leqslant C|x|^2_{D(A^\gamma)} \leqslant \delta |x|^2_{D(A^\alpha)} + C_\delta |x|^2_{D_A(\frac{1}{2},2)}, \quad x \in D(A).$$

REMARK 4.5. It is an open question whether and when

$$u \in L^2(\Omega, \mathscr{F}, P; \mathscr{C}(0, T; D_A(\tfrac{1}{2}, 2))).$$

This property holds in a special case of X being a Hilbert space and the pair (A, B) being coercive, see [34]. See also [27].

The problem of the continuity of paths of a process given by formula (4.8) is also treated in [10], again in the Hilbert space setting, but with w being an infinite dimensional Wiener process, however with $g =$ identity, i.e. with additive noise.

5. Decreasing of Regularity by Complex Interpolation Method

Let X be any Banach space and $-A$ is an infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t \geqslant 0}$ on X, such that it satisfies the conditions **(H1)–(H2)**. In particular A^{-1} exists and is bounded.

On the space X we define a scale of norms (parameterized by $\alpha \in (0, 1]$)

$$\rho_\alpha(x) := |A^{-\alpha} x|, \quad x \in X, \tag{5.1}$$

where, for $\alpha \in (0, 1)$ (see [36] §2.6),

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha}(tI + A)^{-1}\, dt = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-tA}\, dt. \tag{5.2}$$

We denote by Y_α the completion of X in the norm ρ_α; obviously X is dense in Y_α. Let $\Phi_\alpha \in \mathscr{L}(X, Y_\alpha)$ be the unique bounded extension of A^α; in fact Φ_α is an isometric isomorphism. We have the following

PROPOSITION 5.1. *Let $\alpha \in (0, 1)$ be fixed. Let S_t^α be the unique bounded extension of e^{-tA} to Y_α. Then $\{S_t^\alpha\}_{t \leqslant 0}$ is an analytic semigroup on Y_α with an infinitesimal generator denoted by $-\tilde{A}_\alpha$. Moreover the following properties hold.*

(i) $A \subset \tilde{A}_\alpha$, *i.e.*, $D(A) \subset D(\tilde{A}_\alpha)$ *and* $Ax = \tilde{A}_\alpha x$ *for* $x \in D(A)$;
(ii) $\tilde{A}_\alpha$ *satisfies the conditions* **(H1)**–**(H2)**;
(iii) $D(\tilde{A}_\alpha) = D(A^{1-\alpha})$, $D_{\tilde{A}_\alpha}(\vartheta, p) = \Phi_\alpha(D_A(\vartheta, p))$, *for* $0 < \vartheta < 1$, $1 < p < \infty$ *and* $D_{\tilde{A}_\alpha}(\vartheta, p) = D_A(\vartheta - \alpha, p)$ *if also* $\vartheta > \alpha$;
(iv) $D(\tilde{A}_\alpha^{1-\alpha}) = X$;
(v) $X \subseteq D_{\tilde{A}_\alpha}(\vartheta, p)$ *if* $\alpha < 1 - \vartheta$.

Proof. The proof follows from an observation that S_t^α exists and is given by $S_t^\alpha = \Phi_\alpha \circ e^{-tA} \circ \Phi_\alpha^{-1}$5 From this we easily derive the following sequence of equalities:

$$\tilde{A}_\alpha = \Phi_\alpha \circ A \circ \Phi_\alpha^{-1},\ \tilde{A}_\alpha^{-\beta} = \Phi_\alpha \circ A^{-\beta} \circ \Phi_\alpha^{-1},\ D(\tilde{A}_\alpha) = \Phi_\alpha(D(A)),$$
$$D(\tilde{A}_\alpha^\beta) = \Phi_\alpha(D(A^\beta)),\ D_{\tilde{A}_\alpha}(\vartheta, p) = \{y \in Y_\alpha : \Phi_\alpha^{-1} y \in D_A(\vartheta, p)\}.$$

From this and Remark 4.2(i) follow (i), two first parts of (iii), (iv) (as $\Phi_\alpha(D(A^\alpha)) = X$) and (v). Finally the last part of (iii) follows from Reiteration Theorem, see [45]. To prove (ii) we first observe first that $\rho(S_t x) \leqslant Me^{-t}$, so for $\lambda > 0$ $R(\lambda) := \int_0^\infty e^{-t\lambda} S_t \, dt \in \mathscr{L}(Y)$. But, see Theorem 3.1 in [36] $R(\lambda) = (\lambda I + \tilde{A})^{-1}$ and hence $\tilde{A}_\alpha$ satisfies the condition **(H1)**. We see also that $(\lambda I + \tilde{A})^{-1} \supset (\lambda I + A)^{-1}$.

To prove that $\tilde{A}_\alpha$ satisfies also the condition **(H2)** we use **(A.10)** from [16]. For $s \in \mathbb{R}$ we have $A^{-\alpha} A^{is} \subseteq A^{is} A^{-\alpha}$ and so for $x \in X$

$$\rho(A^{is} x) = |A^{-\alpha} A^{is} x| = |A^{is} A^{-\alpha} x| \leqslant K_A e^{\vartheta_A |s|} |A^{-\alpha} x| = K_A e^{\vartheta_A |s|} \rho(x).$$

Hence A^{is} possesses a unique extension, denoted by $(A^{is})^\sim$ for the time being, to a bounded linear operator on Y_α. And it is not difficult to see that $\tilde{A}^{is}$ and $(A^{is})^\sim$ coincides on X and hence $\tilde{A}_\alpha$ satisfies **(H2)**. □

REMARK 5.1. All the results of Proposition 5.1 are true if the condition **(H2)** is replaced by the following weaker one:

(H4) $A^{is} \in \mathscr{L}(X)$ and $|A^{is}| \leqslant C$ for all $s \in \mathbb{R}$ with $|s| \leqslant \delta$ for some $\delta > 0$, $C > 0$.

See Theorem 1.15.3 in [45]. In fact from **(H4)** it follows that $A^{is} \in \mathscr{L}(X)$ for all $s \in \mathbb{R}$ and $|A^{is}| \leqslant Me^{\gamma|s|}$ for some $M, \gamma > 0$, but now not necessarily with $\gamma < \pi/2$.

THEOREM 5.2. *Assume that X is a UMD and M-type 2 Banach space. Let $\{w_t\}_{t \geqslant 0}$ be a d-dimensional Wiener process with respect to the filtration $\{\mathscr{F}_t\}_{t \geqslant 0}$. Let $B^1, \ldots, B^d$ be linear operators in X that are bounded from $D(A^\alpha)$ to X, for some $\alpha \in (0, \frac{1}{2})$. Let the Banach space $Y_{1-\alpha}$ and the operator $A_{1-\alpha}$ be as in Proposition* 5.1.

Then for any $u_0 \in L^2(\Omega, \mathscr{F}_0, P; D_{\tilde{A}_{1-\alpha}}(\frac{1}{2}, 2))$, $f \in M^2(0, T; Y_{1-\alpha})$ problem (1.1) *has a unique strict solution u in $M^2(0, T; D(A^\alpha)) \cap \mathscr{C}(0, T; L^2(\Omega, \mathscr{F}, P; D_{\tilde{A}_{1-\alpha}}(\frac{1}{2}, 2)))$. In particular, if $u_0 \in L^2(\Omega, \mathscr{F}_0, P; X)$ then the strict solution exists (in the same class as above) and is unique.*

Proof. This result is a direct application of Proposition 5.1 and Remark 4.4. □

6. Hilbert Space Case

We start with the following (basic for our further considerations) result.

PROPOSITION 6.1. *Assume that H is a Hilbert space and $-A$ a generator of a $\mathscr{C}_0$ semigroup on H satisfies the conditions* **(H1)** *and* **(H4)**. *Then*

$$D_A(\vartheta, 2) = [H, D(A)]_\vartheta = D(A^\vartheta), \text{ for } 0 < \vartheta < 1, \tag{6.1}$$

where $[H, D(A)]_\vartheta$ denotes the complex interpolation space between H and $D(A)$ of order ϑ, see [45].

Proof. From Remark 3, 1.18.10 in [45] we have that (generally) $[H, D(A)]_\vartheta = (H, D(A))_{\vartheta,2}$. From Theorem 1.15.3 in [45], in view of **(H4)**, we get $[H, D(A)]_\vartheta = D(A^\vartheta)$. The equality $(H, D(A))_{\vartheta,2} = D_A(\vartheta, 2)$ concludes the proof. □

The next result strengthens Proposition 5.1 in the Hilbert space setting.

COROLLARY 6.2. *Let $\alpha \in (0,1)$ be fixed. Let $\{S_1^\alpha\}$ be the unique bounded extension of $\{e^{-tA}\}$ to Y_α. Then all conclusions of Proposition 5.1 hold and moreover $D(\tilde{A}_\alpha^{1-\alpha}) = H = D_{\tilde{A}_\alpha}(1-\alpha, 2)$. In particular $H = D_{\tilde{A}_{\frac{1}{2}}}(\frac{1}{2}, 2)$ and the norm in H is equivalent to the following*

$$|x|^2 = \int_0^\infty |A^{\frac{1}{2}} e^{-tA} x|^2 \, dt. \tag{6.2}$$

Let us recall that in the Hilbert space setting, Theorem 4.1 remains valid if $-A$ is only a generator of analytic semigroup, see [16] and [3]. Therefore, in what follows we drop the assumption **(H2)**. Thus, since $C_2(H) = 1$, from Theorem 4.6 and Proposition 6.1 we have

THEOREM 6.3. *Assume that H is a Hilbert space and $-A$, a generator of an analytic semigroup $\{e^{-sA}\}_{s\geqslant 0}$ in H, satisfies the conditions* **(H1)** *and* **(H4)**.

Then for any linear operators $B^1, \ldots, B^d$ satisfying (4.4) *with $|B| < 1$, for any $u_0 \in L^2(\Omega, \mathscr{F}_0, P; D(A^{\frac{1}{2}}))$ and $f \in M^2(0, T; H)$ the problem* (1.1) *has a unique strict solution u belonging to $Z_T(A)$.*

REMARK 6.1. Although $D(A^{\frac{1}{2}}) = D_A(\frac{1}{2}, 2)$ the proper inequality to be considered is (4.4). The reason for this lies in the fact the exact norm in $D_A(\frac{1}{2}, 2)$ plays an important role in proving Theorem 4.6.

COROLLARY 6.4. *Let H and A be as in Theorem* 6.3. *Assume that for some $C_1 < 1$, $C_2 > 0$ the linear operators $B^1, \ldots, B^d$ satisfy*

$$\sum_{j=1}^{d} |B^j x|^2 \leqslant C_1 |x|^2_{D(A^{\frac{1}{2}})} + C_2 |x|^2, \quad x \in D(A^{\frac{1}{2}}). \tag{6.3}$$

Then for any $u_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$ *and* $f \in M^2(0, T; D(A^{\frac{1}{2}})) \cap \mathscr{C}(0, T; L^2(\Omega, \mathcal{F}, P; H))$ *the problem* (1.1) *has a unique strict solution* u *in the class* $M^2(0, T; D(A^{\frac{1}{2}})) \cap \mathscr{C}(0, T; L^2(\Omega, \mathcal{F}, P; H))$.

