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Slow-Motion Manifolds, Dormant Instability, and Singular Perturbations

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If the coexistence of two phases at the transition temperature is kept under observation for a long time, then one observes that the system is not exactly in equilibrium and a very slow evolution driven by surface tension is taking place. Theoretically, one should eventually see a spatially homogeneous state, but the time for settling down is so long that what one actually observes is "motion towards a stable state." The complexity of the spatial distribution of the two phases keeps decreasing but appears to be stable for very long periods of time with intermittent periods of fast motion when there are small inclusions of one of the two regions embedded in the other phase. For a simple reaction diffusion model, it is shown that this phenomenon can be explained by investigating the flow on the attractor and the unstable manifolds of equilibria.

KEY WORDS: Singular perturbations; transition layers; metastability; integral manifolds.

1. INTRODUCTION

It is almost common experience that two phases of the same substance at the transistion temperature— for instance, liquid and solid at the wetting temperature—may coexist in a region, giving rise to very complicated structures. This is in agreement with the fact that, at thermodynamical equilibrium, the two phases have the same free energy and therefore any spatial distribution of the two phases is a minimizer of the total free energy.

A simple mathematical model for describing this situation is obtained by assuming that the free energy $F(T, u)$ of the substance, besides depending on the temperature T, is also a function of an "order parameter" u that is related to the microscopic structure of the substance. For each value of T, the function $F(T, \cdot)$ is supposed to have two minima at $u = -1$,

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$u=1$, corresponding to the two phases of the substance, and a maximum at $u=0$, representing some unstable microscopic state. It is also assumed that, for some T_0 to be identified with the transition temperature, the following inequalities hold (see Fig. 1):

$$F(T,-1)<F(T,1) \quad \text{for } T<T_0; \qquad F(T,-1)>F(T,1) \quad \text{for } T>T_0.$$

Under these assumptions, the total free energy of the substance contained in a bounded region Ω at constant temperature T,

$$\phi(u)=\int_\Omega F(T,u(x))\,dx \tag{1.1}$$

has, if $T\neq T_0$, a unique global minimizer $u=1$ or $u=-1$ corresponding to $T>T_0$ or $T<T_0$. The situation is completely different in the case $T=T_0$, which is the case of our interest. In fact, in this case, the functional (1.1) has infinitely many minimizers given by

$$\begin{aligned} u(x)&=1 && \text{for } x\in S \\ u(x)&=-1 && \text{for } x\in \Omega\backslash S \end{aligned} \tag{1.2}$$

where $S\subset\Omega$ is any measurable set.

If the coexistence of the two phases at the transition temperature is observed for a long time interval, then one sees that the system is not exactly in equilibrium and that a very slow evolution is taking place. This evolution is driven by the surface tension on the interfaces separating the two phases that tends to reduce the area of the interfaces and therefore slowly drives the system toward a homogeneous state where only one of the two phases exists. Thus, even if the temperature is equal to the transient temperature, what one should finally see is a spatially homogeneous stable state. This may be theoretically true, but the time that the system takes to settle down in homogeneous equilibrium is so long, when measured with the ordinary time scale, that what one actually observes is not the system in a stable state but "motion toward a stable state." Thus, it appears that a thorough understanding of the real behavior of the system cannot be based only on the knowledge of the stable states, but must be dynamic. In

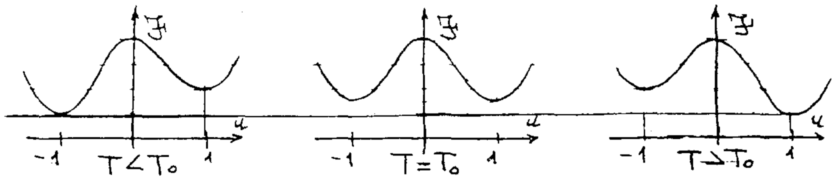

Fig. 1. Free energy as a function of the order parameter u.

understanding this dynamics, unstable equilibria, rather than stable ones, are likely to play the most important role. The need for a dynamical description also is stressed by the fact that the system exhibits a rather interesting phenomena that we may call *dormant instability*. As we have said, if one forgets the effects of the very small driving forces that try to reduce the interfaces, the system appears to be in a stable state. The system will appear to be stable for a very long time, but not forever, because the effects of the slow evolution accumulate so that the system reaches a state where small inclusions of one of the two phases are imbedded in large regions of the other phase. When this happens, the system becomes more unstable and its evolution, which has been extremely slow for a very long time, starts to be much faster. In a time interval that is short when compared with the long period of slow evolution, one of the small inclusions disappears and the system seems to have settled down in a new stable state. But again this state is not exactly an equilibrium, and slow evolution again takes place, leading the system to a situation where small inclusions are present and instability shows up again annihilating some other small inclusion. This intermittent behavior, which may repeat thousands of times, is exactly the mechanism by which the complexity of the spatial distribution of the two phases keeps decreasing until the system, after a very long period of time (which, from a practical point of view, may be regarded as infinitely long), reaches a stable homogeneous state.

From the point of view of applied mathematics, these phenomena are certainly interesting to study and the goal is to understand the nonlinear mathematical mechanism behind them. In this paper, by discussing a simple model, we want to show how ideas from the geometric theory of evolutionary equations may provide a framework for a possible explanation of this mechanism.

To account for the interfacial forces, one needs to add to the total free energy an extra term that penalizes large interfaces. In the setting of the simple mathematical model considered above, a possible description of this interfacial energy that is widely accepted and has good physical foundation (Bongiorno *et al.*, (1976)) is obtained by adding to (1.1) an extra term proportional to $\int_\Omega |\nabla u|^2$ that penalizes high gradients of the order parameter u. With this extra term, the total free energy takes the form

$$\Phi_\varepsilon(u)=\int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u|^2+F(u)\right) \tag{1.3}$$

where $F(u)=F(T_0, u)$ and ε is a parameter that measures the relative importance of the surface energy and it is assumed to be very small ($\varepsilon << 1$). We remark that, regardless of the smallness of ε, the set of absolute minimizers of Φ_ε is quite different from the one defined by (1.2).

In fact, in agreement with the real situation, the only absolute minimizers of Φ_ε are the constants $u = 1$, $u = -1$. If Ω is convex, the only minimizers of Φ_ε are $u = 1$, $u = -1$ (Matano, 1979; Casten and Holland, 1978).

