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Functional Differential Equations of Mixed Type: The Linear Autonomous Case

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Functional differential equations of mixed type (MFDE) are introduced; in these equations of functional type, the time derivative may depend both on past and future values of the variables. Here the linear autonomous case is considered. We study the spectrum of the (unbounded) operator, and construct continuous semigroups on the stable, center, and unstable subspaces.

KEY WORDS: Functional differential equations; infinitesimal generator; semigroup.

1. INTRODUCTION

The purpose of this paper is the study of a special class of linear autonomous functional differential equations (FDE). In the case that we are analyzing, the time derivative can depend on both past and future values of the variable. We call such equations mixed FDE (MFDE) to emphasize this composite nature, of advanced and delayed equations.

The main motivation for such analysis is the study of the dynamics arising from solutions of optimal control problems. It is well known [see, for instance, the classic work of Pontryagin *et al.* (1962)] that the necessary conditions for the solution of an optimal control problem with delays involve a system of functional differential equations with both advanced and delayed terms. [See Chi *et al.* (1986) for an example of an MFDE that does not arise from an optimal control problem.]

It is easy to imagine which will be the single most relevant difficulty that one encounters when dealing with MFDE: the Cauchy problem is, in general, not well posed. Even a linear equation with constant coefficients

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may not have a solution, for positive or negative times, for a given initial condition in the appropriate space (say, as it will always be the case in the following, the space of continuous functions on the delay interval $[-r, r]$, with the sup norm). A solution is here naturally defined to be a differentiable function that satisfies the MFDE and is equal to the initial condition in the initial interval.

The first step in our study of MFDE is an analysis of the simplest case: linear autonomous MFDEs. It is useful to recall here the very broad lines of the treatment of this problem in the case of delay differential equations, as developed for instance in the two fundamental books on the subject [Bellman and Cooke, 1962; Hale, 1977). In this case, the Cauchy problem is well posed, at least for positive times. Furthermore, a simple argument based on Gronwall inequality provides an exponential bound on the growth at infinity of the solution, where the order of growth only depends on the operator norm of the differential operator. If we look at the question from the point of view of the location in the complex plane of the roots of the characteristic equation, the existence of such a bound is strongly related to the fact that all such roots have real part bounded above. The existence of a solution for any initial condition, and the exponential bound on the solutions are the two ingredients for the construction of a C_0 semigroup of operators (defined for positive times) on the phase space. The infinitesimal generator of this semigroup is defined by the differential operator of the delay equation.

We have mentioned above that the first of these two basic ingredients, the existence of a solution, is missing in the case of MFDE. The analysis of the location of the zeroes (which have unbounded positive and negative real parts) proves that the second ingredient, a bound on the order of growth, is also missing. The main content of the first chapter is to develop a treatment, close to the one existing for delay equations, of the theory of C_0 semigroups.

Our line of argument will start, loosely speaking, from the infinitesimal generator rather than from the semigroup. More exactly, we define an unbounded linear operator on a suitable domain in the space of continuous functions over $[-r, r]$. The spectrum of this operator will turn out to be an infinite sequence of eigenvalues, with unbounded real part, in the positive and negative halfspaces. Our aim is to define our semigroup by integration of the resolvent. In order to prove convergence of the Dunford integral, we provide estimates on the norm of the resolvent.

We then construct the semigroup: it is necessary, at this point, to consider separately the case of eigenvalues with positive and with negative real parts, and define for each subspace a proper semigroup. They are shown to be well defined for negative and positive times, respectively; their domain of definition is the closures, in the sup norm, of the two subspaces of eigenfunctions corresponding to the eigenvalues.

We finally point out that one of the main motivations for the study of the linear problem is to prepare the study of the nonlinear mixed differential equations. Nonlinear MFDE are examined in the following paper in this issue [Rustichini (1989)].

1. NOTATION AND DEFINITIONS

$r \geqslant 0$ is any given real number $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space, with norm $|\cdot|$. For two real numbers a, $b \in \mathbb{R}$, $C([a, b], \mathbb{R}^n)$ is the space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$; it is a Banach space when endowed with the natural (sup) norm:

$$\|x\|_C = \|x\| = \sup\{|x(s)|; s \in [a, b]\}.$$

We shall mostly consider the case where $-a = b = r$. If $t_0 \in \mathbb{R}$, $A \geqslant r$, and $x \in C([t_0 - r, t_0 + A], \mathbb{R}^n)$, then, for any $t \in [t_0, t_0 + A - r]$, we let $x_t \in C$ be defined by $x_t(s) \equiv x(t+s)$, $s \in [-r, r]$.

As usual $\dot{x}(t)$ is the (right-hand) derivative of x at t.

$C^k([a, b], \mathbb{R}^n)$ is the space of functions from $[a, b]$ to $\mathbb{R}^n$ that are k-times continuously differentiable.

Consider, as an example, the following differential equation:

$$\dot{x}(t) = x(t+1) + x(t-1).$$

In this equation, the time derivative depends both on a delayed term, $x(t-1)$, and on a forward term, $x(t+1)$, and can therefore be called an equation of mixed type of equations of this type. Let us proceed more formally.

If $f: C \to \mathbb{R}^n$ is a continuous function, then the functional differential equation (FDE):

$$\dot{x}(t) = f(x_t) \tag{1.1}$$

is defined. We shall refer in the following to these MFDE as mixed functional differential equations (MFDE), where the term "mixed" refers to the fact that the time derivative depends both on past and future values of the variable.

As usual, we say that the constant function $c \in C$ is an equilibrium point of (1.1) if

$$f(c) = 0.$$

In the following, we shall consider the case of a continuously differentiable f. If Ω is a neighborhood of zero in C, we let

$$BC'(\Omega)=\{f\colon \Omega\to\mathbb{R}^n \text{ such that } f \text{ is continuous and bounded on } \Omega \text{ and } f \text{ has on } \Omega \text{ a Frechet derivative } f'\}.$$

and we define

$$|f|_{BC'(\Omega)}\equiv\sup\{|f(\phi)|+|f'(\phi)|\colon \phi\in\Omega\}.$$

We are interested in the study of the behavior of solutions of (1.1) around an equilibrium point c that we assume for simplicity to be zero. Then, let $L\colon C\to\mathbb{R}^n$ be defined as

$$L\phi\equiv f'(0)\,\phi,$$

where $f'(0)$ is the Frechèt derivative of f at 0.

From Riesz' representation theorem for $C([-r, r], \mathbb{R}^n)$, we know that there exists a function η, $\eta\colon [-r, r]\to\mathbb{R}^n$ of bounded variation, such that

$$L\phi=\int_{-r}^{r} d\eta(s)\,\phi(s).$$

We shall denote by ℓ the operator norm of L.

We associate with each L the characteristic matrix

$$\Delta(s)\equiv sI-\int_{-r}^{r} e^{s\theta}\,d\eta(\theta).$$

As we mentioned in the introduction, a major difficulty arising in the study of the type of FDE that we are considering is that the Cauchy problem is ill-posed. Consider the initial value problem

$$\dot{x}(t)=f(x_t) \qquad x_0\equiv\phi\in C. \tag{1.1'}$$

A solution to Eq. (1.1′) on an interval $[-r, A]$, with $A>r$, is defined to be a continuously differentiable function $x\colon [-r, A]\to\mathbb{R}^n$ which satisfies (1.1′) for every $t\geqslant 0$. Such equation does not always have a solution, even in the case of a linear L. Consider, for example, $\dot{x}(t)=x(t+1)+x(t-1)$, $x_0\equiv 1$; then, clearly for any $t\in(2j+1, 2j+3)$, $j=1, 2, \ldots$, a solution should satisfy $x(t)=(-1)^{j+1}$; and therefore no everywhere continuous solution can be defined.

