Public Choice **82**: 159–180, 1995.

Third parties in equilibrium*

SIMON HUG
Département de science politique, Université de Genève, 102, Bd. Carl-Vogt, CH-1211 Genève 4

Accepted 5 April 1994

Abstract. Under standard assumptions equilibria with three parties normally do not exist in spatial models of electoral competition. In this paper I show that such equilibria are possible if it is assumed that voters are uncertain about the exact policies parties will adopt once elected. Substantive predictions can be derived from the model, explaining some features of three-party competition. First, the least risk party will always take the most moderate position. Second, this position is also winning. Third, the two riskier parties are always on opposite sides of the median voter and also of the moderate party.

1. Introduction

The number three appears to be a cursed number in the spatial theory of voting. Several authors have shown that under most assumptions equilibria with three parties[1] are not possible (Eaton and Lipsey, 1975; Shaked, 1982; Hermsen and Verbeek, 1992: 160; and Osborne, 1992). Equilibria with two or more than three parties, however, can be obtained under varying assumptions (Eaton and Lipsey, 1975; Greenberg and Shepsle, 1987; and Hermsen and Verbeek, 1992). Only few models exist where three parties are in equilibrium, and most often these are very special cases (Palfrey, 1984; Austen-Smith and Banks, 1988; Feddersen, Sened and Wright, 1990).[2] These results are quite bothersome, since even in strongly majoritarian systems third parties are more often the rule than the exception (e.g., Pinard, 1967; Lemieux, 1977; Rosenstone, Behr and Lazarus, 1984).

In this paper I will present a model which follows almost entirely the standard specifications of models in the spatial theory of voting. The only modifi-

* This is a heavily revised version of the paper "Uncertainty and New Political Parties," prepared for presentation at the Annual Meeting of the Midwest Political Science Association, Chicago 1992. Thanks to Chris Achen, Jeff Banks, Lars-Erik Cederman, Douglas Dion, Joel Kaji, Martin Osborne, Gerald Schneider and an anonymous reviewer for their extremely helpful remarks. While almost all interesting and important ideas of this paper are theirs, the remaining errors, shortcomings and not-so-good ideas are all mine.

cation concerns the information that voters have on parties. Instead of assuming that voters have perfect knowledge of the positions of parties, I model voters as having a common probabilistic assessment of the parties' position. Furthermore voters in my model are risk-averse. Given these changes, equilibria with three parties become possible. Additionally, the model yields some important comparative statics results concerning the position taken by the parties. The party which is the least risky choice for the voters always adopts the most moderate position. This moderate policy stance yields a plurality of votes. The two riskier parties are generally located on opposite sides of the safer choice.

Before presenting the model in detail I will discuss briefly some models in the spatial theory of voting, where voters are uncertain about the parties' position, or where third parties are in equilibrium. Once the model is presented I will derive the results, showing that equilibria are possible under various conditions. Comparative statics analysis will yield some important predictions concerning the positions of parties. Finally, in the discussion I will critically explore the model at the light of its empirical relevance. This suggests some avenues for future research.

2. Uncertainty and third parties in the spatial theory of voting

2.1. Models of uncertainty

In standard models of spatial competition the assumption that voters are perfectly informed on the parties' position plays a central role. Several authors have implicitly shown that one of the central results of the spatial theory of voting, namely the median-voter theorem, does not hold if voters are uncertain (Shepsle, 1972a and 1972b; Enelow and Hinich, 1981; Bernhardt and Ingberman, 1985). The median-voter result, or the principle of minimum differentiation (Eaton and Lipsey, 1975), can only be recovered if either voters are risk-neutral, or the level of uncertainty associated with the different parties is identical. Quite clearly, these are very strong assumptions and likely to be violated.

To recover equilibria with two parties, authors most often impose additional structure. Shepsle (1972a and 1972b), for instance, assumes that the voters experience no uncertainty with respect to the position of the incumbent candidate or party. In face of this situation, the challenger can propose a lottery over several policies, which can prove to be winning if there are enough risk-taking voters.

Bernhardt and Ingberman (1985) propose a similar model, where the incumbent is a Stackelberg-leader and the challenger is a follower. Voters are uncertain about what policy will be carried out, if one party gets elected. They associate a certain probability distribution to the outcome, where the mean corresponds to the position taken by a given party. Bernhardt and Ingbermann argue that the variance of the probability distribution around the position of the incumbent will be lower than the one of the challenger, since voters could observe the incumbent in office and have more information on his or her performance. This Stackelberg game has an equilibrium, where the parties take different positions, as long as the variances associated with their positions differ.

The model developed by Banks (1990) takes up some of these ideas, but develops them a step further. In his model parties have ideal points, but can announce platforms differing from their most preferred policy. Voters, by observing the announced platforms of the parties, try to infer the ideal point of the party and vote according to their estimate. The party with the most votes gets elected, but has to pay a price, which is increasing in the difference between the announced platform and its ideal point. Banks uses as solution concept a refinement of the sequential equilibrium. Different "universally divine electoral equilibria" are possible, depending on the costs a party has to bear for announcing a platform far from its most preferred policy point.

2.2. Models with third parties

Among the models allowing equilibria with three parties, the one developed by Palfrey (1984) plays a central role. He argues that the well known median voter theorem or the principle of minimum differentiation (Eaton and Lipsey, 1975), which holds under the assumption of a fixed two-candidate competition in a one-dimensional space, is flawed, both empirically and theoretically. On the empirical level perfect convergence of two parties can only rarely be observed. On the theoretical level, according to Palfrey, it seems quite unconvincing that the two parties do not consider the potential threat of new competitors. In his model he explicitly includes this potential threat by letting two established parties play a Cournot-Nash game while being simultaneously Stackelberg leaders in respect to a third party. In this specification the median-voter theorem no longer holds. Palfrey uses as solution concept the limit-equilibrium, which Shepsle and Cohen (1990: 33) describe informally in the following way: "In Hotelling-like fashion, competing against an established opponent is a force for convergence. But to remain an established party means protecting against prospective entry by not-established parties seeking to displace a vulnerable established party. This latter concern discourages convergence. The balance of these two opposed forces is a limit-equilibrium."

In equilibrium the two established parties take positions at equal distance from the median voter,[3] while the third party enters at the position of the median voter. One of the established parties, both having equal vote shares, will win the election,[4] while the entrant will always lose the election. These results are considered very critically by Shepsle and Cohen in their review of models on entry: "... [S]ome of the conclusions Palfrey derives are substantively troublesome. First, if the entrant never wins ..., then one must wonder why she bothers entering (or why she bothers vote maximizing). Second, the entrant's vote maximizing location in the Palfrey model is always *between* the established parties ..., at least some American experience with third-party entry would seem to contradict it .... Third, much of the rationale for third-party entry is, contra Palfrey, the alleged *convergence* of the established parties," (Shepsle and Cohen, 1990: 30, emphasis in the original).

A different approach is taken by Austen-Smith and Banks (1988). They argue that voters in PR systems with more than two parties explicitly consider the coalition-bargaining stage in their voting decision. Considering a three-stage model, where parties first take positions, then voters cast their ballot and finally parties form a governmental coalition if necessary, they show that three parties can be in equilibrium.