Several approaches, based on microscopic models (Benassi and Fouque, 1988; Presutti, 1987) or on thermodynamical considerations (Gurtin, 1986a, b) have been used to derive equations for the evolution of the interface between two phases. To describe what we believe is the explanation of the phenomena of dormant instability from the point of view of geometric theory of differential equations, we take the simplest dynamics that one can associate with the functional Φ_ε; namely, we take the gradient system defined by the functional Φ_ε. If, as we shall do in the following, one restricts the discussion to the one-dimensional case $\Omega = [0, 1]$, this amounts to considering the scalar parabolic equation

$$\begin{aligned} u_t &= \varepsilon^2 u_{xx} + f(u), \qquad x \in (0, 1) \\ u_x(0, t) &= u_x(1, t) = 0 \end{aligned} \tag{1.4}$$

where $f = -dF/du$.

We believe that more sophisticated dynamical models like, for instance, the one corresponding to the Cahn–Hilliard equation (Novick-Cohen and Segal, 1984) will share, with the simple Eq. (1.4), the qualitative behavior that we are going to describe.

In the following, after discussing in Section 2 how Eq. (1.4), for a very small value of the parameter ε, may exhibit phenomena like slow evolution and dormant instability, we derive in Section 3 a linear equation for the approximate description of the "slow-motion manifolds." By solving this equation, we obtain explicitly a system of ordinary differential equations for the motion of the "internal layers" that in our unidimensional model play the role of the interfaces.

2. A DYNAMICAL MODEL FOR DORMANT INSTABILITY

We begin by recalling some of the known facts about Eq. (1.4); we shall assume that f is a C^2 function that is odd and has a second derivative that only vanishes at $u = 0$.

Equation (1.4) defines a nonlinear semiflow $\{S_t\colon X \to X,\ t \geqslant 0\}$ on several function spaces; for instance, we can take $X = H^1$ or $X = C^1_B[0, 1]$ where $C^1_B[0, 1]$ is the set of C^1 functions with zero derivatives at $x = 0, 1$. This semiflow admits a "global attractor" $A_\varepsilon \subset X$; that is, a set $A \subset X$ that is compact, connected, and invariant under $S_t (S_t A_\varepsilon = A_\varepsilon)$ and attracts bounded sets of X in the sense that, for any given bounded set $B \subset X$ and $\delta > 0$, there is a $\bar{t}$ depending on B and δ such that $t \geqslant \bar{t}$ implies that $S_t B$

is in a δ-neighborhood of A_ε (Hale, 1988). The set of equilibria of (1.4), that is, the set E_ε of solutions of the boundary value problem,

$$\begin{aligned} \varepsilon^2 u_{xx} + f(u) = 0, \qquad x \in (0, 1) \\ u_x(0) = u_x(1) = 0 \end{aligned} \tag{2.1}$$

depends on ε. Besides the homogeneous equilibria $u_\infty = 0$, unstable, and $u_0 = -1$, $\hat{u}_0 = 1$, stable, if $\varepsilon_i = (f'(0))^{1/2}/2\pi i$, $i = 1, 2, \ldots$, and $\varepsilon_{n+1} \leqslant \varepsilon < \varepsilon_n$, E_ε contains exactly n pairs of nonconstant equilibria u_{ε_i}, $\hat{u}_{\varepsilon_i} = -u_{\varepsilon_i}$, $1 \leqslant i \leqslant n$. The equilibria u_{ε_i}, $\hat{u}_{\varepsilon_i}$ bifurcate from the zero equilibrium at $\varepsilon = \varepsilon_i$ and have exactly i zeros at $x = 1/2i, 3/2i, \ldots, 1 - 1/2i$ (see Fig. 2). We shall assume that $u_{\varepsilon_i}(0) < 0$. When $\varepsilon \to 0$, u_{ε_i} approaches a step function with values -1, 1, $-1, \ldots$ and jumps at the zero of u_{ε_i}. The attractor $\mathbf{A}_\varepsilon$ has a simple characterization in terms of equilibria and their unstable manifolds. In fact, it can be shown that the α-limit set of any orbit defined and bounded in the past is a single equilibrium (the same is true for the ω-limit set of any orbit) (Hale, 1988; Henry, 1983) and this implies

$$\mathbf{A}_\varepsilon = \bigcup_{u \in E_\varepsilon} W(u) \tag{2.2}$$

where, for any equilibrium u, $W(u)$ is the unstable manifold of u. It is also known that the dimension of $W(u_{\varepsilon_i})$ and $W(\hat{u}_{\varepsilon_i})$ is exactly i.

The above characterization of the α- and ω-limit sets of orbits of (1.4) also implies that $\mathbf{A}_\varepsilon$ is the union of orbits connecting pairs of equilibria. Therefore, the main step in the description of the topology of the orbits on $\mathbf{A}_\varepsilon$ consists in solving the following problem: Given $u^-, u^+ \in E_\varepsilon$, is there a solution $\phi(t)$ defined in $(-\infty, \infty)$ such that $\Phi(t) \to u^\pm$ as $t \to \pm\infty$? This problem has been completely solved (Henry, 1985; Brunovsky and Fiedler, 1988a, b) and it is known that u_{ε_i} connects to $u_{\varepsilon_j}(\hat{u}_{\varepsilon_j})$ if and only if $i > j$ (see Fig. 3). The fact that connections between equilibria are always in the direction of reducing the number of zeros is a manifestation of a general property of Eq. (1.4): the number of zeros of solutions of (1.4) is nonincreasing with time (Matano, 1982; Angenent, 1988).

Besides the above results for Eq. (1.4), we also have the following conjectures. We let z_u be the zero set of a function u; $d(x, z_u)$ the distance

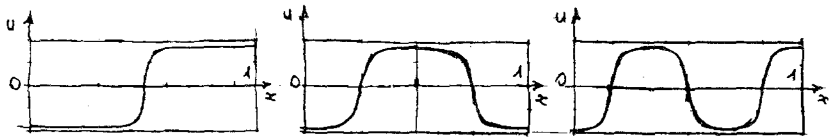

Fig. 2. The equilibria u_1, u_2, and u_3.

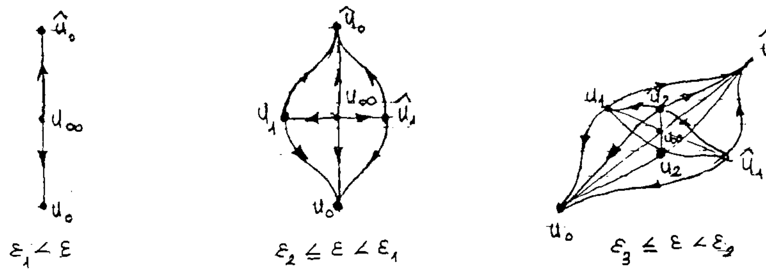

Fig. 3. The connections on the attractor.

between x and z_u; $\Psi \subset C_B^1[0, 1]$ the set of functions that have only simple zeros, $z_u \neq \phi$, and let $\mathbf{N}_{\delta, L^2}(B)$ be an open δ-neighborhood of a set $B \subset C_B^1[0, 1]$ with respect to the L^2-topology.

