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On the formulation of variational theorems involving volume constraints

S.N. Atluri

Center for the Advancement of Computational Mechanics, Georgia Institute of Technology, Atlanta, GA 30332, USA

E. Reissner

Department of Applied Mechanics and Engineering Sciences, University of California at San Diego, LaJolla, CA 92093, USA

Abstract. A continued concern with variational theorems which are suitable for numerical implementation in connection with the analysis of incompressible or nearly incompressible materials has led us to the formulation of five-field, and in one case seven field, theorems for displacements, deviatoric stresses, pressure, distortional strains and volume change. In essence these theorems may be thought of as generalizations of the Hu-Washizu three-field theorem for displacements, stresses and strains and of the earlier two-field theorem for displacements and stresses.

For ease of exposition, what follows is divided into three parts. The first part deals with geometrically linear elasticity. The second part deals with the effect of geometric nonlinearity in terms of Kirchhoff-Trefftz stresses and Green-Lagrange strains. The third part is concerned with results involving generalized Piola stresses and conjugate strains, as well as with results about distinguished (Biot) generalized stresses and their conjugate strains. Also for ease of exposition, attention is limited to statements about volume integral portions, omitting body force and boundary condition terms.

In addition to formulating five field theorems, as well as one seven field theorem, we use these theorems, through the introduction of various constraints, for the deduction of alternate six, five, four, three, and two-field theorems for incompressible or nearly incompressible elasticity.

1 A five-field generalization of the Hu-Washizu theorem for geometrically linear elasticity

With u_i as components of displacement, with the abbreviation,

$$u_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

and with a strain energy density function $U(u_{ij})$ we have for the volume integral portion, in the absence of body forces, of the classical one-field principle of minimum potential energy

$$\delta \int U(u_{ij})\,\mathrm{d}v = 0 \tag{2}$$

With components of stress σ_{ij}, with constraint constitutive equations $\sigma_{ij} = \partial U/\partial u_{ij}$ and with arbitrary variations δu_i, Eq. (2) implies the Euler differential equations of equilibrium $\sigma_{ij,i} = 0$. Furthermore, with components of strain ε_{ij}, and with constraint strain displacement relations $\varepsilon_{ij} = u_{ij}$, Eq. (2) may alternately be written in the form

$$\delta \int U(\varepsilon_{ij})\,\mathrm{d}v = 0 \tag{2'}$$

Given (2′) we obtain, following Hu and Washizu, a three-field variational theorem for displacements, stresses and strains upon changing the character of the constraint relations $\varepsilon_{ij} = u_{ij}$, and $\sigma_{ij} = \partial U/\partial\varepsilon_{ij}$ into Euler equations, with this three-field theorem having the form

$$\delta \int [U(\varepsilon_{ij}) + (u_{ij} - \varepsilon_{ij})\sigma_{ij}]\,\mathrm{d}v = 0 \tag{3}$$

Given the step from the one-field theorem (2) to the three-field theorem (3) we now propose to establish a five-field theorem for displacements u_i, distortional strains ε'_{ij}, a volume change variable $\theta = \varepsilon_{kk}$, deviatoric stresses σ'_{ij} and a mean stress $p = (1/3)\sigma_{kk}$, as follows.

We write, with the introduction of a deviatoric displacement gradient tensor u'_{ij},

$$(u_{ij}; \varepsilon_{ij}; \sigma_{ij}) = (u'_{ij}; \varepsilon'_{ij}; \sigma'_{ij}) + \left(\frac{1}{3} u_{kk}; \frac{1}{3}\theta; p\right) \delta_{ij} \tag{4}$$

We further write

$$U(\varepsilon_{ij}) = U\left(\varepsilon'_{ij} + \frac{1}{3}\theta\,\delta_{ij}\right) = U^*(\varepsilon'_{ij}, \theta) \tag{5}$$

and with this we assert the validity of the five-field variational theorem

$$\delta \int [U^*(\varepsilon'_{ij}, \theta) + (u'_{ij} - \varepsilon'_{ij})\,\sigma'_{ij} + (u_{kk} - \theta)\,p]\,\mathrm{d}v = 0 \tag{6}$$

It is evident that the Euler equations of (6) include the relations $\varepsilon'_{ij} = u'_{ij}$, $\theta = u_{kk}$, and therewith the strain displacement relations $\varepsilon_{ij} = u_{ij}$.

