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# Hopf Bifurcation for Functional Differential Equations of Mixed Type

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We prove Hopf bifurcation and center manifold theorems for functional differential equations of mixed type. An application to the dynamic behavior of a competitive economy (business cycle) is provided.

**KEY WORDS:** Hopf bifurcation; center manifold; periodic solutions; mixed functional differential equations.

## 1. INTRODUCTION

Mixed functional differential equations (MFDE) are here a special class of functional differential equations where the time derivative depends on both past and future values of the variable. The reader is referred to Rustichini (1989) for a brief discussion of the motivations for the study of such equations.

In this paper, we deal with two aspects of the theory of nonlinear MFDEs: Hopf bifurcation and the center manifold theorem. In a final section, we present an application of this theory to a problem arising in economic theory (existence of business cycles in a competitive economy).

We consider first the Hopf bifurcation. We recall that, broadly speaking, Hopf bifurcation theorems prove the existence of periodic solutions of a nonlinear equation, in the vicinity of a stationary solution, when a conjugate pair of distinct eigenvalues of the linearized equation crosses the imaginary axis. In the proof of the Hopf bifurcation theorem for MFDE, a strategy of proof is necessary that does not involve a solution operator. The argument that we adopt is a purely functional analytic one, involving a Lyapunov–Schmidt reduction (LSR). We set our problem in

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the space of periodic functions of fixed period. The linearization of the stationary solution of our MFDE defines a linear operator acting on this space. The key step, in order to set the LSR, is the proof that the linear operator so defined is a Fredholm operator. This difficulty can be overcome thanks to our choice of the space: a linear operator of mixed type, when its action is restricted to the periodic functions, can be in fact identified with an operator of the delay type. Once this is done, the task is reduced to the study of the zeros of the bifurcation functions.

In Section 5, we deal with the existence of a center manifold. Let us recall briefly the main idea of the center manifold theorem (for ordinary, or delay, equations). One again considers the behavior of a nonlinear equation in the vicinity of a stationary solution; if the characteristic equation of the linearized equation at the stationary solution, has, say, a pair of characteristic roots on the imaginary axis, then this same linear equation has a two-dimensional (2D) subspace of solutions that have exponentially bounded growth.

It is natural to ask whether the nonlinear problem has, at least locally, a 2D submanifold of solutions, homeomorphic to the 2D subspace. The affirmative answer is given in the center manifold theorem. It is important to emphasize that the linear subspace and the submanifold mentioned are both defined in the phase space; the center manifold is locally the graph of a function from (a subspace of) the space of continuous functions on the delay interval into itself. This is very natural in a situation where a continuous semigroup is defined, and the map defining the center manifold is found by means of a variations of constants formula built upon this semigroup [see, for instance, Hale (1977)].

In the case of MFDE, this is no longer possible due to the lack of such continuous semigroups [see Rustichini (1989)]. Therefore, a different strategy is necessary, similar in spirit to the Lyapunov–Schmidt decomposition. First of all, it involves the choice of a space different from the (continuous) functions on the interval $[-r, r]$. We consider the space of continuously differentiable functions defined over the entire real line, with norm weighted by an exponential factor. The linear operator, $M$, defined by the linearization of the MFDE, and the MFDE itself both naturally define a map on this space. The image space is the space of continuous functions defined over the entire real line, with a weighted norm. Once we prove that the dimension of the kernel of $M$ is the same as the number of roots (considering multiplicity) located on the imaginary axis, and that $M$ is surjective, then the implicit function theorem will provide the homeomorphism defining the center manifold. It should be emphasized that our center manifold theorem relies on a fairly strong condition on the linearized operator. We now introduce some notation used in the following.

For $r$ positive real number, we denote $C([-r, r], \mathbb{R}^n)$ the Banach space of continuous functions from the interval $[-r, r]$ to $\mathbb{R}^n$, endowed with the sup norm. When no confusion is possible, we simply denote $C$ such space. Given a continuous function $f: C \to \mathbb{R}^n$, the MFDE is defined as

$$\dot{x}(t) = f(x_t) \tag{1.1}$$

where $x_t(s) \equiv x(t+s)$ for $s \in [-r, r]$. When an initial condition $\phi \in C$, and an interval $[a-r, b+r]$ containing 0, are specified, a solution of the MFDE is a continuously differentiable function $x: [a-r, b+r] \to \mathbb{R}^n$ that satisfies (1.1) for every $t \in [a, b]$ and $x_0 = \phi$. Let now the vector $0 \in \mathbb{R}^n$ be a solution (in a naturally defined way) of (1.1); if $f$ is Frechét differentiable at 0, we identify (Riesz representation theorem) its Frechét derivative at zero by $f'(0)$, with the regular measure induced by a function of bounded variation $\eta$, that is,

$$f'(0)\phi = \int_{-r}^{r} d\eta(\theta)\,\phi(\theta) \qquad \text{for every} \quad \phi \in C. \tag{1.2}$$

We associate with this $L$ the characteristic matrix $\Delta(s) \equiv sI - \int_{-r}^{r} e^{s\theta}\, d\eta(\theta)$.

## 2. THE ADJOINT OPERATOR

Our proof of the Hopf bifurcation theorem requires some preliminary construction. Since the analysis closely follows the one for delay equations [as in Hale (1977)], the presentation will be sketchy.

The purpose of this section is to study an operator adjoint to the operator $A$, where $A$ is defined as

$$(A\phi)(s) = \dot{\phi}(s) \qquad s \in [-r, r]$$

$$D(A) = \left\{\phi \in C^1([-r, r], \mathbb{R}^n) \colon \dot{\phi}(0) = \int_{-r}^{r} d\eta(\theta)\,\phi(\theta)\right\}.$$

We follow Hale (1977, Sections 7.3 and 7.6) for this purpose. The domain of such adjoint operators will be contained in the space

$$C^* \equiv C([-r, r], (\mathbb{R}^n)^T)$$

where $T$ denotes transposition.

We define, for any $\alpha \in C^*$

$$(\alpha, \psi) \equiv \alpha(0)\,\psi(0) - \int_{-r}^{r} \int_{0}^{\theta} \alpha(\xi - \theta)\, d\eta(\theta)\, \psi(\xi)\, d\xi.$$

We want to determine an operator $A^*$ with domain $D(A^*)$ that satisfies

$$(\alpha, A\phi) = (A^*\alpha, \phi) \qquad \text{for} \quad \phi \in D(A), \qquad \alpha \in D(A^*).$$

We have that (integration by parts)

$$\begin{aligned}
(\alpha, A\phi) &= \alpha(0) \int_{-r}^{r} d\eta(\theta)\, \phi(\theta) \\
&\quad - \int_{-r}^{r} \left\{ [\alpha(\xi-\theta)\, d\eta(\theta)\, \phi(\xi)]_0^\theta (\xi) + \int_0^\theta \dot\alpha(\xi-\theta)\, d\eta(\theta)\, \phi(\xi)\, d(\xi) \right\} \\
&= \alpha(0) \int_{-r}^{r} d\eta(\theta)\, \phi(\theta) - \int_{-r}^{r} \alpha(0)\, d\eta(\theta)\, \phi(\theta) + \int_{-r}^{r} \alpha(-\theta)\, d\eta(\theta)\, \phi(0) \\
&\quad + \int_{-r}^{r} \int_0^\theta \dot\alpha(\xi-\theta)\, d\eta(\theta)\, \phi(\xi)\, d\xi \\
&= \left[ \int_{-r}^{r} \alpha(-\theta)\, d\eta(\theta) \right] \phi(0) - \int_{-r}^{r} \int_0^\theta [-\dot\alpha(\xi-\theta)]\, d\eta(\theta)\, \phi(\xi)\, d\xi \\
&\equiv (A^*\alpha, \phi).
\end{aligned}$$

It is therefore natural to define $A^*$ as

$$A^*\alpha(s) = -\dot\alpha(s)$$

$$D(A^*) = \left\{ \alpha \in C^1([-r, r], (\mathbb{R}^n)^T) \colon -\dot\alpha(0) = \int_{-r}^{r} \alpha(-\theta)\, d\eta(\theta) \right\},$$

and the formal adjoint equation as

$$\dot y(t) = \int_{-r}^{r} y(t-\theta)\, d\eta(\theta) \equiv L^* y^t$$

where $y^t(u) \equiv y(t+u) \in (\mathbb{R}^n)^T$ for $s \in [-r, r]$. Then we have, as in Hale (1977, p. 175), that, for any complex number $s$, $s \in \sigma(A)$ if and only if $s \in \sigma(A^*)$.

**Remark.** Since $\sigma(A) = P\sigma(A) = \sigma(A^*) = P\sigma(A^*)$, we see that $s \in \sigma(A^*)$ if and only if there exists a $b \neq 0$ such that

$$b \left[ sI + \int_{-r}^{r} e^{-\lambda\theta}\, d\eta(\theta) \right] = b(-\Delta(-s)) = 0.$$

***Lemma 2.1.*** *Let $y$ be a solution of the adjoint equation, and $x$ a solution of $\dot{x}(t) = Lx_t + f(t)$. Then if we define*

$$(y^t, x_t) \equiv y^t(0) \cdot x_t(0) - \int_{-r}^{r} \int_{0}^{\theta} y^t(\xi - \theta)\, d\eta(\theta)\, x_t(\xi)\, d\xi,$$

*one has*

$$(y^t, x_t) = \int_{\sigma}^{t} y(s) f(s)\, ds + (y^\sigma, x_\sigma), \qquad t \geqslant \sigma.$$

**Proof.** In fact, since

$$\dot{y}(s) = -\int_{-r}^{r} y(s - \theta)\, d\eta(\theta), \qquad \dot{x}(s) = \int_{-r}^{r} d\eta(\theta)\, x(s - \theta) + f(s),$$

and since

$$\frac{d}{ds}(y \cdot x)(s) = y(s) f(x) + y(s) Lx_s + L^* y^s x(s),$$

it is enough to prove that in the equality:

$$\begin{aligned}(y^t, x_t) - (y^\sigma, x_\sigma) - \int_{\sigma}^{t} y(s) f(s)\, ds = \int_{\sigma}^{t} [y(s) Lx_s + Ly^s x(s)]\, ds \\ + \int_{-r}^{r} \int_{0}^{t} y^\sigma(\xi - \theta)\, d\eta(\theta)\, x_\sigma(\xi)\, d\xi \\ - \int_{-r}^{r} \int_{0}^{\theta} y^t(\xi - \theta)\, d\eta(\theta)\, x_t(\xi)\, d\xi\end{aligned}$$

the right-hand term is zero, and this follows from an easy calculation. ∎

## 3. THE PROJECTION OPERATORS

We let $\Lambda = \{s_1, ..., s_n\}$ be a finite set of eigenvalues of the linear autonomous equation (1.2); $\Phi_\Lambda = (\Phi_{s_1}, ..., \Phi_{s_n})$ where $\Phi_{s_i} = (\phi_1^i, ..., \phi_{m_i}^i)$ is a basis for the generalized eigenspace of $s_i$; $\Psi_\Lambda = (\Psi_{s_1}, ..., \Psi_{s_n})$ where $\Psi_{s_i} = (\psi_1^i, ..., \psi_{m_i}^i)$ is a basis for the generalized eigenspace of $s_i$ for the formal adjoint; $B = \operatorname{diag}(B_1, ..., B_n)$ the matrix defined by $A\Phi_{s_i} = \Phi_{s_i} B_i$ and $A^* \Psi_{s_i} = B_{s_i}^* \Psi_{s_i}$ where $B^*$ is the transpose of $B$, and assume $\Psi_\Lambda$ to be normalized so that the $(d \times d)$ matrix $(\Phi_\Lambda, \Psi_\Lambda)$ satisfies

$$(\Phi_\Lambda, \Psi_\Lambda) = I_d$$

where $I_d$ is the identity matrix on $\mathbb{R}^d$, with $d = \sum_{i_1}^{n} m_i$.