Finally, Feddersen, Sened and Wright (1990) present a model with sophisticated voters in an election under plurality rule, where the number of competitors is determined endogenously. Their assumptions and solution concept, the Nash-equilibrium, lead to results, interesting on a purely theoretical level, but of little empirical value.[5] In equilibrium all parties that enter locate themselves at the median voter position and the voters vote as if they were sincere. The probability of one party winning the election is equal to $\frac{1}{\text{number of parties}}$ and the number of parties is determined by the cost-benefit-ratio of holding office.[6] This model can under certain restrictions accommodate an equilibrium with three parties.

While all models address important aspects of three-party competition, they are all limited in their own way. The model of Austen-Smith and Banks (1988) relies on the coalition formation to induce an equilibrium with three parties on the electoral stage. While this is of crucial importance in PR systems, it has no relevance in systems where governments are not coalitions. Feddersen, Sened and Wright (1990) find a rather unappealing equilibrium, where all parties enter at the same position. Sophisticated voters maintain this equilibrium by threatening a collective punishment for any deviating party. Finally, Palfrey (1984) imposes a very strong structure, by letting two parties be leaders and the third one a follower. In this respect it is interesting that Osborne (1992:7) finds in a model with much less structure that in equilibrium three parties would never enter together an electoral competition. The question remains, therefore, how a three-party competition would look like.

3. A model with uncertain voters

3.1. Assumptions

My model follows in its basic assumptions the standard specifications. Parties are assumed to be vote maximizers. They propose party platforms which correspond to positions in the policy space. Voters observe the position (μ_k) taken by party k but are uncertain what policy it will adopt once in office. They have a common assessment about the distribution of possible enacted policies, characterized by a probability distribution with mean μ_k and variance σ_k^2.[7] Three parties compete in the election: $k \in \{1,2,3\}$.

Voters have bliss-points in a one-dimensional policy space (R^1), and I assume that their utility functions are quadratic loss-functions:[8]

$$U_i(\mu_k) = -(x_i - \mu_k)^2 \tag{1}$$

where x_i is voter i's bliss point. Since voters are risk-averse and are unsure about the exact policy a party will enact, the expected utility for voter i voting for party k is equal to:[9]

$$E(U_i(\mu_k)) = -(x_i - \mu_k)^2 - \sigma_k^2 \tag{2}$$

Voters vote sincerely[10] for the party to which they attribute the highest expected utility. If several parties yield the same expected utility, they vote for each of those parties with equal probability. Furthermore, voters are distributed according to some continuous, twice differentiable, symmetric density function (f) over the policy space with finite mean and variance. For ease of presentation I assume that the mean is equal to zero. Given these assumptions, one important property can be derived immediately. For every pair of parties (e.g., 1 and 2), located at nonidentical positions, there exists some voter with bliss point x_{12}, who is indifferent between voting for either one of them.[11]

$$x_{12} = \frac{\mu_1 + \mu_2}{2} + \frac{\sigma_2^2 - \sigma_1^2}{2(\mu_2 - \mu_1)} \tag{3}$$

3.2. Results

In the absence of the crucial feature, namely uncertainty, Eaton and Lipsey (1975) as well as Shaked (1982) have proven that no Nash-equilibrium exists in pure strategies for three competitors in a spatial competition.[12] Furthermore, Dasgupta and Maskin (1986a and 1986b) have shown that models of spatial competition violate two necessary conditions for using fixed-point theorems to prove the existence of equilibria in pure strategies. This makes it more difficult to derive equilibria for the model specified, even more so since my model differs in one important aspect, namely uncertainty, from other models of spatial competition.

To derive the results I will adopt the following strategy. Since the variances associated with the three parties are exogenous,[13] they characterize different sets of cases. These are defined by different relationships among the three variances. In increasing order of generality these cases are the following: i) all parties have the same variance; ii) two parties have the same variance, while third has a larger one; iii) one party has a low variance, while the other two have larger ones; iv) all parties have different variances. I will explore all these cases in only three propositions, since I can show that the last two cases do not differ substantially. In this exploration I will show that equilibria with three parties can exist. I will consider, however, only equilibria where all parties win some electoral support. This is an important though innocuous restriction.[14] The first case leads to the following proposition, showing that no equilibrium is possible if the variances associated with three parties are identical.

Proposition 1. No equilibrium exists if three parties all have the same variances: $\sigma_1^2=\sigma_2^2=\sigma_3^2$.

Proof. Suppose the variances associated with the three parties are identical. Then the indifferent voter for each pair of parties ($k,l \in \{1,2,3\}, k \neq l$) becomes the following.

$$x_{kl} = \frac{\mu_k + \mu_l}{2}$$

This implies that the differences in variances become irrelevant and consequently the results from models without voter uncertainty apply. For theses cases Eaton and Lipsey (1975) and Hermsen and Verbeek (1992:160) have shown that no equilibria with three parties exist.[15]

A consequence of the preceding proposition is that if an equilibrium with three parties exists, at most two parties have the same variance. If the highest variance is associated with a single party the following lemma is crucial.[16]

Lemma 1. Suppose one party has a strictly higher variance than the other two ($\sigma_1^2= \sigma_2^2 < \sigma_3^2$), and the voter density function is strictly monotonically increasing on $(-\infty,0)$[17]. Then in equilibrium $\mu_1^* = \mu_2^*$.

Given the particular relationship among the three variances of lemma 1 equilibria with three parties can exist. The necessary conditions for this *single unsafe case*[18] figure in the following proposition.

Proposition 2. (single unsafe case) If $\sigma_1^2 = \sigma_2^2 < \sigma_3^2$, the voter density function is strictly monotonically increasing on $(-\infty,0)$, $F(\frac{\mu_3^* + \mu_1^*}{2}) \leq \frac{1 - F(\mu_3^*)}{2}$, $F(\mu_1^*) = \frac{1 + F(x_{13}^*)}{2}$, and $f(\mu_1^*) \geq 2f(x_{13}^*)$ then there exists a three-party equilibrium. In this equilibrium $\mu_1^* = \mu_2^*$, and $\mu_3^* = \mu_1^* - \sqrt{\sigma_3^2 - \sigma_1^2}$.[19]

The next case to consider is the one where a single party has a strictly lower variance than the other two. As already mentioned, the relationship between the two higher variances does not subdivide this category further.

Lemma 2. Suppose one party has a lower variance than the other two, and the voter density function is strictly monotonically increasing on $(-\infty,0)$. Then the two other parties with higher variances are not located on the same side of the first party. More formally, if $\sigma_1^2 < \sigma_2^2 \leq \sigma_3^2$ then either $\mu_2^* < \mu_1^* < \mu_3^*$ or $\mu_3^* < \mu_1^* < \mu_2^*$.

This lemma shows that the party with the lowest variance will always be in the middle. Given this result, I can show with the next lemma that the parties with higher variances take positions on opposite sides at equal distance from the median voter.

Lemma 3. Suppose one party has a lower variance than the other two, and the voter density function is strictly monotonically increasing on $(-\infty,0)$. Then the two riskier parties are at equal distance from the median voter. More formally, if $\sigma_1^2 < \sigma_2^2 \leq \sigma_3^2$, then equilibrium behavior implies $\mu_2^* = -\mu_3^*$.

Interestingly enough this lemma reflects a result of Palfrey (1984) and Austen-Smith and Banks (1988). In both models two parties also take symmetric positions from the median voter, while the third one is located between the two. While Palfrey (1984) also considers a continuous, symmetric density function, Austen-Smith and Banks (1988:409) base their model on a discrete voter distribution.

