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On the Spectral Density and Asymptotic Normality of Weakly Dependent Random Fields

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For weakly stationary random fields, conditions on coefficients of "linear dependence" are given which are, respectively, sufficient for the existence of a continuous spectral density, and necessary and sufficient for the existence of a continuous positive spectral density. For strictly stationary random fields, central limit theorems are proved under the corresponding "unrestricted ρ-mixing" condition and just finite or "barely infinite" second moments. No mixing rate is assumed.

KEY WORDS: Stationary random fields; spectral density; ρ-mixing; central limit theorem.

1. INTRODUCTION

Suppose d is a positive integer. Suppose $X := (X_k, k \in \mathbf{Z}^d)$ is a centered complex weakly stationary random field on a probability space $(\Omega, \mathscr{F}, P)$. By "centered" we mean that $EX_k = 0$. We allow the X_k's to be complex-valued since the proofs of our results will in any case involve some complex-valued random variables. By "weakly stationary" we mean of course that $E|X_0|^2 < \infty$ and that $EX_k \bar{X}_j$ depends only on the vector $k - j$. (Here the origin in $\mathbf{Z}^d$ is denoted simply by 0; in our context this should not cause any confusion.)

Let us denote the usual Euclidean norm of a vector $k \in \mathbf{Z}$ by $\|k\|$. The "distance" between any two disjoint nonempty subsets $S, D \subset \mathbf{Z}^d$ will be denoted by

$$\operatorname{dist}(S, D) := \min_{j \in S, k \in D} \|j - k\|$$

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For any two disjoint nonempty subsets $S, D \subset \mathbf{Z}^d$, define

$$r(S, D) := \sup EV\bar{W}/(\|V\|_2 \|W\|_2)$$

where this sup is taken over all pairs of random variables V and W of the following forms:

$$V = \sum_{k \in S^*} a_k X_k \qquad \text{and} \qquad W = \sum_{k \in D^*} b_k X_k$$

where S^* is a finite subset of S, D^* is a finite subset of D, and the a_k's and b_k's are complex numbers. Where necessary, 0/0 is interpreted to be 0. For every real number $s \geqslant 1$, define

$$r^*(s) := \sup r(S, D)$$

where this sup is taken over all pairs of nonempty disjoint subsets $S, D \subset \mathbf{Z}^d$ such that $\text{dist}(S, D) \geqslant s$.

In our discussions of spectral densities, we shall use the following notations: Let T denote the unit circle in the complex plane. For each $t \in T^d$, we shall let $\lambda := (\lambda_1, ..., \lambda_d)$ denote the element of $(-\pi, \pi]^d$ such that $t = (\exp i\lambda_1, ..., \exp i\lambda_d)$. The letters t and λ will always be related in this way. Let μ_T denote normalized Lebesgue measure on T [i.e., normalized so that $\mu_T(T) = 1$]. Let $\mu_T^d = \mu_T \times \cdots \times \mu_T$ denote the d-dimensional product measure on T^d. A Borel nonnegative integrable function f on T^d is called a "spectral density" for the centered complex weakly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ if

$$\forall k \in \mathbf{Z}^d \qquad EX_k \bar{X}_0 = \int_{T^d} e^{ik \cdot \lambda} f(t) \, d\mu_T^d(t)$$

Here, of course, $k \cdot \lambda$ denotes the dot product.

Our discussion so far has been restricted to centered random fields. For a "noncentered" complex weakly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$, i.e., with $EX_k = \mu \neq 0$, one of course simply defines $r^*(s)$ and "spectral density" (if it exists) to be equal to $r^*(s)$ and spectral density for the centered random field $Y := (Y_k, k \in \mathbf{Z}^d)$ defined by $Y_k := X_k - \mu$.

We shall prove the following two theorems:

Theorem 1. Suppose d is a positive integer and $X := (X_k, k \in \mathbf{Z}^d)$ is a complex weakly stationary random field such that $r^*(s) \to 0$ as $s \to \infty$. Then X has a continuous spectral density on T^d.

Theorem 2. Suppose d is a positive integer and $X := (X_k, k \in \mathbf{Z}^d)$ is a complex weakly stationary nondegenerate random field. Then the following two statements are equivalent:

1. $r^*(1) < 1$, and $r^*(s) \to 0$ as $s \to \infty$.
2. X has a continuous positive spectral density on T^d.

Part of Theorem 2, the fact that (2) implies $r^*(s) \to 0$, was already shown by Rosenblatt (Ref. 13, p. 73, Theorem 7) in his discussion of stationary Gaussian random fields. Zhurbenko (Ref. 17, Chapter 2, Section 2) gives a number of theorems of the following kind: If a stationary random field satisfies certain moment conditions, together with a "strong mixing" condition with a sufficiently fast "mixing rate," then the random field has a spectral density (or higher-order spectral density) which has higher-order derivatives with certain nice properties.

Now consider for a moment the case $d = 1$, where $X := (X_k, k \in \mathbf{Z})$ is a weakly stationary sequence. For each integer $n \geqslant 1$ define (for the moment)

$$r(n) := r(\{..., -2, -1, 0\}, \{n, n+1, n+2, ...\})$$

This is the standard "linear dependence" coefficient used in prediction theory, and in the case of a stationary Gaussian sequence the condition $r(n) \to 0$ is well known to be equivalent to the standard "ρ-mixing" and "strong mixing" conditions. Obviously $r(n) \leqslant r^*(n)$. Ibragimov and Rozanov (Ref. 11, p. 179, Example 1) give an example of a stationary Gaussian sequence which satisfies $r(n) \to 0$ but which fails to have a continuous spectral density. Comparing this with Theorem 1, we see that for stationary sequences the condition $r(n) \to 0$ is indeed weaker than $r^*(n) \to 0$. The Helson–Sarason Theorem (Ref. 14 or Ref. 11, Chap 5) gives a necessary and sufficient condition for $r(n) \to 0$, based on the spectral density.