The further Euler equations $\sigma'_{ij} = \partial U^*/\partial \varepsilon'_{ij}$, $p = \partial U^*/\partial \theta$ also imply the constitutive relations $\sigma_{ij} = \partial U/\partial \varepsilon_{ij}$ as Euler equations, inasmuch as

$$\frac{\partial U}{\partial \varepsilon_{ij}} = \frac{\partial U^*}{\partial \varepsilon'_{ij}} + \frac{\partial U^*}{\partial \theta}\,\delta_{ij} = \sigma'_{ij} + p\,\delta_{ij} = \sigma_{ij} \tag{7}$$

Finally, in order to verify that (6) also implies the equilibrium equations $\sigma_{ij,i} = 0$ as Euler equations it is sufficient to observe, on the basis of (4), that

$$u'_{ij}\sigma'_{ij} + u_{kk}p = u_{ij}\sigma_{ij} \tag{8}$$

Remarks: Equation (6) may also be obtained upon writing the principle of minimum potential energy in the form $\delta \int U(\varepsilon'_{ij} + 1/3\,\theta\,\delta_{ij})\,\mathrm{d}v = 0$, with an introduction of the the side conditions $\varepsilon'_{ij} - u'_{ij} = 0$ and $\theta = u_{kk}$ by means of Lagrange multipliers σ'_{ij} and p.

Still another way of deriving (6) is to introduce the decompositions (4) into the Hu-Washizu Eq. (3), with the desired result following upon observing the relations $(u'_{ij}; \varepsilon'_{ij}; \sigma'_{ij})\,\delta_{ij} = 0$.

We further note that the linear isotropic materials case of (6) was included in an unpublished 1978 manuscript "Notes on an analysis of nearly or precisely incompressible behavior of elastic-plastic solids" by the first named author in collaboration with H. Murakawa. The present deduction of (6) evolved on the basis of a manuscript "On a modification of the Hu-Washizu variational equation in elasticity" concerning the four-field theorem in Eq. (10) which the senior author submitted in July 1986 to the journal Computational Mechanics.

2 Reductions of the five-field theorem of geometrically linear theory

Our first reduction is to a four-field theorem for displacements, distortional strains, deviatoric stresses and pressure. We change the character of the relation $p = \partial U^*/\partial \theta$ from Euler equation to constraint equation and, with the inversion $\theta = \theta(p, \varepsilon'_{ij})$ of this relation, define a semi-complementary energy density function V^* through the partial Legendre transformation

$$V^*(\varepsilon'_{ij}, p) = p\,\theta(p, \varepsilon'_{ij}) - U^*[\varepsilon'_{ij}, \theta(p, \varepsilon'_{ij})] \tag{9}$$

Equation (9) implies in the usual way that $\theta = \partial V^*/\partial p$ and the introduction of U^* from (9) into (6) gives as the desired four-field theorem

$$\delta \int [(u'_{ij} - \varepsilon'_{ij})\,\sigma'_{ij} + u_{kk}p - V^*(\varepsilon'_{ij}, p)]\,\mathrm{d}v = 0 \tag{10}$$

As regards the problem of determining V^* we note the ease of doing this for materials for which at the outset $U(u_{ij}) = U_0(u'_{ij}) + U_1(u_{kk})$, and also for materials for which $U(u_{ij}) = U_2(u_{ij}) + (u_{kk})^2 \times F(u_{12}, u_{13}, u_{23})$, with U_2 as a second degree polynomial. For other more general cases, and in connection with numerical applications of (10), we expect that it will be possible to combine the discretization of (10) with a determination of the function V^* in an incremental sense as in [1].