Let also $\Gamma_\Lambda$ be any closed contour in the complex plane surrounding the set of eigenvalues $\Lambda$ disjoint from $\sigma(A)$.

Then we define $P_\Lambda: C \to C$ as

$$P_\Lambda(\phi) \equiv \frac{1}{2\pi i} \int_{\Gamma_\Lambda} R(A : s)\phi \, ds.$$

One can see [the argument is similar to the one in Banks and Manitius (1975)]:

***Lemma 3.1.*** *For $P_\Lambda$, $\Phi_\Lambda$, $\Psi_\Lambda$, as above one has*

$$P_\Lambda(\phi) = \Phi_\Lambda(\Psi_\Lambda, \phi).$$

We shall use the notation $\phi^\Lambda \equiv P_\Lambda(\phi)$, for $\phi \in C$.

Let now $x$ be a solution of the nonhomogeneous equation

$$\dot{x}(t) = Lx_t + f(t);$$

and denote

$$y(t) \equiv (\Psi_\Lambda, x_t).$$

Let $T_\Lambda$ be the continuous semigroup associated with the set $\Lambda = \{s_1, ..., s_n\}$ [see Rustichini (1989) for details]. We then have the following:

***Lemma 3.2.*** *Let $x$, $y$ as defined above. Then,*

*1.* $P_\Lambda(x_t)(\theta) = \int_\sigma^t T_\Lambda(t-s)\Phi_\Lambda(\theta)\Psi_\Lambda(0)f(s)\,ds + T_\Lambda(t-\sigma)\Phi_\Lambda(\theta)(\Psi_\Lambda(0), \phi)$, *and*

*2.* $y$ *is the solution of the ordinary differential equation*

$$\dot{y}(t) = B_\Lambda y(t) + \Psi_\Lambda(0) f(t), \qquad \textit{with} \quad y(0) = (\Psi_\Lambda, \phi).$$

**Proof.** The proof is as in Hale (1977):

$$\begin{aligned}
P_\Lambda(x_t) \equiv x_t^\Lambda &= \Phi_\Lambda(\Psi_\Lambda, x_t) \\
&= \Phi_\Lambda e^{B_\Lambda t}\left[\int_\sigma^t e^{-B_\Lambda s}\Psi_\Lambda(0) f(s)\,ds + (e^{-B_\Lambda \sigma}\Psi_\Lambda, x_\sigma)\right] \\
&= \int_\sigma^t \Phi_\Lambda e^{B_\Lambda(t-s)}\Psi_\Lambda(0) f(s)\,ds + \Phi_\Lambda e^{B_\Lambda(t-s)}(\Psi_\Lambda, x_\sigma) \\
&= \int_\sigma^t T_\Lambda(t-s)\,\Phi_\Lambda\Psi_\Lambda(0) f(s)\,ds + T_\Lambda(t-\sigma)\,\phi_\Lambda.
\end{aligned}$$

From the above equalities also follows that

$$y(t) = e^{B_A(t-\sigma)} y(\sigma) + \int_\sigma^t e^{B(t-s)} \Psi_A(0) f(s)\, ds,$$

which proves the second claim. ∎

## 4. HOPF BIFURCATION FOR MFDE

In the following, we shall study Hopf bifurcation for solutions of the equations

$$\dot{x}(t) = F(x_t, \alpha)$$

where $\alpha$ is a real parameter.

In this section, the most natural spaces to work with will be the spaces of periodic functions. We shall denote:

$$\begin{aligned} C_{2\pi} &= C_{2\pi}(\mathbb{R}, \mathbb{R}^n) \\ &\equiv \{f\colon \mathbb{R} \to \mathbb{R}^n\colon f \text{ is continuous and } f(t) = f(t+2\pi) \text{ for every } t\} \end{aligned}$$

and

$$C^1_{2\pi} = C^1_{2\pi}(\mathbb{R}, \mathbb{R}^n) \equiv \{f \in C_{2\pi}\colon f \text{ is continuously differentiable}\}.$$

We shall also consider the inner product defined on $C_{2\pi}$ as

$$\langle f, g\rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} f(s)\, \overline{g(s)}\, ds.$$

Two functions are orthogonal, $f \perp g$, if and only if $\langle f, g\rangle = 0$.

As usual

$$C([-r, r], \mathbb{R}^n) = C.$$

We assume $F\colon C \times \mathbb{R} \to \mathbb{R}^n$ is of class $C^2$ in both variables [see Chow and Hale (1982)]. We denote the Frechèt derivative of the function $F(\cdot, \alpha)$, evaluated at $\phi$, as $F'(\phi, \alpha)$, so that

$$F'(\phi, \alpha)\colon C \to \mathbb{R}^n$$

and we write its representation as

$$F'(\phi, \alpha)(v) = \int_{-\sigma}^{\sigma} d\eta(\phi, \alpha)(\theta)\, v(\theta) \equiv L(\phi, \alpha)(v).$$

We shall study the behavior of the solution in the neighborhood of a stationary solution $c$ at $\alpha = 0$, that is, a constant $c$ such that $F(c, 0) = 0$. We assume that there exists a continuous function $\hat{c}$: $(-\alpha_0, \alpha_0) \to \mathbb{R}^n$ such that $F(\hat{c}(\alpha), \alpha) = 0$, $\alpha \in (-\alpha_0, \alpha_0)$. Now simply redefining $x$ and $F$, we may assume without loss of generality that

$$F(0, \alpha) = 0 \quad \text{for} \quad \alpha \in (-\varepsilon, \varepsilon), \qquad \varepsilon > 0.$$

We shall therefore simplify the notation above, dropping the dependence on $\phi$ of the derivative $F'$; so $\eta(0, \alpha)$ will be written as $\eta(\alpha)$, and $L(0, \alpha)$ as $L(\alpha)$; analogously, when $\alpha = 0$, we shall write $\eta(0)$ as $\eta$ and $L(0)$ as $L$.

We also recall a few standard definitions. Let $X$ and $Y$ be Banach spaces. A bounded linear operator $M$: $X \to Y$ is called a Fredholm operator if the following two conditions hold:

(i) The Kernel of $M$, Ker $M$, is finite dimensional.

(ii) The range of $M$, is a subspace that is both closed and of finite codimension.

Also we say that, if $M$ is Fredholm, the index of $M$ is the integer

$$i(M) \equiv \dim \operatorname{Ker} M - \operatorname{codim} \operatorname{range} M.$$

Our proof will use a Lyapunov–Schmidt reduction (LSR); we recall here, for convenience of the reader and to clarify the notation, the major steps of the LSR [see, for example, Chow and Hale (1982) or Golubitsky and Schaeffer (1985)].

Let $X$, $Y$ be Banach spaces, and $\Phi$ a $C^2$ map $\Phi \cdot X \times \mathbb{R} \to Y$, $\Phi(0, 0) = 0$. The LSR is used to solve the equation

$$\Phi(u, \alpha) = 0 \tag{4.1}$$

for $u$ as a function of the parameter $\alpha$ near $(0, 0)$. We denote $M = M(0)$ the Frechèt derivative of $\Phi$ with respect to $u$ at $\alpha = 0$; we assume that $M$ is Fredholm of index zero, and $Y$ has an inner product $\langle\ ,\ \rangle$ with respect to which an orthogonality $\perp$ is defined.

*Step 1.* Decompose $X$ and $Y$ as

$$X = \operatorname{Ker} M \oplus P$$

$$Y = Q \oplus \operatorname{range} M.$$

*Step 2.* With $E$ the projection of $Y$ onto the range $M$ along $Q$, we write Eq. (4.1) as the equivalent system:

$$\begin{aligned} E\Phi(u, \alpha) &= 0 \\ (I - E)\, \Phi(u, \alpha) &= 0. \end{aligned} \tag{4.1a}$$

*Step 3.* Write $u \in X$ as $u = v + w$, $v \in \operatorname{Ker} M$, $w \in P$; solve $E\Phi(v + w, \alpha) = 0$ by the implicit function theorem to determine the map $W$: $\operatorname{Ker} M \times \mathbb{R} \to P$ such that

$$E\Phi(v + W(v, \alpha)) = 0.$$

*Step 4.* Define $\phi$: $\operatorname{Ker} M \times \mathbb{R} \to Q$ by

$$\phi(v, \alpha) \equiv (I - E)\, \Phi(v + W(v, \alpha), \alpha).$$

*Step 5.* Choose a basis $v_1, ..., v_n$ for $\operatorname{Ker} M$ and a basis $v_1^*, ..., v_n^*$ for $(\text{range } M)^\perp$, and let $g$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be defined by

$$g: (x_1, ..., x_n; \alpha) \mapsto (\langle v_i^*, \phi(x_1 v_1 + \cdots + x_n v_n, \alpha)\rangle)_{i=1}^n.$$

Since $\phi(v, \alpha) \in Q$ for every $v$, we have that $\phi(v, \alpha) = 0$ if and only if $g_i(x, \alpha) = 0$, $i = 1, ..., n$. We conclude the following:

***Theorem 4.1*** *(LSR). Let M be a Fredholm operator of index zero. Then the solutions of (1.1) are locally in a one-to-one correspondence with solutions of the system*

$$g_i(x, \alpha) = 0 \qquad i = 1, ..., n,$$

*for the map g defined in Step 5 above.*

We also recall that the group $S^1$ is said to act on the space $C_{2\pi}$ as

$$(\theta \cdot f)(t) = f(t - \theta) \qquad \text{for} \quad \theta \in [0, 2\pi) \simeq S^1.$$

A subspace $P$ of $C_{2\pi}$ is said to be invariant under this group action if $f \in P$ implies $\theta \cdot f \in P$ for all $\theta$. A map $\phi$ defined from such invariant subspace to $C_{2\pi}$ is said to commute with the action of $S^1$ on $C_{2\pi}$ if $\phi(\theta, f) = \theta \cdot \phi(f)$ for all $\theta$.