Our proof of Theorem 1 (and part of Theorem 2) will mimic the proof of the following result of Ibragimov and Rozanov (Ref. 11, p. 182, Lemma 17, and p. 190, Note 2): If $X := (X_k, k \in \mathbf{Z})$ is a weakly stationary random sequence satisfying $\sum_{n=1}^{\infty} r(2^n) < \infty$, then X has a continuous spectral density. Their argument can be summarized roughly as follows: For each $m \geqslant 1$ define the function f_m on T by

$$f_m(e^{i\lambda}) := (1/m)\, E \left| \sum_{k=1}^{m} e^{-ik\lambda} X_k \right|^2$$

Show that for positive integers $m < n$ one has

$$|f_m(e^{i\lambda}) - f_n(e^{i\lambda})| \leqslant a_m$$

where a_m depends only on m (not on λ) and converges to 0 as $m \to \infty$. [This is the cumbersome part, involving extensive calculations based on the assumption $\sum r(2^n) < \infty$; the rest is easy.] The existence of $f(e^{i\lambda}) := \lim_{m \to \infty} f_m(e^{i\lambda})$ follows. The bound $|f_m(e^{i\lambda}) - f(e^{i\lambda})| \leqslant a_m$ follows. For each $m \geqslant 1$ the function f_m is a trigonometric polynomial. The continuity of f on T follows. A final elementary calculation shows that f is in fact a spectral density for X.

For random sequences (the case $d = 1$), one has the following "growth of variances" result related to Theorem 1:

Theorem 3. Suppose $X := (X_k, k \in \mathbf{Z})$ is a centered complex weakly stationary random sequence such that $r^*(s) \to 0$ as $s \to \infty$. Let f denote its continuous spectral density on T. If $\lambda \in (-\pi, \pi]$ is such that $E |\sum_{k=1}^{n} e^{-ik\lambda} X_k|^2 \to \infty$ as $n \to \infty$, then $f(e^{i\lambda}) > 0$.

Now let us turn our attention to central limit theory for random fields. First, for any two σ-fields $\mathscr{A}$ and $\mathscr{B}$ in our probability space, define the maximal correlation:

$$\rho(\mathscr{A}, \mathscr{B}) := \sup \frac{|EVW - EVEW|}{(\|V\|_2 \, \|W\|_2)} \tag{1.1}$$

where the sup is taken over all square-integrable random variables V and W which are $\mathscr{A}$-measurable and $\mathscr{B}$-measurable, respectively. By Ref. 16, p. 512, Theorem 1.1, and a trivial calculation, the RHS of (1.1) is the same whether the random variables V and W are restricted to be real or allowed to be complex, and remains the same even if V and W are restricted to have mean 0.

Suppose $X := (X_k, k \in \mathbf{Z}^d)$ is a *real strictly* stationary random field. For any two nonempty disjoint sets S and $D \subset \mathbf{Z}^d$, we shall use the abbreviation

$$\rho(S, D) := \rho(\sigma(X_k, k \in S), \sigma(X_k, k \in D))$$

Here and in what follows, $\sigma(\cdots)$ denotes the σ-field generated by $(\cdots)$. For each real number $s \geqslant 1$, define

$$\rho^*(s) := \sup \rho(S, D)$$

where this sup is taken over all pairs of nonempty disjoint subsets $S, D \in \mathbf{Z}^d$ such that $\text{dist}(S, D) \geqslant s$. Obviously, in the case where the X_k's are square-integrable, one has $r(S, D) \leqslant \rho(S, D)$ and $r^*(s) \leqslant \rho^*(s)$. As is well known, in both equations one has equality if the random field is Gaussian.

We shall give two central limit theorems for random fields $X := (X_k, k \in \mathbf{Z}^d)$. First we need some notations for d-dimensional "block sums."

Suppose $j = (j_1,..., j_d)$ and $l := (l_1,..., l_d)$ are elements of $\mathbf{Z}^d$ such that $j_u \leqslant l_u$ $\forall u = 1,..., d$. Define

$$S(X: j_1,..., j_d: l_1,..., l_d) := \sum_k X_k$$

where this sum is taken over all $k := (k_1,..., k_d) \in \mathbf{Z}^d$ such that $j_u \leqslant k_u \leqslant l_u \forall u = 1,..., d$. If $l_1,..., l_d$ are each positive, then we use the abbreviations

$$S(X:l) := S(X:l_1,..., l_d) := S(X:1,..., 1:l_1,..., l_d)$$

Let $\mathbf{N}$ denote $\{1, 2, 3,...\}$. We shall study the asymptotic normality of $S(X:L)$ as $L \in \mathbf{N}^d$ becomes "large," with no extra restriction on its "direction."

Theorem 4. Suppose d is a positive integer and $X := (X_k, k \in \mathbf{Z}^d)$ is a centered, real, strictly stationary random field such that $0 < EX_0^2 < \infty$, $\rho^*(s) \to 0$ as $s \to \infty$, and the continuous spectral density f of X on T^d satisfies $f(1,..., 1) > 0$. Then as $\|L\| \to \infty$ $(L \in \mathbf{N}^d)$, one has that $\|S(X:L)\|_2 \to \infty$ and that $S(X:L)/\|S(X:L)\|_2 \to N(0, 1)$ in distribution.

Theorem 5. Suppose d is a positive integer and $X := (X_k, k \in \mathbf{Z}^d)$ is a centered, real, strictly stationary, nondegenerate random field such that

$$H(c) := EX_0^2 I(|X_0| \leqslant c) \text{ is slowly varying as } c \to \infty \tag{1.2}$$

and

$$\rho^*(1) < 1 \qquad \text{and} \qquad \rho^*(s) \to 0 \qquad \text{as} \quad s \to \infty \tag{1.3}$$

Then as $\|L\| \to \infty$ $(L \in \mathbf{N}^d)$, one has that $\|S(X:L)\|_1 \to \infty$ and that $S(X:L)/[(\pi/2)^{1/2} \|S(X:L)\|_1] \to N(0, 1)$ in distribution.

It is well known that (1.2) implies $E|X_0| < \infty$.

Now consider for a moment the case $d = 1$, where X is a centered real strictly stationary sequence. For each integer $n \geqslant 1$ define (for the moment)

$$\rho(n) := \rho(\{..., -2, -1, 0\}, \{n, n+1, n+2,...\})$$

The standard "ρ-mixing" condition is $\rho(n) \to 0$ as $n \to \infty$. Obviously $\rho(n) \leqslant \rho^*(n)$. Under the assumptions of $\rho(n) \to 0$, finite second moments and $\mathrm{Var}(X_1 + \cdots + X_n) \to \infty$ as $n \to \infty$, the mixing rate $\sum_{n=1}^{\infty} \rho(2^n) < \infty$ is essentially the slowest that implies the CLT. Compare Ibragimov (Ref. 9, Theorem 2.2) and the author (Ref. 2, Theorem 2), or see the survey by Peligrad.[12] The author[3] proved a CLT for strictly stationary sequences under the assumptions of (1.2), $\sum_{n=1}^{\infty} \rho(2^n) < \infty$, and $\rho(1) < 1$. Using a 1986 preprint of Ref. 3 as a starting point, Shao[15] extended the result to a weak invariance principle. Note that, by Theorems 3 and 4, in order for a centered real strictly stationary square-integrable sequence $X := (X_k, k \in \mathbf{Z})$ to satisfy $S(X, n)/\|S(X, n)\|_2 \to N(0, 1)$, it is sufficient to have just $\rho^*(n) \to 0$ and $\operatorname{Var}(X_1 + \cdots + X_n) \to \infty$.