Our second reduction of (6), to a three-field theorem for displacements, volume change and pressure, is obtained upon changing the character of the relations $\varepsilon'_{ij} = u'_{ij}$ and $\sigma'_{ij} = \partial U^* / \partial \varepsilon'_{ij}$ from Euler equation to constraint equation. With this, Eq. (6) evidently reduces to the form

$$\delta \int [U^*(u'_{ij}, \theta) + (u_{kk} - \theta)\, p]\, \mathrm{d}v = 0 \tag{11}$$

Equation (11) has been stated previously in [5]. An implicit version of (11) occurs also in [2], not for it's own sake but as a stepping stone towards a two-field theorem for displacements and pressure which is obtained by again changing the Euler equation $p = \partial U^* / \partial \theta$ into a constraint equation where now $\theta = \theta(p, u'_{ij})$ and therewith

$$V^*(u'_{ij}, p) = p\,\theta(p, u'_{ij}) - U^*\,[u'_{ij}, \theta(p, u'_{ij})] \tag{12}$$

The introduction of U^* from (12) into (11) leaves the two-field theorem

$$\delta \int [u_{kk} p - V^*(u'_{ij}, p)]\, \mathrm{d}v = 0 \tag{13}$$

with (13) being equivalent, upon writing $U^*(u'_{ij}, \theta) = U(u'_{ij}) + U'(u'_{ij}, \theta)$ and $V' = p\,\theta(p, u'_{ij}) - U'\,[u'_{ij}, \theta(p, u'_{ij})$ so that $V^*(u'_{ij}, p) = V'(u'_{ij}, p) - U(u'_{ij})$, to the result in [2].

3 The five-field theorem of geometrically nonlinear theory for Green-Lagrange strains and Kirchhoff-Trefftz stresses

We again depart from a statement of the volume integral portion of the classical variational principle for displacements (1), with the u_{ij} now being the components of the Green-Lagrange strain tensor

$$u_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}) \tag{14}$$

and with the principle of Hu and Washizu for displacements, strains and (Kirchhoff-Trefftz or second Piola-Kirchhoff) stresses being again Eq. (3).

The additive decomposition of geometrically linear theory in (4) into distortional and dilatational contributions is known to be replaceable in geometrically nonlinear theory by a multiplicative decomposition which is dependent on the determinental relation

$$J_u = \|\delta_{ij} + 2u_{ij}\| = (1 + \Delta)^2 \tag{15}$$

where Δ is the relative change of volume due to deformation, so that $J_u = 1$ for an incompressible material.

Given Eq. (15) we have the possibility of defining equantities u'_{ij} which correspond to the quantities $u'_{ij} = u_{ij} - 1/3\, u_{kk} \delta_{ij}$ in (4) by writing

$$\delta_{ij} + 2u_{ij} = (\delta_{ij} + 2u'_{ij}) J_u^{1/3} \tag{16}$$

inasmuch as, evidently,

$$J'_u = \|\delta_{ij} + 2u'_{ij}\| = \|\delta_{ij} + 2u_{ij}\| J_u^{-1} = 1 \tag{17}$$

We now again make use of the artifice of replacing the basic Eq. (2) by (2′), with constraint strain displacement relations $\varepsilon_{ij} = u_{ij}$, and we define distortional strain components ε'_{ij} through the relations

$$\delta_{ij} + 2\varepsilon_{ij} = (\delta_{ij} + 2\varepsilon'_{ij}) \mathrm{J}_\varepsilon^{1/3} \tag{18}$$

where

$$J_\varepsilon = \|\delta_{ij} + 2\varepsilon_{ij}\| \tag{19}$$

Having (18) and (19) we next write, in analogy to Eq. (5) for the geometrically linear case,

$$U(\varepsilon_{ij}) = U\left[\varepsilon'_{ij} J_\varepsilon^{1/3} + \frac{1}{2}(J_\varepsilon^{1/3} - 1)\,\delta_{ij}\right] = U^*(\varepsilon'_{ij}, J_\varepsilon) \tag{20}$$

and we stipulate, in analogy to the result for the geometrically linear case, that the five-field thoerem of geometrically nonlinear theory, involving Lagrange multipliers σ'_{ij} and p, be of the form

$$\delta \int \{ U^* (\varepsilon'_{ij}, J_\varepsilon) + (u'_{ij} - \varepsilon'_{ij})\sigma'_{ij} + [f(J_u) - f(J_\varepsilon)]\, p \}\, \mathrm{d}v = 0 \tag{21}$$

with f as a function the choice of which remains at our disposal.