Still under the assumption that one party has a strictly smaller variance than the other two, proposition 3 states necessary conditions, which together are sufficient to guarantee an equilibrium. Apart from the positions of the three parties, which are determined by the differences in variances associated with them, it also imposes certain restrictions on the voter distribution.

Proposition 3. (single safe case) If the voter density function is strictly monotonically increasing on $(-\infty,0)$, if $\sigma_1^2 < \sigma_2^2 \leq \sigma_3^2$ and if $1 - 2F(\mu_2) \geq F(\frac{\sigma_2^2 - \sigma_1^2}{2(\mu_3^* - \mu_2^*)})$ and $1 - 2F(\mu_2^*) \geq F(\frac{\sigma_3^2 - \sigma_1^2}{2(\mu_3^* - \mu_2^*)})$ then there is a three-party equilibrium. In this equilibrium $\mu_2^* = -\mu_3^*$, $\mu_2^* = \mu_1^* - \sqrt{\sigma_2^2 - \sigma_1^2}$ and $\mu_3^* = \mu_1^* + \sqrt{\sigma_3^2 - \sigma_1^2}$.[20]

Of the two propositions defining equilibria proposition 3 is the more general one. Proposition 2 applies only to very knife-edge situations. Voters have to be uncertain at exactly the same degree with respect to two parties, while the third party must inspire an even higher level of uncertainty.

4. Equilibria and comparative statics

To illustrate possible equilibria I will use a standard normal curve as voter distribution.[21] I will present for both propositions 2 and 3 the set of equilibria. Additionally, examples drawn from these sets will stress some important properties of the equilibria. Those properties will be analyzed in more detail in the section on comparative statics.

4.1. Possible equilibria

Under the conditions of proposition 2, equilibria require quite considerable differences in variances between the parties. Figure 1 shows the set of possible equilibria for the *single unsafe case*. First, equilibria only exist if the third party is quite far away from the median voter. Second, the difference in variances between the unsafe party (3) and the two safer parties (1 and 2) must be quite large. With increases in this difference the third party is forced to move to more extreme positions. Finally, for a given difference in variances, by symmetry, either no or two positions exist, which are part of an equilibrium.

Figure 2 illustrates an example of such an equilibrium.[22] In that example the differences in variances associated with parties 3 and 1, and 3 and 2, are at their lowest level, for which an equilibrium is possible.[23] The differences in variances prove to be the crucial element in equilibrium. More precisely, they determine the distances between the parties. If the differences in variances decrese, the party which creates the most uncertainty among voters moves closer to the two other parties to maximize its support. However, such a position is vulnerable. More precisely, one of the other two parties would have an incentive to move toward the position of the third party. Since voters are risk-averse, party 3 would lose most of its support, while the moving party would increase its vote share.

The major problem with equilibria under proposition 2 is that the required differences in the variances are larger than the variance of the voter distribution. So whatever the variance associated with the other two parties, the third party must be a riskier choice than a random draw from the voter distribution.

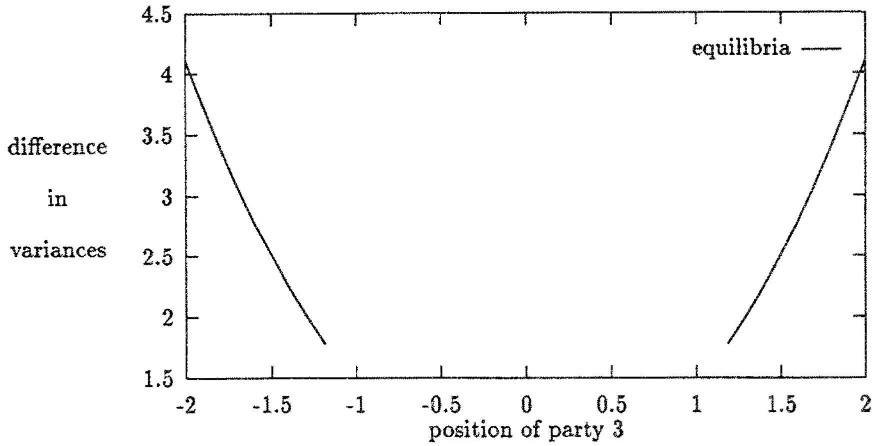

Figure 1. Set of equilibria for single unsafe case.

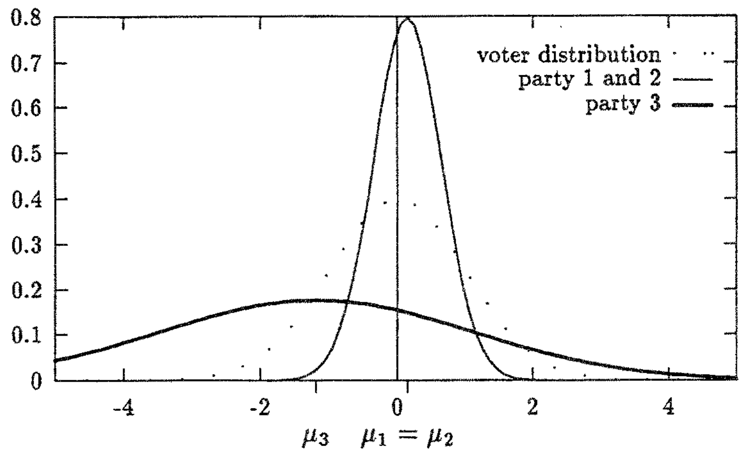

Figure 2. One possible equilibrium with single risky party.

Proposition 3 is much more general and allows for equilibria in a wider range of situations. Its characteristic feature is that the two parties that are riskier alternatives take positions at equal distance from the median voter. The third party, invariably, is located somewhere between its two competitors. If the variances associated with the two riskier parties are equal, the third party will take the position of the median voter.[24] Since two parties are at equal distance from the median voter and their respective distance from party 1 is determiend by the differences in variances, the subsequent result is easy to derive:

$$\mu_3^* = \frac{\sqrt{\sigma_2^2 - \sigma_1^2} + \sqrt{\sigma_3^2 - \sigma_1^2}}{2} = \mu_2^* \qquad (4)$$

This shows that for given levels of uncertainty associated with the three parties, the vote-maximizing positions of party 2 and 3 are determined (4). As proposition 3 shows, this is however not sufficient to guarantee an equilibrium, since it might not imply maximizing behavior of party 1. The implication of this behavior is a complex relationship between the differences in variances and the positions of the two riskier parties. I illustrate this relationship in Figure 3.[25]

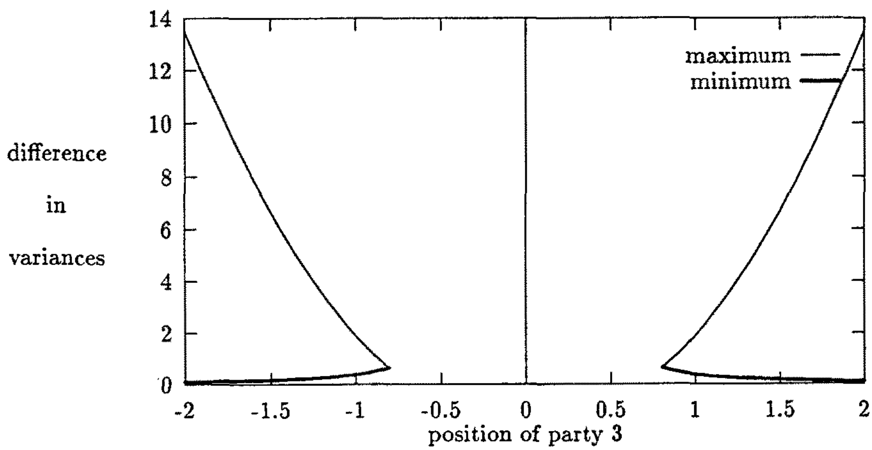

Figure 3. Set of equilibria for single safe case.