In the study of central limit theory for random fields satisfying $\rho^*(s) \to 0$, the mixing rate $\sum_{n=1}^{\infty} \rho^*(2^n) < \infty$ has sometimes been used. See, e.g., Goldie and Greenwood,[6] Goldie and Morrow,[7] and the references therein. As indicated in the results here, the condition $\rho^*(s) \to 0$ seems to give sufficient leverage to obtain a lot of information, without the assumption of a mixing rate or higher-order moments. There are numerous directions in which to pursue this, e.g., in connection with topics discussed in Refs. 6, 7, and 8, or Ref. 17, Chapter 2, Section 2, or even Ref. 3, Lemma 2.2, in order to weaken the assumption $\rho^*(1) < 1$ in (1.3). These directions will not be pursued here.

Theorems 1, 2, and 3 will be proved in Section 2, and Theorems 4 and 5 will be proved in Section 3. Throughout these proofs, the dimension d will be regarded as fixed; the dependence of positive constants on d in certain lemmas in Sections 2 and 3 will be suppressed. For convenience, we shall treat only random fields which are centered. The phrases "centered complex weakly stationary" and "centered real strictly stationary" will be abbreviated CCWS and CRSS, respectively. The notation $a_n \sim b_n$ means $\lim(a_n/b_n) = 1$, and the notation $a_n \ll b_n$ means $a_n = O(b_n)$. If an expression like a_b is a subscript, it will be written $a(b)$. For a given random field the quantities $r^*(s)$ and $\rho^*(s)$ are each monotonic (nonincreasing) as s increases; hence for convenience we can restrict our attention to $r^*(n)$ and $\rho^*(n)$, $n = 1, 2, 3, \ldots$.

2. PROOF OF THEOREMS 1, 2, AND 3

The proofs of these theorems will be based on a series of lemmas.

Lemma 1. Suppose $0 < r < 1$. Suppose $X_1, \ldots, X_n$ is a family of centered complex absolute-square-integrable random variables with the following property: For any two nonempty disjoint subsets $S, D \subset \{1, 2, \ldots, n\}$, one has that

$$\left| E\left(\sum_{k \in S} X_k\right)\left(\overline{\sum_{k \in D} X_k}\right) \right| \leqslant r \cdot \left\| \sum_{k \in S} X_k \right\|_2 \cdot \left\| \sum_{k \in D} X_k \right\|_2$$

Then

$$\frac{(1-r)}{(1+r)}\sum_{k=1}^{n} E\,|X_k|^2 \leqslant E\left|\sum_{k=1}^{n} X_k\right|^2 \leqslant \frac{(1+r)}{(1-r)}\sum_{k=1}^{n} E\,|X_k|^2$$

Proof. Let $(W_1,\dots, W_n)$ be a random vector independent of $(X_1,\dots, X_n)$, such that $W_1,\dots, W_n$ are i.i.d. with $P(W_1=1)=P(W_1=-1)=\frac{1}{2}$. Define the random sets $Q, Q^* \subset \{1,\dots, n\}$ by $Q := \{k\colon W_k=1\}$ and $Q^* := \{k\colon W_k=-1\}$. Define the (complex) random variables Y and Z by

$$Y := \sum_{k\in Q} X_k \qquad \text{and} \qquad Z := \sum_{k\in Q^*} X_k$$

In the sums below, the letter S will range over all subsets of $\{1,\dots, n\}$. For any given S, we denote its complement $S^* := \{1,\dots, n\}-S$. We have

$$\begin{aligned}
|EY\bar{Z}| &= \left|2^{-n}\sum_S E\Big(\sum_{k\in S} X_k\Big)\Big(\sum_{k\in S^*} \bar{X}_k\Big)\right| \\
&\leqslant 2^{-n}\sum_S r\cdot \left\|\sum_{k\in S} X_k\right\|_2 \left\|\sum_{k\in S^*} X_k\right\|_2 \\
&\leqslant 2^{-n} r\left(\sum_S \left\|\sum_{k\in S} X_k\right\|_2^2\right)^{1/2} \left(\sum_S \left\|\sum_{k\in S^*} X_k\right\|_2^2\right)^{1/2} \\
&= r\cdot E\,|Y|^2
\end{aligned}$$

Hence there exists a real number c with $-r\leqslant c\leqslant r$, such that

$$EY\bar{Z}+E\bar{Y}Z=2cE\,|Y|^2$$

By a simple calculation,

$$2(1-c)\,E\,|Y|^2 = E\,|Y-Z|^2 = E\left|\sum_{k=1}^{n} W_k X_k\right|^2 = \sum_{k=1}^{n} E\,|X_k|^2$$

Also, $E\,|Y+Z|^2=2(1+c)\,E\,|Y|^2$. We thus have

$$E\left|\sum_{k=1}^{n} X_k\right|^2 = E\,|Y+Z|^2 = (1+c)(1-c)^{-1}\sum_{k=1}^{n} E\,|X_k|^2$$

Since $-r\leqslant c\leqslant r$, Lemma 1 holds. □

Lemma 2. Suppose $q := (q_1, q_2, q_3,\dots)$ is a nonincreasing sequence of numbers in $[0, 1]$ such that $\lim_{n\to\infty} q_n < 1$. Then there exists a positive number $A := A(q)$ such that the following holds: If $X := (X_k, k\in \mathbf{Z}^d)$ is a

CCWS random field for which $r^*(m) \leqslant q_m \forall m \geqslant 1$, then for any finite set $S \subset \mathbf{Z}^d$ one has

$$E\left|\sum_{k \in S} X_k\right|^2 \leqslant A \cdot (\text{card } S) \cdot E|X_0|^2 \tag{2.1}$$

Proof. Suppose the sequence q is as in the hypothesis of Lemma 2. Our first task is to define the constant $A = A(q)$. Let J denote the least positive integer such that $q_J < 1$. Define the constant A by $A := J^{2d}(1+q_J)/(1-q_J)$.

