Potential Analysis **2**: 199–218, 1993.

On Non-Linear Rayleigh Quotients

PETER LINDQVIST
Helsinki University of Technology, Institute of Mathematics, SF-02150 Espoo, Finland.
Current address: Norwegian Institute of Technology (NTH), Department of Mathematics, N-7034 Trondheim, Norway.

(Received: 8 April 1992; accepted: 28 September 1992)

Abstract. The stability with respect to p of the non-linear eigenvalue problem $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$ is studied.

Mathematics Subject Classifications (1991). 35B35, 35B32, 35J70.

Key words. Non-linear Rayleigh quotients, stability of non-linear eigenvalues, degenerate elliptic partial differential equations.

1. Introduction

There has been a considerable mathematical interest in the partial differential equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u) = 0$ and its immediate generalizations. The so-called p-harmonic differential operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ also appears in many contexts in physics: non-Newtonian fluids (dilatant fluids have $p > 2$, pseudoplastics have $1 < p < 2$), reaction-diffusion problems, non-linear elasticity (for example torsional creep), and glaceology ($p = 4/3$), just to mention a few applications. We are interested in an eigenvalue problem, apparently first studied by F. de Thélin in 1984, cf. [17]. Little is known about the non-linear cases $p \neq 2$ compared to the vast amount of knowledge for the Laplace operator ($p = 2$).

The first eigenvalue $\lambda_p = \lambda_p(\Omega)$ of the p-harmonic operator is here defined as the least real number λ for which the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0 \tag{1.1}$$

has a nontrivial solution u with zero boundary data in a given bounded domain Ω in the n-dimensional Euclidean space. The first eigenvalue is the minimum of the Rayleigh quotient

$$\lambda_p = \inf_u \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}. \tag{1.2}$$

Here $1 < p < \infty$, and in the linear case $p = 2$ one obtains the principal frequency of a vibrating membrane, cf. [15]. We shall often use the term *principal frequency* for the non-linear cases as well. If u is a solution to Equation (1.1) then the function

$$v(x, t) = u(x)e^{-\lambda t/(p-1)}$$

satisfies the evolution equation

$$\frac{\partial |v|^{p-2}v}{\partial t} = \operatorname{div}(|\nabla v|^{p-2}\nabla v) \tag{1.3}$$

at least formally. (The function $w = |v|^{p-2}v$ is a solution to the more well-known parabolic equation $\partial w/\partial t = (p-1)^{1-p}\operatorname{div}(|w|^{2-p}|\nabla w|^{p-2}\nabla w)$ describing a kind of nonlinear diffusion, cf. [3].)

The differential equation (1.1) is interpreted in the weak sense and all functions in the Sobolev space $W_0^{1,p}(\Omega)$ are admissible in (1.2). See Definition 2.1. The solutions to Equation (1.1) are known to be of class $C_{loc}^{1,\alpha}(\Omega)$, i.e., their gradients are locally Hölder continuous. The first eigenfunctions are essentially unique in any bounded domain: they are merely constant multiples of each other. Moreover, they have no zeros in the domain and they are the only eigenfunctions not changing signs. The uniqueness for *arbitrary* bounded domains was proved in [8]. The radial case has been studied by F. de Thélin in [18] and a good reference for C^2-domains is ([16] Theorem A.1).

The main objective of our paper is to study the convergence of the first eigenfunctions in connection with the inequalities

$$\lim_{s \to p-} \lambda_s \leqslant \lambda_p = \lim_{s \to p+} \lambda_s,$$

proved in Theorem 3.5 and Corollary 3.4. In other words, we explore the behaviour of the positive solution $u_p \in W_0^{1,p}(\Omega)$ to the equation $\operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p) + \lambda_p|u_p|^{p-2}u_p = 0$, as p varies continuously.

This is all the more interesting there being some anomaly when $p < n$ (n is the dimension of the underlying Euclidean space). In very irregular domains the situation $\lim_{s \to p-} \lambda_s < \lambda_p$ is possible as a consequence of a strange convergence phenomenon. The eigenfunctions converge to a positive solution of Equation (1.1) having boundary values zero, yet in a sense that is slightly too poor. The limit function is in the Sobolev space $W^{1,p}(\Omega)$ and in every $W_0^{1,p-\epsilon}(\Omega)$, $\epsilon > 0$, but not in the required $W_0^{1,p}(\Omega)$. It is not admissible in (1.2). Needless to say, a delicate balance is needed to construct a domain causing such an effect. Our example in Section 7 is based on a well-known Cantor set. However, in giving a proof that the above mentioned phenomenon really occurs, we have needed the Wiener criterion and the Kellogg property*. Therefore Section 4 is devoted to these advanced concepts. As a byproduct we mention a result about uniform convergence in Wiener regular domains (Theorem 6.1).

Some of our results are immediate. For example, the convergence

$$\lim_{s \to p+} \int_{\Omega} |\nabla u_s - \nabla u_p|^p \, dx = 0$$

for the gradients of the properly normalized eigenfunctions is proved in Section 3 merely by the aid of Functional Analysis in Sobolev spaces, that is without using the differential equation in any essential way. Yet we think that such proofs are of some interest, the fascinating feature being that 'the ground is cut from under one's feet': as p varies the appropriate L^p-space or Sobolev space changes. There is no fixed convenient energy norm to use, at least not for p growing. Indeed, it is possible that $\|\nabla u_p\|_{p+\epsilon} = \infty$ for every $\epsilon > 0$. (We have used similar methods before in a simpler case [7].)

Section 6 is about uniform convergence, as p varies. Deep results in regularity theory (see [2] and [19]) are used to show that the eigenfunctions and even their gradients converge locally uniformly to a positive solution of Equation (1.1). However, this can be the 'wrong' solution! See Theorem 6.3 for the exact formulation.

Let us finally mention that the best constant in the Poincaré–Friedrichs inequality

$$\int_{\Omega} |\varphi|^p \, dx \leqslant C \int_{\Omega} |\nabla \varphi|^p \, dx \quad (1 < p < \infty)$$

is the reciprocal of the principal frequency: $C = 1/\lambda_p$. Here $\varphi \in C_0^{\infty}(\Omega)$, Ω being a bounded domain. (See [4], Eqn (7.44), p. 164.)

2. The Differential Equation

In defining the eigenvalues for the p-harmonic operator in a given bounded domain $\Omega \subset \mathbf{R}^n$ we shall interpret Equation (1.1) in the weak sense.

2.1. DEFINITION. We say that λ is an eigenvalue, if there exists a continuous function $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \eta \, dx = \lambda \int_{\Omega} |u|^{p-2} u \eta \, dx \tag{2.2}$$

whenever $\eta \in C_0^{\infty}(\Omega)$. The function u is called an eigenfunction.

* *Note added in proof:* The referee has observed that the Kellogg property is not really needed here. It can be replaced by a direct calculation. We take the opportunity to thank him for this remark.

The Sobolev space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\eta\| = \left\{ \int_\Omega (|\eta|^p + |\nabla\eta|^p)\,\mathrm{d}x \right\}^{1/p}.$$

See [22]. The continuity of u is a redundant requirement in the definition: the weak solutions of Equation (2.2) can be made continuous after a redefinition in a set of measure zero. This is standard elliptic regularity theory. One even has $u \in C_{loc}^{1,\alpha}(\Omega)$ for some $\alpha > 0$, cf. [2] and [19], but this Hölder continuity of the *gradient* is a deep result.