2. INFINITESIMAL GENERATOR AND RESOLVENT

For a given linear operator L, we define the linear unbounded operator A as follows:

$$D(A) \equiv \{f \in C^1([-r, r], \mathbb{R}^n): \dot{f}(0) = Lf\}$$
$$Af \equiv \dot{f} \qquad \text{for every} \quad f \in D(A).$$

It is easily seen that A is a closed operator, and $\bar{D}(A) = C$, with the closure taken with respect to the sup norm; i.e., A is densely defined.

We shall now study under which conditions there exists a solution $\phi \in C$ to the equation

$$(A - sI)\,\psi = \phi, \qquad \phi \in D(A)$$

for a $\psi \in C$ and $s \in \mathbb{C}$. We denote as usual $\sigma(A)$ the spectrum of A, $\rho(A)$ the resolvent of A, and $P\sigma(A)$ the point spectrum of A:

Lemma 2.1. *Let A be defined as above. Then,*

1. $\sigma(A) = P\sigma(A)$.

2. $s \in \sigma(A)$ *if and only if* $\det \Delta(s) = 0$.

Proof. Let $s \in \rho(A)$; we shall determine the function ϕ such that

$$(A - sI)\,\phi = \psi, \qquad \phi \in D(A). \tag{2.1}$$

Let $\phi(0)$ be a fixed element in $\mathbb{R}^n$, and let ϕ: $[-r, r] \to \mathbb{R}^n$ be defined by

$$\phi(t) \equiv e^{st}\phi(0) + \int_0^t e^{s(t-u)}\psi(u)\,du. \tag{2.2}$$

Clearly $\phi \in C^1([-r, r], \mathbb{R}^n)$, and $\dot{\phi}(t) - s\phi(t) = \psi(t)$, $(t \in [-r, r])$.

The value of $\phi(0)$ will now be determined to satisfy the boundary condition:

$$\dot{\phi}(0) = \int_{-r}^{r} d\eta(\theta)\,\phi(\theta) = L\phi, \tag{2.3}$$

that is, $\phi(0)$ will be fixed so that $\phi \in D(A)$. From (2.1), (2.2), and (2.3),

$$\begin{aligned}\psi(0) + s\phi(0) = \dot{\phi}(0) &= \int_{-r}^{r} d\eta(t)\,\phi(t) \\ &= \int_{-r}^{r} d\eta(t)\left[e^{st}\phi(0) + \int_0^t e^{s(\theta-u)}\psi(u)\,du\right] \end{aligned} \tag{2.4}$$

and therefore one has

$$\left[sI-\int_{-r}^{r} d\eta(\theta)\, e^{s\theta}\right]\phi(0)=\int_{-r}^{r} d\eta(\theta)\left[\int_{0}^{\theta} e^{s(\theta-u)}\psi(u)\,du\right]-\psi(0). \quad (2.5)$$

We want to rewrite (2.5) in a more compact form. We let $\alpha: [-r, r] \to \mathbb{C}^{n^2}$, continuous, be defined as $\alpha(\theta) \equiv e^{-s\theta}I$, $(\theta \in [-r, r])$, I the identity matrix; and, for any $\psi \in C$, we define (α, ψ) by

$$(\alpha, \psi) \equiv \alpha(0)\,\psi(0) - \int_{-r}^{r}\int_{0}^{\theta} \alpha(u-\theta)\, d\eta(\theta)\, \psi(u)\, du. \quad (2.6)$$

Then (2.5) can be rewritten as

$$\Delta(s)\,\phi(0) = -(\alpha, \psi). \quad (2.5')$$

Now [as in Hale (1977), p. 169] we conclude that, if $\det \Delta(s) = 0$, then there exists a nonzero solution of (2.1) with $\psi = 0$; that is, $s \in P\sigma(A)$. On the other hand, (2.1) has a solution for any $\psi \in C$ only if $\det \Delta(s) \neq 0$; only in this case in fact can we solve in (2.5) for $\phi(0)$, and substitute in (2.2) to find the required ϕ. Therefore, $\rho(A) = \{s: \det \Delta(s) \neq 0\}$, and $\sigma(A) = P\sigma(A)$. ∎

We define now the resolvent of the operator A, $R(s: A)$ as

$$R(s: A): C \to D(A)$$

where $R(s: A)\,\psi$ is the solution of (2.1) above for $s \in \rho(A)$.

We want to determine some basic properties and estimates for this operator. We denote $N_k(s) = N_k \subset C$ the null space of $(A - sI)^k$, $\|\cdot\|$ the operator norm.

Lemma 2.2. *Let $s \in \rho(A)$, $|\text{Re}(s)| \geqslant c > 0$, for some $c > 0$. Then*

1. *There exists a constant $K = K(L)$, such that*

$$\|R(s: A)\| \leqslant \frac{e^{|\text{Re}(s)|r}}{|\text{Re}(s)|} + \|\Delta(s)^{-1}\| \frac{e^{2|\text{Re}(s)|r}}{|\text{Re}(s)|} K.$$

2. *$R(s: A)$ is a compact operator*
   Let $s_0 \in \sigma(A)$ then
3. *For $k > 0$, $N_k(s) = N_k$ is always of positive finite dimension, and there exists an integer $n_0(s) = n_0 > 0$ such that*

$$N_k = N_{n_0} \quad \text{for every} \quad k \geqslant n_0 \quad \text{and}$$

$$N_k \subsetneqq N_{k+1} \quad \text{for every} \quad k < n_0.$$

Proof. We consider the case $\operatorname{Re}(s) \leqslant -c < 0$. The other case is analogous.

Recall from the proof of Lemma 2.1 that the solution ϕ can be written as

$$\phi(t) = e^{st}\phi(0) + \int_0^t e^{s(t-u)}\psi(u)\,du \qquad (t \in [-r, r]) \tag{2.7}$$

so that

$$\begin{aligned} |\phi(t)| &\leqslant e^{\operatorname{Re}(s)t}\left[\,|\phi(0)| + \int_0^t e^{-\operatorname{Re}(s)\theta}\psi(\theta)\,d\theta\right] \\ &\leqslant e^{\operatorname{Re}(s)t}\,|\phi(0)| + \left|\frac{e^{\operatorname{Re}(s)t} - 1}{\operatorname{Re}(s)}\right|\,\|\psi\|_c. \end{aligned} \tag{2.8}$$

Recall now that

$$\phi(0) = -\Delta(s)^{-1}\,(\alpha, \psi) \tag{2.9}$$

so that, from (2.6) of Lemma 2.1, one has

$$\begin{aligned} |(\alpha, \psi)| &\leqslant |\psi(0)| + \left|\int_{-r}^{r} d\eta(\theta)\left[\int_0^\theta e^{s(\theta-u)}\psi(u)\,du\right]\right| \\ &\leqslant \|\psi\|_C + \ell\,\|E(\psi)\|_C \end{aligned}$$

where ℓ is the operator norm of L, and

$$\|E(\psi)\|_C \equiv \sup_{\theta\in[-r,r]}\left|e^{s\theta}\int_0^\theta e^{-su}\psi(u)\,du\right| \leqslant \|\psi\|_C \sup_{\theta\in[-r,r]}\left|\frac{1-e^{\operatorname{Re}(s)\theta}}{-\operatorname{Re}(s)}\right|.$$