REMARK 6.2. In view of Corollary (6.2) the inequality (6.3) reads

$$\sum_{j=1}^{d} \int_0^{\infty} |A^{\frac{1}{2}} e^{-tA} B^j x|^2 \, dt \leqslant C_1 |A^{\frac{1}{2}} x|^2 + C_2 |x|^2. \tag{6.4}$$

It follows that if the operator $A - m$ satisfies **(H1)** and **(H4)** for some $m > 0$ and $B^j = (A - m_j)^{\frac{1}{2}}$ with $m_j \geqslant 0$, $\Sigma_j m_j \leqslant m$ then the condition (6.4) is satisfied with all its consequences.

We have the same conclusion if the operators B^j satisfy $|(A - m_j)^{-\frac{1}{2}} B^j| \leqslant 1$ with m_j as before.

7. Some Examples

The examples given here are of some interest but they are not the most general. The reason is that besides the Hilbert space case, the complex interpolation method we have used so far gives not the best results. In Sections 10 and 11 we present more general treatment based on real interpolation method.

EXAMPLE 7.1. Let $X = L^p(\mathbb{R}^n)$ with $p \geqslant 2$ and $A = \sqrt{-\Delta + m^2}$ where $m > 0$ and Δ is the usual Laplacian. We put $D(A) = W^{1,p}(\mathbb{R}^n)$, where $W^{k,p}(D)$ is the Sobolev space of $L^p(D)$ functions such that all their derivatives up to order k belong to $L^p(D)$. Applying the Marcinkiewicz–Mihlin Interpolation Theorem, see [43], one sees that the operator A satisfies the conditions **(H1)**–**(H3)**. We consider also the following linear operators $(B^j u)(x) = b_j(x) u(x)$, $x \in \mathbb{R}^n$ for $b_j \in \mathscr{C}^1(\mathbb{R}^n)$ satisfying appropriate growth condition at infinity. Thus we consider the following Stochastic Partial Differential Equation

$$\begin{gathered} du + \sqrt{-\Delta + m^2}\, u \, dt + \sum b_j(x) u \, dw^j(t) = 0, \; t > 0, \\ u(0) = u_0, \end{gathered} \tag{7.1}$$

where $u_0 \in D_A(\frac{1}{2}, 2) = B^{\frac{1}{2}}_{p,2}(\mathbb{R}^n)$, see [45] for the definition and properties of the Besov spaces $B^s_{p,k}$. This equation is similar to the equation of free field found by Hida and Streit in [24], see also [40].

Since the operators B^j are bounded from $D(A)$ to $D(A^\alpha) = W^{\alpha,p}(\mathbb{R}^n)$ for some $\alpha \in (\frac{1}{2}, 1)$, in view of Remark 4.4 the problem (7.1) is well posed in the class $Z_T(A)$ for all $T > 0$, for any $u_0 \in L^2(\Omega, \mathscr{F}_0, P; B^{\frac{1}{2}}_{p,1})$. Observe also that now $Z_T(A) = M^2(0, T; W^{1,p}(\mathbb{R}^n)) \cap \mathscr{C}(0, T; L^2(\Omega, \mathscr{F}, P; B^{\frac{1}{2}}_{p,2}(\mathbb{R}^n)))$. In a similar way, since B^j are bounded operators from $D(A^\alpha) = W^{\alpha,p}(\mathbb{R}^n)$ to $X = L^p(\mathbb{R}^n)$, we can apply Theorem 5.2. In particular, for any $u_0 \in L^2(\Omega, \mathscr{F}_0, P; L^p(\mathbb{R}^n))$ there exists a unique solution $u \in M^2(0, T; W^{\alpha,p}(\mathbb{R}^n)) \cap \mathscr{C}(0, T; L^2(\Omega, \mathscr{F}, P; D_{\tilde{A}_{1-\alpha}}(\frac{1}{2}, 2)))$. In this moment however, we cannot identify the space $D_{\tilde{A}_{1-\alpha}}(\frac{1}{2}, 2)$ with some known functional space (besides with $\Phi_{1-\alpha}(D_A(\frac{1}{2}, 2))$, what follows from Proposition 5.1).

EXAMPLE 7.2. Let D be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary or let D be the whole $\mathbb{R}^n$. Let $-A$ be a uniformly elliptic operator of order $2m$, with smooth coefficients. We take $X = L^p(D)$ for $p \geqslant 2$ and $D(A)$ is a subset of $W^{k,p}(D)$ satisfying some appropriate boundary conditions. It is proven in [41] then the operator A satisfy the conditions **(H1)** and **(H4)**.

For the operator B^j we take any differential operators of order $\leqslant m - 1$ or integro-differential operators of order strictly smaller than m. Then for some $\alpha \in (0, \frac{1}{2})$, B^j map continuously $D(A^\alpha)$ (which is a subset of $W^{\alpha k,p}(D)$) into X. It means that Theorem 5.2 is applicable. In particular, we get existence and uniqueness result for stochastic partial differential equations of the form (1.1).

As a byproduct, also in all previous Theorems and Examples, we have continuous dependence of solution on the data.

8. Decreasing of Regularity by a Real Interpolation Method

We assume that the linear operator A, in a Banach space X satisfies the condition **(H1)** from Section 4. By $\{e^{-tA}\}_{t\geqslant 0}$ we denote the analytic semigroup on X, generated by $-A$. As in Section 5 we introduce a norm $|\cdot|_{-1}$ on the space X defined as follows

$$|x|_{-1} = |A^{-1}x|.$$

Let $Y = Y_{-1}$ be the completion of X in this norm. Thus X is a dense subspace of Y and the operator A extends to linear and bounded from X to Y. Moreover this extension, denoted in what follows by Φ, is a linear isometric isomorphism between X and Y.

Putting

$$S(t) = \Phi \circ e^{-tA} \circ \Phi^{-1} \tag{8.1}$$

we easily see that

1° $S(t)$ is an analytic semigroup on Y, its generator $-A_Y$ satisfies

$$\begin{gathered} A_Y = \Phi \circ A \circ \Phi^{-1}, \\ A \subset A_Y, \quad D(A_Y) = X, \\ D(A_Y^2) = D(A), \quad A_Y^2 = \Phi \circ A^2 \circ \Phi^{-1}. \end{gathered} \tag{8.2}$$

2° The following formulae hold

$$(Y, D(A))_{\vartheta,p} = \begin{cases} (Y, X)_{2\vartheta,p}, & \text{if } 0 < \vartheta < \frac{1}{2} \text{ and } < p < \infty, \\ (X, D(A))_{2\vartheta-1,p}, & \text{if } \frac{1}{2} < \vartheta < 1 \text{ and } 1 < p < \infty. \end{cases} \tag{8.3}$$

The first part of (8.3) follows from ([45], Theorem 1.15.2f), since $D(A) = D(A_Y^2)$ and $D(A_Y) = X$. The second one follows from the first one by applying Theorem 1.15.2(e) from [45]. Indeed

$$\begin{aligned}(Y, D(A))_{\vartheta,p} &= (Y, D(A_Y^2))_{\vartheta,p} = A_Y^{-1}(Y, D(A_Y^2))_{\vartheta-\frac{1}{2},p} \\ &= A_Y^{-1}(Y, D(A_Y))_{2\vartheta-1,p} = (D(A_Y), D(A_Y^2))_{2\vartheta-1,p} = (X, D(A))_{2\vartheta-1,p}.\end{aligned}$$

In particular, by putting $p = 2$ and $\vartheta = \frac{1}{4}$ or $\vartheta = \frac{3}{4}$ we obtain

$$\begin{aligned}(Y, X)_{\frac{1}{2},2} &= (Y, D(A))_{\frac{1}{4},2}, \\ (X, D(A))_{\frac{1}{2},2} &= (Y, D(A))_{\frac{3}{4},2}.\end{aligned} \tag{8.4}$$

Next by using the Reiteration Theorem, see [4], Theorem 3.5.4, from (8.4) we infer that

$$((Y, X)_{\frac{1}{2},2}, (X, D(A))_{\frac{1}{2},2})_{\frac{1}{2},2} = (Y, D(A))_{\frac{1}{2},2} =: (X, D(A))_{0,2},$$

where the last equality should be viewed as a definition of $(X, D(A))_{0,2}$.

In order to be able to proceed further we need the following

LEMMA 8.1. *Let $X \subset Y$ are two Banach spaces with X being densely and continuously imbedded into Y. Let $S(t)$ be a $\mathscr{C}_0$ (resp. analytic) semigroup acting on both spaces X and Y. Assume that Z is an interpolation space between X and Y of order $\vartheta \in (0, 1)$, i.e. there is a constant $C > 0$ such that if a linear operator $T: Y \to Y$, $X \to X$ is bounded in both spaces then T is also bounded in Z and*

$$|T|_{\mathscr{L}(Z)} \leqslant C |T|^{\vartheta}_{\mathscr{L}(X)} |T|^{1-\vartheta}_{\mathscr{L}(Y)}. \tag{8.5}$$

Let $T(t)$ be restriction of $S(t)$ to Z. Then $T(t)$ is a $\mathscr{C}_0$ (resp. analytic) semigroup on Z.

Moreover, if A_X and A_Y denote respectively the infinitesimal generator of $S(t)$ acting respectively X or Y, and A_Z denotes the infinitesimal generator of $T(t)$ then $A_X \subset A_Z \subset A_Y$ and $D(A_Z) = (\lambda - A_Y)^{-1}(Z)$ for $\lambda \in \rho(A_Z)$.

If $A|_Y$ and $A|_X$ satisfy the condition **(H4)** *(resp.* **(H3)**) *from Section 4, then also $A|_Z$ satisfies* **(H4)** *(resp.* **(H3)**).

Proof. Obviously $T(t)$ is a semigroup of bounded operators on Z. From (8.5) it follows that $\{T(t) - I\}_{0 \leqslant t \leqslant 1}$ are uniformly bounded in $\mathscr{L}(Z)$. Therefore, as X is dense in Z, to prove that the semigroup $T(t)$ is strongly continuous, it is sufficient to prove it for $z \in X$. Since $X \subset Z$ continuously, we have

$$|T(t)z - z|_Z \leqslant C |S(t)z - z|_X \to 0 \text{ as } t \to 0^+.$$

Another method of proving the above result would be by applying the Hille–Yosida Theorem. This method is even better suited for proving analyticity of $T(t)$ (in case when the semigroup $S(t)$ is analytic), one has only to use an extension of the Hille–Yosida Theorem to analytic semigroups, for example Theorem 5.2 in Chapter 2 of [36].