The eigenvalues are positive and the least of them*, say λ_p, is obtained as the minimum of the *Rayleigh quotient*

$$\lambda_p = \inf_v \frac{\|\nabla v\|_{L^p(\Omega)}^p}{\|v\|_{L^p(\Omega)}^p} = \inf_v \frac{\int_\Omega |\nabla u|^p\,\mathrm{d}x}{\int_\Omega |v|^p\,\mathrm{d}x}, \tag{2.3}$$

the infimum being taken among all $v \in W_0^{1,p}(\Omega)$, $v \not\equiv 0$. Alternatively, one can further restrict the class of admissible functions to $C_0^\infty(\Omega)$. The minimization problem is equivalent to Equation (2.2) with $\lambda = \lambda_p$.

To any bounded domain Ω there is a first eigenfunction $u_p > 0$ corresponding to the least eigenvalue λ_p. The existence is standard Calculus of Variations, cf. [18], and the strict positivity follows from the Harnack inequality ([20] Theorem 1.1) applied to the non-negative minimizing function $|u_p|$. The first eigenfunctions are essentially unique in any bounded domain: they are merely constant multiples of each other. The uniqueness for *arbitrary* bounded domains was proved in [8]. The first eigenfunctions are the only eigenfunctions not changing signs.

Throughout this paper the first eigenfunction u_p is *normalized* by

$$\|u_p\|_p = \left\{ \int_\Omega |u_p|^p\,\mathrm{d}x \right\}^{1/p} = 1 \tag{2.4}$$

and required to be *positive*. By the above mentioned uniqueness this determines an unambiguous u_p.

In very irregular domains the boundary values (zero) are not attained in the classical sense, when $p \leqslant n$, but one always has $u_p \in W_0^{1,p}(\Omega)$. (This is an essential point in the situation $\lim_{s \to p-} \lambda_s < \lambda_p$.)

Note that, if $\Omega_1 \subset \Omega_2$, then we have $\lambda_p(\Omega_1) \geqslant \lambda_p(\Omega_2)$ for the corresponding principal frequencies. This can be read off from the Rayleigh quotient. Equality is possible, although $\Omega_1 \neq \Omega_2$. The elementary bounds

* *Note added in proof:* In the non-linear cases it does not seem to be known whether *all* higher eigenvalues can be obtained through a variational principle.

$$\left(\frac{v_n}{\operatorname{mes}\Omega}\right)^{1/n} \leqslant \sqrt[p]{\lambda_p} \leqslant \frac{(n+1)\operatorname{mes}\Omega}{r^{n+1}v_n} \tag{2.5}$$

are convenient (v_n is the volume of the unit ball in $\mathbf{R}^n$, i.e. $2\pi^{n/2}/n\Gamma(n/2)$, and r is the radius of the largest ball contained in Ω), but any estimates of this kind will do for our purpose.

3. Some Fundamental Properties

Most results in this section are derived in an elementary way. That is, we use only Functional Analysis in Sobolev spaces but no deep properties of the eigenfunctions (except their uniqueness). The eigenfunctions are unique when normalized by $\|u_p\|_p = 1$ and $u_p > 0$ so that

$$\lambda_p = \int_\Omega |\nabla u_p|^p \,\mathrm{d}x. \tag{3.1}$$

The Hölder inequality yields the following monotony for the principal frequency.

3.2. THEOREM. *For any bounded domain Ω we have $p\lambda_p^{1/p} \leqslant s\lambda_s^{1/s}$, when $1 < p < s < \infty$.*

Proof. To see this, choose any $\psi \in C_0^\infty(\Omega)$, $\psi \geqslant 0$. Then $\varphi = \psi^{s/p}$ is admissible in the Rayleigh quotient for λ_p. Hölder's inequality yields

$$\begin{aligned}
\lambda_p^{1/p} &\leqslant \frac{\left(\int |\nabla\varphi|^p \,\mathrm{d}x\right)^{1/p}}{\left(\int \varphi^p \,\mathrm{d}x\right)^{1/p}} = \frac{s}{p}\frac{\left(\int \psi^{s-p}|\nabla\psi|^p \,\mathrm{d}x\right)^{1/p}}{\left(\int \psi^s \,\mathrm{d}x\right)^{1/p}} \\
&\leqslant \frac{s}{p}\frac{\left\{\left(\int \psi^s \,\mathrm{d}x\right)^{1-p/s}\left(\int |\nabla\psi|^s \,\mathrm{d}x\right)^{p/s}\right\}^{1/p}}{\left(\int \psi^s \,\mathrm{d}x\right)^{1/p}} \\
&= \frac{s}{p}\frac{\left(\int |\nabla\psi|^s \,\mathrm{d}x\right)^{1/s}}{\left(\int \psi^s \,\mathrm{d}x\right)^{1/s}}.
\end{aligned} \tag{3.3}$$

Taking the infimum over all admissible $\psi \geqslant 0$, we arrive at the inequality $p\lambda_p^{1/p} \leqslant s\lambda_s^{1/s}$. (One can avoid the restriction to positive test-functions by choosing $\varphi = |\psi|^{s/p-1}\psi$ from the beginning.) □

REMARK. It is intuitively clear that *the inequality in the theorem is strict.* One way to establish that $p\lambda_p^{1/p} < s\lambda_s^{1/s}$ is to perform the previous calculations with $\psi = u_s > 0$. Then an equality $p\lambda_p^{1/p} = s\lambda_s^{1/s}$ with $s > p$ would imply that $\varphi = u_s^{s/p}$ is an eigenfunction Cu_p. A direct, but lengthy calculation, shows that this does not agree with the differential equation: $u_s^{s/p}$ is not a solution. Another way of obtaining a contradiction is to observe that equality holds in the Hölder inequality used in (3.3) only if $|u_s|^s$ and $|\nabla u_s|^s$ are proportional in Ω. Such a proportionality between u_s and $|\nabla u_s|$ is out of the question.

3.4. COROLLARY. $$\lim_{s\to p-} \lambda_s \leqslant \lambda_p \leqslant \lim_{s\to p+} \lambda_s.$$

Proof. As a monotone function in s the expression $s\lambda_s^{1/s}$ has one-sided limits. So does λ_s. □

Observe that if $\lim_{s\to p} \lambda_s$ exists, then this limit must be equal to λ_p. As we will see, the cases $s \to p-$ and $s \to p+$ can be very different. This dichotomy is prevalent in irregular domains.

3.5. THEOREM. *For any bounded domain*

$$\lim_{s\to p+} \lambda_s = \lambda_p.$$

Proof. For any $\varphi \in C_0^\infty(\Omega)$ we have

$$\lambda_s \leqslant \frac{\int_\Omega |\nabla\varphi|^s \,dx}{\int_\Omega |\varphi|^s \,dx}.$$

Thus

$$\lim_{s\to p+} \lambda_s \leqslant \frac{\int_\Omega |\nabla\varphi|^p \,dx}{\int_\Omega |\varphi|^p \,dx}.$$

Taking the infimum over all admissible φ we find that $\lim_{s\to p+} \lambda_s \leqslant \lambda_p$. Now the result follows by Corollary 3.4. □

Some cautiousness is needed in the next result, since the possibility that

$$\int_\Omega |\nabla u_p|^s \,dx = \infty \quad (s > p)$$

for every $s > p$ cannot be excluded in very irregular domains.