Since $\operatorname{Re}(s) < 0$, $\sup_{\theta\in[-r,r]}|e^{-\operatorname{Re}(s)\theta} - 1/\operatorname{Re}(s)| = e^{-\operatorname{Re}(s)r} - 1/-\operatorname{Re}(s) < e^{-\operatorname{Re}(s)r/-\operatorname{Re}(s)}$. We have therefore

$$\begin{aligned} |\phi(0)| &\leqslant \|\Delta(s)^{-1}\|\;|(\alpha, \psi)| \\ &\leqslant \|\Delta(s)^{-1}\|\;\{\|\psi\|_C + \ell\,\|E(\psi)\|_C\} \\ &\leqslant \|\Delta(s)^{-1}\|\left\{1 + \ell\,\frac{e^{-\operatorname{Re}(s)r}}{-\operatorname{Re}(s)}\right\}\|\psi\|_C \end{aligned}$$

and therefore

$$\begin{aligned} \|R(s\colon A)\| &\leqslant \sup_{t\in[-r,r]}\left\{e^{-\operatorname{Re}(s)t}\,\|\Delta(s)^{-1}\|\left(1 + \ell\,\frac{e^{-\operatorname{Re}(s)r}}{-\operatorname{Re}(s)}\right) + \frac{e^{\operatorname{Re}(s)t} - 1}{\operatorname{Re}(s)}\right\} \\ &\leqslant \|\Delta(s)^{-1}\|\,\frac{e^{-2\operatorname{Re}(s)r}}{-\operatorname{Re}(s)}\,K + \frac{1}{-\operatorname{Re}(s)} \end{aligned} \tag{2.10}$$

with $K = K(L, r) = \sup\{\ell + xe^{-xr}\colon x \geqslant c\}$.

Note that, for a fixed s, such that $\det \Delta(s) \neq 0$, $R(s\colon A)$ is a bounded linear operator, with norm C_1 say, (depending on s). Therefore, for $\phi = R(s\colon A)\,\psi$,

$$|\dot{\phi}(t)| = |s\phi(t) + \psi(t)| \leqslant |s|\; C_1\, \|\psi\|_c + \|\psi\|_c \qquad (t \in [-r, r])$$

so

$$\|\dot{\phi}\|_C \leqslant C_2\, \|\psi\|_C;$$

where $C_2 = |s|\; C_1 + 1$. From the Ascoli–Arzelà theorem, it now follows that $R(s\colon A)$ is compact. Now the results in Hille–Phillips (1957, p. 211, Theorem 5.14.3) apply to get the conclusion 3 of the Lemma. ∎

Fix now, any number $c > 0$, such that $\sigma(A) \cap \{s\colon |\mathrm{Re}(s)| = c\} = \varnothing$. Then we may split $\sigma(A)$ as follows:

$$\Lambda = \Lambda_S \cup \Lambda_C \cup \Lambda_U$$

$$\Lambda_S = \{s \in \sigma(A)\colon \mathrm{Re}(s) < -c\}$$

$$\Lambda_C = \{s \in \sigma(A)\colon -c < \mathrm{Re}(s) < c\}$$

$$\Lambda_U = \{s \in \sigma(A)\colon \mathrm{Re}(s) > c\}.$$

We also denote

$$M_S = \overline{\mathrm{span}}\{M_s \mid s \in \Lambda_S\}$$

where M_S is the eigenspace spanned by the eigenfunctions associated with s. M_C, M_U are defined analogously. Here, the subscripts S, C, U refer to "stable," "center," "unstable" spaces. The closure is in the top topology on C. Suppose that $\Lambda_1 \equiv \{s_1, ..., s_p\}$ is a finite subset of Λ, and let $\Phi_{\Lambda_1} = (\Phi_{\Lambda_{s_1}}, ..., \Phi_{\Lambda_{s_p}})$, $B_{\Lambda_1} = \mathrm{diag}(B_{\Lambda_{s_1}}, ..., B_{\Lambda_{s_p}})$ where Φ_{s_j} is a basis for the generalized eigenspace $N_{n_0}(s_j)$ of s_j and B_{s_j} is the matrix defined by $A\Phi_{s_j} = \Phi_{s_j} B_{s_j}$, $j = 1, ..., p$. Then we have that the following:

Theorem 2.1. *For $\{s_1, ..., s_p\} \subset \Lambda$, the only eigenvalue of B_{s_j} is s_j, and for any vector a of the same dimension in Φ_{Λ_1}, the solution $T_{\Lambda_1}(t)\,\Phi_{\Lambda_1} a$ with initial value $\Phi_{\Lambda_1} a$ at $t = 0$ is given on $(-\infty, +\infty)$ by the relation*

$$T_{\Lambda_1}(t)\,\Phi_{\Lambda_1} a = \Phi_{\Lambda_1} e^{B_{\Lambda} t} a, \qquad t \geqslant 0;$$

$$\Phi_{\Lambda_1}(\theta) = \Phi_{\Lambda_1}(0)\, e^{B_{\Lambda_1}\theta}, \qquad \text{for} \quad -r \leqslant \theta < r.$$

Proof. As in Hale (1977, p. 170). Only notational changes are necessary. ∎

3. DISTRIBUTION OF THE ROOTS OF THE CHARACTERISTIC EQUATION

Our aim is to define three distinct operators on the closure of the eigenfunctions associated with Λ_S, Λ_C, Λ_U, respectively. We shall do this by integrating the resolvent along a suitable path. In order to do this, we need a better knowledge of the distribution of the spectrum $\sigma(A)$ in the complex plane, together with estimates of $\|\Delta(s)^{-1}\|$. This is the purpose of the present section.

We shall also need more restrictive assumptions on the operator L. More specifically, we define the following condition:

$$L(\phi) = A_{-r}\phi(-r) + B_r\phi(r) + \int_{-r}^{r} d\eta(\theta)\,\phi(\theta) \tag{A1}$$

where A_{-r}, B_r are nonsingular matrices and η is continuous at $\pm r$, has only a finite number of jumps, and the induced measure has no continuous part.

This condition (A1) is a standing assumption for the rest of the paper. In Section 4, we shall consider cases in which such condition is not satisfied.

One first piece of information is easy to get:

Lemma 3.1. *Let c be any real number, $c > 0$. Then, there exists a real number $M = M(c) > 0$ such that*

1. *$\det \Delta(s) \neq 0$ for every $s \in D(c)$, where*

$$D(c) \equiv \{s\colon \mathrm{Re}(s) \in [-c, c], |\mathrm{Im}(s)| \geq M(c)\}.$$

2. *There are a finite number of roots of* det $\Delta(s)$ *in $S(c)$, where*

$$S(c) \equiv \{s\colon \mathrm{Re}(s) \in [-c, c], \mathrm{Im}(s) \in [-M(c), M(c)]\}.$$

Proof. $\Delta(s) = sI - A(s)$ where $A(s) = (a_{ij}(s))$ is an $(n \times n)$ matrix with entries

$$a_{ij} = \sum_k b_{ij}^k e^{s\theta_{ij}^k}, \qquad \theta_{ij}^k \in [-r, r].$$

Therefore,

$$\det \Delta(s) = s^n + \sum_{0 \leq j < n} s^j \cdot \left(\sum_i c_{ij} e^{s\theta_{ij}}\right) \equiv s^n + R(s)$$

where $R(s)=0(|s|^{n-1})$ as $|s| \to +\infty$ in the strip $S(c) \equiv \{z: \mathrm{Re}(s) \in [-c, c]\}$; from this, 1 follows obviously; 2 now follows from the fact that the characteristic function $\det \Delta(s)$ is an entire function. ▮

We now study the asymptotic distribution of the zeroes of the characteristic equation $\det \Delta(s)=0$; we can also derive estimates on the magnitude of $\|\Delta(s)^{-1}\|$. Recall that this factor appears in the resolvent, and we shall need this estimate later to show convergence of integrals involving the resolvent. Firstly we need some definitions.