I. Almost all solutions with a nontrivial zero set develop transition layers and approach a step function with values -1 *and* $+1$ *in the sense that, given* $\psi \in \Psi$ *and a number* $\delta > 0$, *there exist* $\bar{\varepsilon} > 0$ *and* $\bar{t} \geq 0$ *such that, for* $\varepsilon < \bar{\varepsilon}$,

$$d(x, z_\psi) > \delta, \qquad \psi(x) > 0 \Rightarrow |1 - (S_{\bar{t}}\psi)(x)| < \delta.$$

Similarly, for $\psi(x) < 0$ *and* 1 *replaced by* -1, *the same statement holds.*

This conjecture is quite reasonable because, in order that diffusion can become effective when ε is very small, it is necessary that high gradients exist. Therefore, one can expect that the first thing that happens is that the evolution of u at any point x is approximately ruled by the equation $\dot{u}(x) = f(u(x))$, $u(x)(0) = \psi(x)$ (where x appears as a parameter); this clearly implies the above statement for some $\bar{t}$ independent of ε in $(0, \bar{\varepsilon})$.

II. For any integer i, $0 < \varepsilon < \varepsilon_i$, *and any number* $\delta > 0$, *let* $\partial W(u_{\varepsilon_i})$ *be the boundary of* $W(u_{\varepsilon_i})$ *and let*

$$W_\delta(u_{\varepsilon_i}) = W(u_{\varepsilon_i}) \backslash \mathbf{N}_{\delta, L^2}(\partial W(u_{\varepsilon_i})).$$

Then there is an $\bar{\varepsilon} > 0$ *such that, for* $0 < \varepsilon < \bar{\varepsilon}$ *and* $\phi \in W_\delta(u_{\varepsilon_i})$,

(i) ϕ *has exactly i zeros in* $(0, 1)$
(ii) ϕ *is near to a step function with values* -1 *and* $+1$,

$$d(x, z_\phi) > \delta \Rightarrow 1 - |\phi(x)| < \delta.$$

This conjecture is suggested from the fact that, like u_{ε_i}, all functions in $W(u_{\varepsilon_i})$ near u_{ε_i} must be approximately step functions with values -1 and 1 and exactly i transition layers. It is also to be expected that, for $\varepsilon << 1$, the fact that -1 and 1 are stable fixed points for the equation $\dot{u} = f(u)$ will imply that the only possible evolution for a function of this type is through "motion of transition layers" driven by diffusion. This type of evolution should last until either one of the transition layers approaches a small L^2-neighborhood of $W(u_{\varepsilon_i})$.

III. The energy Φ_ε is almost constant on $W_\delta(u_{\varepsilon_i})$ in the sense that, if M_ε is the maximum of the difference $\Phi_\varepsilon(u_{\varepsilon_i}) - \Phi_\varepsilon(u)$ for $u \in W_\delta(u_{\varepsilon_i})$ and m_ε is the minimum of the same difference for $u \in \partial W(u_{\varepsilon_i})$, then

$$\lim_{\varepsilon \to 0} \frac{M_\varepsilon}{m_\varepsilon} = 0.$$

This conjecture says that, for $\varepsilon << 1$, the decreasing of Φ_ε on $W(u_{\varepsilon_i})$ is almost all concentrated near the boundary of $W(u_{\varepsilon_i})$ and is based on the fact that, for functions in $W_\delta(u_{\varepsilon_i})$, the main contribution to the energy comes from the transition layers and depends upon their number rather than upon their positions. Thus, the energy should have very small variations on $W_\delta(u_{\varepsilon_i})$ and, relatively, very large variations near $\partial W(u_{\varepsilon_i})$ where two or more transition layers come together and disappear. Therefore, we can expect the motion to be very slow on $W_\delta(u_{\varepsilon_i})$ and relatively much faster near $\partial W(u_{\varepsilon_i})$, which should act like a "waterfall" for the flow on $W(u_{\varepsilon_i})$ (see Fig. 4). We shall call $W_\delta(u_{\varepsilon_i})$ a "slow-motion manifold."

Carr and Pego (1988) have provided detailed estimates on the energy that should give a justification of this conjecture.

On the basis of the previous conjectures, one can derive the following dynamical interpretation for the phenomena considered in the introduction.

(a) For $\varepsilon << 1$ and for a large set of initial conditions in Ψ, the solution, before approaching one of the two stable equilibria, comes close

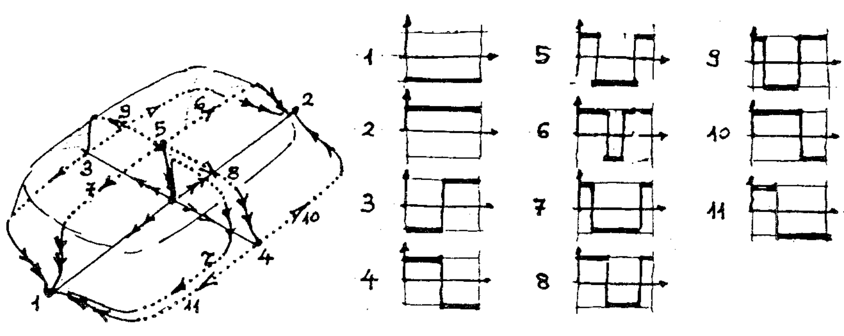

Fig. 4. $W(\hat{u}_{\varepsilon_2})$ for $\varepsilon << 1$. Dotted lines correspond to slow motion, solid lines to fast motion.

to the attractor near the unstable manifold of some equilibrium u_{ε_i}. At this point, the solution is approximately a step function with values -1 and 1 and, in the physical situation that we want to model, corresponds to a distribution of $i+2$ homogeneous regions of the two coexisting phases.

(b) Once the solution is near $W(u_{\varepsilon_i})$, the motion follows that of the slow manifold of u_{ε_i} passing through a sequence of quasi equilibrium states until it finally reaches a small neighborhood of $\partial W(u_{\varepsilon_i})$.

(c) At this point, instability shows up and one or more transition layers disappear in a time interval that is very small when compared with the time that the solution has been dormant on the slow manifold of u_{ε_i}. This quick reduction of the number of transition layers near the boundary of $W(u_{\varepsilon_i})$ corresponds dynamically to the fact that the solution approaches the unstable manifold of an equilibria u_{ε_j} with a lower number of zeros.