3.6. THEOREM. *The strong convergence*

$$\lim_{s \to p+} \int_\Omega |\nabla u_s - \nabla u_p|^p \, dx = 0 \tag{3.7}$$

is valid for any bounded domain Ω.

Proof. By the Hölder inequality and the normalization $\|u_s\|_s = 1$ we have

$$\int_\Omega |\nabla u_s|^p \, dx \leqslant (\text{mes}\,\Omega)^{p/s} \lambda_s^{1-p/s} \tag{3.8}$$

for $s \geqslant p$. This uniform bound for the L^p-norm implies that some sequence $u_{s_1}, u_{s_2}, \ldots$ converges weakly in $W^{1,p}(\Omega)$ to a function u in $W^{1,p}(\Omega)$. Here $s_j \longrightarrow p+$. The limit function u is in $W_0^{1,p}(\Omega)$, as every u_{s_j} is in this space. By the Rellich–Kondrachov Compactness Theorem $\|u_{s_j} - u\|_{p+1/n} \longrightarrow 0$ (the actual convergence is better than this, but the exponent $p + 1/n$ will do). See [22]. The normalization $\|u\|_p = 1$ follows.

Let us identify u. By the weak lower semicontinuity and the Hölder inequality we obtain

$$\int_\Omega |\nabla u|^p \, dx \leqslant \varliminf_{j \to \infty} \int_\Omega |\nabla u_{s_j}|^p \, dx \leqslant \varliminf_{j \to \infty} \int_\Omega |\nabla u_{s_j}|^{s_j} \, dx \tag{3.9}$$

and taking the normalization into account we have

$$\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx} \leqslant \varliminf_{j \to \infty} \lambda_{s_j} = \lambda_p$$

by Theorem 3.5. But $u \in W_0^{1,p}(\Omega)$, that is, u is admissible in the Rayleigh quotient for λ_p. By the uniqueness of first eigenfunctions we have that $u = u_p$. The limit function is the same for all weakly convergent (sub)sequences. Thus $u_s \longrightarrow u_p$ at least in $L^p(\Omega)$ as $s \longrightarrow p+$.

For the strong convergence (3.7) we use Clarkson's inequalities related to uniform convexity, cf. ([1] Theorem 2.28, p. 37). Consider the case $p \geqslant 2$ first. The desired result follows from Clarkson's inequality

$$\int_\Omega \left| \frac{\nabla u_p + \nabla u_s}{2} \right|^p dx + \int_\Omega \left| \frac{\nabla u_p - \nabla u_s}{2} \right|^p dx \leqslant \frac{1}{2} \int_\Omega |\nabla u_p|^p \, dx + \frac{1}{2} \int_\Omega |\nabla u_s|^p \, dx,$$

since

$$\lambda_p \leqslant \frac{\displaystyle\int_\Omega \left|\frac{\nabla u_p + \nabla u_s}{2}\right|^p \mathrm{d}x}{\displaystyle\int_\Omega \left|\frac{u_p + u_s}{2}\right|^p \mathrm{d}x}$$

and

$$\overline{\lim_{s\to p+}} \int_\Omega |\nabla u_s|^p \,\mathrm{d}x \leqslant \lambda_p$$

by (3.8). Remember that

$$\lim_{s\to p+} \int_\Omega \left|\frac{u_p + u_s}{2}\right|^p \mathrm{d}x = \int_\Omega |u_p|^p = 1$$

by the normalization.

In the case $1 < p < 2$ one uses Clarkson's inequality

$$\left\{\int_\Omega \left|\frac{\nabla u_p + \nabla u_s}{2}\right|^p \mathrm{d}x\right\}^{\frac{1}{p-1}} + \left\{\int_\Omega \left|\frac{\nabla u_p - \nabla u_s}{2}\right|^p \mathrm{d}x\right\}^{\frac{1}{p-1}}$$
$$\leqslant \left\{\frac{1}{2}\int_\Omega |\nabla u_p|^p \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u_s|^p \,\mathrm{d}x\right\}^{\frac{1}{p-1}}$$

in a similar way to obtain (3.7). □

When s approaches p from below the adjusted version

$$\lim_{s\to p-} \int_\Omega |\nabla u_s - \nabla u_p|^s \,\mathrm{d}x = 0 \tag{3.10}$$

of the theorem is false, if $p \leqslant n$. In this case the above method surely produces a positive solution $u \in W^{1,p}(\Omega)$ to the differential equation (1.1) with $\lambda = \lim_{s\to p-} \lambda_s$ and $\|\nabla u_s - \nabla u\| \to 0$, as s approaches p from below through some subsequence. Moreover, u is in $W_0^{1,s}(\Omega)$ for every $s < p$. The failure is that u need not belong to $W_0^{1,p}(\Omega)$. An example is given in Section 7.

A thorough analysis shows that the defect $\lim_{s\to p-} \lambda_s < \lambda_p$ must be ruled out in the proper counterpart to Theorem 3.6. Some care is needed, since the situation that $\|\nabla u_s\|_p = \infty$ for every $s < p$ cannot be excluded. We conjecture that (3.10) holds if and only if $\lim_{s\to p-} \lambda_s = \lambda_p$. We have not been able to prove the full conjecture.

3.11. THEOREM. *For any bounded domain Ω, the convergence (3.10) implies that* $\lim_{s\to p-} \lambda_s = \lambda_p$.

Suppose that (3.10) is true. Then $\nabla u_s \to \nabla u_p$ in $L^{p-\epsilon}(\Omega)$, $\epsilon > 0$, by the Hölder inequality. Using this fact and again the Hölder inequality we have $\|\nabla u_p\|_{p-\epsilon} \leqslant (\text{mes}\,\Omega)^{\epsilon/(p-\epsilon)} \lim_{s\to p-} \|\nabla u_s\|_s$. Hence $\|\nabla u_p\|_p \leqslant \lim_{s\to p-} \|\nabla u_s\|_s$, that is, $\lambda_p^{1/p} \leqslant \lim_{s\to p-} \lambda_s^{1/s}$ by the normalization. Together with Corollary 3.4 this leads to $\lim_{s\to p-} \lambda_s = \lambda_p$.

We conjectured that the assumption about the limit function is superfluous in the next lemma. The difficulty is to obtain the seemingly plain normalization

$$\lim_{s\to p-} \int_\Omega \left|\frac{u_p + u_s}{2}\right|^s dx = 1.$$

3.12. LEMMA. *Suppose that* $\lim_{s\to p-} \lambda_s = \lambda_p$. *Then each sequence of real numbers tending to p from below contains a subsequence such that*

$$\lim_{j\to\infty} \int_\Omega |\nabla u_{s_j} - \nabla u|^{s_j} dx = 0 \quad (s_j \to p-) \tag{3.13}$$

for some function $u \in W^{1,p}(\Omega)$. *If* $u \in W_0^{1,p}(\Omega)$, *then* $u = u_p$. *In any case* $\lambda_p = \int_\Omega |\nabla u|^p dx$ *and* $\|u\|_p = 1$.