For any given set of three parties with their respective levels of uncertainty, the position of party 3 can be directly determined. The additional restriction requires that both of the crucial differences in variances, namely between party 1 and 2, and 1 and 3, lie between the two curves in Figure 3. These two curves represent for each position of party 3 the minimum and maximum differences in variances that can be supported in equilibrium.[26] One difference in variances and a position of party define a point in Figure 3. If this point falls between the two curves, another difference in variances exists which allows the particular position of party 3 to be part of an equilibrium. This difference in variance falls somewhere between the two curves. Similarly a pair of differences in variances by (4) determine the position of party 3. If at that position the two differences in variances fall between the two curves, the position of party 3 is part of an equilibrium.

Two insights are important in this graph. First, as is obvious from the graph the lower curve approaches zero. In its limit, however, when the variances associated with parties 1 and 2 are identical, more equilibria are possible, as I

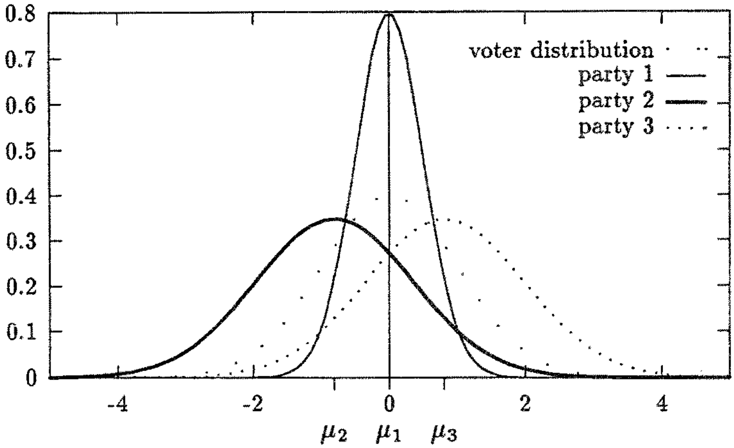

Figure 4. Symmetric equilibrium for singe safe case.

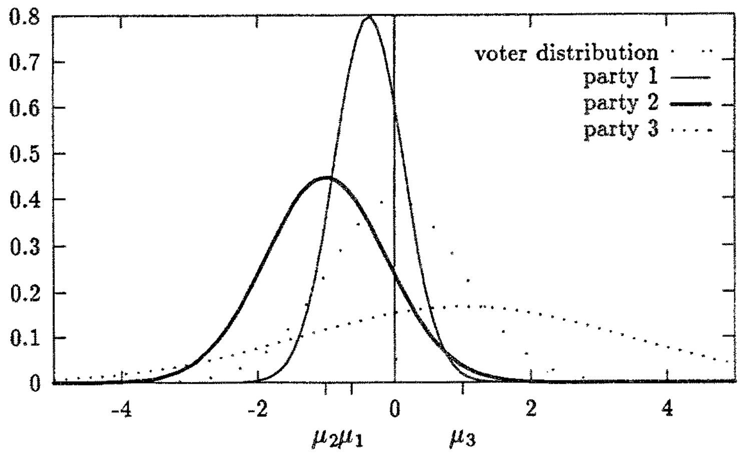

Figure 5. Asymmetric equilibrium for single safe case.

have shown in proposition 2. Second, the two curves intersect. At the intersection point the two riskier parties have to inspire the same level of uncertainty among voters to be at equilibrium. This also represents the case, where party 2 and 3 are closest to the median voter. Figure 4 illustrates this case.[27]

In the single safe case (proposition 3) one of the major problems of the equilibria derived under proposition 2 is averted. The differences in variances required for an equilibrium are considerably lower than the variance of the voter distribution.[28] Additionally, equilibria are also possible where all three parties are risky choices to different degrees. Figure 5 ilustrates one of the cases. Here the two riskier parties are located at -1 and 1, while the third party takes a position to the left of the median voter. This is so, because the party to its left is a less risky choice for the voter than the party to its right.[29] Furthermore, this case represents also the situation where the difference in variances between party 1 and 2 is lowest, given the position of the two riskier parties. If the variance associated with the party to the left were a little lower, the party in the middle would increase its support by moving to the position of its neighbor to the left. This would consequently destroy the equilibrium.

4.2. Comparative statics

As is quite apparent proposition 3 (single safe case) allows for much more "reasonable" equilibria than proposition 2 (single unsafe case). I will confine my analysis of comparative statics to that case. The only interesting aspect of the single unsafe case is to study the impact of increasing levels of uncertainty. If the difference in uncertainty becomes larger, quite clearly the party, which is the riskiest choice for a voter, moves away from the other two parties (Figure 1). The latter simultaneously move closer to the median voter.[30]

In the single safe case a first and central point is that the party in the middle, or equivalently the least risky party, always wins the greatest electoral support. This is quite a logical conclusion and is embedded in proposition 3. If one of the riskier parties had a higher electoral support at its position, nothing would hinder party 1 to take the same position as that party. Party 1, which represents a less uncertain choice for the voters, would win all the previous support of the former winner and some additional votes.

Consequently, if the levels of uncertainty are exogenous, the party with the lowest variance will, in equilibrium, locate itself between the two other parties. Given this position, it will also win the highest electoral support. This result contradicts both Palfrey's (1984) and Austen-Smith's and Banks' (1988) findings. In both models the party in the middle gets strictly fewer votes than the parties on the right and left. In Palfrey's model one of the parties with a higher vote share will win the election by a lottery. In the other model, one party among those with higher vote shares will be designated, again by a lottery,[31] to form a government. Whatever party will be chosen, it will propose a government deal to the party in the middle with the lowest vote share.

Concerning the position of party 1 it is obvious from proposition 3 that it will be closer to the party with the next lowest variance. In equilibrium a party facing two riskier competitors will always be closer to its most credible adversary. If party 1 would fail to approach its most credible competitor, the latter would have an incentive to move closer to the median voter. So by taking a position closer to party 2, party 1 can keep it abreast from the bulk of the voters, which are close to the median voter. If one of the riskier parties can diminish its level of uncertainty[32] both parties 2 and 3 will move closer to the median voter. Simultaneously party 1 will move closer to party 2.[33]

An important concern is also the assumed behavior of the voters, namely their sincerity. It is easy to show that in equilibrium no voter has an incentive to deviate from his or her voting strategy. A voter casting his or her ballot for one of the losing parties could either abstain or vote for another party. Abstaining will not increase his or her expected utility,[34] while voting for the winning party will not change the electoral outcome. This leaves as last alternative the vote for the third party. But since voters are risk-averse, and the third party is both further away and riskier than the winning party, this is a dominated strategy.