Now suppose the random field $X := (X_k, k \in \mathbf{Z}^d)$ is as in the statement of the lemma, and S is any finite set $\subset \mathbf{Z}^d$. For each $l := (l_1,..., l_d) \in \{1,..., J\}^d$, let $S(l)$ denote the set of all elements $k := (k_1,..., k_d) \in S$ such that $\forall u = 1,..., d$, $k_u \equiv l_u \bmod J$. These sets form a partition of S. For each $l \in \{1,..., J\}^d$ one has

$$E\left|\sum_{k \in S(l)} X_k\right|^2 \leqslant (1+q_J)(1-q_J)^{-1} (\text{card } S)\, E|X_0|^2$$

by Lemma 1.

Hence by Minkowski's inequality, (2.1) holds. Lemma 2 is proved. □

We need some more notations. Suppose $X := (X_k, k \in \mathbf{Z}^d)$ is a CCWS random field. For each positive integer m define the nonnegative real number

$$F(X, m) := m^{-d} E\,|S(X : m,..., m)|^2$$

Also, the "linear dependence" coefficients $r^*(m)$ will sometimes be denoted $r^*(X, m)$ to avoid confusion when other random fields are also being treated.

Lemma 3. Suppose $q := (q_1, q_2, q_3,...)$ is a nonincreasing sequence of real numbers in $[0, 1]$ such that $\lim_{n \to \infty} q_n = 0$. Suppose $\varepsilon > 0$. Then there exists a positive integer $M(q, \varepsilon)$ such that the following holds: If $X := (X_k, k \in \mathbf{Z}^d)$ is a CCWS random field for which $r^*(m) \leqslant q_m \forall m \geqslant 1$, then for every integer $M \geqslant M(q, \varepsilon)$ and every positive integer n, one has that $|F(X, M) - F(X, nM)| \leqslant \varepsilon \cdot E|X_0|^2$.

Proof. Suppose the sequence $q := (q_1, q_2,...)$ and the number $\varepsilon > 0$ are as in the statement of the lemma. Our first task is to define the integer $M(q, \varepsilon)$. Let the positive constant $A := A(q)$ be as in Lemma 2. Let L be a positive integer such that $q_L < 1$ and

$$1 - \frac{\varepsilon}{6A} \leqslant \frac{(1-q_L)^{1/2}}{(1+q_L)^{1/2}} \quad \text{and} \quad \frac{(1+q_L)^{1/2}}{(1-q_L)^{1/2}} \leqslant 1 + \frac{\varepsilon}{6A} \tag{2.2}$$

Now let $M(q, \varepsilon)$ be a positive integer such that

$$\forall M \geqslant M(q, \varepsilon) \qquad A[(L+M)^d - M^d]/M^d \leqslant \varepsilon^2/(36A) \tag{2.3}$$

Now suppose the random field $X := (X_k, k \in \mathbf{Z}^d)$ is as in the statement of the lemma. Without loss of generality we assume that $E|X_0|^2 = 1$. Suppose that M and n are positive integers such that $M \geqslant M(q, \varepsilon)$. Our task is to prove that

$$|F(X, nM) - F(X, M)| \leqslant \varepsilon \tag{2.4}$$

For each $j := (j_1, ..., j_d) \in \mathbf{Z}^d$, define the following complex random variable:

$$\begin{aligned} W_j := S(X&:(j_1 - 1)(M+L) + 1, ..., (j_d - 1)(M+L) + 1: \\ &(j_1 - 1)(M+L) + M, ..., (j_d - 1)(M+L) + M) \end{aligned}$$

The random field $W := (W_k, k \in \mathbf{Z}^d)$ is CCWS; its "linear dependence" coefficients obviously satisfy $r^*(W, m) \leqslant r^*(X, m)$ and $r^*(W, 1) \leqslant r^*(X, L)$.

Now, by Minkowski's inequality, the definition of A (satisfying Lemma 2), and Eq. (2.3), we have

$$\begin{aligned} &|(nM)^{-d/2} \|S(X:nM, ..., nM)\|_2 - (nM)^{-d/2} \|S(X:n(L+M), ..., n(L+M))\|_2| \\ &\quad \leqslant (nM)^{-d/2} \|S(X:nM, ..., nM) - S(X:n(L+M), ..., n(L+M))\|_2 \\ &\quad \leqslant (nM)^{-d/2} (A \cdot [(n(L+M))^d - (nM)^d])^{1/2} \leqslant \varepsilon/(6A^{1/2}) \end{aligned} \tag{2.5}$$

Next, using (2.3) again,

$$\begin{aligned} &|(nM)^{-d/2} \|S(X:n(L+M), ..., n(L+M))\|_2 - (nM)^{-d/2} \|S(W:n, ..., n)\|_2| \\ &\quad \leqslant (nM)^{-d/2} \cdot (A \cdot [(n(L+M))^d - (nM)^d])^{1/2} \leqslant \varepsilon/(6A^{1/2}) \end{aligned} \tag{2.6}$$

Next, by Lemma 1,

$$\begin{aligned} (nM)^{-d/2} \|S(W:n, ..., n)\|_2 &= s \cdot (nM)^{-d/2} n^{d/2} \|W_0\|_2 \\ &= s \cdot M^{-d/2} \|S(X:M, ..., M)\|_2 \end{aligned}$$

for some real number s satisfying

$$\frac{(1 - r^*(X, L))^{1/2}}{(1 + r^*(X, L))^{1/2}} \leqslant s \leqslant \frac{(1 + r^*(X, L))^{1/2}}{(1 - r^*(X, L))^{1/2}}$$

Since $r^*(X, m) \leqslant q_m$ (by assumption), we have $1-\varepsilon/(6A) \leqslant s \leqslant 1+\varepsilon/(6A)$ by (2.2). Hence

$$\begin{aligned} &|(nM)^{-d/2}\,\|S(W:n,..., n)\|_2 - M^{-d/2}\,\|S(X:M,..., M)\|_2| \\ &\quad\leqslant [\varepsilon/(6A)]\cdot M^{-d/2}\,\|S(X:M,..., M)\|_2 \\ &\quad\leqslant [\varepsilon/(6A)]\cdot A^{1/2} = \varepsilon/(6A^{1/2}) \end{aligned} \tag{2.7}$$

By (2.5), (2.6), and (2.7),

$$|(nM)^{-d/2}\,\|S(X:nM,..., nM)\|_2 - M^{-d/2}\,\|S(X:M,..., M)\|_2| \leqslant \varepsilon/(2A^{1/2})$$

Using $(a^2-b^2)=(a+b)(a-b)$ and the definition of A again, we have

$$|F(X, nM) - F(X, M)| \leqslant (2A^{1/2})\cdot\varepsilon/(2A^{1/2}) = \varepsilon$$

Thus (2.4) holds. This completes the proof of Lemma 3. □

Lemma 4. Suppose $X := (X_k, k \in \mathbf{Z}^d)$ is a CCWS random field for which $r^*(n) \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} F(X, n)$ exists in $[0, \infty)$.