Proof. The norms $\|\nabla u_s\|_s$ are uniformly bounded and so are *a fortiori* the norms $\|\nabla u_s\|_{s-\epsilon}$, $\epsilon > 0$. A standard diagonalization procedure enables us to find a function $u \in W_0^{p-\epsilon}(\Omega)$ for all $\varepsilon > 0$ and to construct indices $s_1 < s_2 < s_3 < \cdots, \lim s_j = p$, such that (1) $\nabla u_{s_j} \to \nabla u$ weakly in each fixed $L^{p-\epsilon}(\Omega)$ and (2) $u_{s_j} \to u$ strongly in $L^p(\Omega)$ (the Rellich–Kondrachov Compactness Theorem). In particular,

$$\begin{aligned}\|\nabla u\|_{p-\epsilon} &\leqslant \underline{\lim}\, \|\nabla u_{s_j}\|_{p-\epsilon} \leqslant (\text{mes}\,\Omega)^{\epsilon/(p-\epsilon)}\, \underline{\lim}\, \|\nabla u_{s_j}\|_{s_j} \\ &= (\text{mes}\,\Omega)^{\epsilon/(p-\epsilon)} \lambda_p^{1/p}.\end{aligned}$$

Thus $\nabla u \in L^p(\Omega)$ and $\|\nabla u\|_p \leqslant \lambda_p^{1/p}$. The correct normalization $\|u\|_p = 1$ is preserved. Thus $\lambda_p = \|\nabla u\|_p^p$ and

$$\lim_{j\to\infty} \int_\Omega \left|\frac{u + u_{s_j}}{2}\right|^{s_j} dx = 1. \tag{3.14}$$

So far our assumptions have virtually not been needed, but on the other hand, we have not shown that $u \in W_0^{1,p}(\Omega)$, although $u \in W^{1,p}(\Omega)$ and $u \in W_0^{1,p-\epsilon}(\Omega)$, for each $\epsilon > 0$. We can identify u as the right eigenfunction u_p, under the additional assumption that u is in $W_0^{1,p}(\Omega)$, thus proving that in this case the limit is independent of the particular sequence $s_1, s_2, s_3, \ldots$ converging to p.

Using Clarkson's inequality

$$\int_\Omega \left|\frac{\nabla u + \nabla u_s}{2}\right|^s dx + \int_\Omega \left|\frac{\nabla u - \nabla u_s}{2}\right|^s dx \leqslant \frac{1}{2}\int_\Omega |\nabla u|^s dx + \frac{1}{2}\int_\Omega |\nabla u_s|^s dx, \tag{3.15}$$

$s \geqslant 2$, once more and taking the normalization and (3.14) into account, we conclude that

$$\lim_{j\to\infty} \lambda_{s_j} + \overline{\lim_{j\to\infty}} \int_\Omega \left|\frac{\nabla u - \nabla u_{s_j}}{2}\right|^{s_j} \mathrm{d}x \leqslant \frac{1}{2}\lambda_p + \frac{1}{2}\lim_{j\to\infty} \lambda_{s_j}$$

in the case $p > 2$. By assumption $\lim \lambda_{s_j} = \lambda_p$. Hence $\|\nabla u - \nabla u_{s_j}\|_{s_j} \to 0$.

The case $p \leqslant 2$ is similar, the only change being in Clarkson's inequality. □

4. The Maximum of the Eigenfunctions

It is intuitively clear that the eigenfunction u_p is globally bounded in Ω and the proof is evident, if Ω possesses some geometric regularity at the boundary. However, there are continuous functions in $W_0^{1,p}(\Omega)$, $1 < p \leqslant n$, that are unbounded! Therefore it is worth mentioning that a proof can be produced by the well-known method given in ([9] Lemma 5.1, p. 71). Unfortunately the knowledge that $\max|u_p| < \infty$ is not enough for us. We need a bound that is uniform in p, when $\|u_p\|_{p,\Omega} \leqslant 1$. To achieve it we must keep track of various 'constants' and we had better write down a proof.

4.1. LEMMA. *The inequality*

$$\|u_p\|_{\infty,\Omega} \leqslant 4^n \lambda_p^{n/p} \|u_p\|_{1,\Omega} \tag{4.2}$$

is valid for the eigenfunction u_p in any bounded domain Ω in $\mathbf{R}^n$.

Proof. The function $\eta_p(x) = \max\{u_p(x) - k, 0\}$ is in the Sobolev space $W_0^{1,p}(\Omega)$ for any constant k. Hence η_p will do as test-function in (2.2) and so we obtain

$$\int_{A_k} |\nabla u_p|^p \,\mathrm{d}x = \lambda_p \int_{A_k} u_p^{p-1}(u_p - k)\,\mathrm{d}x \tag{4.3}$$

where

$$A_k = \{x \in \Omega \mid u_p(x) > k\}.$$

Of course A_k depends on p. Clearly $k \cdot \operatorname{mes} A_k \leqslant \|u_p\|_1$ and $\operatorname{mes} A_k \to 0$, as $k \to \infty$.

By the elementary inequality $a^{p-1} \leqslant 2^{p-1}(a-k)^{p-1} + 2^{p-1}k^{p-1}$ we have

$$\int_{A_k} u_p^{p-1}(u_p - k)\,\mathrm{d}x \leqslant 2^{p-1}\int_{A_k} (u_p - k)^p \,\mathrm{d}x + 2^{p-1}k^{p-1}\int_{A_k} (u_p - k)\,\mathrm{d}x. \tag{4.4}$$

The Poincaré–Friedrichs inequality yields

$$\int_{A_k} (u_p - k)^p \,\mathrm{d}x \leqslant \left(\frac{1}{2}\operatorname{mes} A_k\right)^{p/n} \int_{A_k} |\nabla u_p|^p \,\mathrm{d}x, \tag{4.5}$$

when applied to each component of the open set A_k. See ([4] Eqn (7.44), p. 164), where the better constant $n\Gamma(n/2)2^{-1}\pi^{-n/2}$ is given instead of 1/2.

Combining (4.3) and (4.5) and then using (4.4) we arrive at

$$[1 - 2^{p-2}\lambda_p(\text{mes}\, A_k)^{p/2}] \int_{A_k} (u_p - k)^p \, dx \leqslant 2^{p-2}k^{p-1}(\text{mes}\, A_k)^{p/n} \int_{A_k} (u_p - k) \, dx.$$

Here $2^{p-2}\lambda_p(\text{mes}\, A_k)^{p/n} \leqslant 1/2$, when $k \geqslant k_1 = 2^{n(p-1)/p}\lambda_p^{n/p}\|u_p\|_1$. Using Hölder's inequality and dividing out we finally obtain the estimate

$$\int_{A_k} (u_p - k) \, dx \leqslant 2\lambda_p^{1/(p-1)} k (\text{mes}\, A_k)^{1 + \frac{p}{(p-1)n}} \tag{4.6}$$

for $k \geqslant k_1$. This is the inequality needed in ([9] Lemma 5.1, p. 71) to bound ess sup u_p.