In the following, we shall consider the bifurcation that takes place when a pair of eigenvalues of the characteristic equation crosses the imaginary axis, as the parameter $\alpha$ varies. More precisely, let one pair of the eigenvalues satisfy $z(\alpha) = x(\alpha) \pm iy(\alpha)$ and $x(0) = 0$, $y(0) \neq 0$. Note that, with a time rescaling, we can assume $y(0) = 1$.

As usual, we shall reduce the search for a periodic solution to the search of a solution in a class of functions with fixed period $(2\pi)$, by introducing an additional parameter. For a real number $\tau$, let $u(\cdot) = x(\cdot/(1 + \tau))$; then we have that, if $\dot{x}(t) = F(x(t + \cdot), \alpha)$, then $\dot{u}(t) = (1/(1 + \tau))\, F((u(t + \cdot(1 + \tau)), \alpha)$. Note that, if $x$ is a periodic function of period $2\pi/(1+\tau)$, then $u$ is a periodic function of period $2\pi$. We shall therefore study the operator $\Phi$: $C^1_{2\pi} \times \mathbb{R} \times \mathbb{R} \to C_{2\pi}$, defined by

$$\Phi(u, \alpha, \tau)(t) \equiv (1+\tau)\,\dot{u}(t) - F(u_t(\cdot(1+\tau)), \alpha)$$

where $u_t(\cdot(1+\tau)) \in C([-\sigma/(1+\tau), \sigma/(1+\tau)], \mathbb{R}^n)$ is defined by naturally restricting $u$ to the interval $[t-\sigma, t+\sigma]$; i.e.,

$$u_t(s(1+\tau)) \equiv u(t+s(1+\tau)), \qquad \left(s \in \left[-\frac{\sigma}{1+\tau}, \frac{\sigma}{1+\tau}\right]\right).$$

It is clear that $u$ is a $2\pi$-periodic solution of

$$(1+\tau)\,\dot{u}(t) = F(u_t(\cdot(1+\tau)), \alpha) \qquad \text{for every} \quad t \in \mathbb{R}$$

if and only if

$$\Phi(u, \alpha, \tau) = 0.$$

Rescaling $(\theta \cdot u)(t) \equiv u(t-\theta)$, we have

$$\begin{aligned}\Phi(\theta \cdot u, \alpha, \tau)(t) &= (1+\tau)\,\dot{u}(t-\theta) - F(u_{t-\theta}(\cdot(1+\tau)), \alpha)\\ &= \theta \cdot \Phi(u, \alpha, \tau)(t).\end{aligned}$$

That is, the operator $\Phi$ commutes with the group action of $S^1$. This observation will play an important role in what follows. In fact, if $\Phi$ satisfies such conditions and the subspaces $P$ and $Q$ are invariant, under the action of $S^1$, then the map $\phi$: Ker $M \times \mathbb{R} \to Q$, defined in Step 4, also commutes with the action of $S^1$. [See Propositiion VII, 3.3, of Golubitsky and Schaeffer (1985).]

We denote the characteristic matrix, at $\alpha = 0$, as

$$\Delta(s) \equiv sI - \int_{-\sigma}^{\sigma} d\eta(0)(\theta)\, e^{s\theta} = sI - Le^{s\cdot},$$

and the characteristic equation: $\det(\Delta(s)) = 0$.

The simple eigenvalue assumptions are

(E1) The characteristic equation has simple eigenvalues $\pm i$.

(E2) There are no other eigenvalues with $\text{Re}(s) = 0$.

We now apply the LSR to the system and we derive the following:

***Theorem 4.2.*** *If the MFDE system*

$$\dot{x}(t) - F(x_t, \alpha) = 0 \tag{4.2}$$

*satisfies E1 and E2, then there exists a continuous function* $g(x, \alpha)$ *of the form*

$$g(x, \alpha) = r(x^2, \alpha)x, \qquad r(0, 0) = 0$$

*such that locally solutions to* $g(x, \alpha) = 0$ *with* $x \geqslant 0$ *are in a one-to-one correspondence with orbits of solutions to (4.2).*

**Proof.** The proof of the theorem is organized in a sequence of auxiliary lemmas and theorems. We first study the Frechèt derivative of $\Phi(\cdot, 0, 0)$ evaluated at the constant function 0:

$$\Phi_1(0, 0, 0) \equiv M$$

where $M\colon C^1_{2\pi} \to C_{2\pi}$ is defined by

$$(Mf)(t) \equiv \dot{f}(t) - \int_{-\sigma}^{\sigma} d\eta(\theta)\, f(t+\theta).$$

A basic condition for the LSR process to apply is that the operator $M$ is Fredholm. This will follow quite simply from the fact that operators of the delay type have such property [see Hale (1977), Chapter 6]. We begin therefore with the observation that, on the space $C^1_{2\pi}$, operators of mixed type can be identified with delay operators.

***Lemma 4.1.*** *Let* $M$, $\eta$ *be defined as above. Then there exists a real number* $R > 0$ *and a matrix valued function of bounded variation* $\bar{\eta}$ *such that*

$$(Mf)(t) = \dot{f}(t) - \int_{-R}^{0} d\bar{\eta}(\theta)\, f(t+\theta)$$

*for any* $f \in C^1_{2\pi}$.

**Proof.** Find $K \equiv \min\{k\colon \pi k \geqslant \sigma, \quad k = 1, 2, \ldots\}$, set $R = 2\pi k$ then $f(R + t + \theta) = f(t + \theta)$ and $-R + \sigma < -\sigma$ while

$$\begin{aligned}(Mf)(t) &\equiv \dot{f}(t) - \int_{-\sigma}^{\sigma} d\eta(\theta)\, f(t+\theta) \\ &= \dot{f}(t) - \int_{-\sigma}^{0+} d\eta(\theta)\, f(t+\theta) - \int_{-R}^{-R+\sigma} d\eta(R+\theta)\, f(t+\theta) \\ &\equiv \dot{f}(t) - \int_{-R}^{0} d\bar{\eta}(\theta)\, f(t+\theta),\end{aligned}$$

where $d\bar{\eta}$ is defined by

$$\int_{-R}^{0+} d\bar{\eta}(\theta)\, f(t+\theta) = \int_{-\sigma}^{0} d\eta(\theta)\, f(t+\theta) + [\eta(0+) - \eta(0-)]\, f(0)$$
$$+ \int_{-R}^{-R+\sigma} d\eta(R+\theta)\, f(t+\theta).$$

Let us now define the formal adjoint of $M$:

$$M^*: C^1_{2\pi} \to C_{2\pi}$$
$$M^*(f)(t) \equiv \dot{f}(t) + \int_{-R}^{0} f(t-\theta)\, d\bar{\eta}(\theta).$$

We now proceed to define the direct sum decomposition of the two spaces $C^1_{2\pi}$ and $C_{2\pi}$.

***Theorem 4.3.*** *Assume E1 and E2; then,*

*1.* dim Ker $M = 2$.

*2.* *$M$ is a Fredholm operator of index 0.*

*3.* *There exists a basis $v_1$, $v_2$ for* Ker $M$ *such that, after we identify* Ker $M$ *with $\mathbb{R}^2$ (with the map $(x, y) \mapsto xv_1 yv_2$), then the action of $S^1$ on* Ker $M$ *is given by $\theta \cdot (x, y) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.*

*4.* $C^1_{2\pi} =$ Ker $M \oplus$ *(range $M \cap C^1_{2\pi}$)* $\equiv$ Ker $M \oplus P$
$C^0_{2\pi} = Q \oplus$ *range* $M \equiv Q \oplus$ (Ker $M^*)^{\perp}$.

**Proof.** Recall that we have defined

$$\Delta(s) = sI - \int_{-\sigma}^{\sigma} d\eta(0, \theta)\, e^{s\theta}.$$

When $s = \pm i$, we have by assumption E1 det $\Delta(\pm i) = 0$. Note that, in this case,

$$\Delta(\pm i) = \pm iI - \int_{-R}^{0} d\bar{\eta}(0, \theta)\, e^{\pm i\theta}$$

because $e^{\pm i\theta}$ is a $2\pi$-periodic function. By assumption, there exist two (conjugate) eigenvectors $c$, $\bar{c}$ such that (after suitable normalization)

$$\Delta(i)c = 0 = \Delta(-i)\bar{c}.$$

Define now the two functions $v_j: \mathbb{R} \to \mathbb{R}^n$, $j = 1, 2$ as follows:

$$v_1(s) \equiv \mathrm{Re}(e^{is}c), \qquad v_2(s) = \mathrm{Im}(e^{is}c).$$

By assumption, $\pm i$ are simple eigenvalues; then, from Lemma 3.5, Section 7.3, of Hale (1977), and Statement 1 above, it now follows that $\dim \mathrm{Ker}\, M = 2$. (Recall that we are here considering the restriction of $M$ to $C^1_{2\pi}$, so the eigenfunctions associated with the other eigenvalues are not in $\mathrm{Ker}\, M$.) Statement 3 is now proved as in Golubtsky–Schaeffer (1985, p. 346).

We now proceed to the proof of the second statement. $M$ is clearly a continuous and linear operator from $C^1_{2\pi}$ to $C_{2\pi}$.

From Theorem 1.2, Section 9.1, of Hale (1977), we have that, given $f \in C_{2\pi}$, there exists a $\phi \in C^1_{2\pi}$ such that $M\phi = f$, if and only if $\langle f, \psi \rangle = 0$ for every $\psi \in \mathrm{Ker}\, M^*$. In other words, we have the Fredholm alternative equivalent for our case:

$$\text{range } M = (\mathrm{Ker}\, M^*)^{\perp}.$$

In addition, from the same theorem, we derive the existence of a continuous projection $J: C_{2\pi} \to C_{2\pi}$ such that

$$\text{range } M = (I - J)\, C_{2\pi}.$$

This implies that range $M$ is closed. Finally, in order to prove that $i(M) = 0$, we recall the definition of the projection $J$. Let $V = \mathrm{col}(v_1, ..., v_d)$, a basis for the $2\pi$-periodic solutions of the adjoint equation. Then ($T$ denotes transposition),

$$Jf \equiv (v_1, ..., v_d) \left( \int_0^{2\pi} V(s)\, V^T(s)\, ds \right)^{-1} \int_0^{2\pi} V(s)\, f(s)\, ds.$$

Now the fact that $d = 2$ follows from the fact that the characteristic equation for the adjoint operator:

$$\det \left( sI + \int_{-R}^{0} e^{-s\theta}\, d\bar{\eta}(\theta) \right) = \det(-\Delta(-s)) = (-1)^n \det \Delta(-s)$$

has roots $\pm i$ that are simple. Now a second appeal to Lemma 3.5, Section 7.3, of Hale (1977) proves the claim.