Given Eq. (21) we note in particular the Euler constitutive equations

$$\sigma'_{ij} = \frac{\partial U^*}{\partial \varepsilon'_{ij}}, \qquad p = \frac{1}{f'(J_\varepsilon)}\, \frac{\partial U^*}{\partial J_\varepsilon} \tag{22a, b}$$

In view of (20) and in view of the relation $\sigma_{ij} = \partial U / \partial \varepsilon_{ij}$ we have as expressions for σ'_{ij} and p in terms of Kirchhoff-Trefftz stresses and Green-Lagrange strains:

$$\sigma'_{ij} = \frac{\partial U}{\partial \varepsilon_{mn}}\, \frac{\partial \varepsilon_{mn}}{\partial \varepsilon'_{ij}} = \sigma_{ij} J_\varepsilon^{1/3}, \qquad p = \frac{(2\varepsilon_{ij} + \delta_{ij})}{6 f'(J_\varepsilon) J_\varepsilon}\, \sigma_{ij} \tag{23a, b}$$

and, alternately, as expressions for the σ_{ij}, in terms of the σ'_{ij}, ε'_{ij} and p

$$\sigma_{ij} = \frac{\sigma'_{ij}}{J_\varepsilon^{1/3}}, \qquad f'(J_\varepsilon) J_\varepsilon = \frac{(\delta_{ij} + 2\varepsilon'_{ij})\sigma'_{ij}}{6p} \tag{24}$$

In order to verify that (21) has the appropriate equilibrium equations as Euler equations as well, it is only necessary to verify that

$$\sigma'_{ij} \delta u'_{ij} + p f'(J_u)\, \delta J_u = \sigma_{ij} \delta u_{ij} \tag{25}$$

This is readily accomplished on the basis of (16) in conjunction with the Euler strain displacement equations of (21).

4 Reductions of the five-field theorem (21)

As for the geometrically linear problem we obtain a four-field theorem which no longer involves the volume change measure J_ε by inverting (22b) and by then defining a function

$$V^*(\varepsilon'_{ij}, p) = p f[J_\varepsilon(p, \varepsilon'_{ij})] - U^* [\varepsilon'_{ij}, J_\varepsilon(p, \varepsilon'_{ij})] \tag{26}$$

with which the desired four-field equation follows from (21) in the form

$$\delta \int [(u'_{ij} - \varepsilon'_{ij})\sigma'_{ij} + f(J_u) p - V^*(\varepsilon'_{ij}, p)]\, \mathrm{d}v = 0 \tag{27}$$

Our second reduction, to a three-field theorem, again depends on considering the relations $\varepsilon'_{ij} = u'_{ij}$ and $\sigma'_{ij} = \partial U^* / \partial \varepsilon'_{ij}$ as constraint equations rather than Euler equations. With this we obtain from (21) as a three-field theorem involving u_i, J_ε, and p

$$\delta \int \{ U^*(u'_{ij}, J_\varepsilon) + [f(J_u) - f(J_\varepsilon)]\, p \}\, \mathrm{d}v = 0 \tag{28}$$

which corresponds to an equivalent theorem in [5], upon stipulating $f(J_u) = J_u$ and $f(J_\varepsilon) = J_\varepsilon$.

