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Reflecting Brownian Motions: Quasimartingales and Strong Caccioppoli Sets

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Abstract. A notion of *strong* Caccioppoli set is defined for bounded Euclidean domains. It is shown that stationary (normally) reflecting Brownian motion on the closure of a bounded Euclidean domain is a quasimartingale on each compact time interval if and only if the domain is a strong Caccioppoli set. A similar result is shown to hold for symmetric reflecting diffusion processes.

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1. Introduction

Let D be a bounded domain in $\mathbb{R}^d$ and let m denote Lebesgue measure on D, normalized so that $m(D) = 1$. The Euclidean closure of D will be denoted by $\bar{D}$. Let $H^1(D)$ denote the set of functions f in $L^2(D, m)$ that have distributional derivatives $\partial f/\partial x_i$, $i = 1, \ldots, d$, that are also in $L^2(D, m)$. Define the symmetric positive definite bilinear form $\mathscr{E}$ on $H^1(D)$ by

$$\mathscr{E}(f, g) = \frac{1}{2}\int_D \nabla f \cdot \nabla g \, \mathrm{d}m, \quad f, g \in H^1(D), \tag{1.1}$$

where ∇ denotes gradient and $\cdot$ denotes vector dot product. By Fukushima ([12] Example 1.2.3), the pair $(H^1(D), \mathscr{E})$ is a symmetric Dirichlet space on $L^2(D, m)$. There is an associated stationary Markov process with continuous sample paths in $\bar{D}$, which is called stationary (normally) reflecting Brownian motion on $\bar{D}$ (cf. [18]). In this paper we give a necessary and sufficient condition on D for this stationary process to be a quasimartingale on each compact time interval. To describe this more precisely, we shall define quasimartingales in this context and the notion of *strong* Caccioppoli set.

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DEFINITION 1.1. A continuous d-dimensional process $Z = \{Z_t, t \geqslant 0\}$ is a *quasimartingale* on $[0, T]$, $T < \infty$, if and only if $E[|Z_t|] < \infty$ for each $t \in [0, T]$ and

$$\sup_{\pi} E\left[\sum_{t_i, t_{i+1} \in \pi} |E[Z_{t_{i+1}} - Z_{t_i} | \mathscr{F}_{t_i}]| \right] < \infty,$$

where the sup is over all finite partitions $\pi: 0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$, and $\{\mathscr{F}_t\}$ denotes the filtration generated by Z. We shall say Z is a quasimartingale if it is a quasimartingale on $[0, T]$ for each $T > 0$. (Here we depart slightly from the conventional terminology of Dellacherie and Meyer ([7] VI.38).)

PROPOSITION 1.1. *A continuous d-dimensional process Z is a quasimartingale if and only if it has a decomposition of the form*

$$Z_t = Z_0 + M_t + Y_t, \quad t \geqslant 0, \tag{1.2}$$

where M and Y are continuous d-dimensional processes starting from zero and relative to the filtration generated by Z, M is a local martingale satisfying

$$\sup \{E[|M_{\tau \wedge T}|] : \tau \text{ is a stopping time}\} < \infty \quad \text{for each } T \geqslant 0, \tag{1.3}$$

and Y is an adapted process whose total variation on any compact time interval is integrable. This decomposition is unique.

Sketch of proof. The decomposition is unique because it is a continuous semimartingale decomposition. Using this uniqueness and our definition of a quasimartingale, one sees that it suffices to consider the process Z stopped at $T \in [0, \infty)$. For this process, the "if" part follows from the Krickeberg–Kazamaki decomposition of an L^1-bounded local martingale into a difference of positive local martingales [7, VI.35], together with Rao's characterization of quasimartingales as differences of positive supermartingales ([7] VI.40). For the "only if" part, one uses Rao's characterization again and the Doob–Meyer decomposition of positive supermartingales. ■

DEFINITION 1.2. The bounded domain D is called a *strong Caccioppoli* set if there is a positive constant C such that

$$\int_D \frac{\partial g}{\partial x_i} dm \leqslant C \|g\|_\infty, \quad i = 1, \ldots, d, \tag{1.4}$$

for all $g \in H^1(D) \cap C_b(D)$, where $\|g\|_\infty = \sup_{x \in D} |g(x)|$ and $C_b(D)$ denotes the space of real-valued, bounded continuous functions on D.

REMARK. Our notion of *strong* Caccioppoli set is a refinement of the geometric measure theoretic notion of *Caccioppoli set* [13]. The bounded domain D is a Caccioppoli set if it satisfies (1.4) for all $g \in C_c^1(\mathbb{R}^d)$ with $\|g\|_\infty = \sup_{x \in \mathbb{R}^d} |g(x)|$, where

$C_c^1(\mathbb{R}^d)$ denotes the set of all once continuously differentiable real-valued functions defined on $\mathbb{R}^d$ that have compact support. An example in Section 5 shows that the strong Caccioppoli sets are a proper subclass of the Caccioppoli sets.

The main result of this paper, which is proved in Sections 2 and 3, is Theorem 1.1 below. A similar result is proved for symmetric reflecting diffusions in Section 4.

THEOREM 1.1. *The stationary reflecting Brownian motion on $\bar{D}$ is a quasimartingale if and only if D is a strong Caccioppoli set.*

To put Theorem 1.1 in perspective, we now briefly review previous work on the problem of when reflecting Brownian motion in a bounded domain is a quasimartingale.

When the boundary of D is C^2-smooth, it is well known that there is a continuous strong Markov process X associated with the Dirichlet space $(H^1(D), \mathscr{E})$ and it has a Skorokhod semimartingale decomposition starting from any point in $\bar{D}$:

$$X_t = X_0 + B_t + \int_0^t n(X_s)\,\mathrm{d}L_s, \quad t \geqslant 0, \tag{1.5}$$

where B is a d-dimensional Brownian motion martingale additive functional of X, n is the inward unit normal vector field on ∂D, and L is a positive continuous additive functional of X with associated (Revuz) measure proportional to surface measure σ on ∂D. When the initial distribution of X is set equal to m, this yields a realization of the stationary process referred to in Theorem 1.1, which is a quasimartingale since the measure σ is finite.

Conditions for reflecting Brownian motion in a non-smooth domain to be a quasimartingale have only recently begun to be investigated. When the boundary of D is non-smooth there need not be a continuous strong Markov process on $\bar{D}$ associated with the Dirichlet space $(H^1(D), \mathscr{E})$. However ([12] Theorems 6.2.1, 6.2.2), there is such a process if the Dirichlet space is regular on $\bar{D}$ (see Definition 1.3 below). Moreover, Fukushima [11] has shown that there is always a suitable compactification of D, called the Martin–Kuramochi compactification, on which there is a continuous strong Markov process associated with the Dirichlet space $(H^1(D), \mathscr{E})$. On the other hand, as mentioned above, there is always a continuous *stationary* Markov process on $\bar{D}$ associated with this Dirichlet space (cf. [18]). Some of the work described below deals with quasimartingale/semimartingale properties of the strong Markov process associated with $(H^1(D), \mathscr{E})$, whilst some deals with the stationary Markov process.

DEFINITION 1.3. The Dirichlet space $(H^1(D), \mathscr{E})$ is *regular* on $\bar{D}$ if $H^1(D) \cap C(\bar{D})$ is dense both in $(H^1(D), \sqrt{\mathscr{E}_1})$ and in $(C(\bar{D}), \|\cdot\|_\infty)$, where $\mathscr{E}_1(f,f) = \mathscr{E}(f,f) + (f,f)_m$, $(\cdot,\cdot)_m$ is the inner product on $L^2(D,m)$, and $\|\cdot\|_\infty$ denotes the uniform norm on the space $C(\bar{D})$ of continuous functions on $\bar{D}$. Convergence in $H^1(D)$ with respect to the norm $\sqrt{\mathscr{E}_1}$ will be described as convergence in $\mathscr{E}_1$-norm.

When D is a bounded Lipschitz domain, the Dirichlet space $(H^1(D), \mathscr{E})$ is regular on $\bar{D}$ ([1] pp. 54, 66–67). In [2], Bass and Hsu showed that for a bounded Lipschitz domain D, the continuous strong Markov process on $\bar{D}$ associated with $(H^1(D), \mathscr{E})$ has a decomposition of the form (1.5) for all starting points in $\bar{D}$ except those in a set of capacity zero. Since the boundary of a bounded Lipschitz domain has finite surface measure, we again have in this case that the *stationary* reflecting Brownian motion is a quasimartingale.