To prove the second part of the Lemma, let us first observe that the inclusion of domains is obvious. Next, we may assume that $0 \in \rho(A_X) \cap \rho(A_Y)$, i.e. $A_X^{-1}: X \to X$ and $A_Y^{-1}: Y \to Y$ are bounded. Since also $A_Y^{-1}|_X = A_X^{-1}$, from (8.5) we infer that $A_Y^{-1}|_Z: Z \to Z$ is bounded. As obviously $A_Z^{-1} = A_Y^{-1}|_Z$, we finally have

$$D(A_Z) = A_Z^{-1}(Z) = A_Y^{-1}(Z).$$

To prove the last part of the Theorem, it is sufficient to notice that $A_X^{it} \subset A_Z^{it} \subset A_Y^{it}$ and so by (8.5)

$$|A_Z^{it}|_{\mathscr{L}(Z)} \leqslant C|A_X^{it}|^{\theta}_{\mathscr{L}(X)}|^{it}_{Y}|^{1-\theta}_{\mathscr{L}(Y)}. \qquad \square$$

We now return to the situation from before Lemma 8.1. Putting

$$Z = (Y, X)_{\frac{1}{2},2} \text{ and } T(t) = S(t)|_Z \tag{8.6}$$

we easily have

COROLLARY 8.2. 1° *$T(t)$ is an analytic semigroup on Z (with a generator denoted by $-A_Z$), $T(t)$ is extension of e^{-tA} and $A \subset A_Z \subset A_Y$.*
2°

$$D(A_Z) = (X, D(A))_{\frac{1}{2},2} = (Y, D(A_Y^2))_{\frac{3}{4},2} \tag{8.7a}$$

$$(Z, D(A_Z))_{\frac{1}{2},2} = (Y, D(A))_{\frac{1}{2},2} = (Y, D(A_Y^2))_{\frac{1}{4},2} =: (D(A), Y)_{0,2}. \tag{8.7b}$$

3° *$A_Z^{\frac{1}{2}}$ is a bounded operator from $D(A_Z)$ to $(Z, D(A_Z))_{\frac{1}{2},2}$ and from $(Z, D(A_Z))_{\frac{1}{2},2}$ to Z.*

Proof. 1° and the first part of 2°, i.e. the equality (8.7a) follow directly from the precedent results. Indeed, in view of Theorem 1.15.2(e) in [45] we have by (8.3)

$$D(A_Z) = A_Y^{-1}(Z) = A_Y^{-1}(Y, D(A_Y^2))_{\frac{1}{4},2} = (Y, D(A_Y^2))_{\frac{3}{4},2} = (X, D(A))_{\frac{1}{2},2}.$$

The equality (8.7b) follows from the Reiteration Theorem. In fact, we have the following sequence of equalities

$$(Z, D(A_Z))_{\frac{1}{2},2} = (Y, D(A_Y^2))_{\frac{1}{4},2}, ((Y, D(A_Y^2))_{\frac{3}{4},2})_{\frac{1}{2},2} = (Y, D(A_Y^2))_{\frac{1}{2},2}.$$

To prove 3° let us first observe that in view of Lemma 8.1, from Theorem 1.15.2(e) in [45] it follows that

$$A_Y^{-\frac{1}{2}}: (Y, D(A_Y^2))_{\frac{1}{4},2} \to (Y, D(A_Y^2))_{\frac{1}{2},2},$$
$$A_Y^{-\frac{1}{2}}: (Y, D(A_Y^2))_{\frac{1}{2},2} \to (Y, D(A_Y^2))_{\frac{3}{4},2}$$

are linear isomorphisms. Thus, in view of (8.7a), (8.7b), the equality $A_Y^{-\alpha}|_Z = A_Z^{-\alpha}$ (which follows directly from the definition of fractional powers) concludes the proof of Corollary 8.2. □

REMARK 8.1. The infinitesimal generator A_Z of the semigroup $T(t)$ has the following (important) property

$$D(A_Z^{\frac{1}{2}}) = (Z, D(A_Z))_{\frac{1}{2},2}. \tag{8.8}$$

Indeed, $A_Z^{\frac{1}{2}}: D(A_Z) \to D(A_Z^{\frac{1}{2}})$ and $A_Z: D(A_Z^{\frac{1}{2}}) \to Z$ isomorphically, so (8.8) follows from part 3° of Corollary 8.2.

In other words, we have constructed an extension Z of the Banach space X with an appropriate extension $T(t)$ of the semigroup e^{-tA}, such that the generator A_Z of $T(t)$ satisfies (8.8).

COROLLARY 8.3. *Under the assumptions of Corollary* 8.2 *if* $(X, D(A))_{\frac{1}{2},2} \subset D(A^{\frac{1}{2}})$, *then the linear operator*

$$A^{\frac{1}{2}}: (X, D(A))_{\frac{1}{2},2} = D(A_Z) \to (Z, D(A_Z))_{\frac{1}{2},2}$$

is bounded.

Proof. Since $A^{-\frac{1}{2}} \subset A_Z^{-\frac{1}{2}}$ and $A^{\frac{1}{2}}|_{D(A_Z)} = A_Z^{\frac{1}{2}}$, the proof follows from Corollary 8.2 in view of the assumptions. A detailed proof is as follows.

Since $(X, D(A))_{\frac{1}{2},2} \subset D(A^{\frac{1}{2}})$

$$A^{-\frac{1}{2}}: (X, D(A))_{\frac{1}{2},2} \to A^{-\frac{1}{2}}((X, D(A))_{\frac{1}{2},2}) \subset X.$$

By 3° of Corollary 8.2,

$$A_Z^{-\frac{1}{2}}: D(A_Z) = (X, D(A))_{\frac{1}{2},2} \to (Z, D(A_Z))_{\frac{1}{2},2},$$

and finally, as $A^{-1} \subset A_Z^{-1}$,

$$A^{-\frac{1}{2}} \subset A_Z^{-\frac{1}{2}}.$$

Hence the following equality holds

$$A^{-\frac{1}{2}}((X, D(A))_{\frac{1}{2},2}) = (Z, D(A_Z))_{\frac{1}{2},2}$$

and $A^{-\frac{1}{2}}: (X, D(A))_{\frac{1}{2},2} \to (Z, D(A_Z))_{\frac{1}{2},2}$ boundedly. □

Since the spaces Z and $(Y, D(A))_{\frac{1}{2},2}$ play such an important role in our paper they deserve greater attention. We shall prove

PROPOSITION 8.4. *The norm in Z is equivalent to the following one*

$$\int_0^\infty |Ae^{-tA}x|^2 t^{\frac{5}{4}}\,dt, \tag{8.9}$$

while the norm in $(Y, D(A))_{\frac{1}{2},2}$ *is equivalent to the following*

$$\int_0^\infty |Ae^{-tA}x|^2 t\,dt. \tag{8.10}$$

Proof. Let us fix any $\theta \in (0,1)$ and let us denote by $|\cdot|$ the norm in $(Y, D(A))_{\theta,2} = (Y, D(A_Y^2))_{\theta,2}$. From the formula (4.1) we have, for $y \in D(A_Y^2)$,

$$\begin{aligned}|y|^2 &= \int_0^\infty |t^{2(1-\theta)} A_Y^2 e^{-tA_Y} y|_Y^2 \frac{dt}{t}\\ &= \int_0^\infty t^{3-4\theta} |A_Y^2 e^{-tA_Y} y|_Y^2\,dt = \int_0^\infty |\Phi A^2 e^{-tA} \Phi^{-1} y|_Y^2 t^{3-4\theta}\,dt\\ &= \int_0^\infty |A^{-1}\Phi A^2 e^{-tA} A y|_X^2 t^{3-4\theta}\,dt = \int_0^\infty |e^{-tA} A y|_X^2 t^{3-4\theta}\,dt.\end{aligned}$$

We conclude the proof by putting $\theta = \frac{1}{2}$ and $\theta = \frac{1}{4}$ and applying formulae (8.4) □

REMARK 8.2. Although the interpolation spaces can be (equivalently) normalized in many different ways, one of these norms plays a special role, see Section 4. It is the one defined by means of the integral, as in formula (4.1). If the norm in Z is defined as above, then the norm in $D_{A_Z}(\frac{1}{2}, 2) = (Z, D(A_Z))_{\frac{1}{2},2}$ is given by

$$|x|_{D_{A_Z}(\frac{1}{2},2)}^2 = \int_0^\infty |Aze^{-tA_Z}x|_Z^2\,dt = \int_0^\infty \int_0^\infty |Ae^{-(s+t)A}x|^2\,ds\,dt. \tag{8.11}$$

By a similar argument to that one used in the proof of Proposition 8.4 we can prove the following (always with norms given by (4.1)).

1° If $\frac{1}{2} \leqslant \vartheta < 1$ then

$$(Y, D(A_Y^2))_{\vartheta,2} = (X, D(A))_{2\vartheta-1,2} \quad \text{isometrically.}$$

2° If $0 < \vartheta < \frac{1}{2}$ then for $x \in X$

$$|x|_{(Y,D(A_Y^2))_{\vartheta,2}}^2 = \int_0^\infty |t^{2(1-\vartheta)} Ae^{-tA}x|^2 \frac{dt}{t},$$

$$|x|_{(Y,D(A_Y))_{\vartheta,2}}^2 = \int_0^\infty |t^{1-\vartheta} e^{-tA}x|^2 \frac{dt}{t}, \quad \text{for } x \in X.$$

Hence

$$A_Y:(X,D(A))_{\vartheta,2}\to(Y,D(A_Y))_{\vartheta,2}$$

is an isometric isomorphism.

In the framework developed so far Theorem 4.6 takes the following form.

THEOREM 8.5. *Assume that X is an M-type 2 Banach space, $-A$ is an infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ on X. Assume that A satisfies the conditions* **(H1)–(H3)** *from Section* 4. *Let Z and A_Z be the spaces described before. Assume that*

$$(X,D(A))_{\frac{1}{2},2}\subset D(A^{\frac{1}{2}})$$

and that the linear operators B_j, $j=1,\dots,d$, satisfy

$$B_j:(X,D(A))_{\frac{1}{2},2}\to(X,D(A))_{0,2} \text{ is bounded.} \tag{8.12}$$

Then there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$ the problem (1.1) *with B_j replaced by εB_j possesses a unique solution $u\in M^2(0,T;D_A(\frac{1}{2},2))\cap\mathscr{C}(0,T;L^2(\Omega,D_A(0,2)))$ for arbitrary $u_0\in L^2(\Omega,\mathscr{F}_0;D_A(0,2))$ and $f\in M^2(0,T;Z)$.*

As we have already noted, Theorem 8.5 is a direct consequence of Theorem 4.6. The importance of Theorem 8.5 lies in its applicability to Stochastic PDE of the type (1.2). This will be seen in the next section. Below we give some sufficient conditions for the condition (8.12) to hold.