Given Eq. (27) we may further deduce a two-field theorem for p and the u_i by changing the Euler relations $\varepsilon'_{ij} = u'_{ij}$ into constraints so as to have this two-field theorem, in generalisation of the result in [2] for the geometrically linear problem, in the form

$$\delta \int [p f(J_u) - V^*(u'_{ij}, p)]\, \mathrm{d}v = 0 \tag{29}$$

5 A five-field theorem of geometrically nonlinear theory in terms of generalized Piola stresses and displacement gradient components

With stress vectors $\boldsymbol{\sigma}_i$ as forces per unit of *un*deformed area acting over the surfaces of deformed material elements defined by position vectors $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{u}$ the equations of equilibrium for deformed material elements are, except for body force terms which are omitted in this account,

$$\boldsymbol{\sigma}_{i,i} = 0, \qquad \boldsymbol{z}_{,i} \times \boldsymbol{\sigma}_i = 0 \tag{30a, b}$$

Given that, in terms of the unit vectors $\boldsymbol{e}_i$ in the directions of the coordinate axes x_i, we have as defining relations for Piola components of stress

$$\boldsymbol{\sigma}_i = s_{ij}\boldsymbol{e}_j \tag{31}$$

we have earlier [3] defined *generalized* Piola components of stress τ_{ij} by writing

$$\boldsymbol{\sigma}_i = \tau_{ij}\boldsymbol{t}_j \tag{32}$$

with the unit vectors $\boldsymbol{t}_j$ given by

$$\boldsymbol{t}_j = \alpha_{jk}\boldsymbol{e}_k, \qquad \alpha_{ik}\alpha_{jk} = \delta_{ij} \tag{33 a, b}$$

with (33) implying that

$$\boldsymbol{e}_j = \alpha_{kj}\boldsymbol{t}_k, \qquad \alpha_{ki}\alpha_{kj} = \delta_{ij} \tag{34 a, b}$$

With (33) and (34) we define *generalized* displacement gradient components

$$w_{ik} = (\delta_{ij} + u_{j,i})\,\alpha_{kj} - \delta_{ik} \tag{35}$$

on the basis of deducing with the help of (34 a) that,

$$\boldsymbol{z}_{,i} = (\delta_{ij} + u_{j,i})\,\boldsymbol{e}_j = (\delta_{ik} + w_{ik})\,\boldsymbol{t}_k \tag{36}$$

Given the representations (32) and (36) we have from (30 a, b) as component equilibrium equations

$$(\alpha_{jk}\tau_{ij})_{,i} = 0, \qquad (\delta_{ik} + w_{ik})\,e_{kjm}\tau_{ij} = 0 \tag{37 a, b}$$

As long as we stipulate that $\delta\alpha_{ij} = 0$, we have on the basis of (32) and (36) that

$$\boldsymbol{\sigma}_i \cdot \delta\boldsymbol{z}_{,i} = \tau_{ij}\delta w_{ij} \tag{38}$$

which assures the conjugacy of τ_{ij} and w_{ij}. If, with (38) and with

$$\delta w_{ij} = \alpha_{jk}\delta u_{k,i} \tag{39}$$

we stipulate constitutive relations of the form

$$\tau_{ij} = \frac{\partial U_\tau}{\partial w_{ij}} \tag{40}$$

we then have that the variational equation

$$\delta \int U_\tau \,\mathrm{d}v = 0 \tag{41}$$

with arbitrary δu_i, has the force equilibrium Eq. (37 a) as Euler equations.

For (41) to be meaningful, it is necessary to restrict the form of U_τ in such a way that the moment equilibrium Eq. (37 b) is also satisfied. It is readily shown that Eq. (37 b) *will be satisfied*, identically, upon stipulating that

$$U_\tau(w_{ij}) = U(u_{ij}) \tag{42}$$

with u_{ij} as in (14), where we note that on the basis of the relations $u_{ij} = 1/2\,(\boldsymbol{z}_{,i}\cdot\boldsymbol{z}_{,j} - \delta_{ij})$, we have as expression for the u_{ij} in terms of the w_{ij},

$$u_{ij} = \frac{1}{2}(w_{ij} + w_{ji} + w_{ik}w_{jk}) \tag{43}$$

With (42) and (43) it is then established that (41) is a one-field variational theorem of geometrically nonlinear theory in terms of generalized displacement gradient components as defined by (35) which directly corresponds to the theorem in Eq. (2) for geometrically linear theory.