In [18], Williams and Zheng used a weak convergence construction of stationary reflecting Brownian motion on $\bar{D}$ to obtain a sufficient condition for this process to be a semimartingale (in fact, their results imply it is a quasimartingale, although they did not explicitly state this). A consequence of their results is that if the boundary ∂D of D has finite $(d-1)$-dimensional upper Minkowski content ([9] §3.2.37), i.e., if

$$\limsup_{\varepsilon \downarrow 0} \frac{m(\{x \in \mathbb{R}^d : d(x, \partial D) < \varepsilon\})}{\varepsilon} < \infty, \tag{1.6}$$

then the stationary reflecting Brownian motion on $\bar{D}$ is a quasimartingale whose local martingale part is a Brownian motion. This result has been extended by Pardoux and Williams [15] to stationary symmetric reflecting diffusion processes. Recently, Chen [5] has given more general sufficient conditions for reflecting Brownian motions (and symmetric reflecting diffusion processes) to be quasimartingales. He considers bounded domains D such that there is an increasing sequence $\{D_n\}$ of smooth subdomains of D with uniformly bounded surface measures such that $D = \cup_n D_n$. In particular, domains having boundaries of finite $(d-1)$-dimensional lower Minkowski content (replace sup with inf in (1.6) for the definition) satisfy this condition. Under the bounded surface measure condition described above, Chen shows that when the Dirichlet space $(H^1(D), \mathscr{E})$ is regular on $\bar{D}$, the associated continuous strong Markov process on $\bar{D}$ has a decomposition of the form (1.5) starting from all points in $\bar{D}$, except for those in a set of capacity zero, when n is a generalized normal vector field and σ is a generalized surface measure on ∂D. Whether the Dirichlet form is regular on $\bar{D}$ or not, Chen's results imply that under the bounded surface measure condition the stationary reflecting Brownian motion on $\bar{D}$ is a quasimartingale. We note here that the results described in this paragraph still hold when "$x \in \mathbb{R}^d$" is replaced with "$x \in D$" in the definitions of upper and lower Minkowski content.

All of the above results concern *sufficient* conditions for reflecting Brownian motion on a bounded domain to be a quasimartingale/semimartingale. Recently, Williams [17] considered the problem of finding a geometric *necessary* condition for reflecting Brownian motion on $\bar{D}$ to be a semimartingale. Assuming the Dirichlet space $(H^1(D), \mathscr{E})$ to be regular on $\bar{D}$, she showed that a necessary condition for the associated continuous strong Markov process on $\bar{D}$ to be a semimartingale (for all starting points except those in a set of capacity zero), whose locally bounded variation part has an associated smooth vector measure with finite energy integral, is that D be a Caccioppoli set (see the remark following Definition 1.2). The finite energy integral condition in fact ensures that the stationary reflecting Brownian motion is a quasimartingale. The present paper grew out of efforts to refine and prove a converse of Williams' [17] result.

Our paper is organized as follows. Sections 2 and 3 deal with necessary and sufficient conditions on D for the stationary reflecting Brownian motion on $\bar{D}$ to be a quasimartingale. In Section 4, we generalize these results to symmetric reflecting diffusion processes. In Section 5, some sufficient conditions for a bounded domain to be a strong Caccioppoli set and for a Caccioppoli set to be a strong Caccioppoli set are given. An example is also given to illustrate that D being a Caccioppoli set is not necessary for the stationary reflecting Brownian motion in $\bar{D}$ to be a semimartingale. This leaves open the natural question: What is a necessary and sufficient condition for stationary reflecting Brownian motion on $\bar{D}$ to be a semimartingale?

For the terminology of quasi-continuous, quasi-everywhere (q.e. in abbreviation), excessive function, capacity, smooth measure, additive functional, etc., used in this paper, we refer the reader to Fukushima [12]. A finite signed measure will be called smooth if and only if its total variation measure is smooth. The reader will note that coordinates of processes are indexed by superscripts so as not to conflict with the use of subscripts for the time index. The coordinates of other vector objects are indexed with subscripts.

2. Necessary Condition

We are concerned in this section with the "necessity" portion of Theorem 1.1. The key to our discussion is a convenient realization of the stationary reflecting Brownian motion, which we now describe.

As noted in Section 1, the Dirichlet space $(H^1(D), \mathscr{E})$ need not be regular on $\bar{D}$. However one can embed D as a dense open subset of a suitable compact metric space D^* in such a way that $(H^1(D), \mathscr{E})$ becomes a regular Dirichlet space on D^*, i.e., $H^1(D) \cap C(D^*)$ is dense in $(H^1(D), \sqrt{\mathscr{E}_1})$ and in $(C(D^*), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ denotes the supremum norm on $C(D^*)$. We use a particular compactification D^* here, namely the Martin–Kuramochi compactification introduced by Fukushima in [11]. Associated with $(H^1(D), \mathscr{E})$ is a strong Markov process

$$X^* = (X_t^*, t \geqslant 0; P_x, x \in D^*).$$

The "association" between X^* and $(H^1(D), \mathscr{E})$ is expressed as follows: Writing $(T_t, t \geqslant 0)$ for the transition semigroup of X^*, we have

$$\mathscr{E}(f,f) = \lim_{t \to 0} t^{-1}(f, f - T_t f)_m, \quad f \in H^1(D),$$

and $H^1(D)$ consists precisely of those functions in $L^2(D, m)$ for which the indicated limit exists. We extend m to a measure on D^* by defining $m(D^* \setminus D) = 0$, thereby identifying $L^2(D, m)$ with $L^2(D^*, m)$. We shall recall below a few facts about X^* that are relevant to the present discussion; for full details the reader is referred to [11] and [12].

The transition probabilities of X^* are absolutely continuous with respect to m. Consequently the notions "set of capacity zero" and "polar set" coincide. Thus, a property or statement holding quasi-everywhere holds outside some polar set.

The process X^* is a diffusion. More precisely, if we adjoin a cemetery state ∂ to D^* as an isolated point, then $t \mapsto X_t^*$ takes values in $D^* \cup \{\partial\}$ and is continuous P_x-a.s. for all $x \in D^*$. In fact, there is a polar set $\Delta_0 \subset D^* \setminus D$ such that

$$P_x(X_0^* = x, X_t^* \in D^* \text{ for all } t \geqslant 0) = 1, \quad \text{for all } x \in D^* \setminus \Delta_0.$$

(The elements of Δ_0 are "branch points" and have only nuisance value.)

We set $P_m = \int_{D^*} P_x m(\mathrm{d}x)$, and we denote expectations with respect to P_x, P_m by E_x, E_m, respectively. It follows from the discussion in the last paragraph that the paths of X^* are D^*-valued and continuous P_m-a.s. The lifetime of X^* is infinite for q.e. starting point, so that $T_t 1 = 1$ q.e. This together with the symmetry of T_t on $L^2(D^*, m)$ implies that m is a stationary distribution for X^*.

Each $f \in H^1(D)$ has an m-equivalent version $\tilde{f}$ that is quasi-continuous on D^*. As such, $\tilde{f}$ is uniquely determined q.e., and $t \mapsto \tilde{f}(X_t^*)$ is continuous on $[0, \infty)$ P_x-a.s. for q.e. starting point $x \in D^*$. In what follows, each element of $H^1(D)$ is understood to be represented by its quasi-continuous version; for this reason we suppress the tilde from our notation.

Crucial to our discussion is Fukushima's decomposition ([12] §5.2): Given $f \in H^1(D)$, there is a martingale additive functional M satisfying $M_0 = 0$ for q.e. x, and a continuous additive functional of zero energy N, such that

$$f(X_t^*) = f(X_0^*) + M_t + N_t, \quad t \geqslant 0, P_x\text{-a.s. for q.e. } x. \tag{2.1}$$

Here and elsewhere the term "additive functional" means additive functional of X^*, and in saying that N is of zero energy we mean

$$\lim_{t \to 0} t^{-1} E_m(N_t^2) = 0.$$

The coordinate maps $D \ni x = (x_1, \ldots, x_d) \mapsto x_i$, $i = 1, 2, \ldots, d$, are elements of $H^1(D)$. We write φ_i for the quasi-continuous extension of $x \mapsto x_i$ to all of D^* and set $\varphi := (\varphi_1, \ldots, \varphi_d)$. Evidently

$$m(\{x \in D : \varphi(x) \neq x\}) = 0, \tag{2.2}$$

and $t \mapsto X_t := \varphi(X_t^*)$ is continuous on $[0, \infty)$ P_x-a.s. for q.e. $x \in D^*$. Since m is a stationary distribution for X^*,

$$\begin{aligned} P_m(X_r \notin D \text{ for some rational } r \geqslant 0) &\leqslant \sum_{r \text{ rational}} P_m(X_r \notin D) \\ &\leqslant \sum_{r \text{ rational}} P_m(X_r^* \in \{x \in D : \varphi(x) \neq x\} \text{ or } X_r^* \notin D) = 0. \end{aligned}$$

Combining this with the path-continuity of X we deduce that

$$P_m(X_t \in \bar{D} \text{ for all } t \geqslant 0) = 1.$$

The process (X_t, P_m) is stationary in view of (2.2) and the fact that m is the stationary distibution of X^*. Moreover, appealing to (2.1) we obtain the decomposition

$$X_t = X_0 + B_t + V_t, \quad t \geqslant 0, \tag{2.3}$$

where B is an $\mathbb{R}^d$-valued continuous martingale additive functional satisfying $B_0 = 0$, and V is an $\mathbb{R}^d$-valued zero-energy continuous additive functional. The covariation of B is simply $\langle B^i, B^j \rangle_t = t \cdot \delta_{ij}$, (cf. [12] 5.2.32). Thus B is a d-dimensional Brownian motion martingale in the natural filtration of X^*. Also, by ([12] Theorem 5.3.4), the process V is constant during those intervals of time for which X^* is in D.