Let now $b$ be a vector in the left null space of the matrix $\Delta(i)$, and let $V^* = (v_1^*(s), v_2^*(s)) = \mathrm{col}(\mathrm{Re}\, be^{-is})$. These two functions provide a basis for $\mathrm{Ker}\, M^*$, and Statement 4 now follows from the Fredholm alternative theorem. ∎

**Remark.** Note that the $\phi$ defined in Step 4 of LSR is in our case defined as $\phi$: Ker $M \times \mathbb{R}^2 \to$ Ker $M$, a map between spaces of finite dimension. For a fixed basis, a fixed $\alpha$ and $\tau$, it will be identified with a map between Euclidean spaces; in our case, $\phi(\cdot, \alpha, \tau): \mathbb{R}^2 \to \mathbb{R}^2$.

Furthermore $P$ and $Q$ are clearly $S^1$-invariant subspaces. Therefore, as mentioned above, the commutativity property of the operator $\Phi$ with respect to the action of $S^1$ is inherited by $\phi$; that is,

$$\phi(\theta \cdot v, \alpha, \tau) = \theta \cdot \phi(v, \alpha, \tau) \qquad \text{for} \quad \theta \in S^1.$$

We now proceed to show some basic property of this function $\phi$.

***Lemma 4.2.*** *Let $x$, $y$ be the coordinates on* Ker $M$ *defined in proposition. Then there exist functions $p_1, p_2: \mathbb{R}^3 \to \mathbb{R}$ such that $\phi(\cdot; \alpha, \tau): \mathbb{R}^2 \to \mathbb{R}^2$ is of the form*

$$\phi(x, y; \alpha, \tau) = p_1(x^2 + y^2, \alpha, \tau) \begin{pmatrix} x \\ y \end{pmatrix} + p_2(x^2 + y^2; \alpha, \tau) \begin{pmatrix} -y \\ x \end{pmatrix}$$

*with*

*(a)* $p_1(0, 0, 0) = 0$

*(b)* $p_2(0, 0, 0) = 0$

*(c)* $p^1_\tau(0, 0, 0) \equiv (\partial/\partial\tau)\, p_1(0, 0, 0) = 0$

*(d)* $p^2_\tau(0, 0, \tau) \equiv (\partial/\partial\tau)\, p_2(0, 0, 0) = -1.$

**Proof.** The first statement, together with (a) and (b), are proved in Golubitsky–Schaeffer (1985, pp. 347–348). Here, the property of $\phi$ that is being used is the commutativity with the action of $S^1$.

We now proceed with the proof of (c) and (d). Notice that

$$\begin{aligned} \phi_j(x, 0, \alpha, \tau) &= \langle v_j^*, \Phi(xv_1 + W(xv_1, \alpha, \tau), \alpha, \tau)\rangle \\ &= p_j(x^2, \alpha, \tau)x \qquad j = 1, 2. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{x \to 0} \phi_j(x, 0, \alpha, \tau)/x &= \left\langle v_j^*, \lim_{x \to 0} \frac{\Phi(xv_1 + W(xv_1, \alpha, \tau), \alpha, \tau)}{x} \right\rangle \\ &= p_j(0, \alpha, \beta). \end{aligned}$$

We now compute the limit inside the inner product. By definition of $\Phi$,

$$\begin{aligned} \Phi(xv_1 + W(xv_1, \alpha, \tau)\alpha, \tau) &= (1 + \tau)\, x\dot{v}_1 + (1 + \tau)\, DW(xv_1, \alpha, \tau) \\ &\quad - F([xv_1 + W(xv_1, \alpha, \tau)]_{t,\tau}, \alpha) \end{aligned}$$

where $D$ denotes derivative with respect to time. Therefore, the limit that we are studying is

$$(1+\tau)\,\dot{v}_1 - F'(W(0,\alpha,\tau),\alpha)\,v^1_{t,\tau} - F'(W(0,\alpha,\tau))(W_1(0,\alpha,\tau))_{t,\tau}$$
$$= (1+\tau)\,L(0)\,v^1_t - L(\alpha)\,v^1_{t,\tau} + o(\alpha).$$

In the last equality, we have used the fact that $\dot{v}_1(t) = L(0)\,v^1_t$, because $v_1$ is a solution of the linear autonomous equation; that $W(0,\alpha,\beta)=0$; and finally that

$$W_1(0,0,0)=0.$$

We conclude that

$$\lim_{x\to 0}\frac{\Phi(xv_1 + W(xv_1,\alpha,\tau)\alpha,\tau)}{x} = (1+\tau)\,L(0)\,v^1_t - L(\alpha)\,v^1_{t,\tau} + O(\alpha).$$

The proof can be concluded with an easy computation, identical to the one in Hale (1979, p. 165). ∎

We can now conclude the proof of Theorem 4.2. At this stage, we only need to recall the argument of Golubitsky–Schaeffer (1985, Chapter 8).

From the explicit form of $\phi$ given in Lemma 4.2, we have that $\phi=0$ if and only if either $x=y=0$ or $p_1=p_2=0$. We can restrict ourselves so the case $y=0$, $x\geqslant 0$, with a rotation, and therefore look for solutions of

$$x\geqslant 0, \qquad p_1(x^2,\alpha,\tau)=p_2(x^2,\alpha,\tau)=0.$$

Near the origin, we can solve for $\tau=\tau(x^2,\alpha)$ from the equation $p_1(x^2,\alpha,\tau)=0$, by using implicit function theorem and Lemma 4.2. Then define $r(t,\alpha)\equiv p_1(t,\alpha,\tau(t,\alpha))$ and $g(x,\alpha)\equiv r(x^2,\alpha)x$. Clearly all solutions of $\phi=0$ are locally in one-to-one correspondence with solutions of $g(x,\alpha)=0$ and in turn with solutions of $\Phi(u,\alpha,\tau)=0$; that is, periodic solutions of the original nonlinear equation. This completes the proof of Theorem 4.2.

We introduce now our final condition:

(E3) Eigenvalue crossing condition: $\operatorname{Re}\lambda'(0)\neq 0$.

We now note that we may assume without loss of generality that there exist two vector-valued functions $d(\alpha)$, $c(\alpha)$ such that

(i) $b(0)=b, \qquad c(0)=c$

(ii) $b(\alpha)\,\Delta(\alpha,\lambda(\alpha))\equiv 0$

(iii) $\Delta(\alpha,\lambda(\alpha))\,c(\alpha)\equiv 0$

(iv) $b(\alpha)\,\Delta_\lambda(\alpha,\lambda(\alpha))\,c(\alpha)\equiv 1.$

***Theorem 4.4.*** *Let the system of MFDE $\dot{x}(t) = F(x_t, \alpha)$ satisfy*

1. *the simple eigenvalue conditions (E1) and (E2), and*
2. *the eigenvalue crossing condition (E3);*

*then there is a one parameter family of periodic solutions bifurcating from the steady-state solution at $\alpha = 0$.*

**Proof.** We have reduced the study of the existence of periodic solutions to the study of the zeros of the map $g(x, \alpha)$.

We recall that the map $g(x, \alpha)$ is of the form $r(x^2, \alpha)x$ for some function $r$; the nontrivial solutions of $g(x, \alpha) = 0$ are given by $r(x^2, \alpha) = 0$. If we now prove that

$$r_\alpha(0, 0) \neq 0, \tag{4.3}$$

then, from the implicit function theorem, we shall be able to solve

$$\alpha = \mu(x^2), \tag{4.4}$$

which determines a one-parameter family of periodic solutions.

We have now only to check Statement 1 above. From the definition

$$r(t, \alpha) = p_1(t, \alpha, \tau(t, \alpha)),$$

we find

$$\begin{aligned} r_\alpha(0, 0) &= p_{1\alpha}(0, 0, 0) + p_{1\tau}(0, 0, 0)\, \tau_\alpha(0, 0, 0) \\ &= p_{1\tau}(0, 0, 0) \end{aligned}$$

because we have seen above that $p_{1\tau}(0, 0, 0) = 0$.

On the other hand,

$$\frac{\partial^2 \phi_1}{\partial \alpha\, \partial x} = \langle v_1^*, \Phi'_\alpha(v_1) - \Phi''(v_1, L^{-1} E \Phi_\alpha) \rangle.$$

We now proceed to show that the inner product above has, in our case, a simpler form. Our operator $\Phi$ is defined as $\Phi(u, \alpha, \tau)(t) = (1 + \tau)\, \dot{u}(t) - F(u_t(\cdot (1 + \tau)), \alpha)$ and therefore

$$\Phi(u, \alpha, \tau)(t) = \frac{\partial F}{\partial \alpha}(u_t(\cdot (1 + \tau)), \alpha).$$

Now, since $F(0, \alpha) = 0$ in an open interval $(-\alpha_0, \alpha_0)$,

$$\Phi(u, \alpha, \tau)(t) = 0 \qquad \text{and} \qquad \frac{\partial F'}{\partial \alpha} v_1(t) = \int_{-\sigma}^{\sigma} d\eta_\alpha(\theta)\, v_1(t + \theta) \equiv L_\alpha v_1(t)$$

so that

$$\frac{\partial^2 \phi_1}{\partial \alpha \, \partial x} = \langle v_1^*, L_\alpha v_1 \rangle.$$

We now proceed to compute this last term in our case:

$$\langle v_1^*, L_\alpha(\alpha) \, v_1 \rangle = \frac{1}{4} \left\{ b, \int_{-\sigma}^{\sigma} d\eta_\alpha(\alpha, \theta) \, e^{-i\theta} \bar{c} + b, \int_{-\sigma}^{\sigma} d\eta_\alpha(\alpha, \theta) \, e^{i\theta} c \, dt \right\}$$

so $p_\alpha(0, 0, 0) = \frac{1}{2} \operatorname{Re}\{b, \int_{-\sigma}^{\sigma} d\eta_\alpha(\alpha, \theta) \, e^{-i\theta} \bar{c}\} = -\frac{1}{2} \operatorname{Re} \lambda'(0) \neq 0$ by assumption. ▮

## 5. CENTER MANIFOLD FOR MFDE

We now consider the situation in which the characteristic equation $\Delta(z)$ has a finite number of roots in a strip $\{s \in \mathbb{C}: |\operatorname{Re} s| < \gamma\}$. Our purpose is to prove the existence of a center manifold for MFDE.

The standard approach to this problem is based on a suitable variation of constants formula, and a decomposition of the space in stable, unstable, and center subspaces. Such decomposition is presented in Rustichini (1989), but unfortunately a variation of constants formula analogous to the one valid for ordinary differential equations, or delay equations [see, for instance, Hale (1977), Section 6.2], does not seem possible in our case. The fundamental reason is that our problem is, in general, ill-posed.

The strategy that we shall follow will be of working with functions defined are the entire real line, and defining our center manifold map through an application of the implicit function theorem. Before we proceed, we introduce some additional notation.