PROPOSITION 8.6. *Assume that $D_A(\frac{1}{2},2)=(X,D(A))_{\frac{1}{2},2}\subset D(A^{\frac{1}{2}})$ and let a linear operator B satisfy the following conditions.*

$$A^{-\frac{1}{2}}B:X\to X \text{ is bounded,}$$

$$B:D(A^{\alpha})\to D(A^{\alpha-\frac{1}{2}}), \text{ is bounded, for some } \alpha\in(\tfrac{1}{2},1).$$

Then B satisfies the condition (8.12), *i.e.*

$$B:(X,D(A))_{\frac{1}{2},2}\to(X,D(A))_{0,2} \text{ is bounded.}$$

Proof. We first show that the linear operator $A^{-\frac{1}{2}}B:(X,D(A))_{\frac{1}{2},2}\to(X,D(A))_{\frac{1}{2},2}$ is bounded. Because, see [45], p. 101, $(X,D(A))_{\frac{1}{2},2}=(X,D(A^{\alpha}))_{1/2\alpha,2}$, it is enough to show that $A^{-\frac{1}{2}}B$ is a bounded operator both in X and $D(A^{\alpha})$. Obviously, only the latter fact is to be proven. Since accordingly to [45] (Theorem 1.15.2), $A^{-\frac{1}{2}}$ is isomorphism from $D(A^{\alpha-\frac{1}{2}})$ onto $D(A^{\alpha})$, this statement follows from the assumptions.

We conclude the proof by writing down $B=B\circ A^{\frac{1}{2}}\circ A^{-\frac{1}{2}}\circ B$ and applying Corollary 8.2 point 3°. □

We finish this section by stating a result which is a direct corollary of Theorem A.7.

PROPOSITION 8.7. *Assume that X is a Banach space of M-type 2 (resp. type 2). Then the spaces Y and Z defined above are of M-type 2 (resp. type 2). Moreover,* $\max\{L_2(Y), L_2(Z)\} \leqslant L_2(X)$ *(resp.* $\max\{K_2(Y), K_2(Z)\} \leqslant K_2(X)$. *In particular* $\max\{C_2(Z), C_2(Y)\} \leqslant C_2(X)$.

9. Duality Theory

In this short section we give another interpretation of the spaces Y and Z introduced in the previous section, this time in terms of the dual operator A^*.

LEMMA 9.1. *Assume that $-A$ is an infinitesimal generator of a $\mathscr{C}_0$ semigroup on a Banach space X. Then the norms in Y and $(D(A^*))^*$ are equivalent and Y can be identified with $(D(A^*))^*$.*

Proof. Since A^{-1} is bounded also $(A^*)^{-1}$ is bounded and $D(A^*)$ endowed with the norm $|\cdot|_{D(A^*)} = |A^*(\cdot)|$ is a Banach space and for $x \in X$ we have the following

$$\begin{aligned} |x|_{D(A^*)^*} &= \sup_{|\varphi|_{D(A^*)}=1} |\langle x, \varphi\rangle| \\ &= \sup_{|A^*\varphi|=1} |\langle A^{-1}x, A^*\varphi\rangle| = |A^{-1}x| = |x|_Y. \end{aligned} \tag{9.1}$$

□

If X is a reflexive Banach space, in view of Theorem 10.5 in Chapter 1 of [36] $D(A^*)$ is a dense subspace of X^* and $-A^*$ is an infinitesimal generator of a $\mathscr{C}_0$ semigroup on X^*. Thus we have

$$D(A^*) \subset X^*, \text{ and } X^{**} = (X^*)^* \subset (D(A^*))^*. \tag{9.2}$$

If now $\theta \in (0,1)$ and $1 < p < \infty$ then by Theorem 3.7.1 in [4], we have (after identifying X^{**} with X)

$$(X^*, D(A^*))^*_{1-\vartheta,q} = ((D(A^*))^*, X^{**})_{\vartheta,p} = (Y, X)_{\vartheta,p},$$

where $1/p + 1/q = 1$. Therefore, we have proven the first part of

COROLLARY 9.2. *If X is a reflexive Banach space, then under the assumptions of Lemma* 9.1, *for any $\theta \in (0,1)$ and $1 < p < \infty$ the following equality holds:*

$$(X^*, D(A^*))^*_{1-\vartheta,q} = (Y, X)_{\vartheta,p}. \tag{9.3}$$

In particular,

$$Z = (Y, X)_{\frac{1}{2},2} = (X^*, D(A^*))^*_{\frac{1}{2},2} \tag{9.4}$$

and for any $\alpha \in (0,1)$

$$(Y, D(A))_{\frac{1}{2},2} = ((Y, X)_{1-\alpha,2}, (X, D(A))_{\alpha,2})_{\frac{1}{2},2}. \tag{9.5}$$

Proof. It remains only to prove the formula (9.5). It is an easy consequence of the Reiteration Theorem and (9.3). Indeed, we have

$$\begin{aligned}(Y, D(A))_{\frac{1}{2},2} &= ((Y, D(A))_{\frac{1}{2}-(\alpha/2),2}, (Y, D(A))_{\frac{1}{2}+(\alpha/2),2})_{\frac{1}{2},2}\\ &= ((Y, X)_{1-\alpha,2}, (X, D(A))_{\alpha,2})_{\frac{1}{2},2}.\end{aligned}$$

□

REMARK 9.1. At the end of this section let us observe that in view of (9.3) we also have

$$(Y, D(A))_{\frac{1}{2},2} = ((X^*, D(A^*))^*_{\alpha,2}, (X, D(A))_{\alpha,2})_{\frac{1}{2},2}. \tag{9.6}$$

Thus we infer that if for some $\alpha > 0$ both the spaces $(X, D(A))_{\alpha,2}$ and $(X^*, D(A^*))_{\alpha,2}$ do not depend on the operator A, then neither depends on $(Y, D(A))_{\frac{1}{2},2}$.

10. More Examples

We begin this section with a simple example, which however has some of the difficulties of the general case and suggests ways of overcoming them.

EXAMPLE 10.1. Consider the following Stochastic Partial Differential Equation in whole domain $\mathbb{R}^n$ (with ε being a real parameter and $b_j\colon \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth vector fields):

$$\begin{gathered} du(t,x) + (1-\Delta)u(t,x)\,dt + \varepsilon \sum_{j=1}^{d} (b_j(x)\cdot\nabla)u(t,x)\,dw^j(t) = \sum_{j=1}^{d} g_j(t,x)\,dw^j(t) + f(t,x)\,dt,\\ t>0, \quad x\in\mathbb{R}^n \qquad (10.1)\\ u(0,x) = u_0(x), \text{ for } x\in\mathbb{R}^n \end{gathered}$$

As the basic space we take $X = L^p(\mathbb{R}^n)$ with $p \geqslant 2$. Moreover we put

$$\begin{aligned} &A = -\Delta + 1 \text{ with } D(A) = W^{2,p}(\mathbb{R}^n),\\ &B_j = b_j(x)\nabla. \end{aligned}$$

According to [45] or [4] the space $D_A(\frac{1}{2}, 2)$ is equal to the Besov space $B^1_{p,2}(\mathbb{R}^n)$. Moreover, the space Z can be given by $Z = A(D_A(\frac{1}{2},2)) = (1-\Delta)(B^1_{p,2}) = B^{-1}_{p,2}$ and so $D_A(0,2) = (Z, D(A_Z))_{\frac{1}{2},2} = (B^{-1}_{p,2}, B^1_{p,2})_{\frac{1}{2},2} = B^0_{p,2}(\mathbb{R}^n)$.

Since $p \geqslant 2$, by [45] we have that $D_A(\frac{1}{2}, 2) \subset D(A^{\frac{1}{2}}) = W^{1,p}(\mathbb{R}^n)$, with equality only for $p = 2$.

Now we want to show that B_j is a bounded operator from $B^1_{p,2}$ to $B^0_{p,2}$. This can be done directly (in fact by repeating the proof from Section 8) or by verifying the sufficient conditions from Proposition 8.6. We choose the latter method. In the following two lemmata we omit the subscript j. We have the following

LEMMA 10.1. *Assume that the vector field* $b: \mathbb{R}^n \to \mathbb{R}^n$ *satisfies*

$$b \in L^\infty(\mathbb{R}^n; \mathbb{R}^n), \tag{10.2a}$$

$$\nabla \cdot b = \operatorname{div} b \in L^\infty(\mathbb{R}^n; \mathbb{R}). \tag{10.2b}$$

Let the linear operator B be given by

$$Bu = (b \cdot \nabla u) = \sum_{j=1}^n b_j(x) \frac{\partial u(x)}{\partial x_j},$$

a priori for $u \in \mathscr{C}^1(\mathbb{R}^n; \mathbb{R})$. *Let us denote* $J = (1 - \Delta)^{\frac{1}{2}}$ *and assume that* $p \in (1, \infty)$.

Then the linear operator $J^{-1}B$ *has a unique extension to a bounded operator from* $L^p(\mathbb{R})$ *to* $L^p(\mathbb{R}\ )$.

Proof. Let us fix p satisfying $1 < p < \infty$ and let q be given by $1/p + 1/q = 1$. We need to show that for some $C > 0$ and for all $\phi, \psi \in \mathscr{C}_0^\infty(\mathbb{R})$ the following inequality holds,

$$|\langle J^{-1}B\phi, \psi\rangle| \leqslant C|\phi|_{L^p}|\psi|_{L^q}.$$

This inequality follows from the following three facts:

$$J^{-1}: L^q(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n) \quad \text{continuously,}$$
$$B: W^{1,q}(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \quad \text{continuously,}$$
$$\langle J^{-1}B\phi, \psi\rangle = -\langle \phi, \operatorname{div} bJ^{-1}\psi\rangle - \langle \phi, BJ^{-1}\psi\rangle, \quad \phi, \psi \in \mathscr{C}^1(\mathbb{R}^n).$$

The first and the second are obvious, while the third one follows from the integration by parts formula and symmetricity of J^{-1} on $\mathscr{C}_0^1$. □

LEMMA 10.2. *Using the notation of Lemma* 10.1, *if* $p > n$, $1 < \alpha < 1 + 1/p$ *and* $b \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ *then the operator B is bounded from* $W^{\alpha,p}$ *to* $W^{\alpha-1,p}$.

Proof. It follows from the definition of Sobolev spaces, see [4], that $D_j = \partial/\partial x_j$ is a bounded operator from $W^{\alpha,p}$ to $W^{\alpha-1,p}$. Since b_j belongs to $W^{1,p}$ and $p > n$ it follows from the Sobolev imbedding theorem that the multiplication operator $v \to b_j v$ is continuous in both spaces $W^{1,p}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$. Hence, as $W^{\alpha-1,p}(\mathbb{R}^n) = [L^p(\mathbb{R}^n); W^{1,p}(\mathbb{R}^n)]_{1-\alpha}$, it is also continuous in $W^{\alpha-1,p}(\mathbb{R}^n)$ by the interpolation principle. This concludes the proof of Lemma 10.2. □

Now we see that from Proposition 8.6, Lemmata 10.1, 10.2 it follows

COROLLARY 10.3. *Assume that* $p > n$ *and*

$$b \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ with } \nabla \cdot b = \operatorname{div} b \in L^\infty(\mathbb{R}^n). \tag{10.3}$$

Then the linear operator B defined in Lemma 10.1 *maps continuously* $B_{p,2}^1(\mathbb{R}^n)$ *into* $B_{p,2}^0(\mathbb{R}^n)$.