(d) Once the solution is near $W(u_{\varepsilon_j})$, one has a long period of slow motion followed by a short period in which the solution jumps from the unstable manifold of u_{ε_j} to the unstable manifold of u_{ε_k} and so on until, after this intermittent unstable behavior has repeated i times, the solution ends up with at most two zeros with the next fast motion leading to one of the two homogeneous stable equilibria. See Fig. 5.

(e) When $\varepsilon \to 0$, the dimension of the attractor approaches ∞ and the attractor contains equilibria with unstable manifolds of larger and larger dimension. Thus, for $\varepsilon << 1$, i can be very large and the sequence slow motion–fast motion may repeat itself thousands of times.

The behavior of solutions of (1.4) described in (a)–(e) has been confirmed by numerical experiments performed by B. Pego and successively by M. McKinney. These numerical experiments have also indicated that, for $\varepsilon << 1$, the speed on the slow-motion manifold should behave like $\exp(-c/\varepsilon)$.

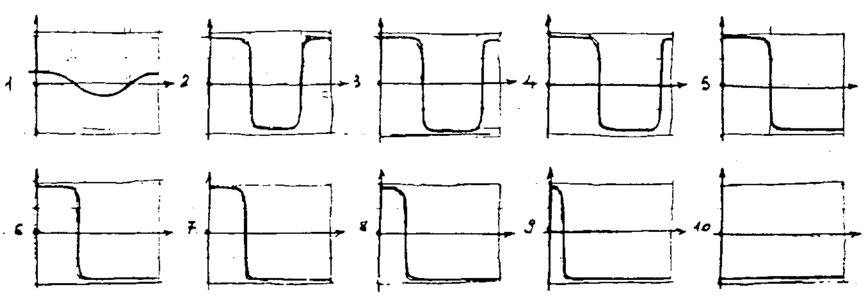

Fig. 5. Qualitative time evolution of spatial pattern along the orbit τ in Fig. 5. Fast motion from 1 to 2, 4 to 5, and 9 to 10; slow motion from 2 to 4, and 5 to 9.

3. A MATHEMATICAL APPROACH TO SLOW-MOTION MANIFOLDS

From the dynamical description of the phenomena of dormant instability discussed above, one sees that rigorous results on this subject require a description of the unstable manifold of u_{ε_i} for $\varepsilon \to 0$. There are several features of this problem that make it very interesting, but also quite difficult. First of all, discussing Eq. (1.4) for $\varepsilon \to 0$ is a *singular perturbation problem*. Moreover, one needs to describe the complete unstable manifold $W(u_{\varepsilon_i})$; that is, one has to solve a *global problem* that also is complicated by the fact that, as we have seen, one expects quite different behaviors of $W(u_{\varepsilon_i})$ depending on the distance from $\partial W(u_{\varepsilon_i})$. The aim of this section is to suggest a possible geometric theory for showing the existence of slow-motion manifolds and deriving a system of ordinary differential equation for describing the flow on them; that is, the dynamics of transition layers.

It is natural to try to set up the problem so that it looks like a perturbation problem. The basic observation is the following:

It is possible to construct a vector field that, for $\varepsilon << 1$, *is close in* $X = H^1$ *to the vector field defined by* (1.4) *and has a slow-motion manifold that can be explicitly computed.*

To construct this vector field, we let v: $(-\infty, \infty)$ be the solution of the problem

$$\begin{aligned} &\varepsilon^2 v_{xx} + f(v) = 0, && x \in (-\infty, \infty) \\ &\lim_{x \to \pm\infty} v(x) = \pm 1, && v(0) = 0. \end{aligned} \tag{3.1}$$

This solution exists and is unique as a consequence of the assumption $F(-1) = F(+1)$, which implies the existence of a heteroclinic orbit connecting $(-1, 0)$ to $(1, 0)$ in the phase plane (v, v_x). Clearly v depends on ε and is actually a function of x/ε. To keep the notation simple, we don't indicate this dependence on ε of v and of all other functions that we shall consider in the following.

Let $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$ be n variables; $\xi = (\xi_1, \ldots, \xi_n)$; $\Gamma = \{\xi \mid 0 < \xi_1 < \cdots < \xi_n < 1\}$ and $\xi_0 = -\xi_1$, $\xi_{n+1} = 2 - \xi_n$. Let $\eta_i = (\xi_{i+1} + \xi_i)/2$, $\zeta_i = (\xi_{i+1} - \xi_i)/2$ for $0 \leqslant i \leqslant n$, and let $U(\cdot, \cdot)$: $\Gamma \times [0, 1] \to \mathbb{R}$ be defined by $U(\xi, x) = (-1)^{i+1} v(x - \xi_i)$, $\eta_{i-1} \leqslant x \leqslant \eta_i$, $1 \leqslant i \leqslant n$.

This definition implies that $U(\xi, \cdot)$ is a continuous function with a piecewise continuous first derivative that jumps at η_i, $1 \leqslant i \leqslant n-1$ (Fig. 6).

The map $\xi \to U(\xi, \cdot)$ defines an n-dimensional manifold $\bar{W} \subset H^1$. The jumps $[U_x(\xi, \eta_i)]$ of $U_x(\xi, \cdot)$ at η_i, $1 \leqslant i \leqslant n-1$, and the derivatives $U_x(\xi, 0)$, $U_x(\xi, 1)$ define $n+1$ functions ϕ_i: $\bar{W} \to \mathbb{R}$

$$\begin{aligned}
&\phi_0(U(\xi, \cdot)) = U_x(\xi, 0) = v_x(\zeta_0)\\
&\phi_n(U(\xi, \cdot)) = U_x(\xi, 1) = (-1)^{n+1}\, v_x(\zeta_n) \qquad (3.2)\\
&\phi_i(U(\xi, \cdot)) = [U_x(\xi, \eta_i)] = 2(-1)^i\, v_x(\zeta_i) \qquad 1 \leqslant i \leqslant n-1
\end{aligned}$$

where we have used the definition of U and the fact that v is an odd function. We can assume that these functions have been extended to all of H^1 and consider the following formal evolutionary problem

$$\begin{aligned}
u_t &= \varepsilon u_{xx} + f(u), && x \in (0, 1)\\
u_x(0, t) &= \phi_0(u), && u_x(1, t) = \phi_n(u) \qquad (3.3)\\
[u_x(\eta_i, t)] &= \phi_i(u), && 1 \leqslant i \leqslant n-1.
\end{aligned}$$