We again consider the equation

$$\dot{x}(t) = f(x_t, \alpha). \tag{5.1}$$

We assume $f$ to be Frechét continuously differentiable, and this derivative to be continuous with respect to the parameter $\alpha$. We can therefore rewrite (5.1) as

$$\dot{x}(t) = L(\alpha) \, x_t + F(x_t, \alpha), \tag{5.1$'$}$$

where, with $x_t$, we denote a function in $C([-\tau, \tau], \mathbb{R}^n)$. We assume $f(0, \alpha) = 0$, for $\alpha \in (-\alpha_0, \alpha_0)$, and shall denote, for $\alpha = 0$,

$$L\phi = L(0)\phi = \int_{-\tau}^{\tau} d\eta(\theta) \, \phi(\theta)$$

where $L = F'(0, 0)$, the Frechét derivative of $f(\cdot, 0)$ evaluated at 0.

We are interested in the case in which, for $\alpha = 0$, the set of roots of the characteristic equation, $\Lambda$, can be partitioned into $\Lambda = \Lambda_U \cup \Lambda_C \cap \Lambda_S$, where $\mathrm{Re}(s) > 0$ in $\Lambda_U$, $\mathrm{Re}(s) = 0$ in $\Lambda_C$, $\mathrm{Re}(s) < 0$ in $\Lambda_S$. In particular, $\Lambda_C = \{z_1(\alpha),..., z_m(\alpha)\}$ is finite. By the continuity of the roots with respect to the parameter $\alpha$, which follows from the continuity in $\alpha$ of the Frechèt derivative of $f$, we may assume that, for a small enough $\alpha_0$, there exist continuous functions $z_i$, $i = 1,..., m$ and a real positive number $\gamma$ such that $z_i$: $(-\alpha_0, \alpha) \to \mathbb{C}$, $z_i(0) = z_i$, $|\mathrm{Re}\, z_i(\alpha)| < \gamma$ for $a \in (-\alpha_0, \alpha_0)$, $i = 1,..., m$, and $\det \Delta(z_i(\alpha), \alpha) = 0$, where $\Delta(z, \alpha) \equiv sI - \int_{-r}^{r} e^{s\theta}\, d\eta(\alpha, \theta)$. Furthermore, if $\det \Delta(z, \alpha) = 0$, $z \neq z_i(\alpha)$, $i = 1,..., m$, then $|\mathrm{Re}\, z| > \gamma$.

The sets $\Lambda_U(\alpha)$, $\Lambda_S(\alpha)$ are defined in the natural way. Before we proceed, we need to introduce some additional definitions and notation.

Let $x$: $\mathbb{R} \to \mathbb{R}^n$ be a continuous function; we define the weighted norm

$$\|x\|_\gamma \equiv \sup_{t \in \mathbb{R}} |x(t)|\, e^{-\gamma |t|}$$

for any $\gamma \in \mathbb{R}$. For any integer $k \geqslant 0$ and $\gamma \in \mathbb{R}$, let

$$C^k(\mathbb{R}, \mathbb{R}^n, \gamma) = C^k_\gamma = \left\{ \begin{matrix} x: \mathbb{R} \to \mathbb{R}^n: x \text{ is } k\text{-times continuously differentiable} \\ \text{and } \|D^j x\|_\gamma < +\infty, j = 0, 1,..., k \end{matrix} \right\}$$

where $D^j$ is the $j$th derivative. When no confusion is possible, we shall drop the superscript 0 in $C^0_\gamma$. The functionals

$$p_{\gamma k}(x) \equiv \|D^k x\|_\gamma$$

are seminorms on $C^k_\gamma$, and

$$\|x\|_{\gamma, k} \equiv \sum_{j=0}^{k} p_{\gamma j}(x)$$

is a norm. We can then define the linear operator $H$: $C^1(\mathbb{R}, \mathbb{R}^n, \gamma) \to C^0(\mathbb{R}, \mathbb{R}^n, \gamma)$ as

$$H(x)(t) \equiv \dot{x}(t) - Lx_t.$$

Note that, for a function $x \in C^0_\gamma$, the two Laplace transforms

$$\hat{x}(x) \equiv \int_0^\infty x(t)\, e^{-st}\, dt$$

$$\check{x}(s) \equiv \int_{-\infty}^0 x(t)\, e^{-st}\, dt$$

are analytic functions in the regions $\operatorname{Re} s > \gamma$, $\operatorname{Re} s < -\gamma$, respectively. As usual, for a function $g: \mathbb{C} \to \mathbb{C}$, we define

$$\int_{(c)} g(s)\, ds \equiv \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} g(s)\, ds$$

whenever the limit exists. We shall also use the following inner product:

$$(\alpha, \phi) \equiv \alpha(0)\, \phi(0) - \int_{-\tau}^{\tau} \int_{0}^{\theta} \alpha(\xi - \theta)\, d\eta(\theta)\, \phi(\xi)\, d\xi.$$

**Remark.** Recall that, if $s$ is a root of $\det \Delta(s)$, then $-s$ is a root of the characteristic equation of the formal adjoint. In fact, if $b \in (\mathbb{R}^n)^T$, $b\Delta(-s) = 0$, the function $y(t) \equiv be^{-st}$ is a solution of the formal adjoint equation. Consider now a function $x \in \operatorname{Ker} H$; i.e., $x \in C^1(\mathbb{R}, \mathbb{R}^n, \gamma)$ and $\dot{x}(t) = Lx_t$ for every $t \in \mathbb{R}$. Then we have

$$(y^t, x_t) - (y^0, x_0) = \int_0^t y(s)\, f(s)\, ds = 0 \qquad (\text{because } f \equiv 0).$$

Also if $\operatorname{Re}(s) > \gamma$, then for $\gamma$ defined as above we have $\lim_{t \to \infty} (y^t, x_t) = 0$, since

$$(y^0, x_0) = (ble^{-s}, x_0) = 0.$$

We now present, in Lemma 5.1, an estimate of the behavior of $\|\Delta^{-1}(s)\|$ over special contours in the complex plane, denoted $C_l$ and described in Bellman and Cooke (1963, p. 100). [See also Banks and Manitius (1975).]

Before we proceed, we need to introduce two different assumptions [see Rustichini (1989)] on the Frechèt derivative of the function $f(\cdot, 0)$ at 0, which we denoted as $L$. The first one is

(A1) $L(\phi) = A_{-r}\phi(-r) + B\phi(r) + \int_{-r}^{r} d\eta(\theta)\, \phi(\theta)$ *for any* $\phi \in C$ *where* $A_{-r}$, $B_r$ *are nonsingular matrices,* $\eta$ *is continuous at* $\pm r$, *has only a finite number of jumps, and the induced measure has no continuous part.*

The second assumption is introduced to deal specifically with systems of MFDE of a special type, namely, of Hamiltonian type. We use this name because systems that characterize optimal solutions of a control problem with delays have such a structure: the final section of this paper is indeed an extensive discussion of one of these cases.

In this case, the state variable is $2n$-dimensional, and is denoted $(x(t), p(t)) \in \mathbb{R}^{2n}$; the assumption we shall need is

(A2) *The linearized equation at* 0 *is*

$$\dot{x}(t) = \int_{-r}^{0} d\eta_1(\theta)\, x(t+\theta) + Ap(t) \equiv Mx_t + A_1$$

$$\dot{p}(t) = Bx(t) + \int_{0}^{r} d\eta_2(\theta)\, p(t+\theta) \equiv Bx(t) + Np^t$$

*where*

$$\int_{-r}^{0} d\eta_1(\theta)\, x(t+\theta) = Sx(t-r) + \int_{-r}^{0} d\eta_1^*(\theta)\, x(t+\theta),$$

$\int_0^r d\eta_2(\theta)\, p(t+\theta) = Rp(t+\theta) + \int_0^r d\eta_2(\theta)\, p(t+\theta)$, *S and R are nonsingular matrices,* $\eta_1^*$ *and* $\eta_2^*$ *are functions of bounded variation continuous at* $\pm r$, *with only a finite number of jumps, and the induced measure has no continuous part.*

We can now prove the following:

***Lemma 5.1.*** *Assume either (A1) or (A2), as defined above ; then,*

$$\lim_{l \to \infty} \int_{C_l} |e^{st}|\, |\Delta^{-1}(s)|\, ds = 0 \qquad \text{for} \quad t > -\tau.$$

*The convergence is uniform in t for t in bounded sets, i.e.,* $t \in [a, b]$, $-\tau < a < b < +\infty$.

**Proof.** The proof follows the lines of the proof of Lemma 4.2 in Bellman and Cooke (1963, pp. 122–123), once the estimate (4.6.5) of such lemma is given. Such estimate is provided in Sections 3 and 4 of Rustichini (1989). ∎

We have now completed the exposition of preliminary concepts and results; we may therefore proceed with the construction of the center manifold. Let $E_s$ be the eigenspace corresponding to the eigenvalue $s$.

***Lemma 5.2.*** *Assume either (A1) or (A2). Then,*

$$\operatorname{Ker} H = \operatorname{span}\{E_s : s \in \Lambda_C\}.$$

**Proof.** Consider any $x \in \operatorname{Ker} H$. Since the behavior at $\pm\infty$ is, in the space that we are considering, prescribed by the definition of the space itself, we can consider the Laplace transform of both sides of the equation

$$\dot{x}(t) = Lx_t \tag{5.2}$$

on the two intervals $t \geqslant 0$ and $t \leqslant 0$. The two functions $\hat{x}(s)$ and $\check{x}(s)$ are analytic in the regions $\operatorname{Re} s > \gamma$ and $\operatorname{Re} s < -\gamma$, respectively, because $x \in C^0(\mathbb{R}, \mathbb{R}^n, \gamma)$. For the interval $t \geqslant 0$, we obtain

$$(sI - Le^{s\cdot})\,\hat{x}(s) = x(0) - \int_{-1}^{1} \int_{0}^{\theta} e^{-s(u-\theta)}\, d\eta(\theta)\, x(u)\, du, \qquad \operatorname{Re}(s) > \gamma$$

or

$$\Delta(s)\,\hat{x}(s) = (e^{-s\cdot} I, x_0) \qquad (\operatorname{Re}(s) > \gamma). \tag{5.3}$$

For the interval $t \leqslant 0$, we obtain

$$(sI - Le^{s\cdot})\,\check{x}(s) = \left( x(0) - \int_{-1}^{1} \int_{0}^{\theta} e^{-s(u-\theta)}\, d\eta(\theta)\, x(u)\, du \right)$$

or, by the definition of the inner product $(\cdot, \cdot)$,

$$\Delta(s)\,\check{x}(s) = (e^{-s\cdot} I, x_0). \tag{5.4}$$

We notice that, since $\hat{x}$, $\check{x}$ are analytic on $\{s \in \mathbb{C} : \operatorname{Re}(s) > \gamma\}$ and $\{s \in \mathbb{C} : \operatorname{Re}(s) < -\gamma\}$, respectively, the function $F: \mathbb{C} \to \mathbb{C}^n$,

$$F(s) = \Delta(s)^{-1}(e^{-s\cdot} I, x_0) \qquad |\operatorname{Re}(s)| > \gamma, \qquad s \notin \sigma(A),$$

is analytic on $|\operatorname{Re}(s)| > \gamma$. Also, the function $(e^{s\cdot} I, x_0)$ is clearly analytic on $\mathbb{C}$, so $F$ is analytic on $\{s \in \mathbb{C} : |\operatorname{Re}(s)| \leqslant \gamma\} \setminus \{s : \Delta(s) = 0\}$; on the finite set of roots of $\Delta$, $F$ has poles of finite order.