THEOREM 10.4. *Assume that each vector field b_j satisfy the condition* (10.3) *and that* $p > n$, $p \geqslant 2$.

Then there exists $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, *there exists a unique solution*

$$u \in M^2(0, T; (B^1_{p,2}(\mathbb{R}^n))) \cap \mathscr{C}(0, T; L^2(\Omega; B^0_{p,2}(\mathbb{R}^n)))$$

to the problem (10.1) *for arbitrary*

$$u_0 \in L^2(\Omega, \mathscr{F}_0; B^0_{p,2}(\mathbb{R}^n)), \; f \in M^2(0, T; B^{-1}_{p,2}(\mathbb{R}^n)) \text{ and } g \in M^2(0, T; B^0_{p,2}(\mathbb{R}^n)^d).$$

Proof. The operator $-A = 1 - \Delta$ satisfies the conditions **(H1)**–**(H3)** from Section 4 and hence Theorem 10.4 is a direct consequence of Theorem 8.5, Corollary 10.3 and finally Theorem 4.6. □

The above result can be restated in a similar way to that given at the end of Section 8.

THEOREM 10.5. *Assume that* $p > n$, $b_j \in W^{2,p}(\mathbb{R}^n)$, $\mathbb{R}^n$ *for each* j. *Then there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, *there exists a unique solution* $u \in M^2(0, T; (B^2_{p,2}(\mathbb{R}^n))) \cap \mathscr{C}(0, T; L^2(\Omega; B^1_{p,2}(\mathbb{R}^n)))$ *to the problem* (10.1) *for any*

$$u_0 \in L^2(\Omega, \mathscr{F}_0; B^1_{p,2}(\mathbb{R}^n)), \; g \in M^2(0, T; (B^1_{p,2}(\mathbb{R}^n))^d) \text{ and } f \in M^2(0, T; B^0_{p,2}(\mathbb{R}^n)).$$

EXAMPLE 10.2. This example is a modification of the former one. We treat almost the same equation (with $1 - \Delta$ replaced by $-\Delta$), but in a bounded domain $D \subset \mathbb{R}^n$ with Dirichlet boundary conditions (the boundary conditions could as well be more general):

$$\begin{aligned} du(t,x) - \Delta u(t,x)\,dt + \varepsilon \sum_{j=1}^{d} (b_j(x) \cdot \nabla) u(t,x)\,dw^j(t) & \\ = \sum_{j=1}^{d} g_j(t,x)\,dw^j(t) + f(t,x)d,\, t > 0, x \in D & \qquad (10.4) \\ u(t,x) = 0, \text{ for } x \in \partial D,\; t > 0 & \\ u(0,x) = u_0(x), \text{ for } x \in D. & \end{aligned}$$

In order to be able to treat this initial value problem we first should describe the functional spaces necessary for a correct statement of the problem 10.4. Let $p \geqslant 2$ be a real number and

$$\begin{aligned} X &= L^p(D), \\ Au &= -\Delta u, \quad u \in D(A) = W^{2,p}(D) \cap W^{1,p}_0(D), \\ (B_j u)(x) &= (b_j(x) \cdot \nabla) u(x), \quad u \in D(A). \end{aligned}$$

Next we introduce the spaces Y and Z as in Example 10.1. Although it represents no difficulty on the abstract level, to identify them with concrete functional spaces, as it has been done in Example 10.1, requires some additional work. For doing so we apply Corollary 9.2. Since $A^* = -\Delta$ in $X^* = L^q(D)$ $(1/p + 1/q = 1)$ with the same boundary conditions as A, see [36], we have

$$D(A^*) = W^{2,q}(D) \cap W_0^{1,q}(D).$$

Thus, by Corollary 9.2

$$Z = (Y, X)_{\frac{1}{2},2} = (X^*, D(A^*))^*_{\frac{1}{2},2} = (L^q(D), W^{2,q}(D) \cap W_0^{1,q}(D))^*_{\frac{1}{2},2}.$$

But according to Theorem 4.3.3 in [45], see also [21],

$$(L^q(D), W^{2,q}(D) \cap W_0^{1,q}(D))_{\frac{1}{2},2} = \{u \in B^1_{q,2}(D) : u|_{\partial D} = 0\} = \mathring{B}^1_{q,2}(D) = \tilde{B}^1_{q,2}(D),$$

where the last equality follows from Theorem 4.3.2(c) in [45]. Hence, by Theorem 4.8.1 in [45] we obtain

$$Z = B^{-1}_{p,2}(D). \tag{10.5}$$

In order to determine the space $(Y, D(A))_{\frac{1}{2},2}$ we use the second part of Corollary 9.2. Taking α small enough ($\alpha \leqslant \min\{1/p, 1/q\}$, see Theorem 4.3.3 in [45]), we have

$$(X, D(A))_{\alpha,2} = B^{2\alpha}_{p,2}(D), \quad (X^*, D(A^*))_{\alpha,2} = B^{2\alpha}_{q,2}(D).$$

Hence

$$(Y, D(A))_{\frac{1}{2},2} = ((Y, X)_{1-\alpha,2}, (X, D(A))_{\alpha,2})_{\frac{1}{2},2},$$
$$(Y, X)_{1-\alpha,2} = (D(A^*), X^{**})_{1-\alpha,2} = (X^*, D(A^*))^*_{\alpha,2} = (B^{2\alpha}_{q,2}(D))^* = B^{-2\alpha}_{p,2}(D),$$

where the last equality follows from Theorem 4.8.2 in [45]. Hence we infer that

$$(Y, D(A))_{\frac{1}{2},2} = (B^{-2\alpha}_{p,2}(D), B^{-2\alpha}_{p,2}(D))_{\frac{1}{2},2} = B^0_{p,2}(D).$$

THEOREM 10.6. *Let $D \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathscr{C}^2$. Assume that $p > n$, $b \in W^{1,p}(D)$ with $\operatorname{div} b \in L^\infty(D)$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, there exists a unique solution $u \in M^2(0, T; \mathring{B}^1_{p,2}(D)) \cap \mathscr{C}(0, T; L^2(\Omega; B^0_{p,2}(D)))$ to the problem* (10.4) *for arbitrary*

$$u_0 \in L^2(\Omega, \mathscr{F}_0; B^0_{p,2}(D)), g \in M^2(0, T; (B^0_{p,2}(D))^d) \text{ and } f \in M^2(0, T; B^{-1}_{p,2}(D)).$$

Theorem 10.6 should be supplemented by a result, similarly as Theorem 10.4 has been supplemented by Theorem 10.5. But now, contrary to the previous case we not only need some regularity assumptions on the coefficients $b_j(x)$ but also we should be able to control the behaviour of $b_j(x)$ for x near the boundary ∂D, so that $B_j u \in \mathring{B}^1_{p,2}(D)$ if $u \in B^2_{p,2} \cap \mathring{B}^1_{p,2}(D)$. It turns out that the following condition

$$b_j \in W^{2,p}(D, \mathbb{R}^n) \text{ and } \langle b_j(x), \nu_x \rangle = 0, \text{ for } x \in \partial D, \text{ for each } j, \tag{10.6}$$

where ν_x denotes the outer normal to ∂D at point x, is sufficient. We have

THEOREM 10.7. *Under the same assumptions as in Theorem* 10.6 *with the additional one* (10.6), *there exists an* ε_0 *such that for all*

$$u_0 \in L^2(\Omega, \mathscr{F}_0; \mathring{B}^1_{p,2}(D)),$$
$$g \in M^2(0, T; (\mathring{B}^1_{p,2}(D))^d) \text{ and } f \in M^2(0, T; \mathring{B}^0_{p,2}(D)),$$

there exists $u \in M^2(0, T; B^2_{p,2}(D) \cap \mathring{B}^1_{p,2}(D)) \cap \mathscr{C}(0, T; L^2(\Omega; \mathring{B}^1_{p,2}(D)))$, *the unique solution to the problem* (10.4).

11. Applications

This section is devoted to presentation of general applications of the results from Sections 4, 8 and 9. These results are generalizations of the two examples from Section 10 to more general elliptic operators with more general boundary conditions. For the sake of completeness of the exposition, we present the precise results below.

Let D be a bounded open domain in $\mathbb{R}^n$ with a boundary of class $\mathscr{C}^\infty$. We make the following assumptions

(i) The differential operator $-A$

$$-A = \sum_{|\alpha| \leqslant 2k} a_\alpha(x) D^\alpha \tag{11.1}$$

is properly elliptic, see [45], 4.9.1. The coefficients a_α are $\mathscr{C}^\infty$ functions on the closure $\bar{D}$ of D.

(ii) A system $\{C_j\}_{j=1}^k$ of differential operators on ∂D is given,

$$C_j = \sum_{|\alpha| \leqslant m_j} c_{j,\alpha}(x) D^\alpha, \tag{11.2}$$

with the coefficients $c_{j,\alpha}$ being $\mathscr{C}^\infty$ functions on ∂D. The orders m_j of the operators C_j are ordered in the following way

$$0 \leqslant m_1 < m_2 < \cdots < m_k.$$

The system $\{C_j\}$ is normal, i.e. $m_k < 2k$ and

$$\sum_{|\alpha| = m_j} c_{j,\alpha}(x) \nu_x^\alpha \neq 0, \quad x \in D, j = 1, \ldots, k, \tag{11.3}$$

where ν_x is the unit outer normal vector to ∂D at $x \in \partial D$.

(iii)

$$(-1)^k \frac{a(x, \xi)}{|a(x, \xi)|} \neq -1, \quad x \in \bar{D}, \xi \in \mathbb{R}^n \setminus \{0\}, \tag{11.4}$$

where $a(x,\xi) = \Sigma_{|\alpha|=2k} a_\alpha(x)\xi^\alpha$.

(iv) If $c_j(x,\xi) = \Sigma_{|\alpha|=m_j} c_{j,\alpha}(x)\xi^\alpha$ then for all $x \in \partial D$, $\xi \in T_x(\partial D)$, $t \in (-\infty, 0]$ the polynomials

$$\{\tau \to c_j(x, \xi + \tau v_x)\}, \quad j = 1, \ldots, k$$

are linearly independent modulo polynomial $\{\tau \to \Pi_{j=1}^k (\tau - \tau_j^+(t))\}$. Here $\tau_j^+(t)$ are the roots with positive imaginary part of the polynomial $\mathbb{C} \ni \tau \to a(x, \xi + \tau v_x) - t$.

The differential operator A gives rise to a linear unbounded operator A_p in a Banach space $X = L^p(D)$ with a domain $D(A_p)$ defined by

$$D(A_p) = W^{2k,p}_{\{C_j\}}(D) = \left\{ u \in W^{2k,p}(D) : C_j u|_{\partial D} = 0 \text{ for } m_j < 2k - \frac{1}{p} \right\}. \tag{11.5}$$

It has been shown by Seeley in [41], see also [45], 4.9.1, that for any $\gamma > 0$ there is $C = C_\gamma > 0$ such that

$$|A_p^{it}| \leqslant Ce^{\gamma|t|}, \quad t \in \mathbb{R},$$

and therefore the operator A_p satisfies the condition **(H3)** from Section 4 (compare with [16]).