$\bar{W}$ is obviously an invariant manifold for Eq. (3.3) and a slow-motion manifold. In fact, any function in $\bar{W}$ is an equilibrium solution for (3.3). Moreover, the vector field defined by (1.4) can be regarded as a perturbation of the vector field defined by (3.3). In fact, one obtains Eqs. (1.4) by simply replacing in (3.3) the functionals ϕ_i with the zero functional, and it is easy to see that the functionals ϕ_i have, on any compact set $K \subset \bar{W}$, a bound of the type $C\exp(-c/\varepsilon)$ with C, c positive constants, and c depending on K. It is therefore to be expected that, for $\varepsilon \ll 1$, Eq. (1.4) has an invariant manifold W near $\bar{W}$, and W should be identified with the slow-motion manifold $W_\delta(u_{\varepsilon_n})$ of the above discussion. The manifold W should be a graph over $\bar{W}$. Therefore, we try to construct a tubular neighborhood of $\bar{W}$ coordinated by (ξ, V) with V orthogonal to $\bar{W}$ by setting

$$\begin{aligned}
&u = U(\xi, \cdot) + V\\
&\langle V, U_i(\xi, \cdot)\rangle = 0, \qquad 1 \leqslant i \leqslant n
\end{aligned} \qquad (3.4)$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in L^2 and the vectors $U_i(\xi, \cdot)$, the derivatives of $U(\xi, \cdot)$ with respect to ξ_i, span the tangent space to $\bar{W}$ at

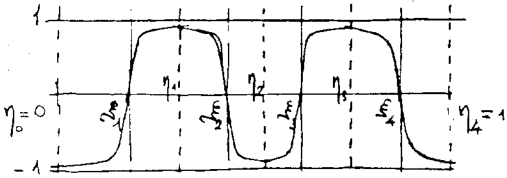

Fig. 6. The graph of the function $U(\xi, \cdot)$.

$U(\xi, \cdot)$. The natural idea at this point would be to derive from Eq. (1.4) a system of two differential equations for the two unknowns (ξ, V) and try to apply integral manifold theory to prove the existence of an invariant manifold of (1.4) near $\bar{W}$. We don't pursue this idea here, but, assuming existence, we compute a first approximation for W and for the differential equations describing the flow on it.

A function $\xi \to V(\xi, \cdot)$, with ξ in some region $\Gamma_\rho \subset \Gamma$, defines an invariant manifold W for (1.4) if and only if the vector field $\varepsilon^2 u_{xx} + f(u)$ computed at $U(\xi, \cdot) + V(\xi, \cdot)$ belongs to the tangent space to W at ξ. Since this tangent space is the span of the vectors $U_i(\xi, \cdot) + V_i(\xi, \cdot)$, this is equivalent to the existence of a function $\xi \to c(\xi) = (c_1(\xi), \ldots, c_n(\xi)) \in \mathbb{R}^n$ such that

$$\begin{aligned}
&\sum_{i=1}^{n} (U_i + V_i) c_i = \varepsilon^2 (U_{xx} + V_{xx}) + f(U + V), \qquad x \in (0, 1),\ \xi \in \Gamma_\rho \\
&V_x(\xi, 0) = -U_x(\xi, 0) \\
&V_x(\xi, 1) = -U_x(\xi, 1) \\
&[V_x(\xi, \eta_i)] = -[U_x(\xi, \eta_i)], \qquad 1 \leqslant i \leqslant n-1 \\
&\langle V(\xi, \cdot), U_i(\xi, \cdot) \rangle = 0 \qquad 1 \leqslant i \leqslant n.
\end{aligned} \tag{3.5}$$

This equation is to be considered as an equation for the two unknowns $\xi \to V(\xi, \cdot)$, $\xi \to c(\xi)$ that describe the invariant manifold and the flow on it. Since $V = 0$, $c = 0$ is a solution of (3.5) with the conditions on V_x replaced by

$$V_x(\xi, 0) = V_x(\xi, 1) = [V_x(\xi, \eta_i)] = 0$$

and, for $\varepsilon << 1$, the functionals ϕ_i are very small, in order to obtain approximate expressions for $\xi \to V(\xi, \cdot)$, $\xi \to c(\xi)$ we shall treat V, c as small quantities. Thus, instead of (3.5), we shall discuss the following problem that can be considered the linear version of (3.5):

$$\begin{aligned}
&\sum_{i=1}^{n} U_i c_i = \varepsilon^2 V_{xx} + f'(U) V, \qquad x \in (0, 1), \qquad \xi \in \Gamma_\rho \\
&V_x(\xi, 0) = -U_x(\xi, 0) \\
&V_x(\xi, 1) = -U_x(\xi, 1) \\
&[V_x(\xi, \eta_i)] = -[U_x(\xi, \eta_i)], \qquad 1 \leqslant i \leqslant n-1 \\
&\langle V(\xi, \cdot), U_i(\xi, \cdot) \rangle = 0 \qquad 1 \leqslant i \leqslant n
\end{aligned} \tag{3.6}$$

where we have used the fact that $\varepsilon^2 U_{xx} + f(U) = 0$. In contrast to the non-linear problem (3.5), the solution of which seems to be problematic because it is not obvious what the right spaces should be and because problems of loss of derivatives arise, the linear problem (3.6) can be explicitly solved. In fact, we have the following theorem in which $\Gamma_\rho \subset \Gamma$ is the open subset of Γ defined, for small $\rho > 0$, by

$$\begin{aligned} \zeta_i > \rho, \qquad &\zeta_i < 2\zeta_{i-1} - \rho \\ &\zeta_i < 2\zeta_{i+1} - \rho \end{aligned} \tag{3.7}$$

g is defined by

$$g(u) = 2(F(u) - F(-1)) = -2\int_{-1}^{u} f \tag{3.8}$$

$\tau(s) = 0(\exp(-(\mu/\varepsilon)s))$, $s > 0$, and $\tau(s^-)$ means $\tau(s) = \tau(\sigma)$ for any $\sigma < s$.