For two given, strictly positive real number c and c_1, we define the following regions in the complex plane. The curvilinear strips V_1, V_2 in the complex plane are defined as follows. Let $(\mu_1, \mu_2)=(1/r, -(1/r))$. Then, we let

$$V_i \equiv \{s \in \mathbb{C}: |\mathrm{Re}(s+\mu_i \log s)| \leqslant c_1\} \qquad i=1, 2;$$

the regions U_0, U_1, U_2 by

$$\begin{aligned} U_1 \equiv \{s \in \mathbb{C}: &|\mathrm{Re}(s+\mu_1 \log s)| > c_1 \\ &|\mathrm{Re}(s+\mu_2 \log s)| < -c_1\} \\ U_0 \equiv \{s \in \mathbb{C}: &|\mathrm{Re}(s+\mu_1 \log s)| < c_1\} \\ U_2 \equiv \{s \in \mathbb{C}: &|\mathrm{Re}(s+\mu_2 \log s)| > c_1\} \end{aligned}$$

and finally the square region

$$R \equiv \{s \in \mathbb{C}: \mathrm{Re}(s) \in [-c, c], \mathrm{Im}(s) \in [-c, c]\}.$$

(See Fig. 2.)

Lemma 3.2. *There are real numbers c and c_1, and a partition of $\mathbb{C}$ as described above such that*

1. *there are no roots of* $\det \Delta(s)$ *in $U_0 \cup U_1 \cup U_2$, i.e., the roots are all contained in $R \cup V_1 \cup V_2$;*
2. *there is a finite number of roots in R;*
3. *there is a real number $c_3 > 0$ such that*

$$\|\Delta(s)^{-1}\| \leqslant c_3 e^{-|\mathrm{Re}(s)| r}$$

on the curves $|\mathrm{Re}(s+\mu_1 \log s)| = c_1$, *and* $|\mathrm{Re}(s+\mu_2 \log s)| = c_1$.

Proof. The proof consists of two parts. In the first, we study the asymptotic distribution of the zeroes of $\det \Delta(s)$, introducing [as in Bellman–Cooke (1963)] an auxiliary function $g(s)$; in the second part, we derive the desired estimates on $\|\Delta(s)^{-1}\|$. A similar analysis is in Banks and Manitius (1975).

We have

$$\Delta(s) = sI - \sum_{j=-m_1}^{m_2} A_j e^{sh_j}$$

with $h_j \in [-r, r]$ and $-h_{-m_1} = h_{m_2} = r$. We also define

$$\begin{aligned} G(s) \equiv e^{sr} \Delta(s) &= se^{sr} I - \sum_{j=-m_2}^{m_2} A_j e^{s(h_j + r)} \\ &= se^{sr} I - A_{m_2} e^{2sr} - A_{m_2 - 1} e^{s(h_{m_2-1}+r)} - \cdots - A_1 e^{s(h_1 + r)} \\ &\quad - A_{-1} e^{s(h_{-1}+r)} - \cdots - A. \end{aligned}$$

We define

$$g(s) = \det G(s).$$

Clearly the roots of $g(s)$ are the same as the roots of $\det \Delta(s)$, and we can therefore equivalently study the distribution of the zeroes of $g(s)$.

The function $g(s)$ is the sum of terms which are products of n factors of the form se^{rs} and e^{wrs}, with $w \in [0, 2]$. So, if $m \in \{0, 1, \dots, n\}$, such a generic term will be of the form

$$s^m e^{[mr + (n-m)kr]s} \equiv s^m e^{\beta s} \qquad (k \in [0, 2])$$

times a constant, which we ignore. Figure 1 illustrates the points in the (β, m) plane that can occur in such terms. Therefore, for a fixed m, we have $\beta \in [mr, mr + 2(n-m)r]$. We conclude that the pairs (β, m) of powers of such factors all lie inside the triangular region drawn in Fig. 1 (including the boundary).

We shall now show that the three vertices of this triangle correspond to nonzero terms of $g(s)$. In fact:

1. From the assumption that A_{-m_1} is nonsingular, $g(s)$ has a nonzero constant term $a_0 = (-1)^n \det A_{-m_1}$. So the pair $(0, 0)$ is one vertex of the triangle.
2. From the assumption that A_{m_2} is nonsingular, $g(s)$ has a nonzero term with e^{2nrs}. So the pair $(2nr, 0)$ is another vertex.

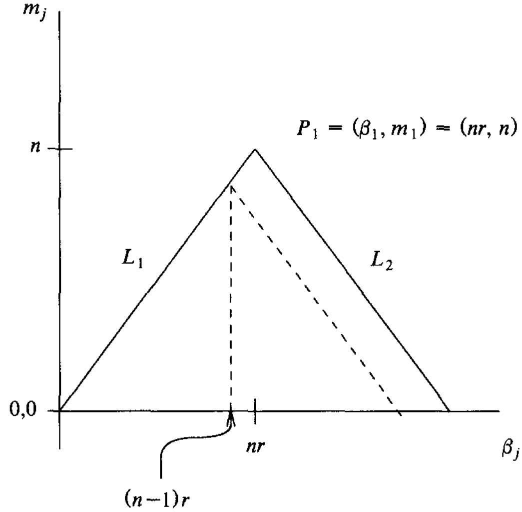

Fig. 1. Distribution diagram of the characteristic equation det $\Delta(s)$. The broken line pictures the distribution diagram for a cofactor of $G(s)$.

3. From the product of elements on the diagonal of $se^{rs}I$, we derive that $g(s)$ has a factor with $s^n e^{nrs}$, so the pair (nr, n) is another vertex.

Now following Bellman–Cooke, we rewrite $g(s)$ as

$$g(s) = \sum_j p_j s^{m_j}(1 + \varepsilon_j(s))\, e^{\beta_j s} \qquad 0 = \beta_0 \leqslant \beta_1 \cdots \leqslant \beta_q,$$

where the ε_j's are functions that satisfy: $|\varepsilon_j(s)| \to 0$ as $|s| \to +\infty$. (The rewriting groups together terms with a common factor $e^{\beta_j s}$, and then chooses among them the one with the highest power m in s^{m_j}.) Now conclusion 1 follows from Bellman–Cooke (1963, Theorem 12.10). See Fig. 2.

Note that, for any pair (β, m): $\beta - mr \geqslant 0$, and $\beta + mr \leqslant 2nr$.

Conclusion 2 is obvious, and the first part of the proof is complete. We proceed to the second part. To prove the desired estimates on $\|\Delta(s)^{-1}\|$, we begin with the study of the matrix $G(s) = (g_{ij}(s))$. Let cof $g_{ij}(s)$ denote the cofactor of $g_{ij}(s)$. Then, by Cramer's rule, $G^{-1}(s) = (\text{cof } g_{ij}(s)/\det G(s))$.