Proof. Suppose $\varepsilon > 0$. Again assume $E|X_0|^2 = 1$. Using Lemma 3 with a fixed sufficiently large positive integer M, one has that $\overline{\lim}_{n\to\infty} F(X, nM)$ and $\underline{\lim}_{n\to\infty} F(X, nM)$ are finite and differ by at most 2ε. By Lemma 2 and an elementary calculation, $F(X, n) - F(X, N) \to 0$ as $n, N \to \infty$ subject to $|n-N| \leqslant M$. Thus $\overline{\lim}_{n\to\infty} F(X, n)$ and $\underline{\lim}_{n\to\infty} F(X, n)$ are finite and differ by at most 2ε. Since $\varepsilon > 0$ was arbitrary, Lemma 4 holds. □

Proof of Theorem 1. Let $X := (X_k, k \in \mathbf{Z}^d)$ denote the random field in the statement of Theorem 1. Without loss of generality we assume that X is centered.

Recall the notations in Section 1. For each $t \in T^d$, define the random field $X^{(t)} := (X_k^{(t)}: k \in \mathbf{Z}^d)$ as follows:

$$\forall k \in \mathbf{Z}^d \qquad X_k^{(t)} := e^{-ik\cdot\lambda} X_k \tag{2.8}$$

For each $t \in T^d$, the random field $X^{(t)}$ is CCWS, and satisfies $E|X_0^{(t)}|^2 = E|X_0|^2$ and $r^*(X^{(t)}, m) = r^*(X, m)$.

From Lemma 4, define for each $t \in T^d$ the nonnegative real number

$$f(t) := \lim_{n\to\infty} F(X^{(t)}, n)$$

By Lemma 3, $|f(t) - F(X^{(t)}, M)|$ converges to 0 uniformly in $t \in T^d$ as $M \to \infty$. Also, for each fixed M, the function $F(X^{(t)}, M)$ is a trigonometric polynomial in the coordinates of t. The continuity of f on T^d follows.

All that remains is to show that f is a spectral density for the random field X. The argument is standard. Because of dominated convergence, it suffices to show that, for each fixed $k \in \mathbf{Z}^d$,

$$EX_k \bar{X}_0 = \lim_{M \to \infty} \int_{T^d} e^{ik \cdot \lambda} F(X^{(t)}, M) \, d\mu_T^d(t) \tag{2.9}$$

If one expands $F(X^{(t)}, M)$ as a sum of terms of the form $M^{-d} EX_j^{(t)} \bar{X}_l^{(t)}$, then by (2.8) the integral in (2.9) remains unchanged if one omits all ordered pairs (j, l) for which $j - l \neq k$. For each $M \geqslant 1$ let $H_{k,M}$ denote the number of ordered pairs (j, l) with $j, l \in \{1,..., M\}^d$ such that $j - l = k$. By a simple calculation the integral is $H_{k,M} \cdot M^{-d} EX_k \bar{X}_0$. For any given k, by a simple calculation, $H_{k,M} \cdot M^{-d} \to 1$ as $M \to \infty$, and hence (2.9) holds. This completes the proof of Theorem 1. □

Proof of Theorem 2. Again assume X is centered. We first prove that $(1) \Rightarrow (2)$. Assume (1) holds. Then the existence of a continuous spectral density comes from Theorem 1. We retain all of the notations and arguments in the proof of Theorem 1. For any given $t \in T^d$, if Lemma 1 is applied to the random field $X^{(t)}$, one sees that $\inf_{n \geqslant 1} F(X^{(t)}, n) > 0$. This forces the (continuous) spectral density f to satisfy $f(t) > 0$ for each $t \in T^d$. This completes the proof that $(1) \Rightarrow (2)$.

The proof that $(2) \Rightarrow (1)$ is a repeat of well-known tricks. Suppose (2) holds. Let f denote the continuous positive spectral density. Suppose S and D are finite nonempty disjoint subsets of $\mathbf{Z}^d$, and V and W are nondegenerate random variables of the form $V := \sum_{k \in S} a_k X_k$ and $W := \sum_{k \in D} b_k X_k$ where the a_k's and b_k's are complex numbers. Multiplying by complex constants if necessary, we may assume that $\|V\|_2 = \|W\|_2 = 1$ and that $EV\bar{W}$ is real and nonnegative. Define the functions g and h on T^d by $g(t) := \sum_{k \in S} a_k e^{ik \cdot \lambda}$ and $h(t) := \sum_{k \in D} b_k e^{ik \cdot \lambda}$. Then $1 = E|V|^2 = \int |g|^2 f$ and $1 = E|W|^2 = \int |h|^2 f$. Here the integrals are taken over T^d, with respect to the measure μ_T^d. Let m and M, respectively, denote the (positive) minimum and maximum of f on T^d. Now $\int g\bar{h} = 0$ and hence

$$2 - 2EV\bar{W} = E|V - W|^2 = \int |g - h|^2 f \geqslant \int |g - h|^2 m$$

$$= \int (|g|^2 + |h|^2) m \geqslant (m/M) \int (|g|^2 + |h|^2) f = 2m/M$$

Hence $EV\bar{W} \leqslant 1 - m/M$. It follows that $r^*(1) \leqslant 1 - m/M < 1$. The proof that $r^*(n) \to 0$ as $n \to \infty$ is already given in Rosenblatt (Ref. 13, p. 73, Theorem 7). The key point is that f can be approximated uniformly very closely by a trigonometric polynomial p on T^d (with maximum error much less than m), and if dist(S, D) is sufficiently large (depending only on p) then $\int g\bar{h}p = 0$ and $EV\bar{W} = \int g\bar{h}(f-p)$ is very small. □