CONVENTION. Henceforth, we will use (X_t, P_m) as our realization of the stationary reflecting Brownian motion on $\bar{D}$ associated with $(H^1(D), \mathscr{E})$.

To study V more closely we require the following lemma, which is essentially Lemma 2.2 of [5]. Consider a continuous additive functional N of zero-energy. In Fukushima's terminology ([12] p. 143), N is of *bounded variation* if for q.e. $x \in D^*$, P_x-a.s. the paths of N are of bounded variation on each compact time interval. If N is of bounded variation, then we can write $N = N^+ - N^-$, where N^+ and N^- are positive continuous additive functionals. This decomposition is unique if we impose the side condition $dN^+ \wedge dN^- = 0$. Choosing this unique decomposition, let μ^+ and μ^- denote the smooth measures associated with N^+ and N^-, respectively. If $\mu^+(D^*) + \mu^-(D^*) < \infty$ then $\mu := \mu^+ - \mu^-$ is a well defined signed measure. This μ will be referred to as the finite smooth (signed) measure associated with N, and its total variation measure will be denoted by $|\mu|$.

LEMMA 2.1. *Given $f \in H^1(D)$, let M and N be as in the decomposition* (2.1). *If N is of bounded variation and has an associated finite smooth signed measure μ, then*

$$\mathscr{E}(f,g) = -\int_{D^*} g(x)\mu(\mathrm{d}x) \quad \text{for all bounded } g \in H^1(D). \tag{2.4}$$

Conversely, if there is a finite smooth signed measure μ such that (2.4) holds, then N is of bounded variation with associated finite smooth measure μ and the total variation of N on each compact time interval is P_m-integrable. In this case, the positive and negative parts of the Jordan decomposition of μ are the smooth measures associated with N^+ and N^- where $dN^+ \wedge dN^- = 0$.

REMARKS. (a) Only the first part of the above lemma is needed in this section. The second part is used in Section 3.

(b) We will see in Section 3 that the validity of the formula in (2.4) for all $g \in H^1(D) \cap C(D^*)$ already implies that the finite measure μ is smooth.

(c) Lemma 2.1 is valid for other Dirichlet spaces $(H^1(D), \mathscr{E})$ that are regular on D^*. In particular, it applies to those treated in Section 4 below.

Proof. By ([12] Lemma 5.1.4(iii)) and an approximation argument, if A is a positive continuous additive functional with associated smooth measure ν, then for any positive Borel function g on D^*,

$$\int_{D^*} g(x)E_x[A_t]m(\mathrm{d}x) = \int_0^t \mathrm{d}s \int_{D^*} E_x[g(X_s^*)]\nu(\mathrm{d}x). \tag{2.5}$$

Thus if N is of bounded variation and has an associated finite smooth measure $\mu = \mu^+ - \mu^-$, then

$$\int_{D^*} g(x)E_x[N_t]m(\mathrm{d}x) = \int_0^t \mathrm{d}s \int_{D^*} E_x[g(X_s^*)]\mu(\mathrm{d}x), \tag{2.6}$$

for all bounded $g \in H^1(D)$. Since we are taking the elements of $H^1(D)$ to be represented by their quasi-continuous versions, we have $\lim_{t \downarrow 0} E_x[g(X_t^*)] = g(x)$ for q.e. $x \in D^*$ by bounded convergence. A second application of the bounded convergence theorem now yields

$$\lim_{t \downarrow 0} t^{-1} \int_{D^*} g(x)E_x[N_t]m(\mathrm{d}x) = \int_{D^*} g(x)\mu(\mathrm{d}x). \tag{2.7}$$

But the left side of (2.7) is equal to $-\mathscr{E}(f,g)$ by ([12] Theorem 5.3.1), so (2.4) obtains.

Under the hypotheses of the converse, by considering the positive and negative parts of g and employing a truncation argument ([12] Theorem 1.4.2), (2.4) can be extended to all $g \in H^1(D)$. Then, modulo a change of sign, the converse assertion is a special case of ([12] Theorem 5.3.2) and (2.5). ■

We are now ready for the main result of this section.

THEOREM 2.1. *Suppose the stationary Brownian motion* (X_t, P_m) *is a quasi-martingale. Then* D *is a strong Caccioppoli set.*

Proof. We should emphasize that the filtration involved in the hypothesis "(X_t, P_m) is a quasimartingale" is the natural filtration of X. In order to bring the decomposition (2.3) to bear, we must first check that X and X^* generate the same filtration, modulo P_m-null sets. The inclusion $\sigma\{X_s: 0 \leqslant s \leqslant t\} \subset \sigma\{X_s^*: 0 \leqslant s \leqslant t\}$ is trivial. On the other hand, because of (2.2) we have

$$P_m(X_r^* = X_r \in D \text{ for all rational } r \geqslant 0) = 1.$$

The path continuity of X^* now yields

$$X_t^* = \lim_{r \uparrow t, r \text{ rational}} X_r, \quad \forall t \geqslant 0, \quad P_m\text{-a.s.,} \tag{2.8}$$

where the limit is taken in D^*. Clearly (2.8) implies the reverse inclusion.

For the remainder of this proof, all martingales and quasimartingales will be under the measure P_m, unless stated otherwise. Let us write $\{\mathscr{F}_t\}$ for the natural filtration of X^*, augmented by the P_m-null sets and made right continuous. By the observation made in the last paragraph, X is an $\{\mathscr{F}_t\}$-quasimartingale. Now, B in (2.3) is an $\{\mathscr{F}_t\}$-martingale, and so it follows that $V = X - X_0 - B$ is an $\{\mathscr{F}_t\}$-quasimartingale. Let

$$V_t = M_t + Y_t, \quad t \geqslant 0, \tag{2.9}$$

be the decomposition (cf. Proposition 1.1) of V into a continuous $\{\mathscr{F}_t\}$-local martingale M satisfying $M_0 = 0$ and (1.3), and a continuous $\{\mathscr{F}_t\}$-adapted process Y that has P_m-integrable total variation on each compact time interval. Now, since V is of zero energy, its quadratic variation at any fixed time t is zero P_m-a.s. [12, (5.2.10)]. But the quadratic variation of V is equal to that of M, so M has zero quadratic variation at each fixed time t and therefore by path continuity for all t, P_m-a.s. This means that M is the zero martingale. Hence $V := (V^1, \ldots, V^d) = Y$ is a continuous additive functional whose total variation on each compact time interval is P_m-integrable.

For $i \in \{1, \ldots, d\}$, let A_t^i denote the total variation of V^i on $[0, t]$ for each $t \geqslant 0$. Then A^i is additive and since m is the stationary measure for X^*, there is a finite constant C_i such that

$$E_m[A_t^i] = C_i t, \quad t \geqslant 0. \tag{2.10}$$

Let U_i denote the (polar) exceptional set for V^i [12, p. 124]. It is easy to check that $\rho_i(x) := P_x(A_t^i = \infty \text{ for some } t \geqslant 0)$ is an excessive function for X^* restricted to $D^* \backslash U_i$. But $\rho_i(x) = 0$ for m-a.e. x because of (2.10), hence $\rho_i(x) = 0$ for q.e. x. In other words, $P_x(A_t^i < \infty, \forall t \geqslant 0) = 1$ for q.e. x. We adjoin to U_i an exceptional set of points such that the last equality holds *everywhere* off the new U_i. Now consider

$$\psi_i(x) := E_x\left[\int_0^\infty e^{-t} A_t^i \, dt\right] = E_x\left[\int_0^\infty e^{-t} \, dA_t^i\right], \quad x \notin U_i. \tag{2.11}$$

The second equality in (2.11) follows by Fubini's theorem because of the finiteness of A_t^i just established. By (2.10),

$$\int_{D^*} \psi_i(x) m(dx) = E_m\left[\int_0^\infty e^{-t} A_t^i \, dt\right] = C_i,$$

and so $\psi_i(x)$ is finite for m-a.e. x. However, it is clear from the second equality in (2.11) that ψ_i is a 1-excessive function for X^* restricted to $D^* \backslash U_i$, and consequently it must be finite for q.e. x. It follows that $E_x[A_t^i] < \infty$ for all t, for q.e. x, and hence the total variation of V^i on each compact time interval is P_x-integrable, for q.e. x. In particularly, V^i is of bounded variation. By (2.10) and (2.5), there is a finite smooth measure associated with the total variation process A^i, and so V^i has an associated finite smooth signed measure μ_i satisfying

$$|\mu_i|(D^*) = E_m[A_1^i] < \infty.$$

Then, by Lemma 2.1,

$$\int_D \frac{\partial g}{\partial x_i} dm = 2\mathscr{E}(\varphi_i, g) = -2\int_{D^*} g(x)\mu_i(dx) \leqslant 2\|g\|_\infty \cdot |\mu_i|(D^*), \tag{2.12}$$

for any bounded $g \in H^1(D)$, and so D is a strong Caccioppoli set. ■

In the following, a vector-valued process is of bounded variation if and only if each of its components is of bounded variation, and a finite vector (signed) measure is said to be smooth if and only if each of its components is smooth.