From Bellman–Cooke (1963, Theorem 12.9), we have that

$$|g(s)\, s^{-n} e^{-nrs}| \geqslant c_3 \qquad s \in U_1.$$

Indeed, from our assumption that $\det A_{-m_1} \neq 0$, $\det A_{m_2} \neq 0$, we have immediately the following estimate

$$|g(s)|^{-1} = [|\det A_{-m_1}| + o(1)]^{-1} \simeq |\det A_{-m_1}|^{-1}$$
$$\text{for} \quad |s| \to +\infty, \qquad \text{Re}(s) < 0$$

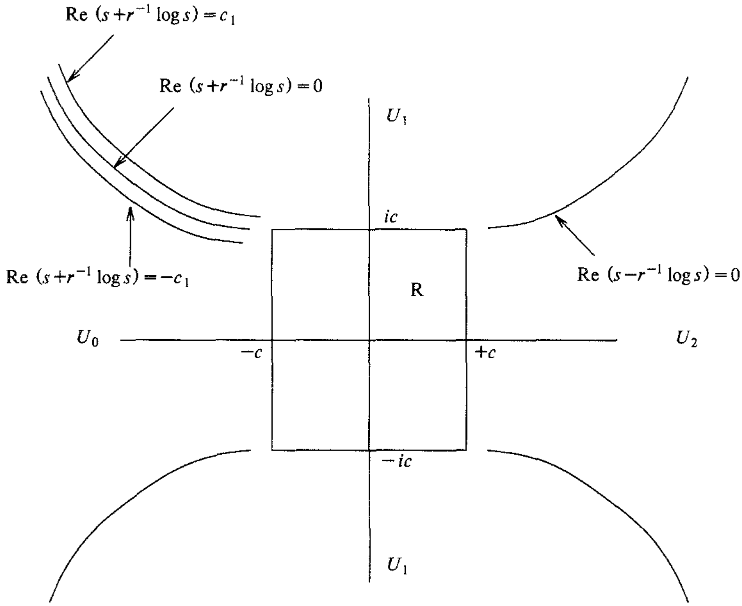

Fig. 2. Asymptotic distribution of the roots of det $\Delta(s)=0$.

because $(-1)^n \det A_{-m_1}$ is the only constant term; and

$$|g(s)^{-1}| = O(e^{-2nr\mathrm{Re}(s)})\ |s| \to +\infty, \qquad \mathrm{Re}(s) > 0$$

because the term $\det A_{m_2} e^{2nrs}$ is nonzero.

We now consider the cofactors of the matrix $G(s)$. We note first of all that, with a reasoning identical to the one for the case of $g(s)$, we can conclude that the distribution diagram of any cofactor is a triangle with vertices $(0, 0)$, $((n-1)\,r, n-1)$, $(2(n-1)\,r, 0)$:

Now we note that, on $\Gamma_1 \equiv \{s\colon |\mathrm{Re}(s+1/\tau \log s)| = c_1\}$, we have $|s| = e^{rc_1}\,|e^{-rs}|$, and therefore the elements of the cofactors satisfy $|s^m e^{\beta s}| = e^{mrc_1}\,|e^{-mrs+\beta s}|$, so that, since $\beta \geqslant mr$,

$$|\mathrm{cof}\ g_{ij}(s)| = O(1) \qquad s \in \Gamma_1.$$

Analogously on $\Gamma_2 \equiv \{s\colon \mathrm{Re}(s+1/\gamma \log s) = c_1\}$, we have $|s| = e^{rc_1}\,|e^{rs}|$, and therefore the elements of the cofactors satisfy: $|s^m e^{\beta s}| = e^{mrc_1}\,|e^{(\beta+mr)s}|$, so that, since $\beta + mr \geqslant 2mr$,

$$|\mathrm{cof}\ g_{ij}(s)| = O(|e^{2(n-1)rs}|).$$

We may now conclude

$$\|G(s)^{-1}\| \leqslant \max_{ij} \{|\operatorname{cof} g_{ij}(s)| \; |g(s)^{-1}|\}$$
$$\leqslant C \max_{ij} \{|\operatorname{cof} g_{ij}(s)| \; |s^{-n}e^{-nrs}\}$$
$$\leqslant O(1)\,|s^{-1}e^{-nrs}| \leqslant C\,|s^{-n}e^{-nrs}|, \qquad (s \in \Gamma_1); \qquad \text{and}$$
$$\|G(s)^{-1}\| \leqslant O(|e^{2(n-1)rs}|)\,|s^{-n}e^{-nrs}| \leqslant C\,|s^{-n}e^{(n-2)rs}|, \qquad (s \in \Gamma_2)$$

and therefore, for $\Delta(s)^{-1} = e^{rs}G(s)^{-1}$, one has

$$\|\Delta(s)^{-1}\| \leqslant \begin{cases} C\,|s^{-n}e^{(n-1)rs}| & (s \in \Gamma_1) \\ C\,|s^{-n}e^{(n-1)rs}| & (s \in \Gamma_2). \end{cases}$$

Now as above, we conclude

$$\|\Delta(s)^{-1}\| \leqslant Ce^{-|\mathrm{Re}\,s|r}. \quad \blacksquare$$

Example. For the one-dimensional equation

$$\dot{x}(t) = ax(t) + bx(t-r) + cx(t+r),$$

where a, b, c are real numbers, we have the characteristic equation $s - a - be^{-sr} - ce^{sr} \equiv \Delta(s) = 0$. Its zeroes are the zeroes of $se^{sr}(\varepsilon(s)) - b - ce^{2sr} = e^{sr}\,\Delta(s) \equiv g(s) = 0$ where $\varepsilon(s) = -a/s$, so that $\varepsilon(s) \to 0$ as $|s| \to +\infty$. The distribution diagram has vertices $(0, 0)$, $(r, 1)$, $(2r, 0)$. Furthermore, the estimates on g and Δ are as follows: for $s \in U_0$, $|g(s)| \geqslant c_3$ and $|\Delta(s)^{-1}| \leqslant c_3^{-1}\,|e^{rs}|$; for $s \in U_1$, $|g(s)| \geqslant c_3\,|se^{rs}|$ and $|\Delta(s)^{-1}| \leqslant c_3^{-1}\,|s^{-1}|$; and for $s \in U_2$, $|g(s)| \geqslant c_3\,|e^{2rs}|$; $|\Delta(s)^{-1}| \leqslant c_3^{-1}\,|e^{-rs}|$.

Note that, in Γ_1, $|s| = c\,|e^{-rs}|$ and $|s^{-1}| = ce^{-r|\mathrm{RE}(s)|}$, so $|\Delta(s)^{-1}| = O(e^{-r|\mathrm{Re}(s)|})$.

4. HAMILTONIAN TYPE SYSTEMS

The nondegeneracy assumption (A1) will not, in general, be satisfied in the case of Hamiltonian-type systems, which therefore need a specific analysis. We introduce first a notation that will be used in this section only. We let $y_t(s) = y(t+s)$ for $s \in [-\tau, 0]$, and $y^t(s) = y(t+s)$ for $s \in [0, \tau]$, whenever $y(t+s)$ is defined. The systems we shall consider are of the form

$$\begin{aligned} \dot{x}(t) &= X(x_t, p(t)), \\ \dot{p}(t) &= P(x(t), pt). \end{aligned} \tag{4.1}$$

Note that, in Hamiltonian systems, the vector p is naturally a column vector; here, we use the transposed form for simplicity of notation. We also assume that the zero vector in $\mathbb{R}^{2n}$ is, in a naturally defined way, a solution of (4.1), and

(A2). *The linearized system at 0 of 4.1 is*

$$\dot{x}(t) = \int_{-r}^{0} d\eta_1(\theta)\, x(t+\theta) + Bp(t) \equiv Mx_t + Bp(t)$$

$$\dot{p}(t) = Cx(t) \int_{0}^{r} d\eta_2(\theta)\, p(t+\theta) \equiv Cx(t) + Np^t$$

where

$$\int_{-r}^{0} d\eta_1(\theta)\, x(t+\theta) = Sx(t-r) + \int_{-r}^{0} d\eta_1^*(\theta)\, x(t+\theta);$$

$$\int_{0}^{r} d\eta_2(\theta)\, p(t+\theta) = Rp(t+r) + \int_{0}^{r} d\eta_2^*(\theta)\, p(t+\theta);$$

S and R are nonsingular matrices; η_1^* and η_2^* are functions of bounded variation.