Proof of Theorem 3. Define the random sequence $Y := (Y_k, k \in \mathbf{Z})$ by $Y_k := e^{-ik\lambda}X_k$. Then Y is CCWS, and $r^*(Y, n) = r^*(X, n) \to 0$ as $n \to \infty$. Let $A := A(r^*(X, 1), r^*(X, 2),...)$ be as in Lemma 2 (for $d = 1$). Let L be a positive integer such that $r^*(X, L) < 1$. Let M be a positive integer such that

$$(1 - r^*(X, L))^{1/2}\,(1 + r^*(X, L))^{-1/2}\,\|S(Y:M)\|_2 - (AL)^{1/2}\,\|Y_0\|_2 > 0 \tag{2.10}$$

For each positive integer j, define the random variable

$$W_j := S(Y:(j-1)(M+L)+1:(j-1)(M+L)+M)$$

The random sequence $W := (W_k, k \in \mathbf{Z})$ is CCWS, with $r^*(W, 1) \leqslant r^*(X, L)$. Using Lemma 1, for each positive integer n one has that

$$\begin{aligned} &\|S(Y:n(M+L))\|_2 \\ &\quad \geqslant \|S(W:n)\|_2 - \|S(Y:n(M+L)) - S(W:n)\|_2 \\ &\quad \geqslant (1 - r^*(X, L))^{1/2}\,(1 + r^*(X, L))^{-1/2}\,n^{1/2}\,\|W_0\|_2 - A^{1/2}(nL)^{1/2}\,\|Y_0\|_2 \\ &\quad = n^{1/2} \cdot [\text{L.H.S. of (2.10)}] \end{aligned} \tag{2.11}$$

Now $f(e^{i\lambda}) = \lim_{n\to\infty} F(Y, n)$ (e.g., as in the proof of Theorem 1). This forces $f(e^{i\lambda}) > 0$ by (2.10), (2.11), and simple arithmetic. Theorem 3 is proved. □

3. PROOF OF THEOREMS 4 AND 5

The proofs of these two theorems will use two preliminary lemmas.

Lemma 5. Suppose r is a real number such that

$$0 < r < 1 \tag{3.1}$$

$$1 - 4 \cdot (2r)^{1/2} \geqslant \tfrac{1}{2} \tag{3.2}$$

$$1 + 4 \cdot (2r)^{1/2} + 3r \leqslant 2 \tag{3.3}$$

and

$$(1 + r)^2/(1 - r)^2 \leqslant 2 \tag{3.4}$$

Suppose $X_1,\ldots, X_n$ are identically distributed, centered, real random variables with finite fourth moments, such that for any two disjoint subsets $S, D \in \{1,\ldots, n\}$ one has $\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in D)) \leqslant r$. Then

$$E\left(\sum_{k=1}^{n} X_k\right)^4 \leqslant 24 \cdot [n \cdot EX_1^4 + n^2(EX_1^2)^2]$$

Proof. From the proof of Lemma 1 we take the random variables $W_1,\ldots, W_n$, the random sets Q and Q^*, the random variables Y and Z, and the conventions on the symbols S and S^*. Without loss of generality (aside from the trivial case $EX_1^2 = 0$), we asume that $EX_1^2 = 1$.

We shall use the fact that for any two random variables U and V with finite (4/3)-norm and 4-norm, respectively, one has

$$|EUV - EUEV| \leqslant [2 \cdot \rho(\sigma(U), \sigma(V))]^{1/2} \, \|U\|_{4/3} \, \|V\|_4$$

This fact, taken from Ref. 4, p. 355, Theorem 4.1(iv), is a consequence of Thorin's multilinear interpolation theorem (Ref. 1, p. 18, Exercise 13).

Now

$$\begin{aligned}
|EY^3Z| &= \left| 2^{-n} \sum_S E\left(\sum_{k \in S} X_k\right)^3 \left(\sum_{k \in S^*} X_k\right)\right| \\
&\leqslant 2^{-n} \sum_S (2r)^{1/2} \left\| \sum_{k \in S} X_k \right\|_4^3 \left\| \sum_{k \in S^*} X_k \right\|_4 \\
&\leqslant 2^{-n}(2r)^{1/2} \left(\sum_S \left\| \sum_{k \in S} X_k \right\|_4^4\right)^{3/4} \left(\sum_S \left\| \sum_{k \in S^*} X_k \right\|_4^4\right)^{1/4} \\
&= (2r)^{1/2} \, EY^4
\end{aligned}$$

Similarly $|EYZ^3| \leqslant (2r)^{1/2} \, EY^4$. Also, using Lemma 1,

$$\begin{aligned}
EY^2Z^2 &\leqslant 2^{-n} \sum_S \left[r \cdot \left\| \sum_{k \in S} X_k \right\|_4^2 \left\| \sum_{k \in S^*} X_k \right\|_4^2 + E\left(\sum_{k \in S} X_k\right)^2 E\left(\sum_{k \in S^*} X_k\right)^2 \right] \\
&\leqslant rEY^4 + 2^{-n} \sum_S [(1+r)/(1-r)]^2 \, (\text{card } S)(\text{card } S^*) \\
&\leqslant rEY^4 + n^2(1+r)^2/(1-r)^2
\end{aligned}$$

Applying Lemma 1 to the centered random variables $U_k := X_k^2 - 1$ and using (3.4) and (3.2), we have

$$3 \cdot [n^2 + 2nEX_1^4] \geqslant 3 \cdot [n^2 + (1+r)(1-r)^{-1} nEU_1^2]$$
$$\geqslant 3 \cdot \left[n^2 + E\left(\sum_{k=1}^{n} U_k \right)^2 \right] = 3 \cdot E\left(\sum_{k=1}^{n} X_k^2 \right)^2$$
$$\geqslant E(Y-Z)^4 \geqslant 2EY^4 - 4\,|EY^3 Z| - 4\,|EYZ^3|$$
$$\geqslant 2[1 - 4 \cdot (2r)^{1/2}]\, EY^4 \geqslant EY^4$$

Hence by (3.3) and (3.4),

$$E\left(\sum_{k=1}^{n} X_k \right)^4 = E(Y+Z)^4$$
$$\leqslant 2[1 + 4 \cdot (2r)^{1/2} + 3r]\, EY^4 + 6n^2(1+r)^2/(1-r)^2$$
$$\leqslant 4EY^4 + 12n^2 \leqslant 24nEX_1^4 + 12n^2 + 12n^2$$