Concerning the choice of voters of the winning party one property of the equilibrium is very helpful. In equilibrium the riskier party never attracts voters that are located on the same side of its position as a less risky party. More precisely, the position of an indifferent voter x_{kl} between two parties is the same as the position of the riskier party. It follows that in equilibrium all voters having bliss-points between the two riskier parties will vote for the winning party. By abstaining a supporter of the winning party could at best force a tie between the parties. But since in equilibrium the riskier parties get exactly the same amount of votes, all three parties must have identical electoral support. A lottery between these three parties will be never preferred by any voter, since they are risk-averse. This leaves as the only potential deviation the vote for a riskier party, when all three parties are already in a tie. But by proposition 3 this can never happen in equilibrium. Consequently, in equilibrium no voter has an incentive to deviate from casting his or her ballot for the party which yields the highest expected utility.

5. Discussion

A very simple modification in the standard setting of spatial models, which is also very intuitive, allows deriving equilibria with three parties. While such equilibria with three parties took the center stage in this paper, one should not forget that with the same basic assumptions of the model, equilibria with more parties are also likely to exist.[35] The major drawback of the model is that equilibria with two parties become possible only under very restrictive assumptions.

While the existence of three-party equilibria is in itself an interesting result, the comparative statics results give some indications on what a three-party competition should look like. Central is the result that the least risky party will always take a position between its competitors. This is also the winning position. Furthermore, the position is always shifted away from the median voter in the direction of the next most credible candidate.[36]

At first sight these properties of the equilibrium seem hardly to find support in empirical cases. Three-party systems, like for instance Germany and Great Britain, see often a small liberal party between a conservative and a socialist party. Some support for my model comes from the American experiences with third parties (Rosenstone, Behr and Lazarus, 1984). As Shepsle and Cohen (1990: 30) note, third parties, which are most likely riskier choices, often enter at extreme positions. Consequently, the winner of the election is frequently a party at the center, since third parties never won a presidential election. A second look at the British case provides also some supporting evidence. The Social Democratic Party (SDP), immediately after its foundation, was credited in several polls a plurality of votes in alliance with the Liberals (Bogdanor, 1981: 287). Both the Tories and Labour appeared to move quickly away from the center and became very uncertain choices at the end of the seventies. But after this early euphoria, the future Alliance lost most of its attraction. Even more strongly against my model speaks the German case, where the center party, the Liberals (FDP), was never close at winning an election. The question here, however, is to what degree the pre-election coalition-promises do not keep artificially in life this party. Modeling explicitly the coalition formation process, as Austen-Smith and Banks (1988) do, might provide more insights into the German case. Another possible explanation for this mixed support for my model might be one of my assumptions, namely that voting is costless. This induces every voter to participate at elections, but this is hardly the case. Introducing voting costs might alter some of the insights of my model, and constitutes, therefore, one if its potential extensions.

Another interesting extension of my model concerns the degree of uncertainty. Given the important advantage for a party to be the most credible competitor, the question arises naturally whether equilibria can exist if the level of uncertainty becomes endogenous. Two perspectives lend themselves to such an extension. On the one hand I could assume that parties have control over their own level of uncertainty. Informing voters in greater detail on proposed policies might reduce the perceived uncertainty among voters. If this is the case it is quite straightforward that a dominant strategy for each party is to reduce this level to zero. But then the uncertainty of voters no longer plays any role in the model, and the standard results of the nonexistence of three-party equilibria apply. On the other hand, it can be argued that parties have control only over the level of uncertainty of their adversaries (Franklin, 1991). Negative ads might make an adversary a much riskier choice in the eyes of the voters. But then again, each party has a dominant strategy, namely to increase the level of uncertainty associated with its competitor to its highest possible level. The consequence is again that the level of uncertainty among voters loses its impact, and the well-knonw results apply again.[37]

Consequently making the level of uncertainty endogenous is feasible only at the price of giving up any equilibrium behavior, except if changing that level is costly. Equilibrium could be recaptured by explicitly modeling the processes which lead to increases or decreases in the levels of uncertainty. Here again, Banks' (1990) model illustrates one possible and very attractive avenue. Its extension to more than two parties would consequently be of great value.

Notes

1. Throughout the paper I will use the term party, although the results apply equally well to candidates.
2. I will discuss these models in the following section.
3. This holds under the assumption of a symmetric voter distribution.
4. The winner is determined randomly in a fair and even lottery.
5. Shepsle (1991: 74ff.) criticizes the model on similar grounds.
6. The expected value of participating in the election must exceed zero in this case: benefits $\times \frac{1}{\text{number of parties}} - \text{costs} \geq 0$.
7. Enelow and Hinich (1981, 1984) assume that voters do not observe the "true" policy position, but attempt to infer it from different informational sources. The resulting formulation is identical to mine, since the important feature is that parties can change positions, but have no control over the variances associated with their positions.
8. I use this type of utility function for reasons of mathematical tractability. Other functional forms are very likely to yield similar results, as long as they reflect the voters' risk-aversion.
9. This follows directly from the quadratic loss function and the uncertainty of the voters (Enelow and Hinich, 1984: 123f.; and Banks, 1990).
10. I will show below that in equilibrium no voter has an incentive to deviate from his or her vote choice.
11. To be absolutely precise, no voter might have this bliss point. This happens, however, only in extreme situations. Below I will use x_{12} to denote the indifferent voter between party 1 and 2, while x_{13} is the one indifferent between 1 and 3. See appendix or Enelow and Hinich (1984: 124) for the derivation of this result.
12. This holds under most types of voter distributions. Shaked (1982) proves explicitly the nonexistence of pure strategy equilibria in spatial models with three firms while computing a mixed strategy equilibrium.
13. I will come back to the exogeneity of the variances in the discussion.
14. If one or even two parties win no votes in equilibrium, one wonders why they bother competing in the election.
15. The remaining proofs are contained in the appendix.
16. Here, as well as in the subsequent parts, I number the parties in the order of increasing variances ($\sigma_1^2 \leq \sigma_2^2 \leq \sigma_3^2$). Party 1 is therefore always the least risky choice for a voter, etc.
17. Since 0 is by assumption the mean of the voter distribution, this simply implies that the density funtion does not have any flat parts. This restriction rules out many density functions. I need it, however, to be able to derive most of the stronger results here and below. Equilibria are possible without such a density function, but exact conditions are not as general.
18. I thank Douglas Dion for suggesting to me this, as well as the *single safe case* label.
19. By symmetry an equilibrium also exists, which is symmetric to the one defined here. The necessary conditions for that one are obtained easily.