Theorem. *Given $\rho > 0$ small, there is an $\varepsilon_\rho > 0$ such that, for any $\varepsilon < \varepsilon_\rho$, eq. (3.6) has a unique solution $\xi \to V(\xi, \cdot)$, $\xi \to c(\xi)$ for $\xi \in \Gamma_\rho$ with $V(\xi, \cdot)$ such that $U(\xi, \cdot) + V(\xi, \cdot)$ is twice continuously differentiable. Moreover, $V(\cdot, \cdot)$, $c(\cdot)$ are C^1 functions and*

$$V(\xi, x) = g^{1/2}\left(\alpha_i + \beta_i \int_0^u g^{-3/2} + \gamma_i \int_0^u \left(g^{-3/2}\int_0^s g^{1/2}\right)\right)\Bigg|_{u = v(x-\xi_i)},$$

$$\eta_{i-1} \leqslant x \leqslant \eta_i, \qquad 1 \leqslant i \leqslant n \tag{3.9}$$

$$c_i(\xi) = \varepsilon \frac{\mu^2}{K} K_1^2 \left\{\exp\left(-\frac{2\mu}{\varepsilon}\zeta_i\right) - \exp\left(-\frac{2\mu}{\varepsilon}\zeta_{i-1}\right)\right\}(1 + \tau(\rho^-)),$$

$$i \leqslant i \leqslant n \tag{3.10}$$

where

$$\alpha_i(\xi) = (-1)^i \frac{\mu K_1^2}{2\varepsilon K}\left\{\zeta_{i-1}\exp\left(-\frac{2\mu}{\varepsilon}\zeta_{i-1}\right) - \zeta_i \exp\left(-\frac{2\mu}{\varepsilon}\zeta_i\right)\right\}(1 + \tau(\rho^-)$$

$$\beta_i(\xi) = (-1)^i \mu^2 K_1^2 \left\{\exp\left(-\frac{2\mu}{\varepsilon}\zeta_{i-1}\right) + \exp\left(-\frac{2\mu}{\varepsilon}\zeta_i\right)\right\}(1 + \tau(\rho^-))$$

$$\gamma_i(\xi) = (-1)^i \frac{c^i(\xi)}{\varepsilon}$$

and

$$\mu^2 = -f'(-1) = -f'(1), \qquad K = \int_{-1}^{0} g^{1/2} = \int_0^1 g^{1/2}$$

$$\log K_1 = \mu \int_0^1 g^{-1/2}\left(1 - \frac{g^{1/2}}{\mu(1-\mu)}\right).$$

Before proving this theorem, we state that, on the basis of Eq. (3.10) and of the definition of ζ_i, the equation $\dot{\xi}=c_i(\xi)$ governing the time evolution of the layers for $\varepsilon \ll 1$ takes the form

$$\dot{\xi}_1=\varepsilon\frac{\mu^2}{K}K_1^2\left\{\exp\left(-\frac{\mu}{\varepsilon}(\xi_2-\xi_1)\right)-\exp\left(-2\frac{\mu}{\varepsilon}\xi_1\right)\right\}$$

$$\dot{\xi}_i=\varepsilon\frac{\mu^2}{K}K_1^2\left\{\exp\left(-\frac{\mu}{\varepsilon}(\xi_{i+1}-\xi_i)\right)-\exp\left(-\frac{\mu}{\varepsilon}(\xi_i-\xi_{i-1})\right)\right\},$$

$$1\leqslant i\leqslant n-1 \qquad (3.11)$$

$$\dot{\xi}_n=\varepsilon\frac{\mu^2}{K}K_1^2\left\{\exp\left(-2\frac{\mu}{\varepsilon}(1-\xi_n)\right)-\exp\left(-\frac{\mu}{\varepsilon}(\xi_n-\xi_{n-1})\right)\right\}.$$

As expected, $\bar{\xi}_i=(2i-1)/2n$ is a fixed point for the dynamics defined by these equations. From Eq. (3.11), we also see that the speed of transition layers is bounded in Γ_ρ by $C\exp(-c/\varepsilon)$ for some C, $c>0$. Moreover, the speed increases with the number of layers. It is also interesting to notice that the dynamics on the slow-motion manifold involves several time scales and $\varepsilon \ll 1$ implies that the closest transition layers will get closer and closer (and eventually annihilate each other) before the other layers can move appreciably. There are other features of Eqs. (3.11) that perhaps should be pointed out. The Jacobian matrix $J(\xi)$ of the right-hand side of (3.11) at any point ξ is a Jacobi matrix with negative off-diagonal elements. Therefore, the number of sign changes in the sequence $\xi_1,\dots,\xi_n$ along solutions of (3.11) cannot decrease [Fusco and Oliva (1988)]. This has obvious implications on the dynamics of the layers. Also, one can derive from Eqs. (3.11) and (3.9) asymptotic formulae for the first n eigenvalues $\lambda_n<\lambda_{n-1}<\cdots<\lambda_1$ and eigenvectors w_n, $w_{n-1},\dots,w_1$ of the linearization of (2.1) at u_{ε_n}. In fact,

$$J(\xi)=\frac{\mu^3}{K}K_1^2\exp\left(-\frac{\mu}{n\varepsilon}\right)A_n \qquad (3.12a)$$

with

$$A_1=4;\quad A_2=\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix};\qquad A_n=\begin{bmatrix}3 & -1 & & 0\\ -1 & 2 & \ddots & \\ & \ddots & 2 & -1\\ 0 & & -1 & 3\end{bmatrix},\qquad \text{if}\quad n\geqslant 3.$$

Therefore, if v_i is the ith eigenvalue of A_n and $h_i=(h_{i1},\dots,h_{in})'$ is a corresponding eigenvector,

$$\lambda_i=\frac{\mu^3}{K}K_1^2\exp\left(-\frac{\mu}{n\varepsilon}\right)v_i,\qquad 1\leqslant i\leqslant n$$

$$w_i=\sum_{j=1}^{n}(U_j+V_j)\Big|_{\xi=\bar{\xi}}h_{ij},\qquad 1\leqslant i\leqslant n$$

where $\xi = \bar{\xi}$ means computed at $\bar{\xi} = (1/2n, 3/2n, ..., 1 - (1/2n))$. It is easy to check that A_n is a positive definite matrix. This implies that $\lambda_i > 0$ as expected because u_{ε_n} is unstable. Also, we expect the λ_i to be distinct and this follows from the fact that the v_i are distinct because A is a Jacobi matrix [Gantmacher (1959)]. Moreover, w_i should have $i-1$ zeros in (0, 1). This follows from the following facts: $U_j|_{\xi=\bar{\xi}}$ is like a pulse centered at $(2j-1)/2$ and it is positive if j is even (negative if j is odd); V_j is a small correction and it is known from the theory of Jacobi matrices that the number of sign changes in the sequence $h_{i1}, h_{i2}, ..., h_{in}$ is exactly $n-1, n-2, ..., 0$.

Figure 7 shows the phase portrait of Eqs. (3.11) for $n = 2$.