Indeed, for

$$f(k) = \int_{A_k} (u_p - k) \, dx = \int_k^\infty \text{mes}\, A_t \, dt$$

we have $f'(k) = -\text{mes}\, A_k$, and hence (4.6) can be written as

$$f(k) \leqslant 2\lambda_p^{1/(p-1)} k (-f'(k))^{1 + p/n(p-1)},$$

when $k \geqslant k_1$. If f is positive in the interval $[k_1, k]$, integration leads to

$$k^{\epsilon/(1+\epsilon)} - k_1^{\epsilon/(1+\epsilon)} \leqslant [2\lambda_p^{1/(p-1)}]^{1/(1+\epsilon)}[f(k_1)^{\epsilon/(1+\epsilon)} - f(k)^{\epsilon/(1+\epsilon)}]$$

where $\epsilon = p/(p-1)n$. This clearly bounds k and hence $f(k)$ is zero sooner or later. The quantitative bound for k is seen to be

$$k \leqslant 2^{1 + 2n(p-1)/p}\lambda_p^{n/p}\|u_p\|_1, \tag{4.7}$$

since $f(k_1) \leqslant f(0) = \|u_p\|_1$. Thus $f(k) = 0$, if (4.7) is not fulfilled, i.e., ess sup u_p is not greater than the right-hand side. This is the desired result (4.2). □

5. The Wiener Criterion and the Kellogg Property

The classical Wiener criterion is necessary and sufficient for a solution to the Laplace equation $\Delta u = 0$ to attain its prescribed continuous boundary values at a given boundary point. It is formulated in terms of electrostatic capacity [21]. The sufficiency of the (appropriately modified) Wiener criterion was proved by Maz'ja [11] for non-linear equations of the type $\text{div}(|\nabla u|^{p-2}\nabla u) = 0$. This result was further extended by Gariepy and Ziemer [5] to fairly general equations including our case $\text{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. The so called p-capacity plays a central role here.

Suppose that A is any open bounded set in $\mathbf{R}^n$ and that $K \subset A$ is compact. Define

$$\operatorname{cap}_p(K, A) = \inf_{\varphi} \int_A |\nabla\varphi|^p \, dx, \tag{5.1}$$

the infimum being taken over all functions $\varphi \in C_0^\infty(A)$ such that $\varphi \geqslant 1$ in K. This is the *p-capacity* of the capacitor (K, A). If $K_1 \subset K_2 \subset A_2 \subset A_1$, then $\operatorname{cap}_p(K_1, A_1) \leqslant \operatorname{cap}_p(K_2, A_2)$. The inequality

$$\left\{\frac{\operatorname{cap}_q(K, A)}{\operatorname{mes} A}\right\}^{1/q} \leqslant \left\{\frac{\operatorname{cap}_p(K, A)}{\operatorname{mes} A}\right\}^{1/p}, \quad \text{if } 1 < q \leqslant p < \infty, \tag{5.2}$$

follows from Hölder's inequality. There is a striking difference between the cases $p > n$ and $p \leqslant n$, namely, the p-capacity is never zero (for a non-empty K), when $p > n$.

For the capacitor consisting of two concentric spheres the expression

$$\operatorname{cap}_p(\overline{B(x_0, r)}, B(x_0, R)) = \frac{\omega_{n-1}}{\left(\displaystyle\int_r^R t^{-\frac{n-1}{p-1}} dt\right)^{p-1}} \tag{5.3}$$

is known, cf. [11] or ([12] p. 106). Here $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere in $\mathbf{R}^n$. Observe that $\operatorname{cap}_p(\overline{B(x_0, r)}, B(x_0, R))$ is a continuous function of p and that

$$\operatorname{cap}_p(\overline{B(x_0, r)}, B(x_0, 2r)) \approx r^{n-p}.$$

Letting $p \to 1+$ and $p \to \infty$ in (5.3) we obtain the sharp bounds

$$\frac{1}{2^n r} < \left\{\frac{\operatorname{cap}_p(\overline{B(x_0, r)}, B(x_0, 2r))}{\omega_{n-1}(2r)^n}\right\}^{1/p} < \frac{1}{r} \tag{5.4}$$

using (5.2).

In order to formulate the Wiener criterion we return to our domain Ω and fix an arbitrary boundary point $x_0 \in \partial\Omega$. The auxiliary quantity

$$\gamma_p(x_0, r) = \frac{\operatorname{cap}_p(\overline{B(x_0, r)} \setminus \Omega, B(x_0, 2r))}{\operatorname{cap}_p(\overline{B(x_0, r)}, B(x_0, 2r))}$$

is proportional to

$$r^{p-n} \operatorname{cap}_p(\overline{B(x_0, r)} \setminus \Omega, B(x_0, 2r)).$$

We have $0 \leqslant \gamma_p(x_0, r) \leqslant 1$. The Wiener integral

$$W_p(x_0) = \int_0^1 \gamma_p(x_0, r)^{1/(p-1)} r^{-1} \, dr$$

measures how much boundary the domain Ω has in a potential theoretic sense near the point x_0. The boundary point x_0 is called regular, if $W_p(x_0) = \infty$. At such boundary points the eigenfunctions take the right boundary value in the classical sense. If $W_p(x_0) = \infty$, then

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} u_p(x) = 0.$$

(The divergence of the integral is also known to be necessary* when $p > n - 1$ or $p = 2$.)

5.5. THEOREM. *Suppose that u_p is the first eigenfunction in Ω, $u_p > 0$. If $x_0 \in \partial\Omega$, then*

$$u_p(x) \leqslant C_1 \exp\left(-C_2 \int_r^R \gamma_p(x_0, t)^{1/(p-1)} t^{-1}\, dt\right), \tag{5.6}$$

when $x \in \Omega$ and $|x - x_0| < r \leqslant R$. Here C_1 and C_2 are positive constants depending only on n, p and the maximum of u_p in $\Omega \cap B(x_0, 2R)$.

If $0 \leqslant u_p \leqslant M$ for each p in the range $[\alpha, \beta]$, $1 < \alpha < \beta < \infty$, then

$$\sup_{\alpha \leqslant p \leqslant \beta} C_1 < \infty \quad \textit{and} \quad \inf_{\alpha \leqslant p \leqslant \beta} C_2 > 0.$$

Proof. This is essentially ([5] Theorem 2.7, p. 31). The constants in the proof of Gariepy and Ziemer are easily calculated in our case and they are seen to depend on p in the desired way. These expressions are not very illuminating themselves. For example one gets

$$C_1 \leqslant 2R(2 + \sqrt[n]{\omega_{n-1}}\lambda_p^{1/(p-1)}) + \max_{|x - x_0| \leqslant 2R} u_p(x).$$

The straightforward calculation of C_2 as in [5] is lengthy and in our special case it seems likely that the procedure could be shortened. However, such improvements are not essential. □

Let us analyze (5.6) in the favourable case $p > n$. Write $B_t = B(x_0, t)$. Then

$$\mathrm{cap}_p(\bar{B}_r \setminus \Omega, B_{2r}) \geqslant \mathrm{cap}_p(\{x_0\}, B_{2r}) = \omega_{n-1}\left(\frac{p-n}{p-1}\right)^{p-1} r^{n-p}$$

by (5.3). Using (5.3) again we see that

$$\gamma_p(x_0, r)^{1/(p-1)} \geqslant 1 - 2^{(p-n)/(1-p)} = c(n, p)$$

* *Note added in proof:* T. Kilpeläinen has kindly informed me that the manuscript "The Wiener Test and Potential Estimates for Quasilinear Elliptic Equations" by T. Kilpeläinen and J. Malý proves the necessity for all values of p, $1 < p < \infty$.

at each boundary point x_0. (Unfortunately, $c(n,p)$ approaches zero, as $p \to n+$.) Integrating we find that (5.6) becomes

$$u_p(x) \leqslant C_1 \left(\frac{|x - x_0|}{R} \right)^{\alpha} \quad (p > n) \tag{5.7}$$

where $x \in \Omega$ and $|x - x_0| < R$. Here α is the positive number $c(n,p)C_2$. The constants C_1 and α do not depend on the boundary point chosen. Needless to say, this uniform Hölder estimate at the boundary can be achieved by simpler tools in this favourable situation, when $p > n$.