When there is no danger of ambiguity, the operator A_p will be denoted simply as A. In order to be able to apply our results from Sections 4 and 8 we need to determine the spaces Z and $(Z, D(A_Z))_{\frac{1}{2},2}$, We start with computation of the latter. This is quite an easy task and is based on formula (9.6). From [45] Theorem 4.3.3 we have

$$(L^p(D), D(A))_{\alpha,2} = B^{2k\alpha}_{p,2;\{C_j\}} \tag{11.6}$$

where $B^{2k\alpha}_{p,2;\{C_j\}} = \{u \in B^{2k\alpha}_{p,2}(D) : C_j u|_{\partial D} = 0 \text{ for } m_j < 2k\alpha - 1/p\}$. Taking a positive number α small enough (for example $0 < \alpha \leqslant 1/2kp$), in view of [45], 4.3.2 Theorem 1 we have in particular

$$(L^p(D), D(A))_{\alpha,2} = B^{2k\alpha}_{p,2}(D) = \mathring{B}^{2k\alpha}_{p,2}(D). \tag{11.7}$$

Now we need some facts about the dual operator A^*. Obviously

$$X^* = L^q(D), \tag{11.8}$$

with $1/p + 1/q = 1$, but to determine $D(A^*)$ we need some results from the theory of boundary value problems. The basic references are [1], [2] and [31], but in the latter only the case $p = 2$ is treated. It follows from the above cited works (Theorem 2.1 p. 114 in [31]) that there exists a system of n boundary operators, let us denote them by $C_1^*, \ldots, C_n^*$ such that

$$C_j^* = \sum_{|\alpha| \leqslant \mu_j} c_{j,\alpha}^*(x) D^\alpha, \tag{11.9}$$

with the coefficients c_j^* being $\mathscr{C}^\infty$ functions. We assume that the orders μ_j of operators C_j^* satisfy $0 \leqslant \mu_1 < \cdots < \mu_k < 2k$ and

$$D(A^*) = W^{2k,q}_{\{C_j^*\}}(D) = \left\{ u \in W^{2k,q}(D) : C_j^* u|_{\partial D} = 0 \text{ for } \mu_j < 2k - \frac{1}{q} \right\}. \tag{11.10}$$

Let us recall that although the operators C_j^* are not defined in a unique way, the space in r.h.s. of formula (11.10) is uniquely determined by the operator A and the boundary operators C_j.

By a similar argument as before we have the following two formulae

$$(L^q(D), D(A^*))_{\alpha,2} = B^{2k\alpha}_{q,2;\{C_j^*\}} = \left\{ u \in B^{2k\alpha}_{q,2}(D) : C_j^*|_{\partial D} = 0 \text{ for } m_j < 2k\alpha - \frac{1}{q} \right\},$$

$$(L^q(D), D(A^*))_{\alpha,2} = B^{2k\alpha}_{q,2}(D) = \mathring{B}^{2k\alpha}_{q,2}(D),$$

where the latter holds for positive α small enough (for example $0 < \alpha \leqslant 1/2kq$). Hence, if $0 < \alpha < 1/2kq$, Theorem 4.8.2(b) in [45] gives

$$(L^q(D), D(A^*))^*_{\alpha,2} = B^{-2k\alpha}_{q,2}(D).$$

Therefore, a repeating use of Section 4.3.2 in [45] gives

PROPOSITION 11.1. *Under the above assumptions and using the above notation*

$$(Y, D(A))_{\frac{1}{2},2} = (Z, D(A_Z))_{\frac{1}{2},2} = B^0_{p,2}(D). \tag{11.11}$$

It remains to compute the space Z. One of the possible methods is to apply formula (9.4). It follows from it and from some of the formulae given before (in this section) that

$$Z = (B^k_{q,2,\{C_j^*\}}(D))^*. \tag{11.12}$$

Yet, we do not see any reasonable general formula for Z (without using a dual space). However let us observe that in Examples 10.1 and 10.2 the space Z is completely determined.

Now we give another example.

EXAMPLE 11.1. In this example D is as before but we take $k = 1$, i.e. the operator A is of second order. We assume that it is given in the following divergence form

$$(Au)(x) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \right) + \sum_{i=1}^n d_i(x) \frac{\partial u(x)}{\partial x_i} + e(x)u(x), \quad x \in D, \tag{11.13}$$

with all coefficients of $\mathscr{C}^\infty$ class on the closure $\bar{D}$ of a bounded domain $\mathscr{C}^\infty$ domain D and the matrix $[a_{ij}(x)]$ not necessary symmetric. The boundary operator C is given by

$$(Cu)(x) = \varepsilon \frac{\partial u}{\partial \nu_A}(x) + \beta(x)u(x), \quad x \in \partial D, \tag{11.14}$$

again with $\mathscr{C}^\infty$ coefficients, where the first term is the "co-normal" derivative with respect to A,

$$\frac{\partial u}{\partial \nu_A}(x) = \sum_{i,j} a_{ij}(x)\nu_x^j \frac{\partial u}{\partial x_i}(x),$$

where $\nu_x = (\nu_x^1, \ldots, \nu_x^n)$ is the unit outer normal vector to ∂D at point $x \in \partial D$.

We take $X = L^p(D)$, $D(A) = W^{2,p}_{\{C\}}(D)$ and the operator A with domain $D(A)$ acting in X via the formula (11.13). Then, see [1], [2] or [23],

$$(C^*u)(x) = \varepsilon \frac{\partial u}{\partial \nu_A}(x) + \beta(x)u(x) - (d(x)\cdot \nu_x)u(x), \quad x \in \partial D. \tag{11.15}$$

In a particular case when $d = 0$ we may take 1°: $\varepsilon = 0$, $\beta = 1$ or 2°: $\varepsilon = 1$, $\beta = 0$. In the former case (Dirichlet boundary conditions) we have that $Z = B^{-1}_{p,2}$ while in the latter (Neumann boundary conditions) the space $Z = (B^1_{p,2})^*$ is even not a space of distributions on D, see [31] §4.7.2 Example 2.

Now we present the main result of this section. The second part of it is concerned with solutions of higher regularity with respect to the first one, compare with Section 10.

THEOREM 11.2. *Assume that D is a bounded domain in* $\mathbb{R}^n$ *with boundary* ∂D *of* $\mathscr{C}^\infty$ *class. Let A be a differential operator satisfying the properties (i)–(iv) above. Let also* $A = A_p$ *denote a linear operator in* $X = L^p(D)$ *with domain as in* (11.5), *where* $p \geqslant 2$. *Assume that* $\{w_t\}_{t\geqslant 0}$ *is a d-dimensional Wiener process. Assume also that* u_0 *is a random field on D and f, g are random processes satisfying*

$$u_0 \in L^2(\Omega, \mathscr{F}_0; B^0_{p,2}(D)), \quad g \in M^2(0,T;(B^0_{p,2}(D))^d), \quad f \in M^2(0,T;Z). \tag{11.16}$$

Finally let us assume that $B_1, \ldots, B_d$ *are linear differential operators of orders* $\leqslant k$,

$$B_j = \sum_{|\alpha|\leqslant k} b_{j,\alpha}(x)D^\alpha, \quad j = 1, \ldots, d, \tag{11.17}$$

with the coefficients $b_{j,\alpha}$ *of* $\mathscr{C}^\infty$ *class. Then there exists* ε_0 *independent of* u_0, f *and* g *such that for all* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ *the problem* (11.18) *below has a unique solution u belonging to* $M^2(0,T;\mathring{B}^k_{p,2;\{C_j\}} \cap \mathscr{C}(0,T;L^2(\Omega;B^0_{p,2}(D))))$.

$$\begin{aligned} du(t,x) + Au(t,x)\,dt + \sum_{j=1}^d (\varepsilon B_j u(t,x) - g_j(t,x))\,dw^j(t) &= f(t,x)\,dt, \quad t > 0,\ x \in D, \\ u(0,x) &= u_0(x), \text{ for } x \in D, \\ C_j u(t,x) &= 0, \text{ for } x \in \partial D,\ t > 0. \end{aligned} \tag{11.18}$$

Moreover, if we assume (for simplicity) that the coefficients $b_{j,\alpha}$ have compact support in D and u_0, f, g satisfy the following stronger conditions

$$\begin{aligned} &u_0 \in L^2(\Omega, \mathscr{F}_0; B^1_{p,2;\{C_j\}}(D)),\\ &g \in M^2(0, T; (B^1_{p,2;\{C_j\}}(D))^d),\\ &f \in M^2(0, T; B^0_{p,2}(D)), \end{aligned} \tag{11.19}$$

then the solution u to (11.18) *satisfies*

$$u \in M^2(0, T; \mathring{B}^{2k}_{p,2;\{C_j\}} \cap \mathscr{C}(0, T; B^1_{p,2,\{C_j\}}(D))).$$

We should stress that the regularity assumptions about the coefficients $b_{j,\alpha}$ (including those about their supports) are only of technical nature and we have imposed them to avoid the discussion similar to that at the end of Section 10.

Appendix A. Interpolation of M-Type 2 Spaces

In this Appendix we show that the interpolation spaces between two Banach spaces of M-type p is again M-type p Banach space. This result, as stated above is not always true. It is true for complex method and for real one with parameter p. The exact statement is given below.

At the beginning we give some equivalent forms of the definition of M-type p Banach spaces. So let us fix a Banach space X, a complete probability space $(\Omega, \mathscr{F}, P)$ and a filtration $\{\mathscr{F}\}_{j=0}^{\infty}$. We put

$$M^p(X) = \{\eta = \{\eta_j\}_{j=0}^{\infty}\text{:satisfying the conditions } 1^\circ, 2^\circ, 3^\circ\}, \tag{A.1}$$

where:

1° η_j is $\mathscr{F}_j$ measurable, for each $j \in \mathbb{N}$,

2° $\mathbb{E}(\eta_j | \mathscr{F}_{j-1}) = 0$, for each $j \in \mathbb{N}$,

3° $\|\eta\|^p_{M^p(X)} := \sum_{j=0}^{\infty} \mathbb{E}|\eta_j|^p < \infty$.

Any element $\eta \in M^p(X)$ is called an X-valued martingale-difference sequence, or shortly, a martingale difference sequence.