Proof. From the definition of $U(\xi, \cdot)$, it follows that

$$\begin{aligned} U_i(\xi, x) &= (-1)^i v_x(x - \xi_i), \qquad && \eta_{i-1} < x < \eta_i \\ U_i(\xi, x) &= 0, && x \notin [\eta_{i-1}, \eta_i]. \end{aligned} \tag{3.12b}$$

Thus, for $\eta_{i-1} < x < \eta_i$, the summation on the left-hand side of (3.6) reduces to $(-1)^i v_x(x - \xi_i)\, c_i(\xi)$. This implies that the coupling between the function V in two subsequent intervals (η_{i-1}, η_i), (η_i, η_{i+1}) is only through the jump condition at η_i which relates $V_x(\xi, \eta_i^+)$ to $V_x(\xi, \eta_i^-)$. These jump conditions are

$$V_x(\xi, \eta_i^+) - V_x(\xi, \eta_i^-) = (-1)^{i+1}[v_x(-\zeta_i) - v_x(\zeta_i)] = 2(-1)^i v_x(\zeta_i).$$

This suggests that one could introduce $n-1$ extra unknowns a_i, $1 \leqslant i \leqslant n-1$ and require that

$$\begin{aligned} V_x(\xi, \eta_{i-1}^+) &= -a_i v_x(\zeta_{i-1}) \\ V_x(\xi, \eta_i^-) &= -b_i v_x(\zeta i) \end{aligned} \tag{3.13}$$

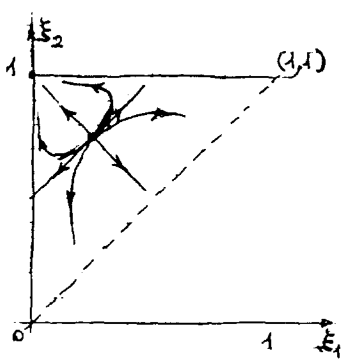

Fig. 7.

with $b_i = a_{i+1} + 2(-1)^{i+1}$. This requirement will automatically imply that the jump conditions are satisfied. Therefore, we obtain in each of the intervals (η_{i-1}, η_i) an equation that yields c_i and V as functions of the extra unknowns that must be then determined by requiring that V is continuous at $x = \eta_i$, $1 \leqslant i \leqslant n-1$. If we make the change of variable $x \to x - \xi_i$, set $\bar{c}_i = (-1)^{i+1} c_i$ and recall (3.12b), (3.13), the equation for V in the interval (η_{i-1}, η_i) takes the form

$$\begin{aligned} -v_x \bar{c}_i &= \varepsilon^2 V_{xx} + f'(v) V, \qquad -\zeta_{i-1} < x < \zeta_i \\ V_x(-\zeta_{i-1}) &= -a_i v_x(\zeta_{i-1}) \\ V_x(\zeta_i) &= -b_i v_x(\zeta_i) \\ \int_{-\zeta_{i-1}}^{\zeta_i} V v_x \, dx &= 0 \end{aligned} \tag{3.14}$$

where we have assumed that $a_1 = b_n = 0$ and have used the fact that $f'(v) = f'(U)$ because f' us even. The $n-1$ conditions expressing the continuity of V at η_i can be written as

$$V(\zeta_i) = \bar{V}(-\zeta_i) \tag{3.15}$$

where V is the solution of (3.14) and $\bar{V}$ denotes the solution of (3.14) with i replaced by $i+1$. To solve (3.14), we let $x(v)$ be the inverse of the function $v(x)$ and change variables to $x = x(v)$. Since v is the solution of (3.1), we have

$$-\frac{\varepsilon^2 x''}{(x')^3} + f(v) = 0 \tag{3.16}$$

where $'$ means derivation with respect to v. Also, from (3.16) and the definition of g, we get

$$\frac{\varepsilon^2}{(x')^2} = g. \tag{3.17}$$

Therefore, taking into account that $g' = -2f$, we obtain

$$\frac{\varepsilon^2}{(x')^2} V'' - \varepsilon^2 \frac{x''}{(x')^3} V' + f'V = gV'' - fV' + f'V = (gV' + fV)'.$$

It follows that, if we let $v_i^0 = v(-\zeta_{i-1})$; $v_i^* = v(\zeta_i)$, Eq. (3.14) can be rewritten as

$$
\begin{aligned}
(gV'+fV)' &= -\frac{\bar{c}_i}{\varepsilon}\, g^{1/2}, \qquad v_i^0 < v < v_i^* \\
V'(v_i^0) &= -a_i \\
V'(v_i^*) &= -b_i \\
\int_{v_i^0}^{v_i^*} V &= 0.
\end{aligned} \tag{3.18}
$$

From this equation and the fact that the fundamental solution at s of the equation $gy'+fy=0$ is given by $(g(v)/g(s))^{1/2}$, we obtain

$$
V = g^{1/2}\left\{\alpha_i + \beta_i \int_0^v g^{-3/2} - \frac{\bar{c}_i}{\varepsilon}\int_0^v \left(g^{-3/2}\int_0^s g^{1/2}\right)\right\}.
$$

where α_i, β_i are integration constants that together with $\bar{c}_1$ are to be determined by requiring V to satisfy the boundary conditions and the integral condition in (3.18). Explicitly, these conditions come out to be

$$
\begin{aligned}
-\alpha_i(fg^{-1/2})^0 + \beta_i P^0 + \frac{\bar{c}_i}{\varepsilon} Q^0 &= -a_i \\
-\alpha_i(fg^{-1/2})^* + \beta_i P^* + \frac{\bar{c}_i}{\varepsilon} Q^* &= -b_i \\
\alpha_i(R^* - R^0) + \beta_i(S^* - S^0) - \frac{\bar{c}_i}{\varepsilon}(T^* - T^0) &= 0
\end{aligned} \tag{3.19}
$$

where the superscripts 0, * mean computated at v_i^0, v_i^* and

$$
P = g^{-1} - fg^{-1/2}\int_0^v g^{-3/2}; \qquad Q = fg^{-1/2}\int_0^v \left(g^{-3/2}\int_0^s g^{1/2}\right) - g^{-1}\int_0^v g^{1/2}
$$

$$
R = \int_0^v g^{1/2}; \qquad S = \int_0^v \left(g^{1/2}\int_0^s g^{-3/2}\right); \qquad T = \int_0^v \left(g^{1/2}\int_0^s \left(g^{-3/2}\int_0^r g^{1/2}\right)\right). \tag{3.20}
$$

We also define

$$
L = g^{1/2}\int_0^v g^{-3/2}; \qquad M = g^{1/2}\int_0^v \left(g^{-3/2}\int_0^s g^{1/2}\right). \tag{3.21}
$$

On the basis of (3.20), (3.21) and the definition of g, it is quite standard to get the following estimates for v near 1.