The characteristic equation now is

$$\Delta(s) = \begin{pmatrix} sI - \int_{-r}^{0} d\eta_1(\theta)\, e^{s\theta} & B \\ C & sI - \int_{0}^{r} d\eta_2(\theta)\, e^{s\theta} \end{pmatrix} \equiv \begin{pmatrix} sI - A(s) & B \\ C & sI - D(s) \end{pmatrix}. \tag{4.2}$$

The asymptotic distribution of the roots of $\det(sI - A(s)) = 0$ and $\det(sI - D(s))$ is well known [see Bellman and Cook (1963)]. In particular, it is known that the roots of norm larger than c, for c large, are contained in the two curvilinear strips:

$$V_1 = \left\{ s \in \mathbb{C} : \left| \operatorname{Re}\left(s + \frac{1}{r} \log s \right) \right| \leqslant c_1 \right\}$$
$$V_2 = \left\{ s \in \mathbb{C} : \left| \operatorname{Re}\left(s - \frac{1}{r} \log s \right) \right| \leqslant c_2 \right\} \tag{4.3}$$

for $\det(sI - A(s)) = 0$, $\det(sI - D(s)) = 0$, respectively. In other words, the matrices B and C do not affect the asymptotic distribution of the zero.

We now prove that the roots of $\det \Delta(s)$ are asymptotically distributed like the union of the roots of the two characteristic equations $\det(sI - A(s)) = 0$, $\det(sI - D(s)) = 0$.

Lemma 4.1. *There exist constants $c_0 > 0$, $c_1 > 0$, and $c_2 > 0$ such that, if $\det \Delta(s) = 0$, and $|s| > c_0$, then*

$$s \in V_1 \cup V_2.$$

Proof. We define the region U_1' as $\{s \in \mathbb{C}: \mathrm{Re}(s + 1/r \log s) \geqslant c_1, \mathrm{Re}(s) < 0\}$, and U_1'' as $\{s \in \mathbb{C}: \mathrm{Re}(s - 1/r \log(s)) \leqslant -c_1, \mathrm{Re} s \geqslant 0\}$.

We prove that, for large $|s|$, no root can be found in the region U_1'. The argument for the other cases (U_0, U_1'', U_2), is similar.

It is easy to see that, on the region U_1',

$$|A(s)| = O(|e^{-sr}|), \qquad |e^{-sr}s^{-1}| \leqslant e^{-rc_1}, \qquad \text{and} \qquad |D(s)| = O(1)$$

so we conclude that, choosing c_1, c_0 large enough, we can make the matrix $\Delta(s)\, s^{-1}$, for $|s| > c_0$, $s \in U_1'$ arbitrarily close in norm to the identity, and therefore nonsingular. ∎

For our future purposes, we need a more detailed analysis of the distribution of the zeroes of $\det \Delta(s)$.

The rest of the notation is as in Section 3. In particular, we shall denote $\tilde{V}_1$ any subregion of V_1 in which s is uniformly bounded away from the roots of $\det \Delta(s)$.

Our $G(s) = e^{sr}\, \Delta(s)$ matrix is now given by

$$G(s) = \begin{pmatrix} se^{sr}I - \int_{-r}^{0} d\eta(\theta)\, e^{s(r+\theta)} & Ae^{sr} \\ Be^{sr} & se^{sr} - \int_{0}^{r} d\eta(\theta)\, d\eta(\theta)\, e^{s(r+\theta)} \end{pmatrix}. \tag{4.4}$$

As in Section 3, the pairs (β, m) of the factors of $\det G(s)$ lie inside the triangle with vertices $(0, 0)$; $(snr, 2n)$, $(4nr, 0)$.

We have now

$$|g(s)^{-1}| \leqslant c_2\, |s^{-n}e^{-nrs}| \qquad \text{on} \quad U_0 \cup \tilde{V}_1, \qquad |s| \geqslant c$$
$$|g(s)^{-1}| \leqslant c_3\, |s^{-2nr}e^{-2nrs}| \qquad \text{on} \quad U_1 \cup \tilde{V}_1, \qquad |s| \geqslant c.$$

From the way the regions U_0, U_1 are defined,

$$|e^{rs}| \leqslant e^{c_1 r} \quad \text{on} \quad U_0; \qquad \text{and} \qquad |e^{sr}s| \geqslant e^{rc_1}, \qquad |e^{sr}s^{-1}| \leqslant e^{-rc_1}.$$

The cofactors $g_{ij}(s)$ of the matrix $G(s)$ consist of combination of elements $s^m e^{\beta rs}$ with $(m, \beta r)$ in the triangle with vertices (nr, n), $(snr-1, 2n-1)$, $(3nr-1, n)$. On U_0, this estimate on the cofactors together with the above gives

$$|g_{ij}(s)\, g^{-1}(s)| \leqslant |s|^{2n-1}\, |e^{(2n-1)rs}|, \qquad \text{and therefore}$$
$$|g_{ij}(s)\, g^{-1}(s)| = O(|s|^{-1}\, |e^{-rs}|) \qquad s \in U_1.$$

From the definition, $\Delta^{-1}(s) = e^{sr}G^{-1}(s)$, and so

$$|\Delta^{-1}(s)| = O(|e^{rs}|), \qquad s \in U_0 \cup \tilde{V}_1, \qquad |s| \geqslant c$$
$$|\Delta^{-1}(s)| = O(|s|^{-1}), \qquad s \in U_1 \qquad |s| \geqslant c.$$

Example. For the system

$$\dot{x}(x) = mx(t-r) + bp(t)$$
$$\dot{p}(t) = cx(t) + np(t+r)$$

we have

$$G(s) = \begin{pmatrix} se^{rs} - m & -be^{rs} \\ -ce^{rs} & se^{rs} - ne^{2rs} \end{pmatrix}$$

and $$g(s) = s^2 e^{2rs} - nse^{3rs} - mse^{rs} + (mn - bc)\, e^{2rs},$$

and, therefore,

$$|g(s)^{-1}| = O(|s|^{-1}\, |e^{-rs}|) \qquad \text{on} \quad U_0;$$
$$|g(s)^{-1}| = O(|s|^{-2}\, |e^{-2rs}|) \qquad \text{on} \quad U_1.$$

Recalling that $|e^{sr}s| \leqslant c$, on U_0; and $|e^{sr}| \leqslant c\,|s|$, $|e^{sr}s| \geqslant c$ on U_1, then $|G^{-1}(s)| = O(1)$ on U_0, and $|G^{-1}(s)| = O(|s|^{-1}\, |e^{-rs}|)$ on U_1, and so $|\Delta^{-1}(s)| = O(|e^{rs}|)$ on U_0, $|\Delta^{-1}(s)| = O(|s|^{-1})$ on U_1.