20. Again by symmetry, a similar equilibrium exists, with parties 2 and 3 exchanging their positions. The necessary conditions can be obtained easily.
21. The advantage of this distribution is that it fulfills the assumptions on the voter distribution. With other distributions equilibria are also possible, as long as they fulfill the necessary conditions of either one of the two propositions.
22. The two parties with lower variances are at 0.15, with a variance of 0.5, while the third party is positioned at -1.187 with a variance of 2.28. Here, as well as in all subsequent figures, I represent the variances of the parties by normal curves. I chose this curve for clearness of presentation. The equilibrium, however, is solely dependent on the variances of the curves, and not their specific shape.
23. This still under the assumption that two parties have the same low variance, while the third party has a higher variance. The smallest difference possible in equilibrium is equal to 1.78. I derived this value numerically, since the formula involves both density functions and their derivatives.
24. This follows quite directly from proposition 3. Since the variance of party 2 an 3 are identical, they will be at equal distance from the party 1. Simultaneously they are at equal distance from the median voter, which implies that party 1 is located at the median.
25. This relationship is easily obtained by setting one of the two inequalities in proposition 3 to equality. For given positions of party 3 the lowest and highest variances follow quite directly. I obtained the exact values, however, only numerically.
26. The minimum and maximum differences imply each other mutually. That is, if one party (e.g., 2) has the minimum level of variance for a given position, the other one (e.g., party 3) has to have the maximum variance in equilibrium.
27. Formally this is the situation where both of the inequalities in condition 3 of proposition 4 become equalities. The two riskier parties are located at -0.806 and 0.806, while the third party takes position at 0. The variance assiociated with the party at the median is equal to 0.5, which implies that the other two variances are equal to 1.15.
28. In the case represented in Figure 4 this difference it equal to 0.6322.
29. The variances, from left to right are 0.8955, 0.5, 2.3799.
30. Both of these results follow easily from proposition 2.
31. This only holds if the over distribution is symmetric (Austen-Smith and Banks, 1988: 416).
32. This is equivalent to an increase of the variance associated with party 1.
33. This holds obviously only if this new set of variances falls into the set of possible equilibria illustrated in Figure 3.
34. Naturally all this holds only if voting is costless.
35. The easist case to see this is the equilibrium with four parties, which combines aspects of propositions 2 and 3. If two parties have the same low variance, they will take identical positions close to the median, while two riskier parties would be on opposite sides from the median voter.
36. Naturally this does not hold if both competitors are equally credible.
37. One has to note that the same conclusions apply to two-party competition, where equilibria only exist if either voters are risk-neutral or the levels of uncertainty are identical.

In appendix

38. Through all the proofs I use a slightly modified version of the notation adopted by Greenberg and Shepsle (1987). $S(\mu_1)$ represents the electoral support of party 1 at position μ_1, given the positions of the other parties.
39. In this proof I will always use party I as deviating party. The different steps apply equally well, however, to party 2.
40. In most cases the left-hand side and the right-hand side are equal, but I need only the inequality to derive the necessary condition.

41. This is not exactly true, since if $f(x_{12})$ is equal to zero, the first order condition for a maximum is satisfied. Since I will look mostly at strictly, monotonically increasing functions on $(-\infty,0)$, $f(x_{12}) = 0$ would imply that the party on the left would gain no votes. Since I exclude these equilibria, the restriction is not crucial, and I will rely on it below.

References

Austen-Smith, D. and Banks, J.S. (1988). Elections, coalitions, and legislative outcomes. *American Political Science Review* 82: 405–422.

Banks, J.S. (1990). A model of electoral competition with incomplete information. *Journal of Economic Theory* 50: 309–325.

Bernhardt, M.D. and Ingberman, D.E. (1985). Candidate reputations and the 'incumbency effect'. *Journal of Public Economics* 27: 47–67.

Bogdanor, V. (1981). The Social Democrats and the Constitution (Difficulties confronting the newly formed party). *Political Quarterly* 52: 285–294.

Dasgupta, P. and Maskin, E. (1986a). The existence of equilibrium in discontinuous economic games, I: Theory. *Review of Economic Studies* 53: 1–26.

Dasgupta, P. and Maskin, E. (1986b). The existence of equilibrium in discontinuous economic games, II: Applications. *Review of Economic Studies* 53: 27–41.

Eaton, B.C. and Lipsey, R.G. (1975). The principle of minimum differentiation reconsidered: Some new developments in the theory of spatial competition. *Review of Economic Studies* 42: 27–49.

Enelow, J.H. and Hinich, M.J. (1981). A new approach to voter uncertainty in the Downsian spatial model. *American Journal of Political Science* 25: 483–493.

Enelow, J.H. and Hinich, M.J. (1984). *The spatial theory of voting: An introduction*. Cambridge: Cambridge University Press.

Feddersen, T.J., Sened, I. and Wright, S.G. (1990). Rational voting and candidate entry under plurality rule. *American Journal of Political Science* 34: 1005–1016.

Franklin, C.H. (1991). Eschewing obfuscation?: Campaigns and the perception of Senate incumbents. *American Political Science Review* 85: 1193–1214.

Greenberg, J. and Shepsle, K.A. (1987). The effect of electoral rewards in multi-party competition with entry. *American Political Science Review* 81: 525–537.

Hermsen, H. and Verbeek, A. (1992), Equilibria in multi-party systems. *Public Choice* 73: 147–166.

Lemieux, P.H. (1977). Political issues and liberal support in the February 1974 British General Election. *Political Studies* 25: 323–342.

Osborne, M.J. (1992). Candidate positioning and entry in a political competition. Hamilton, Ontario: Department of Economics, McMaster University (Working Paper No. 92-09).

Palfrey, T.R. (1984). Spatial equilibrium with entry. *Review of Economic Studies* 51: 139–156.

Pinard, M. (1967). One-party dominance and third parties. *Canadian Journal of Economics and Political Science* 33: 358–373.

Rosenstone, S.J., Behr, R.L. and Lazarus, E.H. (1984). *Third parties in America*. Princeton: Princeton University Press.

Shaked, A. (1982). Existence and computation of mixed strategy nash equilibrium for 3-firms location problem. *Journal of Industrial Economics* 31: 93–96.

Shepsle, K.A. (1972a). Parties, voters, and the risk environment: A mathematical treatment of electoral competition under uncertainty. In R.G. Niemi and H.F Weisberg (Eds.), *Probability models of collective decision-making*, 273–297. Columbus: Merrill.

Shepsle, K.A. (1972b). The strategy of ambiguity: Uncertainty and electoral competition. *American Political Science Review* 66: 555–568.
Shepsle, K.A. (1991). *Models of multiparty competition*. London: Harwood.
Shepsle, K.A. and Cohen, R.N. (1990). Multiparty competition, entry, and entry deterrence in spatial models of elections. In J.M. Enelow and M.J. Hinich (Eds.), *Advances in the spatial theory of voting*. Cambridge: Cambridge University Press.