$$
\begin{aligned}
fg^{-1/2} &= \mu(1+0(1-v)) && \text{odd}\\
P &= \frac{1}{2}(\mu(1-v))^{-2}(1+0(1-v)) && \text{even}\\
Q &= -\frac{K}{2}(\mu(1-v))^{-2}(1+0(1-v)) && \text{odd}\\
R &= K(1+0(1-v)) && \text{odd}\\
S &= -\frac{1}{2\mu^2}(\ln(1-v)+0(1)) && \text{even}\\
T &= -\frac{K}{2\mu^2}(\ln(1-v)+0(1)) && \text{odd}\\
g^{1/2} &= \mu(1-v)(1+0(1-v)) && \text{even}\\
L &= \frac{1}{2\mu^2}(1-v)^{-1}(1+0(1-v)) && \text{odd}\\
M &= \frac{K}{2\mu^2}(1-v)^{-1}(1+0(1-v)) && \text{even}
\end{aligned}
\tag{3.22}
$$

where we have also indicated for each function if it is odd or even. By integrating Eq. (3.17) and by a simple analysis, one obtains

$$
\begin{aligned}
1-v &= K_1 \exp\left(-\frac{\mu}{\varepsilon}x\right)(1+\tau(x)), && x>0\\
1+v &= K_1 \exp\left(\frac{\mu}{\varepsilon}x\right)(1+\tau(-x)), && x<0.
\end{aligned}
\tag{3.23}
$$

From the estimates (3.22), (3.23), it follows that Eqs. (3.19) have a unique solution given by

$$
\begin{aligned}
\alpha_i &= \frac{\mu K_1^2}{2K\varepsilon}\left\{-a_i\zeta_{i-1}\exp\left(-\frac{2\mu}{\varepsilon}\zeta_{i-1}\right)+b_i\zeta_i\exp\left(-\frac{2\mu}{\varepsilon}\zeta_i\right)\right\}\\
&\quad+\tau(\zeta_i+2\zeta_{i-1}^-)+\tau(\zeta_{i-1}+2\zeta_i^-)\\
\beta_i &= -\mu^2K_1^2\left\{a_i\exp\left(-\frac{2\mu}{\varepsilon}\zeta_{i-1}\right)+b_i\exp\left(-\frac{2\mu}{\varepsilon}\zeta_i\right)\right\}\\
&\quad+\tau(\zeta_i+2\zeta_{i-1})+\tau(\zeta_{i-1}+2\zeta_i)\\
\bar{c}_i &= -\frac{\varepsilon\mu^2}{K}K_1^2\left\{a_i\exp\left(-2\frac{\mu}{\varepsilon}\zeta_{i-1}\right)-b_i\exp\left(-\frac{2\mu}{\varepsilon}\zeta_i\right)\right\}\\
&\quad+\tau(\zeta_i+2\zeta_{i-1})+\tau(\zeta_{i-1}+2\zeta_i).
\end{aligned}
\tag{3.24}
$$

By means of the expressions of α_i, β_i, $\bar{c}_i$ and of the estimates (3.22), we can write the $n-1$ Eqs. (3.15) in explicit form:

$$\begin{aligned} V(\zeta_i) &= -b_i K_1 \exp\left(-\frac{\mu}{\varepsilon}\zeta_i\right) + \tau(2\zeta_{i-1}) + \tau(2\zeta_i) = \bar{V}(-\zeta_i) \\ &= a_{i+1} K_1 \exp\left(-\frac{\mu}{\varepsilon}\zeta_i\right) + \tau(2\zeta_i) + \tau(2\zeta_{i+1}), \qquad 1 \leqslant i \leqslant n-1. \end{aligned} \tag{3.25}$$

If, as we have assumed, $\xi \in \Gamma_\rho$, then these equations imply that

$$a_{i+1} = -b_i + \tau(\rho^-), \qquad 1 \leqslant i \leqslant n-1.$$

Therefore, by recalling that $b_i = a_{i+1} - 2(-1)^i$, we have

$$\begin{aligned} a_i &= -(-1)^i + \tau(\rho), \qquad \mathrm{i} = 2, \ldots, n \\ b_i &= -(-1)^i + \tau(\rho), \qquad i = 1, \ldots, n-1. \end{aligned} \tag{3.26}$$

By introducing these values of a_i, b_i into (3.26) and by recalling that $a_1 = b_n = 0$, $\bar{c}_i = (-1)^{i+1} c_i$, the proof is easily concluded. ■

Let us turn to the question of the existence of an invariant manifold for Eq. (1.4) on which the qualitative properties of the flow are similar to those described by the differential Eqs. (3.11). One natural approach is the following. Let $V^0(\xi, \cdot)$ be the function given in the theorem and let

$$W^1 = \{U(\xi, \cdot) + V^0(\xi, \cdot), \xi \in \Gamma_\rho\}.$$

The set W^1 is a manifold in H^1. Let us try to construct a manifold of solutions of (1.4) near W^1. Introducing normal coordinates

$$\begin{aligned} u &= U(\xi, \cdot) + V^0(\xi, \cdot) + V^1 \\ \langle V^1, U_i + V_i^0 \rangle &= 0, \qquad 1 \leqslant i \leqslant n \end{aligned}$$

and proceeding as before, we obtain such an invariant manifold if there exists a function V^1 and constants $d_i = d_i(\xi)$, $1 \leqslant i \leqslant n$, such that

$$\sum_{i=1}^{n} U_i d_i + (V_i^0 + V_i^1)(c_i + d_i) = \varepsilon^2 V_{xx}^1 + f'(U + V^0) V^1 + O(|V^0|^2) + O(|V^1|^2)$$

with the auxiliary conditions

$$\begin{aligned} &V_x^1(\xi, 0) = 0, \qquad V_x^1(\xi, 1) = 0 \\ &\langle V^1, U_i(\xi, \cdot) + V_i^0(\xi, \cdot) \rangle = 0, \qquad 1 \leqslant i \leqslant n. \end{aligned}$$

A proof of the existence of a solution to these equations has not been given. Of course, one also could consider the differential equations for (ξ, V^1) and attempt to use integral manifold theory to obtain an invariant manifold near W^1. This method also has not been carried to completion.

We conclude this paper with some remarks about the work of Carr and Pego (1988). Using the coordinate system (3.4) for a slightly different approximate manifold $\bar{W}$, they discussed properties of the differential equations for ξ, V. More specifically, using the fact that the linearization in the V-equation is exponentially asymptotically stable with a decay rate bounded away from zero with respct to ε, they obtain good bounds on the energy and show that solutions are crossing the lateral boundary of a tubular neighborhood of a part of $\bar{W}$ from the outside to the inside. They also observe that the time evolution of the transition layers have the same qualitative properties as Eq. (3.11).

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NOTE ADDED IN PROOF

Since this paper was written, the first author has been able to show that an exact integral manifold does exist near the refined approximate manifold W_1 by the application of integral manifold theory to the differential equations or ξ, V_1. It is our understanding that Carr and Pego have obtained similar results using their approach.

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