From (41) to (43) we next have as a three-field Hu-Washizu theorem, corresponding to Eq. (3) of the linear theory

$$\delta \int [U_\tau(\gamma_{ij}) + (w_{ij} - \gamma_{ij})\tau_{ij}]\,\mathrm{d}v = 0 \tag{44}$$

where now $U_\tau(\gamma_{ij}) = U(\varepsilon_{ij})$, with $\varepsilon_{ij} = \gamma_{ij} + \gamma_{ji} + \gamma_{ik}\gamma_{jk}$.

The generalization of (44) to a five-field theorem involving volume change and pressure, in addition to deviatoric stresses and strains, involves the relation

$$\delta_{ij} + w_{ij} = (\delta_{ij} + w'_{ij}) J_w^{1/3} \tag{45}$$

where

$$J_w = \| \delta_{ij} + w_{ij} \| = z_{,1} \cdot z_{,2} \times z_{,3} = 1 + \Delta \tag{46}$$

with corresponding formulas for γ'_{ij}, γ_{ij} and J_γ. In this way we then have in analogy to (21) with

$$U_\tau(\gamma_{ij}) = U_\tau[(\gamma_{ij} + \gamma'_{ij}) J_\tau^{1/3} - \delta_{ij}] = U_\tau^*(\gamma'_{ij}, J_\gamma) \tag{47}$$

as a five-field theorem involving displacements, deviatoric strains and stresses, and pressure and volume change variables,

$$\delta \int \{ U_\tau^*(\gamma'_{ij}, J_\gamma) + (w'_{ij} - \gamma'_{ij}) \tau'_{ij} + [f(J_w) - f(J_\gamma)] p \} \mathrm{d}v = 0 \tag{48}$$

From (48) we may again deduce a three-field theorem which corresponds to a result in [5] upon introducing the relations $\gamma'_{ij} = w'_{ij}$ and $\tau'_{ij} = \partial U_\tau^* / \partial \gamma'_{ij}$ as constraints. This three-field theorem is, in analogy to (28)

$$\delta \int \{ U_\tau^*(w'_{ij}, J_\gamma) + [f(J_w) - f(J_\gamma)] p \} \mathrm{d}v = 0 \tag{49}$$

The possibility of deducing four and two-field theorems corresponding to (10) and (13) is questionable to the extent that the invertibility of the constitutive relation $p = [1/f'(J_\gamma)] \partial U_\tau^* / \partial J_\gamma$ is questionable within the framework of the concept of generalized Piola components of stress.

While the practical usefulness of the theorems in (48) and (49) is in doubt for other than the special case of $\alpha_{ij} = \delta_{ij}$, the considerations leading to them furnish a particularly convenient approach to a broadened formulation where the status of the α_{ij} is changed from that of given quantities to that of additional dependent variables.

6 A seven-field theorem involving distinguished generalized Piola or Biot stresses and displacement gradient components

We now consider the directions of the vectors $\boldsymbol{t}_j$ in (33a, b) not as given but as dependent variables which are to be determined as part of the solution of the problem. We then have, as a consequence of (35)

$$\delta w_{ik} = \alpha_{kj} \delta u_{j,i} + (\delta_{ij} + u_{j,i}) \delta \alpha_{kj} \tag{50}$$

An observation of the fact that, as a consequence of (33b), $\alpha_{ik} \delta \alpha_{jk} = - \alpha_{jk} \delta \alpha_{ik}$ and therewith, in terms of the quantites $\delta \omega_m$, $\alpha_{ik} \delta \alpha_{jk} = e_{ijm} \delta \omega_m$ leads, with (34b) and with a consistent change of subscripts, to the relation $\delta \alpha_{ij} = \alpha_{kj} e_{kim} \delta \omega_m$. This, in conjunction with (35), transforms (50) into

$$\delta \omega_{ik} = \alpha_{kj} \delta u_{j,i} + (\delta_{ij} + w_{ij}) e_{jkm} \delta \omega_m \tag{51}$$

With equation (39) replaced by (51), and with τ_{ij} again as in (40) we will now have that the variational equation (41), with arbitrary δu and $\delta \omega_m$, has the character of a two-field principle with not only the force equilibrium Eq. (37a) but also the moment equilibrium Eq. (37b) as Euler equations.