COROLLARY 2.1. *The stationary reflecting Brownian motion (X_t, P_m) is a quasimartingale if and only if V from (2.3) is of bounded variation and has an associated finite smooth vector (signed) measure μ. In this case, (X_t, P_x) is a quasimartingale for q.e. x. Conversely, if (X_t, P_x) is a quasimartingale for q.e. x, then V is of bounded variation.*

Proof. The only if part of the first statement follows from the last paragraph of the proof of Theorem 2.1. The if part of the first statement follows from the fact that the sum of a continuous martingale and a continuous adapted process with paths of integrable total variation on each compact time interval is a quasimartingale. The second statement follows similarly, using the fact from the proof of Theorem 2.1 that for q.e. x, the total variation of V on each compact time interval will be P_x-integrable. For the last statement, observe that since the quadratic variation of V is zero P_m-a.s., it is zero P_x–a.s. for q.e. x, by an argument similar to that given for the finiteness of $E_x[A_t^i]$ in the proof of Theorem 2.1. Thus, for q.e. x, the quasimartingale $V = X - X_0 - B$ has paths of P_x-integrable variation on each compact time interval. This easily implies V is of bounded variation. ■

3. Sufficient Condition

In this section, D^*, m, X^*, P_x, P_m and X are as defined in Section 2. We will show that when D is a strong Caccioppoli set, the stationary reflecting Brownian motion (X_t, P_m) on $\bar{D}$ is a quasimartingale.

LEMMA 3.1. *Suppose D is a strong Caccioppoli set. Then there is a vector (signed) measure $\mu = (\mu_1, \ldots, \mu_d)$ on D^* such that for $i = 1, \ldots, d$, $|\mu_i|(D^*) < \infty$ and*

$$\mathscr{E}(\varphi_i, f) = -\int_{D^*} f(x)\mu_i(\mathrm{d}x) \quad \textit{for all } f \in H^1(D) \cap C(D^*). \tag{3.1}$$

Proof. Since $H^1(D) \cap C(D^*) \subset H^1(D) \cap C_b(D)$, by the definition of a strong Caccioppoli set, there is a positive constant C such that

$$\int_D \frac{\partial f}{\partial x_i} \mathrm{d}m \leqslant C\|f\|_\infty \quad \text{for all } f \in H^1(D) \cap C(D^*), \quad i = 1, \ldots, d. \tag{3.2}$$

Since the Dirichlet space $(H^1(D), \mathscr{E})$ is regular on D^*, $H^1(D) \cap C(D^*)$ is dense in $(C(D^*), \|\cdot\|_\infty)$, and so (3.1) follows from (3.2) and the Riesz representation theorem. ■

THEOREM 3.1. *Fix $g \in H^1(D)$ and suppose there is a finite signed measure ν on D^* such that*

$$\mathscr{E}(g, f) = \int_{D^*} f(x)\nu(\mathrm{d}x) \quad \textit{for all } f \in H^1(D) \cap C(D^*). \tag{3.3}$$

Then ν is a smooth measure on D^*.

Before proving Theorem 3.1, we establish the following two lemmas. For $\alpha > 0$, let G_α denote the α-resolvent of X^*, and $G_\alpha(x, y)$, $x \in D^*$, $y \in D$, denote the resolvent density function relative to the reference measure m, as defined in ([11] §3).

LEMMA 3.2. *Let f be a bounded Borel function with compact support in D. Then $G_\alpha f \in C(D^*)$.*

Proof. By ([11] 1.6), $G_\alpha f$ is continuous on D. Let K denote the compact support of f and let U be an open set containing K such that $\bar{U}$ is compact and contained in D. Then by ([11] Lemma 3.1),

$$\sup_{x \in D^* \setminus U, y \in K} G_\alpha(x, y) < \infty,$$

and

$$G_\alpha(\cdot, y)|_{D^* \setminus \bar{U}} \in C(D^* \setminus \bar{U}), \quad \forall y \in K.$$

It follows by bounded convergence that $G_\alpha f = \int_K G_\alpha(\cdot, y) f(y) m(\mathrm{d}y)$ is continuous on $D^* \setminus \bar{U}$, hence on all of D^*. ■

Let us now record four facts to be used in what follows.

(a) If h is 1-excessive (i.e., if h is a Borel function, $h \geqslant 0$, and $\alpha G_{\alpha+1} h$ is monotone in α and increases pointwise to h as $\alpha \uparrow \infty$), then there is a sequence $\{g_n\}$ of bounded positive Borel functions such that $G_1 g_n \uparrow h$ as $n \to \infty$. This is standard, and can be found in Chapter II of [4].

(b) Cartan's lemma: If $\{h_n\}$ is a monotone sequence of 1-excessive functions in $H^1(D)$ with $\mathscr{E}_1$-norms that are uniformly bounded, then $h_n \to h$ in $\mathscr{E}_1$-norm, where $h := \lim_n h_n$. See Proposition (5.12) in [10], for example.

(c) If h is a bounded 1-excessive function, then $h \in H^1(D)$. This is a simple but important consequence of fact (a) and the symmetry of G_1 on $L^2(D, m)$. Indeed by Lemma 1.3.4 (ii) of [12], it suffices to check that

$$\sup_{\alpha > 0} \alpha(h, h - \alpha G_{\alpha+1} h)_m < \infty.$$

Let $\{g_n\}$ be as provided by fact (a). Then by the resolvent equation, $G_1 g_n - \alpha G_{\alpha+1} G_1 g_n = G_{\alpha+1} g_n$, and so

$$\begin{aligned}
\alpha(h, h - \alpha G_{\alpha+1} h)_m &= \lim_n \alpha(G_1 g_n, G_1 g_n - \alpha G_{\alpha+1} G_1 g_n)_m \\
&= \lim_n \alpha(G_1 g_n, G_{\alpha+1} g_n)_m = \lim_n \alpha(g_n, G_1 G_{\alpha+1} g_n)_m \\
&\leqslant \limsup_n (g_n, G_1 g_n)_m \leqslant \limsup_n \|h\|_\infty (g_n, 1)_m \\
&= \|h\|_\infty \limsup_n (g_n, G_1 1)_m = \|h\|_\infty \limsup_n (G_1 g_n, 1)_m \\
&\leqslant \|h\|_\infty (h, 1)_m < \infty.
\end{aligned}$$

(d) If f and g are bounded 1-excessive functions with $f \leqslant g$, then

$$\mathscr{E}_1(f, f) \leqslant \mathscr{E}_1(g, g).$$

This follows from fact (c) and Theorem 3.2.1 (iv) of [12].

LEMMA 3.3. *Suppose the hypotheses of Theorem 3.1 hold. Then*

$$\mathscr{E}(g, p) = \int_{D^*} p(x)\nu(dx) \quad \textit{for all bounded 1-excessive functions } p. \tag{3.4}$$

Proof. By (3.3), Lemma 3.2, and fact (c), if f is a bounded positive Borel function with compact support in D, then

$$\mathscr{E}(g, G_1 f) = \int_{D^*} G_1 f(x)\nu(dx). \tag{3.5}$$

Next, assume that f is a bounded positive Borel function on D. Let $\{K_n\}$ be an increasing sequence of compact sets whose union is D. Substitute $1_{K_n} f$ for f in (3.5), and pass to the limit as $n \to \infty$. Using Cartan's lemma and fact (d) on the left side and bounded convergence on the right, we see that (3.5) holds without the assumption of compact support for f. Finally, given a bounded 1-excessive function p, we can appeal to fact (a) to find a sequence $\{f_n\}$ of bounded positive Borel functions such that $G_1 f_n \uparrow p$. Upon substituting f_n into (3.5), Cartan's lemma and fact (d) again allow us to pass to the limit, to obtain (3.4). ∎

Proof of Theorem 3.1. First, note that from the definition of a smooth measure ([12] p. 72), since D^* is compact and ν is a finite measure, it suffices to show that $|\nu|$ does not charge sets of capacity zero. However, since subsets of sets of capacity zero are of capacity zero, by the Hahn decomposition of ν it suffices to show that ν does not charge sets of capacity zero. Furthermore, by the equivalence of polar sets to sets of capacity zero (cf. Section 2), it is enough to show that ν does not charge polar sets. Finally, we observe that it suffices to prove that $\nu(K) = 0$ for each compact polar set $K \subset D^*$. Indeed D^* is compact, so $|\nu|$ is Radon. Thus, if H is a Borel subset of D^* then there is an increasing sequence $\{K_n\}$ of compact subsets of H such that $|\nu|(H \setminus K_n) \to 0$ as $n \to \infty$, and then $\nu(H) = \lim_n \nu(K_n)$.