Thus Lemma 5 holds. □

Lemma 6. Suppose $q := (q_1, q_2, q_3, \ldots)$ is a nonincreasing sequence of numbers in $[0, 1]$ such that $\lim_{n \to \infty} q_n = 0$. Then there exists a positive constant $B = B(q)$ such that the following holds: If $X := (X_k, k \in \mathbf{Z}^d)$ is a CRSS random field for which $EX_0^4 < \infty$ and $\rho^*(m) \leqslant q_m \forall m \geqslant 1$, then for any finite set $S \subset \mathbf{Z}^d$, one has that

$$E\left(\sum_{k \in S} X_k \right)^4 \leqslant B \cdot [(\text{card } S) \cdot EX_0^4 + (\text{card } S)^2\, (EX_0^2)^2]$$

Proof. Let J be the least positive integer such that Eqs. (3.1)–(3. 4) all hold with r replaced by q_J. Define the constant $B := 24J^{4d}$ and proceed as in the proof of Lemma 2.

Now we come to Theorems 4 and 5. We shall prove Theorem 5 first, and then at the end just briefly indicate the changes to be made in that proof in order to establish Theorem 4. Theorem 5 will be derived via the following proposition:

Proposition 1. Suppose $X := (X_k, k \in \mathbf{Z}^d)$ is a CRSS random field satisfying the hypothesis of Theorem 5. Suppose that for each $n \geqslant 1$, $L^{(n)}$ is an element of $\mathbf{N}^d$ whose first coordinate is n. Then as $n \to \infty$, one has that $\|S(X:L^{(n)})\|_1 \to \infty$, and that $S(X:L^{(n)})/[(\pi/2)^{1/2}\, \|S(X:L^{(n)})\|_1] \to N(0, 1)$ in distribution.

By permuting coordinates, one obviously has this proposition with the condition "first coordinate is n" replaced by, say, "the 37th coordinate is

n." Using this and essentially just the definition of limit, one can easily derive Theorem 5. Thus to prove Theorem 5 it suffices to prove Proposition 1.

Proof of Proposition 1 (and therefore of Theorem 5). Let the function H be as in (1.2). Then $t^{-2}H(t) \to 0$ as $t \to \infty$. Let M^* be a positive integer such that $\sup_{t>0} t^{-2}H(t) > 1/M^*$. For each integer $n \geqslant M^*$ define the positive number $t_n := \sup\{t > 0: t^{-2}H(t) \geqslant 1/n\}$. These t_n's are standard truncation levels in the proofs of the classic CLT for i.i.d. random variables satisfying (1.2). The following properties of the t_n's are well known:

$$t_n^2 = nH(t_n) \forall n \geqslant M^* \tag{3.5}$$

$$t_n \to \infty \text{ monotonically} \quad \text{as} \quad n \to \infty \tag{3.6}$$

$$n \cdot E\,|X_0|\; I(|X_0| > t_n) = o(n^{1/2} H^{1/2}(t_n)) \quad \text{as} \quad n \to \infty \tag{3.7}$$

See, e.g., Ref. 10, Chapter 2, Section 6, or Ref. 3, Eqs. (3.2)–(3.3) and Lemma 3.1(b).

Lemma 7. One has that

$$EX_0^4 I(|X_0| \leqslant t_n) = o(nH^2(t_n)) \quad \text{as} \quad n \to \infty$$

Proof. This is also standard, but let us go through it. Suppose $\varepsilon > 0$. Using (1.2), let $c > 0$ be such that $\forall x \geqslant c$, $H(x) - H(x/2) \leqslant \varepsilon H(x)$.

Suppose $n \geqslant M^*$ is sufficiently large that $t_n > c$. Let J be the least positive integer such that $2^{-J} t_n < c$. Then

$$\begin{aligned} EX_0^4 I(|X_0| \leqslant t_n) &\leqslant \sum_{j=0}^{J-1} (2^{-j} t_n)^2\, EX_0^2 I(2^{-j-1} t_n < |X_0| \leqslant 2^{-j} t_n) \\ &\quad + EX_0^4 I(|X_0| \leqslant 2^{-J} t_n) \\ &\leqslant t_n^2 \sum_{j=0}^{J-1} 4^{-j} \varepsilon \cdot H(2^{-j} t_n) + EX_0^4 I(|X_0| \leqslant c) \\ &\leqslant t_n^2 \varepsilon H(t_n) \cdot \sum_{j=0}^{\infty} 4^{-j} + EX_0^4 I(|X_0| \leqslant c) \\ &= (4/3)\varepsilon \cdot nH^2(t_n) + EX_0^4 I(|X_0| \leqslant c) \end{aligned}$$

where the last equality comes from (3.5). Since ε can be chosen arbitrarily small, and (with c chosen depending on ε) the very last term is independent of n, Lemma 7 follows.

For each $n \geqslant 1$, the vector $L^{(n)}$ will be represented by $L^{(n)} := (n, L_2^{(n)},..., L_d^{(n)})$. For each $n \geqslant 1$ define the positive integer

$$I_n := n \cdot L_2^{(n)} \cdot \cdots \cdot L_d^{(n)} \tag{3.8}$$

For each $n \geqslant M^*$ [see (3.5)] define the CRSS random field $X^{(n)} := (X_k^{(n)}, k \in \mathbf{Z}^d)$ as follows:

$$\forall k \in \mathbf{Z}^d \qquad X_k^{(n)} := X_k I(|X_k| \leqslant t_{I(n)}) - EX_k I(|X_k| \leqslant t_{I(n)}) \tag{3.9}$$

Now $|EX_0 I(|X_0| \leqslant t_{I(n)})| \to 0$ as $n \to \infty$ (since $E|X_0| < \infty$ and $EX_0 = 0$). Consequently

$$E|X_0^{(n)}|^2 \sim H(t_{I(n)}) \qquad \text{as} \quad n \to \infty \tag{3.10}$$

and

$$I_n H(t_{I(n)}) \ll E|S(X^{(n)} : L^{(n)})|^2 \ll I_n H(t_{I(n)}) \qquad \text{as} \quad n \to \infty \tag{3.11}$$

where (3.11) comes from (3.10) and Lemma 1 [using (1.3)].