5. CONSTRUCTION OF THE SOLUTION OPERATORS

We now determine the paths in the complex plane along which we shall perform the integration (Fig. 3). Note that the curve $\{s\colon \mathrm{Re}(s + 1/r \log s) = c_1\}$, the boundary of U_1 in the left half complex plane, is also described by (for x large enough) $s = x \pm iy(x)$, where we define

$$y(x) \equiv (e^{2r(c_1 - x)} - x^2)^{1/2} = e^{r(c_1 - x)}[1 + o(1)],$$

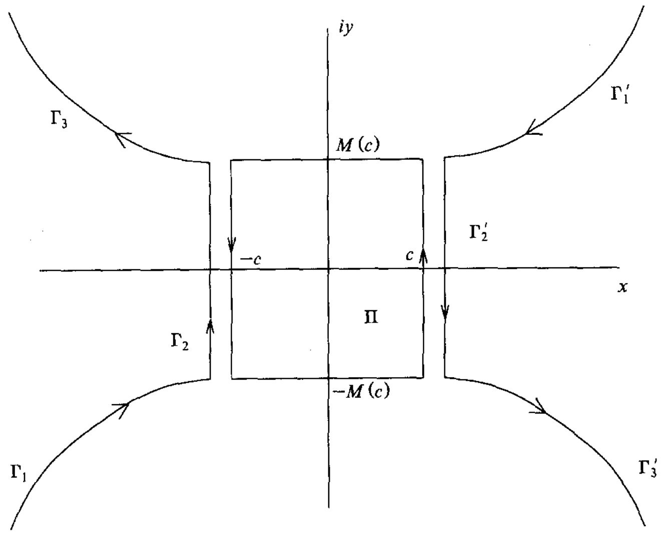

Fig. 3. Paths of integration.

as $|x| \to \infty$. One easily checks

$$|y'(x)| \cong re^{r(c_1 - x)}$$

[where we denote $f(x) \simeq g(x)$ if and only if there exists a pair $c > 0$, $x_0 > 0$ such that $c^{-1}f(x) \leqslant g(x) \leqslant cf(x)$ for $|x| > x_0$].

We now define the following path in the complex plane: $\Gamma = \bigcup_{i=1}^{3} \Gamma_i$, where

$$\Gamma_1 = \left\{s\colon \operatorname{Re}\left(s + \frac{1}{r}\log s\right) = c_1,\ \operatorname{Im}(s) < 0,\ \operatorname{Re}(s) \leqslant -c\right\}$$

$$\Gamma_2 = \{s\colon \quad \operatorname{Re}(s) = -c, \qquad \operatorname{Im}(s) \in [-M(c), M(c)]\}$$

$$\Gamma_3 = \left\{s\colon \operatorname{Re}\left(s + \frac{1}{r}\log s\right) = c_1,\ \operatorname{Im}(s) > 0,\ \operatorname{Re}(s) \leqslant -c\right\}$$

(see Fig. 4). Analogously we define the path $\Gamma' = \bigcup_{i=1}^{3} \Gamma'_i$, in the case $\operatorname{Re}(s) > 0$. Finally we define the rectangular path of integration Π as

$$\Pi = \{s\colon \operatorname{Re}(s) = \pm c, \operatorname{Im}(s) \in [-M(c), M(c)]\}$$
$$\cup \{s\colon \operatorname{Im}(s) = \pm M(c), \operatorname{Re}(s) \in [-c, c]\}.$$

We shall let E_s denote the (generalized) eigenspace associated with the eigenvalue s, and M_s the subspace of $C([-r, r], \mathbb{R}^n)$ given by restrictions of elements of E_s; also we define (after ordering the eigenvalues in Λ_S, Λ_U, and Λ_C in some fashion; for instance, a lexicographic order over the absolute value of real and imaginary part)

$$E_S^r = \text{Span}\{E_{s_j}: s_j \in \Lambda_S,\ j = \pm 1, \pm 2, ..., \pm k\}$$
$$E_U^r = \text{Span}\{E_{s_j}: s_j \in \Lambda_U,\ j = \pm 1, \pm 2, ..., \pm k\}$$
$$E_C^r = \text{Span}\{E_{s_j}: s_j \in \Lambda_C,\ j = \pm 1, \pm 2, ..., \pm k\}$$

and $E_S = E_S^\infty$, $E_U = E_U^\infty$, $E_C = E_C^\infty$. We define M_S^k, M_U^k, M_C^k, and M_S, M_U, M_C, analogously.

Note that, by Lemma 2.2, we know that, for any finite k, the spaces M_S^k, M_U^k, M_C^k are all finite dimensional.

Lemma 5.1. *For any pair t_0, $t_1: t_0 > 2r$, $t_1 < -2r$, there exist three families of operators*

$$T_S(t): M_S \to M_S \quad \textit{for} \quad t \geqslant t_0$$
$$T_U(t): M_U \to M_U \quad \textit{for} \quad t \leqslant t_1$$
$$T_C(t): M_C \to M_C \quad \textit{for} \quad t \in (-\infty, +\infty)$$

such that

$$T_S(t)\,\phi = \phi_t \qquad \textit{for every} \quad \phi \in M_S$$

and analogously for T_U, T_C, $T_U(t)\,\phi = \phi_t$ for $\phi \in M_U$, $T_C(t)\,\phi = \phi_t$ for $\phi \in M_C$. Also,

$$\|T_S(t)\,\phi\|_C \leqslant C_1(t_0)\, e^{-ct}\, \|\phi\|_C \qquad t \geqslant t_0,$$
$$\|T_U(t)\,\phi\|_C \leqslant C_2(t_1)\, e^{ct}\, \|\phi\|_C \qquad t \leqslant t_1.$$

The constant $C_1(t_0)(C_2(t_1))$ tends in general to $+\infty$ as $t_0 \to 2r(t_1 \to -2r)$.

Proof. We firstly notice that, since only a finite number of elements of $\sigma(A)$ are contained in the rectangle R, the definition

$$T_C(t)\,\phi \equiv \frac{1}{2\pi i}\int_R e^{st} R(s\colon A)\,\phi\, ds$$

provides us with an operator that satisfies all of the claims of the lemma.

We now turn to the construction of the two operators $T_S(t)$, $T_U(t)$.

It is clear that we only need to prove our claims for $T_S(t)$, say. For notational convenience, we shall drop in the sequel of the proof the subscript S.

The first step is the definition:

$$T_k(t)\,\phi \equiv \frac{1}{2\pi i}\int_\Gamma e^{st}R(s\colon A)\,\phi\,ds \qquad \text{for any} \quad \phi\in M^k.$$

Since M^k is finite dimensional, no problem of convergence of the integral arises.

The following clearly hold:

1. $T_k(t)\,\phi_0=\phi(t+\cdot)$ for every $\phi\in E^k$ (because M_k is a finite dimensional subspace),
2. $T_k(t)\,\phi=T_{k+1}(t)\,\phi$ for every $\phi\in M^k$,
3. $k>j$ implies $\|T_j(t)\|\leqslant\|T_k(t)\|$.

We now extend the domain of our operator to the entire space M. We let $I_j=1/2\pi i\int_{\Gamma_j} e^{st}R(s\colon A)\,\phi\,ds$, $j=1,2,3$. Then, one has

$$T_k(t)\,\phi\equiv I_1+I_2+I_3;$$

clearly $\|I_2\|_C\leqslant c\,\|\phi\|_C$.

We now consider I_3; we parametrize the curve with $s\colon s\mapsto x+iy(x)$, $y(x)$ defined at the beginning of the section. From the remarks following its definition, we have $s'(x)=1+iy'(x)$, $y'(x)=O(e^{-rx})$, for $|x|\to+\infty$. Then, for any $t_0>2r$,

$$\begin{aligned}\left\|\frac{1}{2\pi i}\int_{\Gamma_3} e^{st_0}R(s\colon A)\,\phi\,ds\right\|_C &\leqslant\frac{1}{2\pi}\int_{\Gamma_3} e^{\operatorname{Re}(s)t_0}\,\|R(s\colon A)\|\;|ds|\;\|\phi\|_C\\ &\leqslant\frac{1}{2\pi}\int_{-\infty}^{0} e^{xt_0}\left\{\frac{e^{-xr}}{-x}+\frac{e^{-2xr}}{-x}\cdot e^{xr}K\right\}Ce^{-xr}\,dx\;\|\phi\|_C\\ &\equiv C(t_0)\,\|\phi\|_C.\end{aligned}$$

Clearly, for any $t_0>2r$, $C(t_0)$ is finite. The same argument holds for I_1.