We now invert Eq. (5.3), with $\gamma' > \gamma$, and obtain

$$x(t) = \int_{(\gamma')} e^{st} \Delta^{-1}(s)(e^{-s\cdot} I, x_0)\, ds \qquad (t > 0). \tag{5.5}$$

We consider now the special contours $C_l$ mentioned above. Since the function $F$ is analytic on the region $\operatorname{Re}(s) > \gamma$, we can modify the contour $C_l$ to the union of $C_{l-}$, the part of $C_l$ to the left of $\operatorname{Re}(s) = \gamma$, and a segment on this line. Then, arguing as in Bellman and Cooke (1963, pp. 102–104), we have

$$x(t) = \lim_{l \to +\infty} \left\{ \sum_{C_l} \operatorname{Res} e^{st} F(s) - \frac{1}{2\pi i} \int_{C_{l-}} e^{st} F(s)\, ds \right\} \tag{5.6}$$

where $\sum_{C_l} \operatorname{Res}$ is the sum over the residues of $e^{st}F(s)$ inside $C_l$. We examine the first term:

$$\int_{C_l-} e^{st} \Delta^{-1}(s)(e^{s\cdot}, \phi)\, ds$$

$$= \int_{C_l-} e^{st} \Delta^{-1}(s) \left[ x(0) - \int_{-1}^{1} \int_{0}^{\theta} e^{-s(\xi-\theta)}\, d\eta(\theta)\, x(\xi)\, d\xi \right] ds$$

$$= \int_{C_l-} e^{st} \Delta^{-1}(s)\, x(0)\, ds - \int_{-1}^{1} \int_{C_l-} \int_{0}^{\theta} \Delta^{-1}(s)\, e^{s(t-\xi+\theta)}\, d\eta(\theta)\, x(\xi)\, d\xi\, ds.$$

From Lemma 5.1, we conclude that the first term is zero.

The second term will only have a finite number of terms, corresponding to the roots of det $\Delta(s)$ with $|\text{Re}(s)| < \gamma$. Consequently,

$$x(t) = \sum_{s_j \in \Lambda_C} \text{Re}\, se^{st} F(s) = \sum_{s_j \in \Lambda_C} p_j(t)\, e^{s_j t} \qquad t > 0. \tag{5.7}$$

where $p_j(t)$ is a polynomial of degree $m_j - 1$, $m_j$ the multiplicity of $s_j$. Since a perfectly symmetric statement holds for $t < 0$, we have completed the proof. ▌

We can now proceed with the proof that the operator $H$ is onto.

In the final part of this section, we concentrate our attention on systems of Hamiltonian type. It will be obvious from the proofs of the following results that similar statements hold for systems that satisfy the nonsingularity assumption (A1). We first recall a result from the theory of retarded functional differential equations. In the nonhomogeneous equation of delay type,

$$\dot{x}(t) - Mx_t = f(t) \tag{5.8}$$

where $M$ is a continuous linear operator on $C([-r, 0], \mathbb{R}^n)$ for any $\delta > 0$ such that no root of the characteristic equation is on the lines $|\text{Re}(s)| = \delta$, we can decompose the set $\Lambda = \{s \in \mathbb{C}: \det \Delta(s) = 0\}$ as $\Lambda = \Lambda_U \cup \Lambda_C \cup \Lambda_S$ with $\Lambda_U \equiv \{s \in \Lambda: \text{Re}(s) > \delta\}$, $\Lambda_C \equiv \{s \in \Lambda: |\text{Re}(s)| < \delta\}$ and $\Lambda_S \equiv \{s \in \Lambda: \text{Re}(s) < -\delta\}$.

For any $f \in C(\mathbb{R}, \mathbb{R}^n, \gamma)$, there exists a solution $x(t, f)$ of (5.2), with $x(\cdot, f) \in C^1(\mathbb{R}, \mathbb{R}^n, \gamma)$, given by

$$x_t(\theta) \equiv \int_0^t T(t-x)\, X_0^C(\theta)\, f(s)\, ds + \int_{-\infty}^t T(t-s)\, X_0^S(\theta)\, f(s)\, ds$$
$$- \int_t^{\infty} T(t-s)\, X_0^U(\theta)\, f(s)\, ds. \tag{5.9}$$

We shall denote this solution $x(t, f)$. [See Hale (1977), Chapter 7, for the definition of $X_0^C$, $X_0^S$, $X_0^U$.]

We recall in particular that the following exponential estimates hold, for some constant $K>0$,

$$\|T(t)\,X_0^C\| \leqslant Ke^{\delta t}, \qquad t \geqslant 0;$$
$$\|T(t)\,X_0^U\| \leqslant Ke^{\delta t}, \qquad t \leqslant 0;$$
$$\|T(t)\,X_0^S\| \leqslant Ke^{-\delta t}, \qquad t \geqslant 0$$

where the norm is the operator norm. Let now $f=(f_1, f_2)$, $f_i \in C^0(\mathbb{R}, \mathbb{R}^{2n}, \gamma)$, $i=1,2$. We consider the nonhomogeneous system corresponding to

$$\begin{aligned} \dot{x}(t) &= Mx_t + Ap(t) + f_1(t) \\ \dot{p}(t) &= Bx(t) + Np^t + f_2(t) \end{aligned} \tag{5.10}$$

where $M\phi = \int_{-\tau}^{0} d\eta(\theta)\,\phi(\theta)$, $N\phi = \int_0^\tau d\eta_2(\theta)\,\phi(\theta)$; more compactly, (5.10) can be written as $(\dot{x}(t), \dot{p}(t)) = H(x_t, p^t) + f(t)$. Setting $q(t) \equiv p(-t)$, (5.10) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= Mx_t + Ap(-t) + f_1(t) \\ \dot{q}(t) &= -Bx(-t) + N^*q_t + f_2(-t) \end{aligned} \tag{5.11}$$

where $N^*\phi \equiv \int_{-\tau}^{0} d\eta_2(-\theta)\,\phi(\theta)$.

We now assume, w.l.o.g., that no root of the characteristic equation of the two delay equations $\dot{x}(t) = Mx_t$ and $\dot{q}(t) = N^*q_t$ is on the lines $|\mathrm{Re}(s)| = \gamma$. We therefore partition the two spectral sets, $\Lambda_1$ and $\Lambda_2$, say, according to $\gamma$: $\Lambda_1 \equiv \{s \in \Lambda: \mathrm{Re}(s) < \gamma\}$, $\Lambda_2 \equiv \{s \in \Lambda: \mathrm{Re}(s) > \gamma\}$.

Let now $K_1$, $K_2$ be the constants associated with the exponential estimates above, for $L_1$ and $L_2$, respectively. Let also $d_1$ and $d_2$ be the distance of the spectra of $L_1$, $L_2$, respectively from the line $|\mathrm{Re}(s)|| = \gamma$.

Then, we assume the following gap condition:

(G) $$\frac{K_1 K_2}{d_1 d_2} \|A\|\;\|B\| < 1.$$

***Lemma 5.3.*** *Assume (A1), (A2), the gap condition G; then, H is onto.*

**Proof.** Let $f=(f_1, f_2) \in C^0(\mathbb{R}, \mathbb{R}^{2n}, \gamma)$. We want to prove the existence of $(x, p) \in C^1(\mathbb{R}, \mathbb{R}^{2n}, \gamma)$ such that

$$(\dot{x}, \dot{p}) = H(x_t, p^t) + f(t). \tag{5.12}$$

We define the operator:

$$T_1 : C^1(\mathbb{R}, \mathbb{R}^n, \gamma) \to C^1(\mathbb{R}, \mathbb{R}^n, \gamma)$$
$$T_1(q) = x(\cdot, q)$$

where $x(\cdot, q)$ is the solution of $\dot{x}(t) - Mx_t = Aq(-t) + f_1(t)$ in $C^1(\mathbb{R}, \mathbb{R}^n, \gamma)$ given by (5.9) above. Also define $T_2$: $C^1(\mathbb{R}, \mathbb{R}^n, \gamma) \to C^1(\mathbb{R}, \mathbb{R}^n, \gamma)$ as

$$T_2(q) \equiv y(\cdot, T_1(q))$$

where $y(\cdot, q)$ denotes the solution of $\dot{y}(t) - N^* y_t = -BT_1(q)(-t) - f_2(-t)$ in $C^1(\mathbb{R}, \mathbb{R}^n, \gamma)$ given by (5.9). Clearly a solution of (5.2) is given by a fixed point of $T_2$. The existence of this fixed point follows from a standard application of Banach's fixed point. ▮

We now denote

$$\mathbf{C} \equiv \{(x, p) : (x, p) \text{ solve } (5.12),\ (x, p) \in C^1(\mathbb{R}, \mathbb{R}^{2n}, \gamma)\}$$

and we adopt the notation, for two metric spaces $X$ and $Y$, that $X \approx Y$ if $X$ and $Y$ are homeomorphic. Then we have the following:

***Theorem 5.1.*** *Assume (A2) and the gap condition G. Then,*

$$\text{Ker } H \approx \mathbf{C}$$

*in some neighborhood of the origin of* $C^1(\mathbb{R}, \mathbb{R}^n, \gamma)$.

**Proof.** This follows immediately from Lemmas 5.2 and 5.3, and the implicit function theorem. ▮

## 6. AN APPLICATION TO ECONOMIC DYNAMICS

As we mentioned before, a reason for the interest in studying MFDE is an application to the analysis of the dynamic behavior of a competitive economy. Before we present a formal model, we discuss briefly the main lines of the problem.

One can characterize a competitive economy (or, rather, an abstract model of it) as a set of consumers, producers (firms), and a list of given data: interest rate, rate of depreciation of the capital stock, initial endowment of capital stock. Each consumer has a preference ordering, represented by a utility function, and the technology of firms by a transformation function that gives, for every pair of capital stock and investment goods, the (maximum) amount of consumption good that can be produced.

Given a price of consumption and investment goods, and a retail price of capital goods, consumers maximize profit, and finally market chooses between acquisition of capital goods on one side and acquisition of financial activities on the other, with a profit maximization criterion. As an overall result, the optimizing behavior of the agents completely determines the dynamics of the economy, which is fully characterized by the time evolution of two variables: the vector of stocks of capital goods, and the vector of prices of investment goods. The time derivative of the capital stock equals aggregate investment minus the depreciation, where the aggregate investment is the result of the profit-maximizing decisions of the firm. On the other hand, the possibility of buying and selling on the market for capital goods insures a nonarbitrage condition: the rate of change in prices of investment goods plus the rental price of capital must be equal to the interest rate.