Essentially the same method yields a similar boundary Hölder estimate for any $p > 1$ in domains satisfying the *exterior cone property.* A sharper result is given in the proof of Theorem 6.1.

Those boundary points x_0 for which the Wiener integral converges, i.e. $W_p(x_0) < \infty$, are called *irregular.* They form the irregular set F_p. It may be empty (as is always the case, when $p > n$), but it can never be large, according to the Kellogg property proved by Hedberg and Wolff. See [6] for this deep result. The Kellogg property means that the Sobolev p-capacity of F_p is zero.

It does not seem to be known whether F_p is a Borel set or not. Fortunately, a weaker version of the Kellogg property, avoiding the complications in the structure of F_p, is sufficient for our counter example in Section 7. Namely, *every compact set K in F_p has zero capacity: given any $\epsilon > 0$, there is a function $\varphi \in C_0^\infty(\Omega)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in K, and*

$$\int_{R^n} (|\varphi|^p + |\nabla \varphi|^p)\, dx < \epsilon. \tag{5.8}$$

The integrand in the Wiener integral, practically speaking, grows with p. Consequently, $W_{p+\epsilon}(x_0) = \infty$ *for every* $\epsilon > 0$ *if* $W_p(x_0) = \infty$. This is included in the well-known inequality below.

5.9. LEMMA. *For $1 < q < p < \infty$ we have*

$$\gamma_q(x_0, r)^{\frac{1}{q-1}} \leqslant 2^{\frac{nq}{q-1}} \gamma_p(x_0, r)^{\frac{1}{p-1}}. \tag{5.10}$$

Proof. Denote $D_r = \overline{B(x_0, r)} \setminus \Omega$ and $B_r = B(x_0, r)$. By (5.2) we have

$$\left\{ \frac{\operatorname{cap}_q(D_r, B_{2r})}{\omega_{n-1}(2r)^n} \right\}^{\frac{1}{q-1}} \leqslant \left\{ \frac{\operatorname{cap}_p(D_r, B_{2r})}{\omega_{n-1}(2r)^n} \right\}^{\frac{1}{p-1} + \frac{p-q}{p(q-1)(p-1)}}$$
$$\leqslant \left\{ \frac{\operatorname{cap}_p(D_r, B_{2r})}{\omega_{n-1}(2r)^n} \right\}^{\frac{1}{p-1}} \left\{ \frac{\operatorname{cap}_p(\bar{B}_r, B_{2r})}{\omega_{n-1}(2r)^n} \right\}^{\frac{p-q}{p(q-1)(p-1)}}.$$

Dividing by $[\operatorname{cap}_q(\bar{B}_r, B_{2r})/\omega_{n-1}(2r)^n]^{1/(q-1)}$ we arrive at

$$\gamma_q(x_0,r)^{\frac{1}{q-1}} \leqslant \gamma_p(x_0,r)^{\frac{1}{p-1}} \left(\frac{\left\{\dfrac{\operatorname{cap}_p(\bar{B}_r, B_{2r})}{\omega_{n-1}(2r)^n}\right\}^{\frac{1}{p}}}{\left\{\dfrac{\operatorname{cap}_q(\bar{B}_r, B_{2r})}{\omega_{n-1}(2r)^n}\right\}^{\frac{1}{q}}} \right)^{\frac{q}{q-1}}$$

after some arithmetic. Now the inequalities (5.4) yield the constant $2^{nq/(q-1)}$. □

6. Uniform Convergence

The normalized eigenfunctions u_s need not converge *uniformly* in the whole domain, not even in the case $s \to p+$, when $p \leqslant n$. A simple counterexample in the case $p = n$ is provided by the punctured unit ball $\Omega = \{x \in \mathbf{R}^n \mid 0 < |x| < 1\}$. Now $\lim_{x\to 0} u_s(x) = 0$ for $s > n$ but $\lim_{x\to 0} u_n(x) \neq 0$.

In regular domains the situation is better. The analogue of the theorem below for equations of the type $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ is credited to Martio. Actually, a much simpler version is sufficient for our example in Section 7.

6.1. THEOREM. *Suppose that $W_p(x_0) = \infty$ at every boundary point x_0 of Ω. Then $u_s \to u_p$ uniformly in Ω as $s \to p+$. The same holds as $s \to p-$, if for some $q < p$, $W_q(x_0) = \infty$ at every boundary point $x_0 \in \partial\Omega$.*

Proof. Suppose first that $p < s < p + 1$. For $x \in \Omega$, $|x - x_0| < r \leqslant 1$, Theorem 5.5 and Lemma 5.9 yield the estimate

$$u_s(x) \leqslant C_1 \exp\left(-2^{np/(p-1)} C_2 \int_r^1 \gamma_p(x_0,t)^{1/(p-1)} t^{-1}\,dt\right), \tag{6.2}$$

where the positive constants C_1 and C_2 depend only on n and p. This shows that the family $\{u_s \mid p < s < p + 1\}$ is uniformly equicontinuous in $\bar{\Omega}$. To be more precise, take $\epsilon > 0$. To each boundary point ξ there is a radius r_ξ such that $u_s(x) < \epsilon/2$, when $x \in B(\xi, r_\xi) \cap \Omega$. The open balls $B(\xi, r_\xi)$ cover the boundary $\partial\Omega$, even their centers do it. By compactness there is a finite subcover. The construction yields a δ_ϵ (= the smallest of the radii) such that $u_s(x) < \epsilon/2$, when $x \in \Omega$ and $\operatorname{dist}(x, \partial\Omega) < \delta_\epsilon$. Restricted to any compact set in Ω (here we have the set $\operatorname{dist}(x, \partial\Omega) \geqslant \delta_\epsilon$ in mind) the function family is uniformly equicontinuous according to the local Hölder-continuity estimates. This is standard elliptic regularity theory. Altogether, this proves the desired equicontinuity. The family is uniformly bounded by Lemma 4.1.

By the theorem of Ascoli the convergence is uniform in Ω, at least for a subsequence. But Theorem 3.6 shows that all subsequences have the same limit, namely u_p.

The case $s \to p-$ is proved in the same way, but with p replaced by q in (6.2). Note that this estimate also implies that the limit function $u = \lim u_s$ of any subsequence in Ascoli's theorem takes the boundary values zero at each boundary point. This shows that $u \in W_0^{1,p}(\Omega)$. According to Lemma 3.12 u is the right function, that is u_p. $\square$

It is not difficult to show that the eigenfunctions u_s converge *locally uniformly*, though they might converge towards the wrong solution of the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, the failure being due to effects near the boundary. However, much deeper knowledge is needed to establish that also their gradients converge locally uniformly.