We conclude that the operator $T(t)$ is defined for any $t\geqslant t_0>2r$, with the integral above providing an upper bound (dependent on t_0) on its norm.

The exponential estimate follows obviously if we note that, for every $s\in\Gamma$, one has $\operatorname{Re}(s)\leqslant -c$, so that

$$\begin{aligned}\|T(t)\,\phi\|_C &\leqslant\frac{1}{2\pi}\int_\Gamma e^{\operatorname{Re}(s)t_0}e^{-c(t-t_0)}\,\|R(s\colon A)\|\;|ds|\;\|\phi\|_C\\ &\leqslant e^{-ct}[C(t_0)\,e^{ct_0}]\,\|\phi\|_C.\quad\blacksquare\end{aligned}$$

For a continuous function ϕ, we denote supp ϕ the support of ϕ. From the assumption that the function of bounded variation η has only a finite number of jumps, we conclude that there exists an $\varepsilon > 0$ such that, for every ϕ with supp $\phi \subset (-\varepsilon, 0)$, we have $\int_{-r}^{r} d\eta(\theta)\, \phi(\theta) = 0$; and the same for every ϕ with supp $\phi \subset (0, \varepsilon)$.

We can now prove the following:

Lemma 5.2. *There exists a constant C depending only on L, such that, for any $\phi \in M_S$ and for any $\psi \in M_U$,*

$$|\phi(t)| \leqslant C\, \|\phi\|_C, \qquad \text{for any} \quad t \in [r, 2r]]$$

$$|\psi(t)| \leqslant C\, \|\psi\|_C, \qquad \text{for any} \quad t \in [-2r, -r].$$

Proof. We only consider the case of $\phi \in M_S$, since the proof for $\psi \in M_U$ is analogous. Choose first a $t_0 > r$ with $t_0 - r < \varepsilon$; using the exponential estimate above, we have that, for any $t \in [r, r+\varepsilon]$, $|\dot{x}(\phi)(t)| \leqslant e(\|\phi\| + C\, \|\phi\|)$ and therefore, for any such t, $|x(\phi)(t)| \leqslant \|\phi\| + \ell(\|\phi\| + C\, \|\phi\|)\, r \equiv C_2\, \|\phi\|$. The statement now follows, using again Lemma 5.1. ∎

Remark. It is obvious now that, by changing if necessary the constant C in the conclusion of the Lemma 5.2, we obtain the exponential decay estimate for every $t \geqslant 0$ in the case of $T_S(t)$ (and for every $t \leqslant 0$ for $T_U(t)$).

The final step is the extension of the (families of) operators defined above to the closure of their domain of definition.

Let $\bar{M}_S(\bar{M}_U)$ denote the closure, in the C norm, of M_S and M_U, respectively. (Recall that M_C is finite dimensional and therefore closed, so no extension is necessary.)

From the above results, we already know that there exist three families of operators $T_S(t)$, $T_U(t)$, $T_C(t)$, defined on M_S, M_U, M_C and for $t \geqslant 0$, 0, $t \in \mathbb{R}$, respectively, and satisfying

$$\|T_S(t)\, \phi\| \leqslant Ce^{-ct}\, \|\phi\|_C, \qquad t \geqslant t \geqslant t_0 > 2r \qquad \text{for every} \quad \phi \in M_S$$

$$\|T_U(t)\, \psi\| \leqslant Ce^{ct}\, \|\psi\|_C, \qquad t \leqslant -t_0 \leqslant -2r \qquad \text{for every} \quad \psi \in M_U$$

for some positive constant C.

Theorem 5.1. *There exist three families of operators, which define C_0 semigroups:*

$$T_S: \bar{M}_S \to C \qquad \textit{defined for} \quad t \geqslant 0$$

$$T_C: M_C \to C \qquad \textit{defined for} \quad t \in (-\infty, \infty)$$

$$T_U: \bar{M}_U \to C \qquad \textit{defined for} \quad t \leqslant 0$$

which satisfy, with the constant C as above,

$$\|T_S(t)\| \leqslant Ce^{-ct}, \qquad t \geqslant 0$$

$$\|T_U(t)\| \leqslant Ce^{ct}, \qquad t \leqslant 0$$

and for $\phi \in \bar{M}_S$, $\psi \in \bar{M}_U$, $\xi \in M_C$ *we have*

$$T_S(t)\,\phi = x(\phi)_t \qquad \textit{where} \quad x(\phi)\colon [-r, +\infty) \to \mathbb{R}^n$$

is the solution of $\dot{x}(t) = Lx_t$ *for* $t \geqslant 0$, *and* $x_0 = \phi$;

$$T_U(t)\,\psi = x(\psi)_t \qquad \textit{where} \quad x(\psi)\colon (-\infty, +r] \to \mathbb{R}^n$$

is the solution of $\dot{x}(t) = Lx_t$, *for* $t \leqslant 0$, *and* $x_0 = \psi$;

$$T_C(t)\,\xi = x(\xi)_t \qquad \textit{where} \quad x(\xi)\colon (-\infty, \infty) \to \mathbb{R}^n$$

is the solution of $\dot{x}(t) = Lx_t$, *for every t; and* $x_0 = \xi$.

Proof. The statement for $T_C(t)$ is obvious.

As usual, we only need to prove our claim for $T_S(t)$, say. The extension is a standard application of an Ascoli–Arzelà line of argument.

Consider any $\phi \in \bar{M}_S$, which is the uniform limit of a sequence $\{\phi_i\} \subset M_S$.

Consider first any $t_1 : r \leqslant t_1 < +\infty$, and define $x(\phi_i)$ by $x(\phi_i)(t) = T(t)\,\phi_i(0)$ where we have dropped the subscript S for notational simplicity. The sequence of derivatives is uniformly bounded. In fact (assuming w.l.o.g. $\|\phi_i - \phi\|_C \leqslant 1$, $i = 1, \cdots$), we have, for $t \in [0, t_1]$ and for any i,

$$|\dot{x}(\phi_i)(t)|_{\mathbb{R}^n} = |L(x(\phi_i)_t)| \leqslant \ell \; \|x(\phi_i)_t\|_C \leqslant \ell C_1.$$

Clearly also the sequence $\{x(\phi_i)\}$ is uniformly bounded, and equicontinuous; let $x(\phi)$ denote the uniform limit of a subsequence.

Now, for every $t \in [0, t_1]$, the limit of the (sub)sequence defined by

$$\dot{x}_i(\phi)(t) = \int_{-r}^{r} x_i(\phi)(t+\theta)\, d\eta(\theta)$$

exists, and defines a continuous function of t; call f such function. From the Lebesgue-dominated convergence theorem, such function is almost everywhere the derivative of the function $x(\phi)$; and it also satisfies the equality

$$\dot{x}(\phi)\, t = \int_{-r}^{r} x(\phi)(t+\theta)\, d\eta(\theta).$$

We can now define a function $x(\phi)$ with analogous properties as $[-r, +\infty)$, by taking a sequence $\{t_1^k\}$, $\lim_k t_1^k = +\infty$; since the exponential estimate $\|T(t)\, \phi_i\| \leqslant Ce^{-ct}\, \|\phi_i\|$ is satisfied as every element of the sequence, it will be satisfied in the limit. Finally,

$$\lim_{t \to 0} T(t)\, \phi = \phi \qquad \text{for any} \quad \phi \in \bar{M}_S$$

follows from the fact that such equality holds for any $\phi \in M_S$, and that $T(t)$ is uniformly (in t) bounded in norm. ∎

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