We can now turn to the derivation of the precise form of "laws of motion" for such a model. The standard procedure, a classical tool of economic analysis, is to analyze a fictitious optimal planning problem, and then proceed to prove that the necessary and sufficient conditions for the optimality are exactly the conditions that characterize a competitive economy. An extensive discussion of this procedure is, for example, in Cass and Shell (1976). In the optimal planning problem (which, from the mathematical point of view, is an optimal control problem), the utility functions of the consumers are aggregated into a single welfare function (a preference ordering for the society), while the production possibility sets of the firms are aggregated into a production possibility set for the entire economy; the nonarbitrage condition is now the necessary condition for optimality. To be more precise, we need first to introduce some notation.

Consumption is denoted $c$, a nonnegative real number, investment $u$, and capital stock $k$, both $n$-dimensional nonnegative vectors. When we want to emphasize the dependence on time, we shall write $c(t)$, $u(t)$, $k(t)$, respectively. Now, given a capital stock $k$, for every level of investment $u$ there is a maximum lever of consumption, $c$, which is (technologically) feasible. We assume that this relationship can be expressed as a function $c = T(u, k)$. It is standard to assume (and we do) that this function is continuous and concave in both variables. To every consumption $c$, we associate a welfare $U(c)$, and the objective is to maximize the integral of the discounted welfare. So far, the model is standard. We now introduce our modification: *both the production of the consumption good and the investment activity take time.* A similar model has been analyzed in Kydland and Prescott (1982). In other words, our model has a time delay.

Various formulations are possible (we shall discuss some of them later), but we shall concentrate our attention on the following optimal control problem:

$$\text{(P)}\quad \max_{c(\cdot)} \int_0^\infty e^{-\rho t} U(c(t))\, dt$$

$$\begin{aligned} \text{subject to}\quad & c(t) = T(u(t-\tau), k(t-\sigma)) \\ & \hat{k}(t) = u(t-\tau) - gk(t-\tau) \\ & k, u, c \geqslant 0 \\ & k(t) = \phi(t) \qquad \text{for} \quad t \in [-\sigma, 0]. \end{aligned}$$

(Here, $\sigma \geqslant \tau \geqslant 0$: such restriction is used here only to fix the ideas; in the discussion of the effect of parametric changes in $\sigma$ and $\tau$, it will be dropped.)

A proof of the existence of a solution to this optimization problem, in some suitably defined functional space, is fairly standard and will not be discussed here. Strict concavity of $U$ and $T$ insures uniqueness of the optimal solution. Further assumptions on $T$ and $U$ [see Benhabib and Nishimura (1979)] insure the existence of an optimal stationary solution, which is interior. Since we are interested in the behavior of the optimal solution close to such stationary solution, we may assume that the Euler equations [together with the transversality condition, see Benveniste and Scheinkmann (1982)] characterize the optimal solution. In particular, a periodic solution of the Euler equations will be optimal. A final remark is in order before we proceed with the proof of the existence of periodic solutions. As mentioned above, the existence of such solutions has been proved, for the nondelayed case, in the important paper of Benhabib and Nishimura (1979), when the number of capital goods $n$ is at least three. It is also well known that, in such models, periodic solutions cannot exist for $n = 1$. In the model with time delays, periodic solutions are possible even for $n = 1$. This is the case that we consider here. The Euler equations [see, for instance, Hughes (1968)] are easily computed to be

$$\begin{aligned} \dot{x}(t) &= u(p(t), k(t-\sigma)) - gk(t-\tau) \\ \dot{p}(t) &= \rho p(t) + ge^{\rho\tau} p(t+\tau) - e^{-\rho\sigma} [U'(T)\, T_k](u(p(t+\sigma), k(t)), k(t)) \end{aligned} \tag{6.1}$$

where $T_k \equiv \partial T/\partial k$; $u(p(t), k(t-\sigma))$ is the optimal control (investment) determined by the maximum principle as the solution of the maximization of the Hamiltonian. In the case of $U$, $T$ concave and continuously differentiable, and $u$, $k$ interior solutions, the optimal investment is given in implicit form by

$$(U'(T)\, T_k)(u(t-\tau), k(t-\sigma)) = p(t). \tag{6.2}$$

It is interesting to compute the values of the steady-state solutions. Setting $\dot{p}=\dot{k}=0$ and letting $\tilde{r}=\rho e^{\rho\sigma}+ge^{\rho(\sigma-\tau)}$, $w(p, k)\equiv(U'(T)\,T_k)(u(p, k), k)$ (the variable $w$ is the control price of capital), one derives the equations for the steady-state solution pair $(p, k)$:

$$\tilde{r}p = w(p, k)$$
$$u(p, k) = gk.$$

In other words, a delay in production and investment has the same effect over the value of the steady-state variables as an increase in the interest rate.

We now linearize the Euler equations. The characteristic equation is

$$\det \Delta(z) = \det \begin{bmatrix} z-[u_k e^{-z\sigma}-ge^{-z\tau}] + e^{-\rho\sigma}D_k S & -u_p \\ & z-[\rho+ge^{-\rho\tau}e^{z\tau}-e^{-\rho\sigma}D_p S e^{z\sigma}] \end{bmatrix} = 0 \qquad (6.3)$$

where we have defined

$$u_k \equiv \frac{\partial u}{\partial k}(\bar{p}, \bar{k}); \qquad u_p \equiv \frac{\partial u}{\partial p}(\bar{p}, \bar{k}); \qquad [U'(T)\,T_k](u(p, k), k) \equiv S(p, k).$$

Notice that we have, from the symmetry properties of the Hamiltonian equations: $D_p S = u_k$ at the equilibrium point. We present now an observation concerning symmetry properties of the roots of characteristic equations of the kind considered above.

Let

$$\Delta(z) = \det \begin{pmatrix} z-\int_{-\sigma}^{0} d\eta(\theta)\, e^{z\theta} & A \\ B & z-\rho+\int_{-\sigma}^{0} e^{-(z-\rho)\theta}\, d\eta(\theta) \end{pmatrix};$$

then we have the following:

***Lemma 6.1.*** *If*

$$\Delta(z_0) = 0,$$

*then*

$$\Delta(-z_0+\rho) = 0.$$

**Proof.** An obvious computation:

$$\Delta(-z_0+\rho)=\det\begin{pmatrix} z_0+\rho-\int_{-\sigma}^{0} d\eta(\theta)\, e^{-(z-\rho)\theta} & A \\ B & z_0+\int_{-\sigma}^{0} e^{z_0\theta}\, d\eta(\theta) \end{pmatrix}$$

$$=(-1)^{2n}\det \Delta(z_0)=0$$

because

$$\det\begin{pmatrix} C & A \\ B & D \end{pmatrix}=\det\begin{pmatrix} C & -B \\ -A & D \end{pmatrix}. \quad \blacksquare$$

It may be useful to consider first the case of no delay. When $\sigma=\tau=0$, the eigenvalues are given by the roots of the characteristic equation

$$\det\begin{pmatrix} z-(u_k-g) & u_p \\ -D_k S & z-(r-u_k) \end{pmatrix}. \tag{6.4}$$

In the special case in which $u_p$, say, is zero, then the two eigenvalues are $r-u_k$, $u_k-g$. But $u_k/g=u_k\bar{k}/u(\bar{k},\bar{p})$ and $u(\cdot,p)$ is a concave function, so $u_k=u_k(\bar{k},\bar{p})\leqslant g$. We conclude that $r-u_k$ is nonnegative and $u_k-g$ is nonpositive. Now, since $u_p\geqslant 0$, $D_k S\leqslant 0$, it is immediate to check that such terms cannot change this (local) saddle-point feature of the equilibrium point.

We now proceed to examine the general case of $\sigma$, $\tau\neq 0$. We shall do this under two special assumptions:

(A5.1) $\quad u_k(\bar{p},\bar{k})=\text{constant} \quad$ (w.r. to $\sigma,\tau$)

(A5.2) $\quad u_p(\bar{p},\bar{k})=0.$

These are, admittedly, special cases; the reason for their introducton is simplicity of the analysis. Many of the arguments below can be adapted easily to a situation in which $u_k$ is a variable, and use of Rouche's theorem extends them to small enough perturbations of $u_p$.

In the following, to simplify the notation, we let $u\equiv u_k(\bar{p},\bar{k})$. We study the roots of equations

$$\begin{aligned} &\text{(i)} && z-ue^{-z\sigma}+ge^{-z\tau}=0 \\ &\text{(ii)} && z-\rho-ge^{-\rho\tau}e^{z\tau}+e^{-\rho\sigma}ue^{z\sigma}=0. \end{aligned} \tag{6.5}$$

From Lemma 6.1, we already know that will be enough to consider the roots of the first equation; for every such root, to say, $-z_0+\rho$ will be a root of the second equation.

***Lemma 6.2.*** *There exists a* $\tau_m > 0$ *such that, if* $0 \leqslant \tau \leqslant \tau_m$, *then* $\mathrm{Re}(z) \neq 0$ *for every* $\sigma \geqslant 0$.

**Proof.** Suppose $\mathrm{Re}(z) = 0$; then, from $u_k \cos y\sigma = g \cos yz$, we have $u_k g^{-1} \geqslant |\cos y\tau|$, so, for some $\sigma > 0$,

$$y\tau \in \{k\pi + \pi/2 - \sigma,\ j\pi + \pi/2 + \sigma \mid j = 0, 1, \ldots\}$$

(we take w.l.o.g. $y \geqslant 0$). Then, $y\tau \geqslant \pi/2 - \sigma$. From $-y = u_k \sin y\sigma - g \sin y\tau$, $y \leqslant u_k + g$; so $\tau_m \equiv (\pi/2 - \sigma)/(u_k + g)$ satisfies the claim. ▮

We also note that, if there are $z = iy$, $\sigma_0$, $\tau_0$ such that $z = ue^{-z\sigma} + ge^{-z\tau} = 0$, then any triple $(iy, \sigma_n, \tau_m)$ with $\sigma_n = \sigma_1 + (2\pi/y)n$, $\tau_m = \tau_0(2\pi/y)m$ for $n$, $m$ integers will also satisfy the equation above.

Our next task will be to determine the values, if any, of $\sigma$ and $\tau$ for which a Hopf bifurcation takes place.

We now proceed to consider the cases $\sigma = 0$ and $\sigma = \tau$. We firstly recall a result of Hayes [see Bellmann and Cooke (1963), p. 444].