LEMMA A.1. *$M^p(X)$ is a closed subspace of $L^p(\Omega \times \mathbb{N}, \mathscr{F} \bar{\times} \mathscr{B}(\mathbb{N}), P \times m)$ and so a Banach space. Moreover, there exists a projection Π from $L^p(\Omega \times \mathbb{N}, \mathscr{F} \bar{\times} \mathscr{B}(\mathbb{N}), P \times m)$ onto $M^p(X)$. Here $\mathscr{B}(\mathbb{N})$ is the σ-algebra of all subsets on $\mathbb{N}$ and m is the counting measure on $(\mathbb{N}, \mathscr{B}(\mathbb{N}))$.*

Proof. We denote the space $L^p(\Omega \times \mathbb{N}, \mathscr{F} \bar{\times} \mathscr{B}(\mathbb{N}), P \times m)$ by $L^p(\Omega \times \mathbb{N})$. For given $\xi \in L^p(\Omega \times \mathbb{N})$ we put

$$\eta_j = \mathbb{E}(\xi(\cdot\,. j) | \mathscr{F}_j) - \mathbb{E}(\xi(\cdot\,, j) | \mathscr{F}_{j-1}),$$

where as usually $\mathscr{F}_{-1} = \{\Omega, \emptyset\}$. Obviously, the sequence $\eta = \{\eta_j\}_{j=0}^{\infty}$ satisfies the conditions 1° and 2° above.

What concerns 3° we have by Jensen Inequality $\mathbb{E}|\eta_j|^p \leqslant 2^p \mathbb{E}|\xi(\cdot, j)|^p$ and hence

$$\sum_{j=0}^{\infty} \mathbb{E}|\eta_j|^p \leqslant 2^p \sum_{j=0}^{\infty} \mathbb{E}|\xi(\cdot, j)|^p = 2^p \|\xi\|^p_{L^p(\Omega \times \mathbb{N})}.$$

Therefore, the just defined mapping $\Pi: L^p(\Omega \times \mathbb{N}) \ni \xi \to \eta \in M^p(X)$ is linear and bounded. Moreover, if $\xi \in M^p(X)$ then from 1° and 2° it follows easily that $\Pi\xi = \xi$, and hence Π is a projection. □

A sequence $\eta \in M^p(X)$ will be called finite iff there exists a natural number n such that $\eta_j = 0$ for all $j > n$. Such a sequence will also be denoted by $\{\eta_j\}_{j=0}^{n}$. The subspace $M^p_{fin}(X)$ of all finite $\eta \in M^p(X)$ is dense in $M^p(X)$.

PROPOSITION A.2. *For a given Banach space X, probability space $(\Omega, \mathscr{F}, P)$, a filtration $\{\mathscr{F}_j\}_{j=0}^{\infty}$ of σ-subalgebras of $\mathscr{F}$ and a real number $p \geqslant 1$ the following three conditions are equivalent.*

(1) *there exists a constant $C > 0$ such that for any X-valued martingale $\{M_n\}_{n \in \mathbb{N}}$,*

$$\sup_n \mathbb{E}|M_n|^p \leqslant C \sum_{n=0}^{\infty} \mathbb{E}|M_n - M_{n-1}|^p. \tag{A.2}$$

(2) *there exists a constant $C > 0$ such that for any finite X-valued martingale difference sequence $\eta = \{\eta_j\}_{j=0}^{n} \in M^p(X)$,*

$$\mathbb{E}\left|\sum_{j=0}^{n} \eta_j\right|^p \leqslant C \sum_{j=0}^{n} \mathbb{E}|\eta_j|^p. \tag{A.3}$$

(3) *there exists a constant $C > 0$ such that for any $\eta = \{\eta_j\}_{j=0}^{\infty} \in M^p(X)$,*

$$\sup_n \mathbb{E}\left|\sum_{j=0}^{n} \eta_j\right|^p \leqslant C \sum_{j=0}^{\infty} \mathbb{E}|\eta_j|^p. \tag{A.4}$$

Moreover, if C_i denotes the smallest constant C such that the condition (i) above holds, then $C_1 = C_2 = C_3$.

Proof. (2) ⇒ (3). It is obvious, also $C_3 \leqslant C_2$.

(3) ⇒ (2). The linear operator $\mathrm{tr}: M^p_{fin}(X) \ni \eta \to \Sigma \eta_j \in L^p(\Omega)$ is bounded, by (3). Since $M^p_{fin}(X)$ is dense in $M^p(X)$, there exists a unique extension of tr to a bounded, linear operator $\mathrm{tr}: M^p(X) \to L^p(\Omega)$. Hence (2) holds, and $C_2 \leqslant C_3$.

(3) ⇒ (1). If M_j is a martingale, we put $\eta_j = M_j - M_{j-1}$ for $j \leqslant n$.

(1) ⇒ (3). If $\{\eta_j\}$ is a martingale-difference sequence, by putting $M_j = \eta_j + M_{j-1}$, $j \leqslant n$ and $M_j = M_n$ for $j > n$ we obtain a martingale. Hence, (3) follows from (1). □

DEFINITION A.1. A Banach space X is called an M-type p iff there exists a constant $C > 0$ such that for any probability space $(\Omega, \mathscr{F}, P)$ and any filtration $\{\mathscr{F}_j\}$ the condition (1) in Proposition A.2 is satisfied. The smallest such constant C is denoted by $C_p(X)$.

COROLLARY A.3. *A Banach space X is of M-type p iff there exists a constant $C > 0$ such that for any probability space $(\Omega, \mathscr{F}, P)$ and any filtration $\{\mathscr{F}_j\}$ the condition* (2) (*or* (3)) *in Proposition* A.2 *is satisfied. The smallest such constant C is denoted by $L_p(X)$.*

THEOREM A.4. *If the Banach spaces $X \subset Y$, with X continuously and densely imbedded in Y, are of M-type p, and $\vartheta \in (0,1)$ then the complex interpolation space $[Y, X]_\vartheta$ and the real interpolation space $(Y, X)_{\vartheta,p}$ are of M-type p.*

Proof. We use the characterization of M-type p Banach spaces in terms of continuity of the operator $\mathrm{tr}: M^p(X) \to L^p(\Omega \times \mathbb{N}, X)$ given by Corollary A.3. Then Theorem A.4 is a direct consequence of Theorem 1.17.1 in [45], Lemma A.1 and Theorem 1.18.4 in [45]. Indeed, from Theorem 1.18.4 we have

$$[L^p(\Omega \times \mathbb{N}; Y), L^p(\Omega \times \mathbb{N}; X)]_\vartheta = L^p(\Omega \times \mathbb{N}; [Y, X]_\vartheta),$$
$$(L^p(\Omega \times \mathbb{N}; Y), L^p(\Omega \times \mathbb{N}; X))_{\vartheta,p} = L^p(\Omega \times \mathbb{N}; (Y, X)_{\vartheta,p}).$$

Therefore, from Theorems 1.17.1, 1.18.4 and Lemma A.1 we have

$$[M^p(Y), M^p(X)]_\vartheta = M^p([Y, X]_\vartheta),$$
$$(M^p(Y), M^p(X))_{\vartheta,p} = M^p((Y, X)_{\vartheta,p}).$$

Applying the interpolation property together with Corollary A.3 concludes the proof. □

REMARK A.1. Similarly as is noted in Remark 1, 1.18.4 [45], in the real interpolation method we cannot replace p by a different number.

QUESTION A.1. It follows from Corollary A.3 that $L_p(X)$ is equal to the norm of the operator tr acting in $\mathscr{L}(M^p(X); L^p(\Omega \times \mathbb{N}; X))$. Can we obtain a nice formula for $L_p((Y, X)_{\vartheta,p})$ and $L_p([Y, X]_\vartheta)$ in terms of $L_p(X)$, $L_p(Y)$ and ϑ? The complex method is exact so we have $L_p([Y, X]_\theta) = L_p(Y)^{1-\theta} L_p(X)^\theta$. See also Theorem A.7.

The following result is not of interpolation type, but with conjunction with the previous one yields that the Besov spaces are of M-type 2.

LEMMA A.5. *Assume that for Banach spaces X and Y there exists a finite family $\{\Lambda_i\}_{i=1}^m$ of bounded linear operators from X to Y such that the following inequalities hold*

$$\|x\|^2 \leqslant K_1 \sum_i |\Lambda_i x|^2 \leqslant K_1 K_2 \|x\|^2, \quad x \in X, \tag{A.5}$$

where $\|\cdot\|$ *and* $|\cdot|$ *denote respectively the norm in* X *and* Y.

Then, if Y *is an M-type 2 Banach space then also* X *is of M-type 2.*

Proof. Assume that M_n is an X-valued martingale. Then, as for any i, $\Lambda_i M_n$ is a Y-valued martingale, we have by (A.5)

$$\begin{aligned}\mathbb{E}\|M_n\|^2 &\leqslant K_1 \sum_i \mathbb{E}|\Lambda_i M_n|^2 \leqslant K_1 C_2(Y) \sum_i \sum_j \mathbb{E}|\Lambda_i M_j - \Lambda_i M_j|^2 \\ &= K_1 C_2(Y) \sum_j \sum_i \mathbb{E}|\Lambda_i (M_j - M_{j-1})|^2 \leqslant K_1 K_2 C_2(Y) \sum_j \mathbb{E}\|M_j - M_{j-1}\|^2.\end{aligned}$$

This allows us to infer that also X is of M-type 2. □

COROLLARY A.6. *The Sobolev spaces* $W^{k,p}(\mathbb{R}^n)$ *with* $k \in \mathbb{Z}$, $p \geqslant 2$ *and the Besov spaces* $B^s_{p,2}(\mathbb{R}^n)$ *with* $s \in \mathbb{R}$, $2 \leqslant p < \infty$ *are all of M-type 2.*

Proof. As it is unambiguous, we will not write in this proof the space $\mathbb{R}^n$. We start with Sobolev space $W^{k,p}$ with $k \in \mathbb{N}$. Since $W^{0,p} = L^p$ we may assume that $k \geqslant 1$. Then putting $\Lambda_\alpha = D^\alpha$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $|a| \leqslant k$ we see that the couple of spaces $X = W^{k,p}$, $Y = L^p$ satisfy the assumptions of Lemma A.5. Indeed, with

$$|u| = |u|_{L^p}, \quad \|u\| = \|u\|_{W^{k,p}} = \left\{ \sum_{|\alpha| \leqslant k} |D^\alpha u|^p \right\}^{\frac{1}{p}}$$

and hence

$$\begin{aligned}\|u\|^2 &= \left\{ \sum_{|\alpha| \leqslant k} |D^\alpha u|^p \right\}^{\frac{2}{p}} \leqslant \left\{ \sum_{|\alpha| \leqslant k} |D^\alpha u|^2 \right\}^{\frac{2}{p}\frac{p}{2}} = \sum_{|\alpha| \leqslant k} |D^\alpha u|^2 \\ &\leqslant \left(\sum_{|\alpha| \leqslant k} \right)^{1-\frac{2}{p}} \left\{ \sum_{|\alpha| \leqslant k} |D^\alpha u|^p \right\}^{\frac{2}{p}} = \left(\sum_{|\alpha| \leqslant k} \right)^{1-\frac{2}{p}} \|u\|^2.\end{aligned}$$

By Applying Lemma A.5 we infer that $W^{k,p}$ is of M-type 2. To proceed further we denote by $J^s = (1 - \Delta)^{s/2}$. Since for negative $k \in \mathbb{Z}$, $W^{k,p} = J^{-2k}(W^{-k,p})$ (by the lift property, see [4], Theorem 6.2.7 or [45] Theorem 2.3.4) we conclude the proof in the case of the Sobolev spaces.