It is necessary at this point to decide to what extent the form of the function $U_\tau(w_{ij})$ cannot be arbitrarily stipulated. We do know that if U_τ is as in (42) that then the moment equilibrium equations are satisfied automatically and that therewith the variational Eq. (41), with or without $\delta \alpha_{ij} = 0$, is a valid one-field equation. An alternate disposition concerning U_τ, leading to a *three*-field variational equation, is as follows. We stipulate, as a physically reasonable restriction, to be satisfied by an appropriate determination of the α_{ij}, the strain symmetry conditions.

$$w_{ij} = w_{ji} \tag{52}$$

and we write

$$U_\tau = U_\tau(w_{11}, w_{12} + w_{21}, \ldots) \tag{53}$$

After this we change the constraint relations (52) into Euler equations with the help of Lagrange multipliers λ_k, and therewith state as a three-field variational equation which has force and moment equilibrium conditions as well as the kinematic relations (52) as Euler equations

$$\delta \int [U_\tau(w_{11}, w_{12}+w_{21}, \ldots) + e_{ijk}(w_{ij}-w_{ji})\lambda_k]\,\mathrm{d}v = 0 \tag{54}$$

The physical significance of the multipliers λ_k is obtained upon deducing from (54), with the abreviation $\hat{w}_{ij} = w_{ij} + w_{ji}$

$$\int \left[\frac{\partial U_\tau}{\partial \hat{w}_{11}}\delta w_{11} + \frac{\partial U_\tau}{\partial \hat{w}_{12}}\delta \hat{w}_{12} + \ldots + \lambda_3 \delta(w_{12}-w_{21}) + \ldots\right]\mathrm{d}v = 0 \tag{55}$$

or

$$\int \left[\frac{\partial U_\tau}{\partial w_{11}}\delta w_{11} + \left(\frac{\partial U_\tau}{\partial \hat{w}_{12}} + 2\lambda_3\right)\delta w_{12} + \left(\frac{\partial U_\tau}{\partial \hat{w}_{12}} - 2\lambda_3\right)\delta w_{21} + \ldots\right]\mathrm{d}v = 0 \tag{56}$$

For (56), with δw_{ik} as in (50), to have the equilibrium Eqs. (37a, b) as Euler equations it is evidently necessary to have

$$\tau_{11} = \frac{\partial U_\tau}{\partial w_{11}}, \qquad \tau_{12} = \frac{\partial U_\tau}{\partial \hat{w}_{12}} + 2\lambda_3, \qquad \tau_{21} = \frac{\partial U_\tau}{\partial \hat{w}_{12}} - 2\lambda_3 \quad \text{etc.} \tag{57}$$

Equation (57), with the further abreviation $\hat{\tau}_{12} = 1/2(\tau_{12} + \tau_{12})$ implies that

$$\hat{\tau}_{12} = \frac{\partial U_\tau}{\partial \hat{w}_{12}}, \qquad \lambda_3 = \frac{\tau_{12} - \tau_{21}}{4} \quad \text{etc.} \tag{58}$$

and accordingly we have that only the sums of the stresses τ_{12} and τ_{21}, etc., enter into the constitutive relations.

Having expressions for the λ_k in accordance with (58), it is possible to write the three-field theorem (54) in the form

$$\delta \int \left[U_\tau(w_{11}, \hat{w}_{12}, \ldots) + \frac{1}{4}(w_{ij} - w_{ji})(\tau_{ij} - \tau_{ji})\right]\mathrm{d}v = 0 \tag{59}$$

with independent variations δu_i; $\delta \omega_i$; and $\delta(\tau_{ij} - \tau_{ji})$, consistant with a result in [4].