For each Borel set $H \subset D^*$, define the stopping times

$$T_H := \inf\{t > 0 : X_t^* \in H\},$$
$$S_H := \inf\{t \geqslant 0 : X_t^* \in H\}.$$

Fix a compact polar set $K \subset D^*$. Let $\{U_n\}$ be a decreasing sequence of open sets in D^* with $U_n \supset \bar{U}_{n+1}$ for all n, and $\cap_n U_n = K$. A simple real variable argument shows that $S_K = \uparrow \lim_n T_{U_n}$. Each of the functions $p_n(x) := E_x[\exp(-T_{U_n})]$ is 1-excessive, and $p_\infty(x) := \downarrow \lim_n p_n(x) = E_x[\exp(-S_K)]$ is 1-supermedian. Clearly $p_\infty(x) = 1$ if $x \in K$. If $x \notin K$, then $p_\infty(x) = E_x[\exp(-T_K)] = 0$ since K is compact and polar. Thus $p_\infty \equiv 1_K$ ($= 0$ m-a.e. on D^*). Consequently

$$\begin{aligned} \nu(K) &= \lim_n \int_{D^*} p_n(x)\nu(\mathrm{d}x) \\ &= \lim_n \mathscr{E}(g, p_n), \quad \text{by Lemma 3.3,} \\ &= \mathscr{E}(g, p_\infty), \quad \text{by Cartan's lemma and fact (d),} \\ &= 0. \end{aligned}$$

Thus ν is a smooth measure. ■

THEOREM 3.2. *Suppose D is a strong Caccioppoli set. Then the stationary reflecting Brownian motion (X_t, P_m) is a quasimartingale.*

Proof. We know from Theorem 3.1 that the finite measures μ_i, $i = 1, \ldots, d$, in Lemma 3.1 are smooth measures. Note that $(H^1(D), \mathscr{E})$ is regular on D^*. Thus for any bounded function $f \in H^1(D)$, there is a sequence $\{f_n\} \subset H^1(D) \cap C(D^*)$ such that $\{f_n\}$ converges to f both in $\mathscr{E}_1$-norm and quasi-everywhere on D^* (cf. [12] Theorem 3.1.4), and by ([12] Theorem 1.4.2(v)) we may assume that each f_n is bounded by $\|f\|_\infty$. Then, on replacing f by f_n in (3.1) and letting $n \to \infty$, we obtain

$$\mathscr{E}(\varphi_i, f) = -\int_{D^*} f(x)\mu_i(\mathrm{d}x) \quad \text{for all bounded } f \in H^1(D). \tag{3.6}$$

Recall the decomposition (2.3) of X. By the converse part of Lemma 2.1 and (3.6), for each i, the total variation of V^i on each compact time interval is P_m-integrable. Thus, X is a quasimartingale under P_m (cf. Proposition 1.1). ■

Now by Theorem 3.2 and Corollary 2.1, V in the decomposition (2.3) of X is of bounded variation with an associated finite smooth vector (signed) measure μ. It follows from the divergence theorem by letting f in (3.6) range over smooth functions with compact support in D that μ is supported on the boundary $\partial D^* = D^* \setminus D$. Let

$$\begin{aligned} \nu &= \sum_{i=1}^{d} |\mu_i|, \\ \phi_i &= \frac{\mathrm{d}\mu_i}{\mathrm{d}\nu}, \quad i = 1, \ldots, d. \end{aligned}$$

We define the "*surface measure*" σ on ∂D^* by

$$\sigma(\mathrm{d}x) = 2\left(\sum_{i=1}^{d} |\phi_i(x)|^2\right)^{1/2} \nu(\mathrm{d}x), \tag{3.7}$$

and the "*unit inward normal*" at $x \in \partial D^*$ by

$$n(x) = (n_1(x), \ldots, n_d(x)) = \begin{cases} \dfrac{(\phi_1(x), \ldots, \phi_d(x))}{\left(\sum_{i=1}^{d} |\phi_i(x)|^2\right)^{1/2}} & \text{if } \sum_{i=1}^{d} |\phi_i(x)|^2 > 0, \\ 0 & \text{if } \sum_{i=1}^{d} |\phi_i(x)|^2 = 0. \end{cases} \tag{3.8}$$

Thus, $\mu_i(\mathrm{d}x) = \frac{1}{2} n_i(x)\sigma(\mathrm{d}x)$, $i = 1, \ldots, d$.

REMARKS. (a) The normalizing factor 2 in the definition of σ ensures that our generalized divergence theorem (3.10) will have a familiar form.

(b) For the precise sense in which n is an inward unit normal see ([8] §5.7.2).

Let L denote the positive continuous additive functional associated with the finite smooth positive measure $\frac{1}{2}\sigma$. Then by ([12] Theorem 5.1.3, (5.3.8)), we have for q.e. x, P_x-a.s.,

$$V_t = \int_0^t n(X_s^*)\,\mathrm{d}L_s, \quad \forall t \geqslant 0.$$

Thus we have the following Skorokhod decomposition for X.

THEOREM 3.3. *If D is a strong Caccioppoli set, then for q.e. $x \in D^*$, we have*

$$X_t = X_0 + B_t + \int_0^t n(X_s^*)\,\mathrm{d}L_s, \quad \forall t \geqslant 0, \quad P_x\text{-a.s.}, \tag{3.9}$$

where B is a Brownian motion martingale additive functional and L is a positive continuous additive functional with associated finite smooth measure $\frac{1}{2}\sigma$. (All of the additive functionals are with respect to the strong Markov process X^* on D^*.)

For a bounded function $f \in H^1(D)$, by (3.6) the following *generalized* divergence formula is valid:

$$\int_D \frac{\partial f}{\partial x_i}\,\mathrm{d}m = -\int_{\partial D^*} f n_i\,\mathrm{d}\sigma, \quad i = 1, \ldots, d. \tag{3.10}$$

REMARK. Consider the case where D is a bounded Lipschitz domain. We will see in Corollary 5.1 that D is then a strong Caccioppoli set. Bass and Hsu [3] showed that D^* for a bounded Lipschitz domain is just $\bar{D}$, the Euclidean closure of D. It follows easily from the divergence theorem (cf. [8] p. 209) that for a bounded Lipschitz domain D our definitions of σ and n coincide with the usual ones, up to a scale factor used to make $m(D) = 1$. In particular, the surface measure on ∂D is a smooth measure (this was also previously observed by Bass and Hsu [3]).

4. Extension to Symmetric Reflecting Diffusion Processes

The results of Sections 2 and 3 will now be extended to a large class of symmetric reflecting diffusions. As before, $D \subset \mathbb{R}^d$ is a bounded Euclidean domain. Each diffusion in the class under study is determined by a symmetric matrix-valued function $\Lambda = (a_{ij}): D \to \mathbb{R}^d \otimes \mathbb{R}^d$. Furthermore, we assume that each of the functions a_{ij} is an element of $H^1(D)$ and that there is a constant $\lambda > 1$ such that

$$\frac{1}{\lambda} \sum_{i=1}^{d} |\xi_i|^2 \leqslant \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leqslant \lambda \sum_{i=1}^{d} |\xi_i|^2 \quad \text{for all } (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \quad x \in D. \tag{4.1}$$

In particular, each a_{ij} is a bounded function on D. Our work in this section concerns the form $\mathscr{E}$ defined on $H^1(D)$ by

$$\mathscr{E}(f, g) = \frac{1}{2} \sum_{i,j=1}^{d} \int_D a_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} m(\mathrm{d}x), \quad f, g \in H^1(D). \tag{4.2}$$

As before, m is the normalized Lebesgue measure on D. Clearly (4.1) implies that

$$\lambda^{-1} \int_D |\nabla f|^2 \, \mathrm{d}m \leqslant 2\mathscr{E}(f, f) \leqslant \lambda \int_D |\nabla f|^2 \, \mathrm{d}m, \quad f \in H^1(D).$$

Hence, the form $\mathscr{E}$ is equivalent to that treated in Section 2. In particular this implies that $(H^1(D), \mathscr{E})$ is a Dirichlet space and that it is regular on the Martin–Kuramochi compactification D^* introduced in Section 2, and the diffusion X^* on D^* associated with $(H^1(D), \mathscr{E})$ has the same classes of polar sets, smooth measures, and quasi-continuous functions as the reflecting Brownian motion of Sections 2 and 3. (As a rule notation introduced in earlier sections for the reflecting Brownian motion will now serve the analogous role for X^*.)