The case of Besov spaces follows immediately from [4] Theorem 6.2.4, the first part of this Corollary and Theorem A.4. □

Let us observe that the above proof gives also that for a bounded domain $D \subset \mathbb{R}^n$ (with boundary sufficiently regular) the Sobolev spaces $W^{k,p}(D)$ with $k \in \mathbb{N}$ and $p \geqslant 2$ are of M-type 2 and by Theorem 4.3.1 in [45] such are the Besov spaces $B^s_{p,q}(D)$ with $s > 0$, $p \geqslant 2$.

REMARK A.2. In the framework of Lemma A.5 the space X can be renormed in such a way that

$$C_2(X) = C_2(Y). \tag{A.6}$$

Indeed, it is sufficient to take

$$|||x||| = \left\{\sum_i |\Lambda_i x|^2\right\}^{\frac{1}{2}}.$$

Then (A.4) implies that $|||\cdot|||$ is equivalent to the original norm in X and the proof of Lemma A.5 yields the equality (A.5).

THEOREM A.7. *Assume that X is an M-type p Banach space and $-A$ is a generator of an analytic semigroup on X such that $0 \in \rho(A)$. Then for any $\theta \in (0,1)$ the space $D_A(\theta, p)$ is also of M-type p and*

$$L_2(D_A(\theta,p)) \leqslant L_2(X). \tag{A.7}$$

Proof. Although the first part follows from Theorem A.4, we observe that if the norm in $D_A(\theta, p)$ is given by (4.1) then the whole statement follows readily from the definition by using (4.1). □

REMARK A.3. In a completely similar way one can study interpolation of type p Banach spaces. All the preceding results remain true if M-type p property is replaced by type p one. It follows from the simple observation that X is of type p iff for any $n \in \mathbb{N}$ and for any Bernoulli sequence $\varepsilon: \Omega \to \{-1, 1\}$ the linear operator

$$T_n: X^n \ni x = (x^1, \dots, x^n) \to \sum_{i=1}^n \varepsilon_i x^i \in L^p(\Omega, X)$$

is bounded. If X^n is endowed with a norm $|\cdot|$ such that $|x|^p = \Sigma_i |x^i|^p$ then obviously $\|T\|^p \leqslant K_p(X)$.

Appendix B. L^p Space with $p \geqslant 2$ is of M-Type 2

In this Appendix we intend to show that the spaces L^p with $2 \leqslant p < \infty$ are of M-type 2. We start with the following

LEMMA B.1. *If (D, dx) is a measure space, $\varepsilon_1, \dots, \varepsilon_n: \Omega \to \{-1, 1\}$, $n \in \mathbb{N}^*$ are symmetric i.i.d. random variables, $p = 2k$, $k \in \mathbb{N}^*$ then*

$$\left\{\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i f_i\right\|_{L^p(D)}^p\right\}^{\frac{1}{p}} \leqslant \left\{\sum_{i=1}^n \|f_i\|_{L^p(D)}^2\right\}^{\frac{1}{2}} \tag{B.1}$$

for any sequence $f_1, \ldots, f_n \in L^p(D)$.

Proof. By the properties of ε_i we have

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i f_i\right\|_{L^p(D)}^p = \int_D \left|\sum_i \varepsilon_i f_i(x)\right|^{2k} dx = \int_D \mathbb{E} \sum_{|\alpha|=2k} \varepsilon_1^{\alpha_1} f_1^{\alpha_1} \ldots \varepsilon_n^{\alpha_n} f_n^{\alpha_n} dx$$
$$= \int_D \sum_{|\alpha|=2k} \mathbb{E}\left(\prod \varepsilon_i^{\alpha_i}\right) \prod f_i^{\alpha_i}(x)\, dx = \int_D \sum_{|\alpha|=k} \prod f_i^{2\alpha_i}(x)\, dx.$$

Next, the Hölder inequality gives $\int_D \prod f_i^{2\alpha_i}(x)\, dx \leqslant \prod \{\int_D |f_i(x)|^{2k} dx\}^{\alpha_i/k}$ and therefore

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i f_i\right\|_{L^p(D)}^p \leqslant \sum_{|\alpha|=k} \prod_i \left\{\int_D |f_i(x)|^{2k} dx\right\}^{\frac{\alpha_i}{k}} = \sum_{|\alpha|=k} \prod_i \|f_i\|_{L^p(D)}^{2\alpha_i}.$$

The equality

$$\left\{\sum_i \|f_i\|_{L^p}^2\right\}^{\frac{p}{2}} = \left\{\sum_i \|f_i\|_{L^p}^2\right\}^k = \sum_{|\beta|=k} \prod_i \|f_i\|_{L^p}^{2\beta_i},$$

concludes the proof. □

Now by putting

$$X_p = L^2(\mathbb{N}, L^p(D)); \quad Y_p = L^p(\Omega; L^p(D))$$

and considering

$$T_p: X_p \ni (f_i)_{i=1}^n \to \sum \varepsilon_i f_i \in Y_p,$$

we see that (B.1) is equivalent to

$$\|T_p\|_{\mathscr{L}(X_p, Y_p)} \leqslant 1 \text{ for any } p \in 2\mathbb{N}^*. \tag{B.2}$$

By means of a simple interpolation argument we may deduce that in fact (B.2) holds for all $p \geqslant 2$. Indeed, if $p \in [2, \infty)$ then $p = \theta q + (1 - \theta)s$ for some $s < q \in 2\mathbb{N}^*$, $\theta \in (0, 1)$. Since the complex interpolation method is exact, see [4], $X_p = [X_s, X_q]_\theta$ and $Y_p = [Y_s, Y_q]_\theta$, the inequalities $\|T_s\| \leqslant 1$, $\|T_q\| \leqslant 1$ yield $\|T_p\| \leqslant 1$.

Thus, we have proven

COROLLARY B.2. *Inequality* (B.1) *holds true for any* $p \in [2, \infty)$.

Now we recall a definition of a type 2 space, see [37] and of UMD space, see [8].

DEFINITION B.1. A Banach space X is of type 2 iff for any finite sequence

$$\varepsilon_1, \ldots, \varepsilon_n : \Omega \to \{-1, 1\}$$

of symmetric i.i.d. random variables and for any finite sequence $x_1, \ldots, x_n$ of elements of X, the following inequality holds,

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\|_X^2 \leqslant K \sum_{i=1}^{n} \|x_i\|_X^2, \tag{B.3}$$

for some constant $K > 0$.

The smallest constant K for which (B.3) holds will be denoted by $K_2(X)$.

DEFINITION B.2. A Banach space X is a UMD space iff for any X-valued martingale difference sequence $\eta = \{\eta_j\}_{j=1}^n$ and for any $\varepsilon \in \{-1, 1\}^n$

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i \eta_i\right\|_X^p \leqslant \beta\, \mathbb{E}\left\|\sum_{i=1}^{n} \eta_i\right\|_X^p, \tag{B.4}$$

for some constant β and some $p \in (1, \infty)$.

The smallest constant β for which (B.4) holds will be denoted by $\beta_p(X)$. This definition is p independent, see [8].

PROPOSITION B.3. *If (D, dx) is a measure space and $p \in [2, \infty)$ then $L^p(D)$ is a Banach space of type 2. Moreover, the constant $K_2(L^p(D))$ is equal to 1, i.e.*

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i f_i\right\|_{L^p(D)}^2 \leqslant \sum_{i=1}^{n} \|f_i\|_{L^p(D)}^2. \tag{B.5}$$

Proof. The Jensen inequality gives

$$\{\mathbb{E}\|\textstyle\sum \varepsilon_i f_i\|_{L^p(D)}^2\}^{\frac{1}{2}} = \|\textstyle\sum \varepsilon_i f_i\|_{L^2(\Omega; L^p(D))} \leqslant \|\textstyle\sum \varepsilon_i f_i\|_{L^p(\Omega; L^p(D))}$$

and we conclude by applying Corollary B.2. □

We would like to underline the fact that for $X = L^p(D)$ the smallest constant for which the inequality (B.5) holds true is 1. Moreover, $\beta_p(L^p(D)) = \beta_p(\mathbb{R})$, see [8].

PROPOSITION B.4. *If a UMD Banach space X is also of type 2 then it is of M-type 2 as well.*

Proof. Let $\eta = \{\eta_i\}_{i=1}^n$ be an X-valued martingale difference sequence. Let $\varepsilon = \{\varepsilon_i\}_{i=1}^n$ be a sequence of independence from η, symmetric i.i.d. $\{-1, 1\}$-valued random variables. Since $\eta_i = \varepsilon_i \varepsilon_i \eta_i$ and $\{\varepsilon_i \eta_i\}_{i=1}^n$ is a martingale difference sequence, the properties of conditional expectation and UMD property of X yield

$$\mathbb{E}(\|\textstyle\sum \eta_i\|^2 | \varepsilon) = \mathbb{E}(\|\textstyle\sum \varepsilon_i \varepsilon_i \eta_i\|^2 | \varepsilon) \leqslant \beta_2 \mathbb{E}(\|\textstyle\sum \varepsilon_i \eta_i\|^2 | \varepsilon) \quad \text{a.s.,}$$

where $\beta_2 = \beta_2(X)$. Taking the mean value gives

$$\mathbb{E}\|\textstyle\sum \eta_i\|^2 \leqslant \beta_2 \mathbb{E}\|\textstyle\sum \varepsilon_i \eta_i\|^2.$$

Now, since X is of type 2, if $K_2 = K_2(X)$ we have

$$\mathbb{E}(\|\sum \varepsilon_i \eta_i\|^2 | \eta) \leqslant K_2 \sum \|\eta_i\|^2, \quad \text{a.s.}$$

Therefore, taking expectation again we get

$$\mathbb{E}\|\sum \eta_i\|^2 \leqslant \beta_2 K_2 \sum \|\eta_i\|^2,$$

which completes the proof. □

From Propositions B.3 and B.4 we immediately have

THEOREM B.5. *If (D, dx) is a measure space and $p \in [2, \infty)$ then $L^p(D)$ is an M-type 2 Banach space.*

Moreover, the following inequality holds

$$C_2(L^p(D)) \leqslant \beta_2(L^p(D)). \tag{B.6}$$

QUESTION B.1. What is the best value of $C_2(L^p)$?

Because of its role in the preceding sections, we believe this question to be of some importance.

Acknowledgements

The research presented in this paper was performed while the author was visiting Dipartimento di Matematica, Università di Torino and was supported by CNR-GNAFA. The author is grateful to Professor Franco Flandoli for his interest, numerous suggestions and corrections of this paper. The author also would like to thank Professors Sergio Albeverio and Edgert Dettweiler for helpful discussion. The present paper is a revised version of preprints [5] and [6].

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