Given equation (59) it is again possible to increase the number of fields and now obtain a *five* field principle in the sense of Hu-Washizu, by introducing components of strain γ_{ij}, with strain displacement relations $\gamma_{ij} = w_{ij}$ as additional Euler equations. In this way we then have, in place of (59)

$$\delta \int \left[U_\tau(\gamma_{11}, \hat{\gamma}_{12}, \ldots) + (w_{11} - \gamma_{11})\tau_{11} + (\hat{w}_{12} - \hat{\gamma}_{12})\hat{\tau}_{12} + \frac{1}{2}(w_{12} - w_{21})(\tau_{12} - \tau_{21}) + \ldots\right]\mathrm{d}v = 0 \tag{60}$$

with independent variations $\delta\gamma_{11}$, $\delta\hat{\gamma}_{12}, \ldots \delta\tau_{11}$, $\delta\hat{\tau}_{12}, \ldots$, $\delta(\tau_{12} - \tau_{21}), \ldots, \delta u_i$, and $\delta\omega_i$. Finally we may use (60) to again deduce a theorem with supplementary volume change and pressure variables, where we now write in analogy to (16)

$$1 + w'_{11} = (1 + w_{11})J_w^{-1/3}, \qquad \hat{w}'_{12} = \hat{w}_{12}J_w^{-1/3}\ldots \tag{61}$$

and

$$J_w = \begin{vmatrix} 1 + w_{11} & \frac{1}{2}\hat{w}_{12} & \ldots \\ \frac{1}{2}\hat{w}_{12} & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix} \tag{62}$$

with corresponding formulas for $\gamma_{11}, \hat{\gamma}_{12}, \ldots,$ and J_γ and with

$$U_\tau(\gamma_{11}, \hat{\gamma}_{12}, \ldots) = U_\tau[(1+\gamma'_{11})J_\gamma^{1/3} - 1, \hat{\gamma}'_{12}J_\gamma^{1/3}, \ldots] = U_\tau^*(\gamma'_{11}, \hat{\gamma}'_{12}, \ldots, J_\gamma) \tag{63}$$

With (63) and again with a pressure function p, we now deduce a *seven*-field variational theorem

$$\delta \int \Big\{ U_\tau^*(\gamma_{11}, \hat{\gamma}'_{12}, \ldots, J_\gamma) + (w'_{11} - \gamma'_{11})\tau'_{11} + (\hat{w}_{12} - \hat{\gamma}'_{12})\hat{\tau}_{12} + \ldots$$
$$+ \frac{1}{2}(w'_{12} - w'_{21})(\tau'_{12} - \tau'_{21}) + \ldots + [f(J_w) - f(J_\gamma)]\,p \Big\}\,\mathrm{d}v = 0$$

with independent variations $\delta\gamma'_{11}, \delta\hat{\gamma}'_{12}, \ldots, \delta J_\gamma, \delta\tau'_{11}, \delta\hat{\tau}'_{12}, \ldots, \delta(\tau'_{12} - \tau'_{21}), \ldots, \delta p, \delta u_i,$ and $\delta\omega_i$.

As before, we may deduce from this, by introduction of suitable constraints, lower field relations including a five field theorem

$$\delta \int \Big\{ U_\tau^*(w'_{11}, \hat{w}'_{12}, \ldots, J_\gamma) + \frac{1}{2}(w'_{12} - w'_{12})(\tau'_{12} - \tau'_{21}) + \ldots + [f(J_w) - f(J_\gamma)]\,p \Big\}\,\mathrm{d}v = 0 \tag{65}$$

corresponding to the three-field theorem in [5], and a four field theorem

$$\delta \int \Big[\frac{1}{2}(w'_{12} - w'_{21})(\tau'_{12} - \tau'_{21}) + \ldots + pf(J_w) - V_\tau^*(w'_{11}, \hat{w}'_{12}, \ldots, p) \Big]\,\mathrm{d}v = 0 \tag{66}$$

corresponding to the two-field theorem in terms of Green-Lagrange strains and pressure in Eq. (29).

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