6. Appendix

6.1. Indifferent voter in one-dimensional space

An important position in the policy space is the one (x_{kl}), where voters are indifferent between voting for any party of a given pair ($k,l \in \{1,2,3\}, k \neq l$). At this position the following has to be true:

$$E(U(\mu_k)) = E(U(\mu_l))$$
$$\Rightarrow -x_{kl}^2 + 2x_{kl}\mu_k - \mu_k^2 - \sigma_k^2 = -x_{kl}^2 + 2x_{kl}\mu_l - \mu_l^2 - \sigma_l^2$$
$$\Rightarrow 2x_{kl}(\mu_k - \mu_l) = \mu_k^2 + \sigma_k^2 - \mu_l^2 - \sigma_l^2$$
$$\Rightarrow x_{kl} = \frac{\mu_k^2 - \mu_l^2}{2(\mu_k - \mu_l)} + \frac{\sigma_k^2 - \sigma_l^2}{2(\mu_k - \mu_l)} = \frac{\mu_l + \mu_k}{2} + \frac{\sigma_k^2 - \sigma_l^2}{2(\mu_k - \mu_l)}$$

Proof of lemma 1

Suppose not, then I have to consider three alternatives. The first one is trivial since it involves either $\mu_1 = \mu_3$ or $\mu_2 = \mu_3$, but $\mu_1 \neq \mu_2$. In both cases, since $\sigma_1^2 = \sigma_2^2 < \sigma_3^2$ no voter will ever vote for party 3. Consequently, this cannot be an equilibrium. In the second case I have $\mu_1 < \mu_3 < \mu_2$. Then we know that the electoral support for party 1 is equal to $S(\mu_1)$[38] $= \int_{-\infty}^{x_{13}} f(x)dx$ where

$$x_{13} = \frac{\mu_1 + \mu_3}{2} + \frac{\sigma_3^2 - \sigma_1^2}{2(\mu_3 - \mu_1)}$$

It follows that $S(\mu_1) = F(x_{13}) = F\left(\frac{\mu_1 + \mu_2}{2} + \frac{\sigma_3^2 - \sigma_1^2}{2(\mu_3 - \mu_1)}\right)$

Since party 1 will maximize its electoral support the following has to be true:

$$\frac{\partial S(\mu_1)}{\partial \mu_1} = 0; \frac{\partial^2 S(\mu_1)}{\partial \mu_1^2} \leq 0.$$

But

$$\frac{\partial S(\mu_1)}{\partial \mu_1} = f(x_{13})\left[\frac{1}{2} + \frac{2(\sigma_3^2 - \sigma_1^2)}{4(\mu_3 - \mu_1)^2}\right].$$

Since by assumption $\sigma_3^2 > \sigma_1^2$ and the denominator is squared the expression in brackets is always strictly positive. Consequently, as long as $\mu_1 < \mu_3$ party 1 has an incentive to move the right. As is quite clear this can be no equilibrium as $\mu_1 < \mu_2 < \mu_3$. A similar argument rules out equilibria in the third case where $\mu_1 < \mu_2 < \mu_3$. Here, no equilibrium is possible as long as $\mu_1 < \mu_2$. QED

Proof of proposition 2

Suppose there is an equilibrium with $\mu_1^*, \mu_2^*, \mu_3^*$, and $\sigma_1^2 = \sigma_2^2 < \sigma_3^2$. From lemma 1 we know that $\mu_1^* = \mu_2^*$. Maximizing behavior of party 3 by lemma 3 implies that $\mu_3^* = \mu_1^* - \sqrt{\sigma_3^2 - \sigma_1^2}$. These two conditions are both necessary, but even together not sufficient. What I have to consider are potential deviations of one of the safer parties. Party 1[39] can move to four different regions or positions: i) $\mu_1 < \mu_3^*$, ii) $\mu_1 = \mu_3^*$, iii) $\mu_3^* < \mu_1 < \mu_2^*$, iv) $\mu_2^* < \mu_1$. By the proof of lemma 1 we know that party 1 can improve its vote share in case i) by moving to the same position as party 3. In case ii) $F\left(\frac{\mu_3^* + \mu_1^*}{2}\right) \leq \frac{1 - F(\mu_3^*)}{2}$ is a necessary condition to maintain equilibrium. If it is violated party 1 is better off at μ_3^* than at μ_1^*.

Case iii), which provides the next necessary condition, is trickier. By the previous necessary conditions and the assumptions on the voter distributions it follows quite directly that $\lim_{\mu_1 \to \mu_1^*} S(\mu_1) = S(\mu_1^*)$. Hence, at $\mu_1 = \mu_1^*$ $S(\mu_1)$ is continuous, and additionally $\lim_{\mu_1 \to \mu_3^*+} S(\mu_1) \leq S(\mu_1 | \mu_1 = \mu_3^*)$[40]. Finally, one can show that when $\mu_3^* < \mu_1^* < \mu_1^*$, the function $S(\mu_1)$ has at most one critical point, given the assumptions on the voter distribution. Consequently, if the partial of $S(\mu_1)$ with repect to μ_1 is positive at $\mu_1 = \mu_1^*$, it follows that party has no incentive to move from μ_1^* to a position between μ_3^* and μ_1^*. It follows that $f(\mu_1^*) \geq 2f(x_{13}^*)$ is a necessary condition for an equilibrium.

In case iv) party 1 has an incentive to move arbitrarily close to μ_2^*. To make party 1 indifferent between μ_1^* and a position arbitrarily close to μ_2^*, the equality $F(\mu_1^*) = \frac{1 + F(x_{13}^*)}{2}$ has to hold.

To establish sufficiency let μ_1^*, μ_2^*, and μ_3^* fulfill the set of necessary conditions. By the proof of lemma 3, $\mu_3^* = \mu_1^* - \sqrt{\sigma_3^2 - \sigma_1^2}$ implies that μ_3 is a local maximum for $S(\mu_3)$. Since $F\left(\frac{\mu_3^* + \mu_1^*}{2}\right) \leq \frac{1 - F(\mu_3^*)}{2}$, it can be easily shown, that party 1 and party 3 are on opposite sides of the median voter. Consequently a position for party 3 on the other side of party 1 would yield a smaller electoral support. It follows that μ_3^* is a global maximum. From the above discussion of the four possible deviations for party 1, it is obvious that if the derived conditions hold μ_1^*, and by extension μ_2^* are positions that maximize the electoral support for the two parties. Consequently μ_1^*, μ_2^*, and μ_3^* form an equilibrium, and the set of necessary conditions is sufficient for equilibria. QED

Proof of lemma 2

Suppose not, then without loss of generality I can assume that $\mu_1 < \mu_2 < \mu_3$. Then we know from the proof of lemma 1 that the partial of $S(\mu_1)$ in respect to μ_1 is strictly positive as long as $\mu_1 < \mu_2$. If $\mu_2 = \mu_1$ the expression is undefined, but since the variance of party 2 is higher than party 1, at that point it would gain no votes. Party 1, on the other hand would capture all its previous share, and all of what party 2 had before. Since the partial for the first party is positive as long as $\mu_1 < \mu_2$, no equilibrium can exist under this condition. QED

Proof of lemma 3

To prove this lemma I will first define the electoral support for the three parties. These follow directly from the preceding lemma.

$$S(\mu_2) = \int_{-\infty}^{x_{12}} f(x)dx = F(x_{12})$$

$$S(\mu_3) = \int_{x_{13}}^{\infty} f(x)dx = 1 - F(x_{13})$$

$$S(\mu_1) = \int_{x_{12}}^{x_{13}} f(x)dx = F(x_{13}) - F(x_{12})$$

Second, I will derive the conditions, under which the two parties with higher variances are at their equilibrium positions. These require that the following two sets of equalities and inequalities hold:

$$\frac{\partial S(\mu_2)}{\partial \mu_2} = 0; \frac{\partial^2 S(\mu_2)}{\partial \mu_2^2} \leq 0 \text{ and } \frac{\partial S(\mu_3)}{\partial \mu_3} = 0; \frac{\partial^2 S(\mu_3)}{\partial \mu_3^2} \leq 0$$

The first set implies that

$$\frac{\partial S(\mu_2)}{\partial \mu_2} = f(x_{12})\frac{\partial x_{12}}{\partial \mu_2} = f(x_{12})\left[\frac{1}{2} - \frac{\sigma_2^2 - \sigma_1^2}{2(\mu_2 - \mu_1)^2}\right] = 0$$

since $f(x_{12}) \geq 0$ the expression in brackets has to be equal to zero.