Just as before, we let $\varphi = (\varphi_1, \ldots, \varphi_d): D^* \to \bar{D}$ denote the quasi-continuous extension of the identity map on D, and we set $X := \varphi(X^*)$. Fukushima's decomposition of X (cf. (2.3)) now reads

$$X_t = X_0 + M_t + V_t, \quad t \geqslant 0, \tag{4.3}$$

where $M = (M^1, \ldots, M^d)$ is a continuous $\mathbb{R}^d$-valued martingale additive functional of X^* satisfying $M_0 = 0$ and with covariation

$$\langle M^i, M^j \rangle_t = \int_0^t a_{ij}(X_s^*)\,\mathrm{d}s, \quad t \geqslant 0, \tag{4.4}$$

and V is a continuous $\mathbb{R}^d$-valued zero-energy additive functional of X^*. (Formula (4.4) results from a trivial modification of ([12] Example 5.2.1, esp. (5.2.32)).)

The measure m is an invariant measure for X^*, and (X_t, P_m) is a realization of the stationary reflecting diffusion on $\bar{D}$ associated with $(H^1(D), \mathscr{E})$.

We begin with the analogue of Theorem 2.1.

THEOREM 4.1. *Suppose the stationary reflecting diffusion (X_t, P_m) is a quasimartingale. Then D is a strong Caccioppoli set.*

Proof. Arguing exactly as in Section 2, the quasimartingale property of (X_t, P_m) implies that the process V in (4.3) is of bounded variation and that the associated smooth measure $\mu = (\mu_1, \ldots, \mu_d)$ has finite total variation. Appealing to Remark (c) following Lemma 2.1, we see that

$$\mathscr{E}(\varphi_i, f) = -\int_{D^*} f(x)\mu_i(\mathrm{d}x), \quad \text{for all bounded } f \in H^1(D), \quad i = 1, \ldots, d. \tag{4.5}$$

At this point we note that D is a strong Caccioppoli set if and only if there is a positive constant C such that

$$\int_D \operatorname{div}(h)\,\mathrm{d}m \leqslant C\|h\|_\infty, \tag{4.6}$$

for all $h = (h_1, \ldots, h_d)$ with $h_i \in H^1(D) \cap C_b(D)$, $i = 1, \ldots, d$. Here

$$\|h\|_\infty := \sup_{x \in D} \left[\sum_{i=1}^d |h_i(x)|^2 \right]^{1/2}.$$

Fix such an h and define $g = (g_1, \ldots, g_d) := h \cdot \Lambda^{-1}$. Our hypotheses on Λ imply that the components of the matrix Λ^{-1} are (bounded) elements of $H^1(D)$. Thus the components of g are also (bounded) elements of $H^1(D)$, because of ([12] Theorem 1.4.2). Therefore

$$\begin{aligned}\int_D \operatorname{div}(h)\,\mathrm{d}m &= \int_D \sum_{j=1}^d \frac{\partial}{\partial x_j}\left(\sum_{i=1}^d a_{ij}g_i\right)\mathrm{d}m \\ &= \sum_{i=1}^d \sum_{j=1}^d \left[\int_D a_{ij}\frac{\partial g_i}{\partial x_j}\,\mathrm{d}m + \int_D \frac{\partial a_{ij}}{\partial x_j} g_i\,\mathrm{d}m\right] \\ &\leqslant 2\sum_{i=1}^d \mathscr{E}(\varphi_i, g_i) + \sum_{i=1}^d \sum_{j=1}^d \|g_i\|_\infty \int_D \left|\frac{\partial a_{ij}}{\partial x_j}\right|\mathrm{d}m.\end{aligned}$$

The first term in the last line above is bounded by a constant multiple of $\|g\|_\infty$, because of (4.5). The second term is dominated by

$$\sum_{i,j=1}^d \|g\|_\infty \left[\int_D \left|\frac{\partial a_{ij}}{\partial x_j}\right|^2 \mathrm{d}m\right]^{1/2}.$$

Now (4.1) implies that

$$\frac{1}{\lambda}\|g\|_\infty \leqslant \|h\|_\infty \leqslant \lambda\|g\|_\infty .$$

Thus (4.6) holds, and D is a strong Caccioppoli set. ■

We proceed now to the converse of Theorem 4.1, leaving it to the reader to formulate the analogue of Corollary 2.1.

THEOREM 4.2. *Suppose D is a strong Caccioppoli set. Then the stationary reflecting diffusion (X_t, P_m) is a quasimartingale.*

Proof. By the proof of Theorem 3.2, there is a finite vector (signed) smooth measure $\mu = (\mu_1, \ldots, \mu_d)$ on D^* such that

$$\frac{1}{2}\int_D \frac{\partial f}{\partial x_i}\,\mathrm{d}m = -\int_{D^*} f(x)\mu_i(\mathrm{d}x) \quad \text{for all bounded } f \in H^1(D), \quad i = 1, \ldots, d. \tag{4.7}$$

If f is a bounded function in $H^1(D)$ then

$$\begin{aligned}\mathscr{E}(\varphi_i, f) &= \frac{1}{2}\sum_{j=1}^d \int_D a_{ij}\frac{\partial f}{\partial x_j}\,\mathrm{d}m \\ &= \frac{1}{2}\sum_{j=1}^d \int_D \frac{\partial}{\partial x_j}(a_{ij}f)\,\mathrm{d}m - \frac{1}{2}\sum_{j=1}^d \int_D f\frac{\partial a_{ij}}{\partial x_j}\,\mathrm{d}m \\ &= -\sum_{j=1}^d \int_{D^*} a_{ij}f\,\mathrm{d}\mu_j - \frac{1}{2}\sum_{j=1}^d \int_D f\frac{\partial a_{ij}}{\partial x_j}\,\mathrm{d}m \\ &= -\int_{D^*} f\,\mathrm{d}\nu_i,\end{aligned}$$

where

$$\mathrm{d}\nu_i = \sum_{j=1}^{d} a_{ij}\mathrm{d}\mu_j + \frac{1}{2}\sum_{j=1}^{d}\frac{\partial a_{ij}}{\partial x_j}\mathrm{d}m. \tag{4.8}$$

Since $a_{ij} \in H^1(D)$, $\int_D |\partial a_{ij}/\partial x_j|\,\mathrm{d}m \leqslant \|\partial a_{ij}/\partial x_j\|_2 < \infty$. Consequently, the second sum in (4.8) defines a finite smooth signed measure. Thus each ν_i is a finite signed smooth measure. The theorem now follows from Lemma 2.1, Remark (c). ∎

We close this section with the Skorokhod decomposition for X. For the statement of the result, recall the generalized surface measure σ and inward unit normal vector field n defined in Section 3.

THEOREM 4.3. *If D is a strong Caccioppoli set, then for q.e. $x \in D^*$ we have P_x-a.s.,*

$$X_t^i = X_0^i + \sum_{j=1}^{d}\int_0^t \gamma_{ij}(X_s^*)\,\mathrm{d}B_s^j + \frac{1}{2}\sum_{j=1}^{d}\int_0^t \frac{\partial a_{ij}}{\partial x_j}(X_s^*)\,\mathrm{d}s$$
$$+ \sum_{j=1}^{d}\int_0^t a_{ij}(X_s^*)n_j(X_s^*)\,\mathrm{d}L_s, \quad \forall t \geqslant 0, \quad i = 1,\ldots,d. \tag{4.9}$$

Here $B = (B^1,\ldots,B^d)$ is a Brownian motion martingale additive functional of X^*, $(\gamma_{ij}(x))$ is the symmetric positive definite $d \times d$ matrix whose square is $\Lambda(x)$, and L is the positive continuous additive functional of X^* with associated smooth measure $\frac{1}{2}\sigma$.

Proof. Combining (4.8) with (3.7) and (3.8), we obtain

$$\mathrm{d}\nu_i = \frac{1}{2}\sum_{j=1}^{d} a_{ij}n_j\,\mathrm{d}\sigma + \frac{1}{2}\sum_{j=1}^{d}\frac{\partial a_{ij}}{\partial x_j}\mathrm{d}m.$$

Thus by Lemma 2.1 and uniqueness, the zero-energy term V in the decomposition (4.3) can be expressed as

$$V_t = \sum_{j=1}^{d}\int_0^t a_{ij}(X_s^*)n_j(X_s^*)\,\mathrm{d}L_s + \frac{1}{2}\sum_{j=1}^{d}\int_0^t \frac{\partial a_{ij}}{\partial x_j}(X_s^*)\,\mathrm{d}s, \tag{4.10}$$

where the right member is well defined since the associated smooth measure ν is of finite total variation. Let $(\kappa_{ij}(x))$ be the matrix-inverse of $(\gamma_{ij}(x))$ and define

$$B_t^i := \sum_{j=1}^{d}\int_0^t \kappa_{ij}(X_s^*)\,\mathrm{d}M_s^j, \quad t \geqslant 0, \quad i = 1,\ldots,d.$$

Then $B = (B^1,\ldots,B^d)$ is a vector martingale additive functional of X^*, and by (4.4)

$$\langle B^i, B^j\rangle_t = \delta_{ij}t, \quad t \geqslant 0, \quad i,j = 1,\ldots,d.$$

Consequently, B is a d-dimensional Brownian motion, and clearly

$$M_t^i = \sum_{j=1}^{d} \int_0^t \gamma_{ij}(X_s^*)\,\mathrm{d}B_s^j, \quad t \geqslant 0, \quad i = 1,\dots,d. \tag{4.11}$$

The Skorokhod decomposition (4.9) results upon combining (4.10) with (4.11). ∎

REMARK. A trivial time change argument shows that the results of this section are valid for the class of diffusions studied in [5]. In that paper the normalized Lebesgue measure m is replaced (both in the definition (4.2) and in the specification of the associated L^2 space) by $\rho(x)m(\mathrm{d}x)$, where $\rho \in H^1(D)$ is a positive function bounded away from zero and infinity.