***Theorem*** *(Hayes). All the roots of* $pe^z + q - ze^z$, *where* $p$ *and* $q$ *are real, have negative real parts if and only if*

*(a)* $p < 1$ *and*

*(b)* $p < -q < \sqrt{a_1^2 + p^2}$

*where* $a_1$ *is the root of* $a = p \tan a$ *such that* $0 < a < \pi$. *If* $p = 0$, *we take* $a_1 = \pi/2$. *See Fig. 1.*

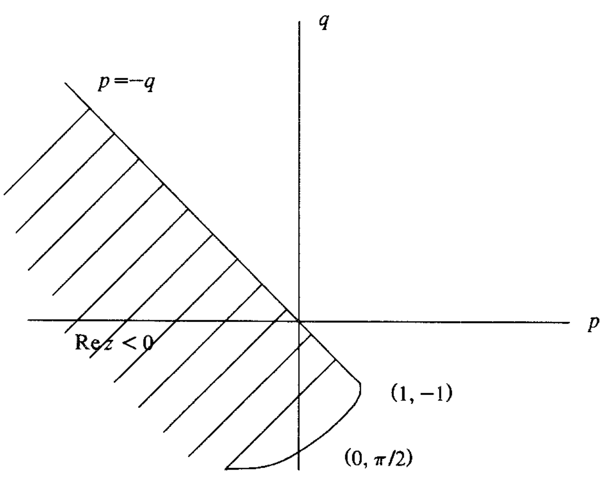


**Fig. 1**

The shaded region is the region in the $(p, 1)$-plane where all the roots have negative real part.

***Lemma 6.3.***

*If* $\sigma = 0$, *then there exists a* $\tau_2$ *such that,*

*if* $\tau < \tau_2$, *then Eq. (6.6) has no eigenvalues with* $\mathrm{Re}(z) > 0$;

*if* $\tau > \tau_2$, *then Eq. (6.6) has a pair of eigenvalues with* $\mathrm{Re}(z) > 0$.

*If* $\sigma = \tau$, *then there exists a* $\tau_3 > 0$ *with analogous properties.*

**Proof.** $z - u + ge^{-z\tau} = 0$ if and only if $-we^{w} + u\tau e^{w} - g\tau = 0$ with $z\tau \equiv w$. So, for $q = -g\tau$, $p = u\tau$, Hayes theorem applies, giving a $\tau_2$ such that the equation has a root with positive real part for $\tau \geqslant \tau_2$. Analogously, if $\sigma = \tau$, $z - ue^{z\tau} + ge^{z\tau} = 0$ if and only if $-we^{w} + \tau(u - g) = 0$; so, with $p = 0$, $q = \tau(u - g) < 0$, we have $\tau_3 > 0$ such that the equation has a root with positive real part for $\sigma = \tau > \tau_3$. In both cases, it is easily checked that the imaginary part is nonzero. (For example, when $\sigma = 0$, we have the two equations $u = g \cos y\tau$, $-y = -g \sin y\tau$; and $y \neq 0$ because $u < g$.) ▮

Now we consider the general situation. Setting $z = x + iy$, Eq. (6.5) is equivalent to the following system $S(s, y, \sigma, \tau)$:

$$
\begin{aligned}
x - ue^{-x\sigma} \cos y\sigma + ge^{-x\tau} \cos y\tau &\equiv f(x, y; \sigma, \tau) = 0 \\
y + ue^{-x\sigma} \sin y\sigma - ge^{-x\tau} \sin y\tau &\equiv g(x, y, \sigma, \tau) = 0.
\end{aligned}
\tag{6.6}
$$

Rarely, when we want to make explicit the dependence of the above system on $u$ and $g$, we shall use the more cumbersome notation $S(s, y; \sigma, \tau; u, g)$.

We shall consider different values of $g$ and $u$, and the corresponding different possible values of $y$. Our analysis will proceed in two steps. We shall firstly determine the values of $\sigma$, $\tau$, $y$ for which $z = iy$ is a solution of (6.6). Then, we shall prove that all such eigenvalues (except for the case $y = 0$) are Hopf bifurcation values, i.e., they also satisfy the crossing with positive speed condition. We shall consider the nonnegative $y$.

Fix now, $g > 0$, and for any $y \in (0, 2g)$, denote $\underline{\theta} = \theta(y) = \min\{y/g; 2 - y/g\}$. Then, we have the following:

***Lemma 6.4.*** *For any* $y \in (0, 2g)$ *and any* $u$, $g$ *with* $\theta \equiv u/g \in [\underline{\theta}, 1)$, *there exists a pair* $(\sigma, \tau) \in \mathbb{R}^2_+$ *satisfying* $S(0, y; \sigma, \tau; u, g) = 0$.

**Proof.** $z = iy$ is a root of $\sigma$ if and only if the two equations

$$-u \cos y\sigma + g \cos y\tau = 0 \tag{6.7}$$

$$-\sin y\sigma + g \sin y\tau - y = 0 \tag{6.8}$$

are both satisfied. Setting $y\sigma \equiv X$, $y\tau = Y(X)$, when $y > 0$, (6.7) and (6.8) are equivalent to

$$Y(X) = \arccos(\theta \cos X) \tag{6.7'}$$

$$y = -u \sin X + g \sin Y(X). \tag{6.8'}$$

Now the values of $y$ for which (6.7), (6.8) have solutions are the image of $F(X) \equiv -u \sin x + g \sin Y(X)$ for $x \in [0, 2\pi]$; this image set depends on $u$, $g$, but it is easily found to be the interval $[g(1-\theta), g(1+\theta)]$. The claim follows. ▌

We refer here to Fig. 2, drawn in the $y$ and $u/g$ plane, for a fixed $g$. Clearly, for $y > 2g$, there are no periodic solutions with period $2\pi/y$; for $0 \leqslant y \leqslant 2g$, the shaded area gives the values of $u$ for which solutions of period $2\pi/y$ are possible. In the figure, we also mention the values of $\sigma$, $\tau$ that are associated with such solutions. We note that, for every point in the region $A$, there exist two pairs $(\sigma, \tau)$, different modulo $2\pi/y$, which give the same $y$. For the region $B$ and the line $y = g(1-\theta)$, such a pair is unique.

The ratio, $\tau/P$, between the length of the delay and the length of the period is of economic interest. Since $\tau/P = Y/2\pi$, we have $\tau/P \in [0, \frac{1}{2}]$, and $\sigma/P \in [0, 1]$. For example, if $\sigma = \tau$, then $\sigma/P = \tau/P = \frac{1}{4}$; if $\sigma = 3\tau$, then $\tau/P = \frac{1}{4}$; if $\sigma = 0$, $\tau/P = \arccos \theta/2\pi$.

Finally, a numerical example. Choose $g = \log 2$, so that the time unit

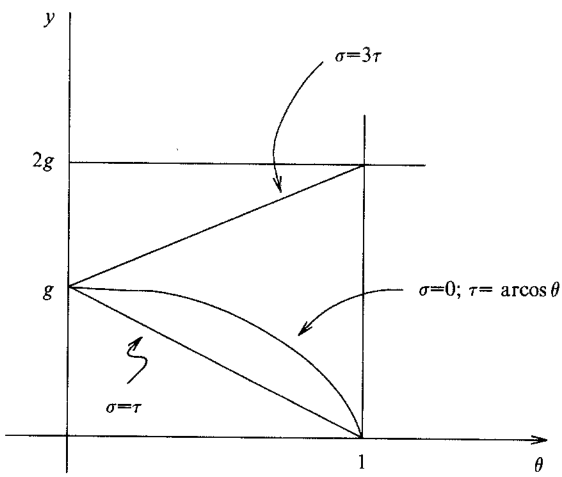


**Fig. 2**

$U$ is the halving time for the capital stock, at investment zero; let $u = \frac{1}{2}$, and $U = 1$ year; then,

| | | | | |
|---|---|---|---|---|
| if | $\sigma = \tau$ | then | $\tau \simeq 4.5$ | $P \cong 18$ |
| if | $\sigma = 3\tau$ | then | $\tau \simeq 1.5$ | $P \cong 6$ |
| if | $\sigma = 0$ | then | $\tau \simeq 1.7$ | $P \cong 10.5$ |

Let us now consider the conditions under which eigenvalues with zero real part are Hopf bifurcation values.

***Lemma 6.5.*** *For any $\theta \in (0, 1)$, and $y \neq 0$, eigenvalues with zero real part are Hopf bifurcation values.*

**Proof.** Let, with $f_x \equiv \partial f/\partial x$, and similarly for $f_y$, $g_x$ and so on:

$$F \equiv \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}; \qquad G = \begin{pmatrix} f_\sigma & f_\tau \\ g_\sigma & g_\tau \end{pmatrix}; \qquad X = \begin{pmatrix} x_\sigma & x_\tau \\ y_\sigma & y_\tau \end{pmatrix},$$

each one evaluated at $(0, y, \sigma, \tau)$ such that $S(0, y, \sigma, \tau) = 0$. If $F$ is nonsingular, then to have $(x_\sigma, x_\tau) \neq 0$, it is sufficient to have $(f_\sigma, g_\sigma) \neq 0$ or $(f_\tau, g_\tau) \neq 0$. But

$$G = \begin{pmatrix} uy \sin y\sigma & -gy \sin y\tau \\ uy \cos y\sigma & -gyt \cos y\tau \end{pmatrix},$$

so, for $u$, $y \neq 0$, we are left to prove that $F$ is nonsingular.

Now

$$F = \begin{pmatrix} 1 + u\sigma \cos y\sigma - g\tau \cos y\tau & u\sigma \sin y\sigma - g\tau \sin y\tau \\ -u\sigma \sin y\sigma + g\tau \sin y\tau & 1 + u\sigma \cos y\sigma - g\tau \cos y\tau \end{pmatrix} \equiv \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

with eigenvalues $a \pm ib$; so $F$ is singular only if

$$u\sigma \sin y\sigma = g\tau \sin y\tau. \tag{6.9}$$

We claim that such an equation is not satisfied, when $0 < u < g$, for all the values $(y, \sigma, \tau)$ such that $S(0, y, \sigma, \tau) = 0$.

We rewrite Eq. (6.9) using the notation of the previous lemma; the dependence of $Y$ on $\theta$ is now made explicit:

$$\theta X \sin X = Y \sin Y \qquad Y = Y(\theta, X) = \arccos(\theta \cos X). \tag{6.9'}$$

Now, for $X \in [0, \pi/2]$, $\sin Y \geqslant \sin X$; and, for $[\pi, 2\pi]$, $\sin X \leqslant 0$, $Y \sin Y > 0$; so, in both cases, (6.9) cannot hold. We now consider

$X \in (\pi/2, \pi)$. Let $\hat{X}$ be the $X \in (\pi/2, \pi)$ where $X \sin x$ has a maximum, so that $\hat{X} = -\sin \hat{X}/\cos \hat{X}$. Since both sides of (6.9′) are concave functions of $X$, it will be enough to prove $\theta\hat{X} \sin \hat{X} < Y(\theta, \hat{X}) \sin Y(\theta, \hat{X})$; and, since $\sin Y(\theta, \hat{X}) > \sin \hat{X}$ for every $\theta \in (0, 1)$, it will be enough to show $Y(\theta, \hat{X}) \geqslant \theta\hat{X}$; and this follows from both functions being concave in $\theta$, $Y(1, \hat{X}) = \hat{X}$ because $Y(1, X) = X$ for every $X \in [0, \pi]$, and

$$Y_\theta(1, \hat{X}) = -\cos \hat{X}/\sin \hat{X} < -\sin \hat{X}/\cos \hat{X} = \frac{\partial}{\partial\theta}(\theta\hat{X}).$$

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