6.3. THEOREM. *Given any sequence converging to p, there is a subsequence* $s_1, s_2, s_3, \dots$ *and a weak solution u to the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, $\lambda = \lim_{j\to\infty} \lambda_{s_j}$, *such that* $u_{s_j} \to u$ *and* $\nabla u_{s_j} \to \nabla u$ *locally uniformly in* Ω. (*It is possible that u is not* u_p*!*)

Proof. By an advanced result in regularity theory u_s is in $C^{1,\alpha}_{loc}(\Omega)$. According to [19] or ([2] Theorems 1 and 2) we have the following estimates: given any compact set K in Ω, there are constants L_K, M_K, and $\alpha(s) > 0$ such that

$$|\nabla u_s(x)| \leqslant M_K, \quad |\nabla u_s(x) - \nabla u_s(y)| \leqslant L_K|x - y|^{\alpha(s)} \tag{6.4}$$

when $x, y \in K$. Here L_K, M_K, and $\alpha(s)$ depend only on s, n, K and the maximum of u_s in Ω. But the uniform bound for $\max|u_p|$ in Section 4 shows that the constants depend only on s, n, K. The dependence of s for the constants in these estimates is of a continuous nature: The family $\{\nabla u_s\}_K$ is equicontinuous when s is restricted to any fixed closed interval $[a, b]$ in $]1, \infty[$. Especially, $\inf_{a\leqslant s\leqslant b} \alpha(s) > 0$, when K is fixed. This requires a thorough analysis of the regularity proofs and some minor adjustments are necessary especially when $n \in]a, b[$. We skip this lengthy routine verification, here. The resulting *local* convergence of gradients is not used later.

Choosing an exhaustion of Ω with compact sets and using a standard process of diagonalization we can construct a sequence $s_1, s_2, s_3, \dots$ of indices converging to p and a function $u \in C^1(\Omega)$ such that $u_{s_j} \to u$ and $\nabla u_{s_j} \to \nabla u$ locally uniformly in Ω. At each step of the construction the theorem of Ascoli yields a subsequence. The normalization prevents the sequence from degenerating: $u \not\equiv 0$.

By Equation (2.2)

$$\lim \lambda_{s_j} = \lim_{j\to\infty} \frac{\displaystyle\int_\Omega |\nabla u_{s_j}|^{s_j-2}\nabla u_{s_j}\cdot\nabla\eta\,dx}{\displaystyle\int_\Omega |u_{s_j}|^{s_j-2}u_{s_j}\eta\,dx} = \frac{\displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\eta\,dx}{\displaystyle\int_\Omega |u|^{p-2}u\eta\,dx} \tag{6.5}$$

whenever $\eta \in C_0^\infty(\Omega)$, because of the uniform convergence in the support of the test-function η. This means that $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$ in the weak sense, where $\lambda = \lim \lambda_{s_j}$. Since each $u_s > 0$, we have $u \geqslant 0$ by the construction and hence

$u > 0$ by the Harnack inequality. This is a specific property of the first eigenfunctions. Only the first eigenfunctions are positive [8]. This means that *if* λ is an eigenvalue, it is λ_p.

We have come to a delicate point. Although u is the solution to the right equation it is not always the right eigenfunction u_p. The example in Section 7 will explain this strange phenomenon as one caused by the boundary values: although u belongs to $W^{1,p}(\Omega)$ and to every $W_0^{1,p-\epsilon}(\Omega)$, $\epsilon > 0$, it may fail to be in $W_0^{1,p}(\Omega)$. Neither does the above construction show that $\lim_{s\to p}\lambda_s$ would exist, the reason being that u depends on the particular sequence $s_1, s_2, s_3, \ldots$. Therefore nothing further comes out of this proof. □

7. A Domain with $\lim_{s\to p-} \lambda_s < \lambda_p$

Given any p in the range $]1,n]$ we shall construct a domain with $\lim_{s\to p-} \lambda_s < \lambda_p$ and analyse what happens to the corresponding eigenfunctions, as $s \to p-$. To this end we need a very small set, yet of positive p-capacity. It will be constructed as a Cantor set according to a precise criterion, originally due to Nevanlinna, cf. [13], and later extended by Ohtsuka, cf. [14], and others. To be on the safe side we have included the following lemma.

7.1. LEMMA. *Suppose that* $1 < p \leqslant n$. *Then there is a compact set* F_p *such that* $\operatorname{cap}_p F_p > 0$ *and* $\operatorname{cap}_s F_p = 0$, *when* $s < p$. *Moreover,* F_p *can be constructed as a Cantor set.*

Proof. Suppose that $l_1, l_2, \ldots$ are positive numbers with $2l_{j+1} < l_j$. Let us first construct a set on the real line. Let $\Delta_1 = [0, l_1]$, $\Delta_2 = [0, l_2] \cup [l_1 - l_2, l_1], \ldots$. The set Δ_j is the union of 2^{j-1} disjoint closed intervals of length l_j. Delete an open segment in the middle of each of these 2^{j-1} intervals so that each of the remaining intervals has length l_{j+1}. The union of these 2^j closed intervals is Δ_{j+1}. The set $\Delta = \cap \Delta_j$ is compact and of linear measure zero.

To obtain a set in $\mathbf{R}^n$, just take the Cartesian product $\Delta \times \Delta \times \cdots \times \Delta$, i.e.,

$$F_p = \bigcap_{j=1}^{\infty} \Delta_j \times \Delta_j \times \cdots \times \Delta_j.$$

This set is compact. We choose

$$l_j = \frac{j^{\frac{p}{n-p}}}{2^{jn/(n-p)}}, \quad (1 < p < n)$$

if $p < n$. Then $2l_{j+1} < l_j$ for $j = 2, 3, \ldots$. The construction for the border-line case $p = n$ is given in ([10] Example 4.7 and Corollary 4.5, p. 116).

By the afore-mentioned extension of Nevanlinna's criterion $\operatorname{cap}_s F_p > 0$ if and only if the series

$$\sum_{j=1}^{\infty} [2^{-jn} l_j^{n-s}]^{\frac{1}{s-1}}$$

converges. Here $1 < s < n$. See ([12] §7.2.3, Proposition 5, p. 358). The above sum is

$$\sum_{j=1}^{\infty} \frac{2^{\frac{jn(p-s)}{(s-1)(n-p)}}}{j^{\frac{p(n-s)}{(s-1)(n-p)}}}$$

for $1 < s \leqslant p$. This is clearly divergent, when $s < p$, but for $s = p$ we have the convergent sum

$$\sum_{j=1}^{\infty} \left(\frac{1}{j}\right)^{\frac{p}{p-1}}.$$

This proves the lemma. □

The set F_p has Lebesgue measure zero. (Actually, the Hausdorff dimension is $n - p$.) The constructed Cantor set F_p lies in the open cube $0 < x_1 < 1, \ldots, 0 < x_n < 1$.

Now we are ready for the counter example. Let Q denote the open cube $|x_1| < 1, \ldots, |x_n| < 1$. Then $\Omega = Q \backslash F_p$ is a domain and the Cantor set F_p is part of the boundary of Ω. We shall show that $\lim_{s \to p-} \lambda_s^{\Omega} < \lambda_p^{\Omega}$ and that the normalized eigenfunctions u_s^{Ω} in Ω converge towards the 'wrong' function. The idea is very simple. First, $u_s^{\Omega} = u_s^{Q}$ in Ω, when $s < p$. Second, u_p^{Ω} and u_p^{Q} are essentially different functions. Third, $u_s^{Q} \to u_p^{Q}$ uniformly in Q, the cube Q being a very regular domain. Combining these facts we have that $u_s^{\Omega} = u_s^{Q} \to u_p^{Q} \neq u_p^{\Omega}$ as $s \to p-$, the convergence being uniform in Ω. Because the eigenfunctions are continuous, $\|u_p^{Q} - u_p^{\Omega}\|_1 > 0$, and so not even in $L^1(\Omega)$ does u_s^{Ω} converge to u_p^{Ω}!