5. Complements

Throughout this section, the Dirichlet form $\mathscr{E}$ is as defined in (1.1). We start by recording several conditions ensuring that a bounded Euclidean domain D is a strong Caccioppoli set.

THEOREM 5.1. *Suppose there is an increasing sequence $\{D_k\}_{k=1}^{\infty}$ of smooth domains with union D, such that the total masses of the associated surface measures are uniformly bounded. Then D is a strong Caccioppoli set.*

Proof. Let σ_k denote the surface measure for D_k, and let n^k be the inward unit normal vector field on ∂D_k. Recall that the Lebesgue measure m on D is normalized so that $m(D) = 1$. Accordingly, the surface measures σ_k are normalized by a constant that does not depend on k such that the divergence theorem on the smooth domain D_k yields

$$\int_{D_k} \frac{\partial f}{\partial x_i}\mathrm{d}m = -\int_{\partial D_k} f n_i^k \,\mathrm{d}\sigma_k \leqslant \sigma_k(\partial D_k)\cdot \|f\|_\infty, \quad i = 1,\dots,d, \tag{5.1}$$

for all $f \in C^2(\bar{D}_k)$. Here $C^2(\bar{D}_k)$ consists of those functions defined on $\bar{D}_k$ that can be extended to be twice continuously differentiable on some domain containing $\bar{D}_k$. The Dirichlet space $(H^1(D_k), \mathscr{E}^k)$, where $\mathscr{E}^k(f,g) = \frac{1}{2}\int_{D_k} \nabla f \cdot \nabla g \,\mathrm{d}m$, is regular on $\bar{D}_k$ and possesses $C^2(\bar{D}_k)$ as a core. Thus, given a bounded function $f \in H^1(D_k)$ there is a sequence $\{f_n\} \subset C^2(\bar{D}_k)$ that converges to f both in $\mathscr{E}_1^k$ norm and q.e. on $\bar{D}_k$. By truncating if necessary with a C^2 cutoff function we can assume that each f_n is bounded by $\|f\|_\infty$ (cf. [12] Theorem 1.4.2 (v)). Substituting f_n into (5.1) and then passing to the limit as $n \to \infty$ we obtain

$$\int_{D_k} \frac{\partial f}{\partial x_i}\mathrm{d}m \leqslant \sigma_k(\partial D_k)\cdot \|f\|_\infty \quad \text{for all bounded } f \in H^1(D_k), \quad i = 1,\dots,d.$$

But if $f \in H^1(D) \cap C_b(D)$, then the restriction of f to D_k lies in $H^1(D_k) \cap C_b(D_k)$. Therefore,

$$\int_D \frac{\partial f}{\partial x_i} dm = \lim_{k \to \infty} \int_{D_k} \frac{\partial f}{\partial x_i} dm \leqslant \sup_{k \geqslant 1} \sigma_k(\partial D_k) \cdot \|f\|_\infty .$$

Hence D is a strong Caccioppoli set. ■

COROLLARY 5.1. *If the bounded domain $D \subset \mathbb{R}^d$ satisfies the condition*

$$\liminf_{\varepsilon \downarrow 0} \frac{m(\{x \in D : d(x, \partial D) < \varepsilon\})}{\varepsilon} < \infty, \tag{5.2}$$

then D is a strong Caccioppoli set. In particular, any bounded Lipschitz domain is a strong Caccioppoli set.

REMARK. Condition (5.2) holds if the boundary of D has finite $(d-1)$-dimensional lower Minkowski content, i.e., if (5.2) holds with $x \in \mathbb{R}^d$ in place of $x \in D$.

Proof. If (5.2) holds, then by ([5] Lemma 2.5) the hypothesis of Theorem 5.1 is satisfied. For the last conclusion, note from Federer ([9] Theorem 3.2.38) that a bounded Lipschitz domain in $\mathbb{R}^d$ has a boundary of finite $(d-1)$-dimensional (lower) Minkowski content and hence (5.2) holds. ■

The results of Sections 2 and 3 lead one to ask whether the semimartingale property of a stationary reflecting Brownian motion implies that the associated domain is a Caccioppoli set. The following example provides a negative answer to this question.

EXAMPLE 5.1.* Let D be the bounded planar domain obtained by smoothing the corners of the domain (presented in polar coordinates)

$$\{(r,\theta): 0 < r < 1, 0 < \theta < \pi\} \Big\backslash \left(\bigcup_{k=1}^{\infty} \left\{ (r,\theta): \frac{1}{4k+1} \leqslant r \leqslant \frac{1}{4k}, 0 < \theta \leqslant \frac{3\pi}{4} \right\} \right.$$
$$\left. \cup \bigcup_{k=1}^{\infty} \left\{ (r,\theta): \frac{1}{4k+3} \leqslant r \leqslant \frac{1}{4k+2}, \frac{\pi}{4} \leqslant \theta < \pi \right\} \right)$$

in such a way that ∂D is C^2 smooth except at the origin. Clearly D is a Jordan domain. Using the boundary correspondence theorem for conformal mappings, it is easy to check that the Dirichlet space $(H^1(D), \mathscr{E})$ is regular on $\bar{D}$ (see [6]). For any positive integer $k \geqslant 1$, let

* Z.Q.C. learned this example from James Jenkins. The domain D is a Jordan domain with a boundary point (the origin) that is not rectifiably accessible from the interior of the domain.

$$D_k = D \cap \left\{(r,\theta): r > \frac{1}{4k}\right\}.$$

Then D_k is a bounded Lipschitz domain, and so $(H^1(D_k), \mathscr{E}^k)$ is regular on $\bar{D}_k$ (cf. [16] p. 181, Theorem 5). By the remark at the end of Section 3, the surface measure for D_k is a smooth measure with respect to $(H^1(D_k), \mathscr{E}^k)$. Note that $\bar{D} \cap \{(r,\theta): r > 1/4k\}$ is relatively open both in $\bar{D}_k$ and in $\bar{D}$. Thus, by ([12] Theorem 4.4.2), $(H^1(D_k), \mathscr{E}^k)$ and $(H^1(D), \mathscr{E})$ have identical classes of zero-capacity subsets of $\bar{D} \cap \{(r,\theta): r > 1/4k\}$. If we let σ_k and n^k denote the restrictions to $\bar{D} \cap \{r > 1/4k\}$ of the surface measure and unit inward normal vector field of D_k, then σ_k is a smooth measure relative to the Dirichlet space $(H^1(D), \mathscr{E})$. Clearly σ_l (resp. n^l) restricted to $\bar{D} \cap \{(r,\theta): r > 1/4k\}$ equals σ_k (resp. n^k), provided $l > k$. Thus we can define

$$\sigma := \lim_{k \to \infty} \sigma_k, \quad n := \lim_{k \to \infty} n^k.$$

It is easy to check that σ (which is the usual surface measure for D) is a smooth measure relative to $H^1(D)$. Moreover, n (which is left undefined at the origin) is the inward unit normal vector field on ∂D. Let X be the continuous strong Markov process on $\bar{D}$ associated with the regular Dirichlet space $(H^1(D), \mathscr{E})$ on $\bar{D}$. We know that X has a decomposition of the form (2.3). Fix $k \in \mathbb{N}$ and $f = (f_1, f_2)$ with $f_i \in C_c^2(\mathbb{R}^2)$ and $\operatorname{supp}(f_i) \subset \{(r,\theta): r > 1/4k\}$, $i = 1, 2$. By the divergence theorem,

$$\int_{D_k} \operatorname{div}(f)\, \mathrm{d}m = -\int_{\partial D_k} f \cdot n^k\, \mathrm{d}\sigma_k = -\int_{\bar{D}_k} f \cdot n\, \mathrm{d}\sigma. \tag{5.3}$$