$$(\mu_2 - \mu_1)^2 = \sigma_2^2 - \sigma_1^2$$

$$\mu_2 = \mu_1 - \sqrt{\sigma_2^2 - \sigma_1^2}$$

The second order condition implies that

$$\frac{\partial^2 S(\mu_2)}{\partial \mu_2^2} = \frac{\partial f(x_{12})}{\partial \mu_2}\frac{\partial x_{12}}{\partial \mu_2} + f(x_{12})\frac{\partial^2 x_{12}}{\partial \mu_2^2} \leq 0$$

At equilibrium, by definition $\frac{\partial x_{12}}{\mu_2} = 0$ and therefore:

$f(x_{12})\frac{\partial^2 x_{12}}{\partial \mu_2^2} \leq 0$ since $f(x_{12}) \geq 0 \; \forall x_{12}$, $\sigma_2^2 - \sigma_1^2 > 0$ and $(\mu_2 - \mu_1) < 0$.

We know from above that

$$\frac{\partial^2 x_{12}}{\partial \mu_2^2} = \frac{\sigma_2^2 - \sigma_1^2}{(\mu_2 - \mu_1)^3} < 0$$

A similar result can be derived for party 3, which is on the right side of party 1. The first and second order conditions imply that

$$\mu_3 = \mu_1 + \sqrt{\sigma_3^2 - \sigma_1^2}$$

Now, to be an equilibrium, the position of party 1 has to be a global maximum for its electoral support.

$$\frac{\partial S(\mu_1)}{\partial \mu_1} = f(x_{13})\frac{\partial x_{13}}{\partial \mu_1} - f(x_{12})\frac{\partial x_{12}}{\partial \mu_1}$$

Since

$$x_{12} = \frac{\mu_2+\mu_1}{2} + \frac{\sigma_2^2-\sigma_1^2}{2(\mu_2-\mu_1)}$$

$$x_{13} = \frac{\mu_3+\mu_1}{2} + \frac{\sigma_3^2-\sigma_1^2}{2(\mu_3-\mu_1)}$$

and

$$\mu_2 = \mu_1 - \sqrt{\sigma_2^2-\sigma_1^2}$$

$$\mu_3 = \mu_1 + \sqrt{\sigma_3^2-\sigma_1^2}$$

the following has to be true:

$$f(x_{13})\left[\frac{1}{2}+\frac{2(\sigma_3^2-\sigma_1^2)}{4(\mu_3-\mu_1)^2}\right] = f(x_{12})\left[\frac{1}{2}+\frac{2(\sigma_2^2-\sigma_1^2)}{4(\mu_2-\mu_1)^2}\right]$$

this implies that

$$f(x_{13})\left[\frac{1}{2}+\frac{(\sigma_3^2-\sigma_1^2)}{2(\sigma_3^2-\sigma_1^2)}\right] = f(x_{12})\left[\frac{1}{2}+\frac{(\sigma_2^2-\sigma_1^2)}{2(\sigma_2^2-\sigma_1^2)}\right]$$

and therefore

$$f(x_{12}) = f(x_{13}).$$

Under the assumption of a symmetric and strictly monotonically increasing density function on $(-\infty,0)$ this implies that $x_{13} = -x_{12}$. Given the formulae for x_{12} and x_{13} this implies in equilibrium that $\mu_2 = -\mu_3$. The second order condition

$$\frac{\partial^2(\mu_1)}{\partial\mu_1^2} \leq 0$$

also holds in equilibrium. In a very similar derivation I can show that

$$\frac{\partial^2(\mu_1)}{\partial\mu_1^2} = f'(x_{13})-f'(x_{12})$$

which is strictly negative, given my assumptions on the voter distribution. QED

Proof of proposition 3

From lemma 3 we know that equilibria exist only if $\mu_2^* = -\mu_3^*$, given the assumptions on the voter distribution and the relationship among the three variances. Similarly $\mu_2 = \mu_1 - \sqrt{\sigma_2^2-\sigma_1^2}$ and $\mu_3 = \mu_1 + \sqrt{\sigma_3^2-\sigma_1^2}$ are both necessary conditions for equilibria, as I have shown in the proof of lemma 3. The symmetry and the two locational conditions are all, however, only necessary and not sufficient to guarantee an equilibrium. Sufficiency is provided by an additional restriction on the voter distribution. This restriction follows from the only possible other position for party 1, which might be an equilibrium. From lemma 2 we know that party 1 will never be to the left or the right of both riskier parties in equilibrium. This leaves as only option for party 1 to improve its vote share to take the same position as either party 2 or 3. In these two cases party 1 would win respectively $F(\frac{\sigma_2^2-\sigma_1^2}{2(\mu_3-\mu_2)})$ or $F(\frac{\sigma_3^2-\sigma_1^2}{2(\mu_3-\mu_2)})$. Consequently, an additional necessary condition os that party 1 has no incentive to deviate from its equilibrium position: $1-2F(\mu_2) \geq F(\frac{\sigma_2^2-\sigma_1^2}{2(\mu_3-\mu_2)})$ and $1-2F(\mu_2) \geq F(\frac{\sigma_3^2-\sigma_1^2}{2(\mu_3-\mu_2)})$. Together, these necessary conditions are sufficient to guarantee an equilibrium, still under the assumption on the voter distribution and the particular relationship among the three variances.

To establish sufficiency let μ_1^*, μ_2^* and μ_3^* fulfill the set of necessary conditions. Then, by the proof of lemma 3 we know that $\mu_2^* = -\mu_3^*$ implies that $\frac{\partial S(\mu_1^*)}{\partial \mu_1^*} = 0$, and $\frac{\partial^2 S(\mu_1^*)}{\partial \mu_1^{*2}} \leq 0$. Consequently, party 1 has locally no incentive to move from its position. By the proof of lemma 2 the only other positions, which might be global maxima for $S(\mu_1^*)$, are either μ_2^* or μ_3^*. But since $1-2F(\mu_2^*) \geq F(\frac{\sigma_2^2-\sigma_1^2}{2(\mu_3^*-\mu_2^*)})$, and $1-2F(\mu_2^*) \geq F(\frac{\sigma_2^3-\sigma_1^2}{2(\mu_3^*-\mu_2^*)})$, party 1 has no incentive to move away from μ_1^*. By the proof of lemma 3 $\mu_2^* = \mu_1^* - \sqrt{\sigma_2^2-\sigma_1^2}$ implies that $\frac{\partial S(\mu_2^*)}{\partial \mu_2^*} = 0$, and $\frac{\partial^2 S(\mu_2^*)}{\partial \mu_2^{*2}} \leq 0$, and $\mu_3^* = \mu_1^* + \sqrt{\sigma_3^2-\sigma_1^2}$ implies that $\frac{\partial S(\mu_3^*)}{\partial \mu_3^*} = 0$, and $\frac{\partial^2 S(\mu_3^*)}{\partial \mu_3^{*2}} \leq 0$. Both party 2 and 3 consequently have no incentive to move. It follows that μ_1^*, μ_2^* and μ_3^* form an equilibrium, and that the set of necessary conditions are sufficient to guarantee an equilibrium. QED