As far as the eigenvalues are concerned, $\lambda_s^{\Omega} = \lambda_s^{Q}$, when $s < p$ and so $\lim \lambda_s^{\Omega} = \lim \lambda_s^{Q} = \lambda_p^{Q}$ since Q is regular. But $\lambda_p^{Q} < \lambda_p^{\Omega}$, as we will see.

To prove these statements, let us first consider the case $s < p$. Now $\mathrm{cap}_s F_p = 0$. We shall first show that u_s^{Q} is in $W_0^{1,s}(\Omega)$, which implies that

$$\lambda_s^{\Omega} \leqslant \frac{\int_{\Omega} |\nabla u_s^{Q}|^s \, dx}{\int_{\Omega} |u_s^{Q}|^s \, dx} = \frac{\int_{Q} |\nabla u_s^{Q}|^s \, dx}{\int_{Q} |u_s^{Q}|^s \, dx} = \lambda_s^{Q}.$$

Of course $\lambda_s^{Q} \leqslant \lambda_s^{\Omega}$. Hence $\lambda_s^{\Omega} = \lambda_s^{Q}$ and by the uniqueness of first eigenfunctions $u_s^{Q} = u_s^{\Omega}$. (Remember the normalization $\|u_s^{Q}\|_s = 1$.) Since $\mathrm{cap}_s F_p = 0$ there is, given $\epsilon > 0$, a function φ in $C_0^{\infty}(Q)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in an open neighbourhood of F_p, and $\|\nabla \varphi\|_s < \epsilon$. Now $(1 - \varphi) u_s^{Q}$ is in $W_0^{1,s}(\Omega)$ and it follows that the $W^{1,s}$-norm

$\|u_s^Q - (1-\varphi)u_s^Q\| = \|\varphi u_s^Q\|$ approaches zero, as $\epsilon \to 0$. As a limit of functions in $W_0^{1,s}(\Omega)$, u_s^Q itself is in $W_0^{1,s}(\Omega)$.

Next we show that $u_p^Q \neq u_p^\Omega$. By the Kellogg property* (Section 5) there must exist regular points in F_p with respect to Ω. Otherwise the condition $\mathrm{cap}_p F_p > 0$ cannot be fulfilled. If $z_0 \in F_p$ is such that $W_p(x_0) = \infty$, then $\lim_{x\to x_0} u_p^\Omega(x) = 0$. But by the Harnack inequality $u_p^Q(x_0) > 0$, x_0 being an interior point in Q. By the uniqueness of first eigenfunctions in any domain, this behaviour clearly prevents u_p^Q from being an eigenfunction also in Ω.

It is now evident that $\lambda_p^Q < \lambda_p^\Omega$. Indeed, one can modify u_p^Ω near x_0 so that the Rayleigh quotient decreases while the modified function is in $W_0^{1,p}(Q)$. (If $m > 0$ is the minimum of u_p^Ω on the cube $\max\{|x_1|, \ldots, |x_n|\} = 3/4$, then replace u_p^Ω by $\max\{u_p^\Omega, m/2\}$ near F_p. The outer boundary values on ∂Q are not affected, but at least near x_0 there will be a favourable change for the Rayleigh quotient. Namely, for the modified function the L^p-norm has increased strictly and the L^p-norm of the gradient has decreased.)

Acknowledgements

We thank A. Touzani for pointing out a flaw in our paper [7]. An unpublished note of O. Martio contains Theorem 6.1 for equations of the type $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$. We thank him for sending us a copy of his work. We thank T. Kilpeläinen for a useful discussion about capacity.

References

1. Adams, R.: *Sobolev Spaces*, Academic Press, New York (1975).
2. Dibenedetto, E.: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Analysis TMA* **7** (1983), 827–859.
3. Esteban, J. and Vazquez, J.: Homogeneous diffusion in **R** with power-like nonlinear diffusivity. *Archive for Rational Mechanics and Analysis* **103** (1988), 39–80.
4. Gilbarg, D. and Trudinger, N.: *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, Springer-Verlag, Berlin (1983).
5. Gariepy, R. and Ziemer, W.: A regularity condition at the boundary for solutions of quasilinear elliptic equations, *Arch. Rat. Mech. Anal.* **67** (1977), 25–89.
6. Hedberg, L. and Wolff, Th.: Thin sets in nonlinear potential theory, *Ann. Inst. Fourier (Grenoble)* **33** (1983), 161–187.
7. Lindqvist, P.: Stability for the solutions of $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f$ with varying p, *Journal of Mathematical Analysis and Applications* **127** (1987), 93–102.
8. Lindqvist, P.: On the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, *Proceedings of the American Mathematical Society* **109** (1990), 157–164. Addendum, *ibid.* **116** (1992), 583–584.

*As the referee has kindly remarked, the Kellogg property is not really needed here. It can be replaced by a direct computation based on the structure of F_p.

9. Ladyzhenskaya, O. and Ural'tseva, N.: *Linear and Quasilinear Elliptic Equations*, Academic Press, New York (1968).
10. Martio, O.: Capacity and measure densities, *Annales Academiae Scientiarum Fennicae, Series A. I. Mathematica* **4** (1978/1979), 109–118.
11. Maz'ya, V.: On the continuity at a boundary point of solutions of quasi-linear elliptic equations, *Vestnik Leningradskogo Universiteta* **25** (1970), 42–55 (in Russian), *Vestnik Leningrad Univ. Math.* **3** (1976), 225–242 (English translation).
12. Maz'ja, V.: *Sobolev Spaces*, Springer-Verlag, Berlin (1985).
13. Nevanlinna, R.: Über die Kapazität der Cantorschen Punktmengen, *Monatshefte für Mathematik und Physik* **43** (1936), 435–447.
14. Ohtsuka, M.: Capasité d'ensembles de Cantor généralisés, *Nagoya Mathematical Journal* **11** (1957), 141–160.
15. Pólya, G. and Szegö, G.: *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton (1951).
16. Sakaguchi, S.: Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems, *Annali della Scuola Normale Superiore de Pisa, Serie IV (Classe de Scienze)* **14** (1987), 403–421.
17. de Thélin, F.: Quelques résultats d'existence et de non-existence pour une E.D.P. elliptique non linéaire, *C.R. Acad. Sc. Paris, Série I* **299** (1984), 911–914.
18. de Thélin, F.: Sur l'espace propre associé à la première valeur propre du pseudo-laplacien, *C.R. Acad. Sc. Paris, Série I* **303** (1986), 355–358.
19. Tolksdorf, P.: Regularity for a more general class of quasi-linear elliptic equations, *Journal of Differential Equations* **51** (1984), 126–150.
20. Trudinger, N.: On Harnack type inequalities and their application to quasilinear elliptic equations, *Communications on Pure and Applied Mathematics* **20** (1967), 721–747.
21. Wiener, N.: The Dirichlet problem, *Journal of Math. and Phys., Massachusetts Institute of Technology* **3** (1924), 127–146.
22. Ziemer, W.: *Weakly Differentiable Functions (Sobolev Spaces and Functions of Bounded Variation)*, Springer-Verlag, New York (1989).