Note that the class of functions f_i occurring above is $\mathscr{E}_1$-dense in $\mathscr{F}_k := \{g \in H^1(D): g = 0$ q.e. on $\bar{D} \setminus D_k\}$, since the latter coincides with $\{g \in H^1(D): g = 0$ q.e. on $\bar{D} \cap \{(r,\theta): r \leqslant 1/4k\}\}$. Using the conformal mapping of D onto the open unit disk (and the fact that this mapping admits a continuous extension to $\bar{D}$), one can check that the origin is of zero capacity for $(H^1(D), \mathscr{E})$ since the image of the origin under the mapping is of zero capacity for the reflecting Brownian motion in the closed unit disk. Thus $\{\bar{D}_k\}$ is a nest in the sense of Fukushima ([12] (5.3.12)–(5.3.13)), so we can apply ([12] Theorem 5.3.2) to deduce that X is a semimartingale with decomposition

$$X_t = X_0 + B_t + \int_0^t n(X_s)\, \mathrm{d}L_s, \quad \forall t \geqslant 0, \tag{5.4}$$

P_x-a.e. for q.e. $x \in \bar{D}$. Here B is a Brownian motion martingale additive functional and L is the positive continuous additive functional associated with the smooth measure $\frac{1}{2}\sigma$. However, D is not a Caccioppoli set. Indeed

$$\lim_{k \to \infty} \sigma(\bar{D}_k) = \infty,$$

while by (5.3)

$$\sup_{f \in \mathscr{C}(k)} \int_D \operatorname{div}(f)\, dm = \sigma(\bar{D}_k),$$

where $\mathscr{C}(k)$ is the class of functions $f = (f_1, f_2)$ with $f_i \in C_c^2(\mathbb{R}^2)$, $\operatorname{supp}(f_i) \subset \{(r, \theta): r > 1/4k\}$, $i = 1, 2$, and $\|f\|_\infty \leqslant 1$.

We conclude this section with a brief comparison of the notions of "Caccioppoli set" and "strong Caccioppoli set". The former concept is measure theoretic in nature: if two domains differ by a set of Lebesgue measure zero and if one is a Caccioppoli set, then so is the other. The latter is more geometric in character, since $H^1(D)$ may change significantly if D is altered by a set of Lebesgue measure zero. Our intuition tells us that the sample path of a reflecting Brownian motion X should "feel" the geometry of the boundary of the underlying domain. One may view the results of Sections 2 and 3 as confirmation of this intuition.

In the next example we exhibit a bounded domain D which is a Caccioppoli set and for which the stationary reflecting Brownian motion is a semimartingale but not a quasimartingale. In view of Theorem 3.2, this example shows that the class of strong Caccioppoli sets is properly contained in the class of Caccioppoli sets.

EXAMPLE 5.2. Let D be the unit disc in the plane from which a spiral curve spinning to the origin has been removed (the origin is considered to be one end of the spiral curve and accordingly is not in D). More precisely, consider

$$D = \{(r, \theta): r < 1\} \backslash (\{(r, \theta): r = \theta^{-1}, \theta \geqslant 1\} \cup \{(r, \theta): r = 0\}).$$

Clearly D is a Caccioppoli set, since the unit disc is a Caccioppoli set and the deleted spiral γ is of Lebesgue measure zero. Note that the length of γ is infinite. To discuss the reflecting Brownian motion in D we need to recall the following concept from ([6] (1.4)).

DEFINITION 5.1. The *manifold distance* $\delta(x, y)$ between two points $x, y \in D$ is the infimum of the lengths of piecewise C^1 curves in D connecting x and y.

Let $\hat{D}$ be the completion of D relative to the manifold metric δ. It is clear that the subspace topology which D inherits from $\hat{D}$ coincides with the Euclidean topology on D, and that $\partial\hat{D} = \hat{D} \backslash D$ is homeomorphic to the unit circle. In the terminology of [6], D is a (simply connected) pseudo Jordan domain. Therefore, by ([6] Theorem 2.3), the Dirichlet space $(H^1(D), \mathscr{E})$ is regular on $\hat{D}$. For $k \in \mathbb{N}$ let

$$D_k = D \cap \{(r, \theta): r > k^{-1}\}.$$

Then D_k is a pseudo Jordan domain whose boundary has finite 1-dimensional lower

Minkowski content and hence by Corollary 5.1, D_k is a strong Caccioppoli set. Let $\hat{D}_k$ denote the completion of D_k relative to the manifold metric in D_k. Then $(H^1(D_k), \mathscr{E}^k)$, where $\mathscr{E}^k(f,g) = \frac{1}{2}\int_{D_k} \nabla f \cdot \nabla g \, dm$, is regular on $\hat{D}_k$ (see [5], [6]). It follows that the surface measure σ_k for $\hat{D}_k$, as defined in Section 3, is a smooth measure. By a similar argument to that presented in Example 5.1, the four points on $\partial\hat{D}_k$ where the spiral strikes the circles of radius one and k^{-1} form a set of zero capacity for $(H^1(D_k), \mathscr{E}^k)$ and hence are not charged by the smooth measure σ_k. From the generalized divergence theorem on D_k (cf. (3.10)), it follows that the surface measure σ_k for $\hat{D}_k$ consists of arclength on the circles of radius one and k^{-1} together with arclength along both sides of the curve $\gamma_k = \{(r,\theta): r = \theta^{-1}, k^{-1} \leqslant r \leqslant 1\}$ (each point on γ_k splits into two different points in $\partial\hat{D}_k$). The inward unit normal vector field n^k on $\partial\hat{D}_k$ has an analogous description. The measures σ_k are consistent, i.e., $\sigma_k = \sigma_l$ on $\hat{D}_k \cap \{(r,\theta): r > k^{-1}\}$ provided $l > k$. Define

$$\sigma = \lim_{k\to\infty} \sigma_k, \quad n = \lim_{k\to\infty} n^k,$$

on $\hat{D}$. Let $\hat{X}$ denote the continuous strong Markov process on $\hat{D}$ associated with the Dirichlet space $(H^1(D), \mathscr{E})$ which is regular on $\hat{D}$. For $i = 1, \ldots, d$, let $X^i = \varphi_i(\hat{X})$, $i = 1, \ldots, d$, where φ_i is the quasi-continuous extension of $D \ni x \mapsto x_i$ to all of $\hat{D}$. Then by the same kind of argument as used in Example 5.1, but with the generalized divergence theorem on $\hat{D}_k$ in place of (5.3), we conclude that $\hat{X}$ has the semimartingale decomposition

$$X_t = X_0 + B_t + \int_0^t n(\hat{X}_s)\, dL_s, \quad t \geqslant 0,$$

P_x-a.s. for q.e. $x \in \hat{D}$, where B is a Brownian motion martingale additive functional of $\hat{X}$ and L is the positive continuous additive functional of $\hat{X}$ associated with the smooth measure $\frac{1}{2}\sigma$. By (2.5)

$$E_m[L_1] = \tfrac{1}{2}\sigma(\hat{D}) \geqslant \text{arclength}(\gamma) = \infty,$$

so X cannot be a quasimartingale under the stationary measure P_m.

Under certain conditions, the notions of Caccioppoli set and strong Caccioppoli set coincide.

THEOREM 5.2. *Let $D \subset \mathbb{R}^d$ be a bounded domain such that $C_c^1(\mathbb{R}^d)$ is $\mathscr{E}_1$-dense in $H^1(D)$. Then D is a Caccioppoli set if and only if it is a strong Caccioppoli set.*

Proof. Suppose that D is a Caccioppoli set. Then there is positive constant C such that

$$\int_D \frac{\partial g}{\partial x_i}\, dm \leqslant C\|g\|_\infty, \quad i = 1, \ldots, d, \tag{5.5}$$

for all $g \in C_c^1(\mathbb{R}^d)$, where $\|g\|_\infty = \sup_{x\in\mathbb{R}^d} |g(x)|$. By hypothesis, $C_c^1(\mathbb{R}^d)$ is $\mathscr{E}_1$-dense in

$H^1(D)$, so that given $g \in H^1(D) \cap C_b(D)$ there is a sequence $\{g_n\} \subset C_c^1(\mathbb{R}^d)$ converging to g in $\mathscr{E}_1$ norm with $\|g_n\|_\infty \leqslant \|g\|_\infty$ for all n. Substituting g_n into (5.5) and letting $n \to \infty$, we see that (5.5) holds for all $g \in H^1(D) \cap C_b(D)$. Therefore D is a strong Caccioppoli set. ∎

REMARK. Let $H^1(D)$ and $H^1(\mathbb{R}^d)$ be endowed with the norms induced by $\mathscr{E}_1$ on D and $\mathbb{R}^d$ respectively. A domain D in $\mathbb{R}^d$ is said to be an *extension domain* for the Dirichlet space $(H^1(D), \mathscr{E})$ if there is a bounded linear operator

$$T: H^1(D) \to H^1(\mathbb{R}^d)$$

such that $Tf|_D = f$ for any $f \in H^1(D)$. See, for example, [14]. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$, it follows that $C_c^\infty(\mathbb{R}^d)$ is dense in $H^1(D)$ whenever D is an extension domain for the Dirichlet space $(H^1(D), \mathscr{E})$. Examples of extension domains are bounded Lipschitz domains and (ε, δ) domains (cf. [14] Theorem 1).

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