Let $q_1, q_2, q_3,...$ be a sequence of positive integers such that

$$q_n \to \infty \qquad \text{as} \quad n \to \infty \tag{3.12}$$

and

$$q_n/n \to 0 \qquad \text{as} \quad n \to \infty \tag{3.13}$$

Let $m_1, m_2, m_3,...$ be a sequence of positive integers such that

$$m_n \to \infty \qquad \text{as} \quad n \to \infty \tag{3.14}$$

$$m_n \leqslant q_n \forall n \geqslant 1 \tag{3.15}$$

$$m_n \cdot q_n/n \to 0 \qquad \text{as} \quad n \to \infty \tag{3.16}$$

$$m_n \cdot \rho^*(X, q_n) \to 0 \qquad \text{as} \quad n \to \infty \tag{3.17}$$

and

$$m_n EX_0^4 I(|X_0| \leqslant t_{I(n)}) = o(I_n H^2(t_{I(n)})) \qquad \text{as} \quad n \to \infty \tag{3.18}$$

[To justify (3.18), use Lemma 7.] For each $n \geqslant 1$ let p_n denote the integer such that

$$m_n(p_n - 1 + q_n) < n \leqslant m_n(p_n + q_n) \tag{3.19}$$

Referring to (3.16), let N^* be a positive integer such that

$$N^* \geqslant M^* \qquad \text{and} \qquad p_n \geqslant 1 \forall n \geqslant N^*$$

For any $n \geqslant N^*$ and any two integers $u \leqslant v$, define the random variable

$$Y^{(n)}(u, v) := S(X^{(n)}: u, 1, \ldots, 1: v, L_2^{(n)}, \ldots, L_d^{(n)})$$

For each $n \geqslant N^*$, define the following random variables:

$$\begin{aligned}
W_1^{(n)} &:= Y^{(n)}(1, p_n) \\
V_1^{(n)} &:= Y^{(n)}(p_n + 1, p_n + q_n) \\
W_2^{(n)} &:= Y^{(n)}(p_n + q_n + 1, 2p_n + q_n) \\
V_2^{(n)} &:= Y^{(n)}(2p_n + q_n + 1, 2p_n + 2q_n) \\
&\cdots\cdots \\
W_{m(n)}^{(n)} &:= Y^{(n)}((m_n - 1)(p_n + q_n) + 1, m_n p_n + (m_n - 1) q_n) \\
U^{(n)} &:= Y^{(n)}(m_n p_n + (m_n - 1) q_n + 1, n)
\end{aligned}$$

[By (3.15) and (3.19), $m_n p_n + (m_n - 1) q_n < n$.] Then for each $n \geqslant N^*$,

$$S(X^{(n)}: L^{(n)}) = \sum_{k=1}^{m(n)} W_k^{(n)} + \sum_{k=1}^{m(n)-1} V_k^{(n)} + U^{(n)} \tag{3.20}$$

Note that $n - [m_n p_n + (m_n - 1) q_n] \leqslant q_n$ by (3.19). By Lemma 1 [with (1.3)], (3.8), (3.10), (3.11), and (3.16), one has that

$$\begin{aligned}
E\left|\sum_{k=1}^{m(n)-1} V_k^{(n)} + U^{(n)}\right|^2 &\ll m_n q_n L_2^{(n)} \cdot \cdots \cdot L_d^{(n)} E\,|X_0^{(n)}|^2 \\
&= o(E\,|S(X^{(n)}: L^{(n)})|^2) \qquad \text{as} \quad n \to \infty
\end{aligned} \tag{3.21}$$

Hence by (3.20), $E\,|\sum_{k=1}^{m(n)} W_k^{(n)}|^2 \sim E\,|S(X^{(n)}: L^{(n)})|^2$ as $n \to \infty$. Hence by Lemma 1, (3.12), and (1.3),

$$m_n E\,|W_1^{(n)}|^2 \sim E\,|S(X^{(n)}: L^{(n)})|^2 \qquad \text{as} \quad n \to \infty \tag{3.22}$$

For each $n \geqslant N^*$ define the positive integer $J_n := p_n \cdot L_2^{(n)} \cdot \cdots \cdot L_d^{(n)}$. By (3.16) and (3.19), $q_n = o(p_n)$ as $n \to \infty$, and $J_n \sim I_n / m_n$ as $n \to \infty$. By Lemma 6 [using $B := B(\rho^*(X, 1), \rho^*(X, 2), \ldots)$], (3.18), (3.10), (3.11), and (3.22), we have that as $n \to \infty$,

$$\begin{aligned}
E\,|W_1^{(n)}|^4 &\ll J_n E\,|X_0^{(n)}|^4 + J_n^2 [E\,|X_0^{(n)}|^2]^2 \\
&\ll J_n m_n^{-1} I_n H^2(t_{I(n)}) + J_n^2 H^2(t_{I(n)}) \\
&\ll J_n^2 H^2(t_{I(n)}) \ll [E\,|W_1^{(n)}|^2]^2
\end{aligned} \tag{3.23}$$

Using (3.17), (3.21), (3.22), and (3.23), one can carry out a standard blocking argument based on Lyapounov's CLT for arrays of independent random variables. See, e.g., Ref. 5, p. 528, Theorem 5.1. One obtains that

$$\|S(X^{(n)}:L^{(n)})\|_2^{-1} S(X^{(n)}:L^{(n)}) \to N(0, 1) \tag{3.24}$$

in distribution as $n \to \infty$. As a standard corollary to this, one has that

$$(\pi/2)^{1/2} \|S(X^{(n)}:L^{(n)})\|_1 \sim \|S(X^{(n)}:L^{(n)})\|_2 \quad \text{as} \quad n \to \infty \tag{3.25}$$

By (3.7) and (3.11), one has that

$$\begin{aligned} E\,|S(X:L^{(n)}) - S(X^{(n)}:L^{(n)})| &= o(I_n^{1/2} H^{1/2}(t_{I(n)})) \\ &= o(\|S(X^{(n)}:L^{(n)})\|_2) \quad \text{as} \quad n \to \infty \end{aligned} \tag{3.26}$$

By (3.25) and (3.26),

$$(\pi/2)^{1/2} \|S(X:L^{(n)})\|_1 \sim \|S(X^{(n)}:L^{(n)})\|_2 \quad \text{as} \quad n \to \infty \tag{3.27}$$

Applying (3.26) and then (3.27) to (3.24), we obtain the conclusion of Proposition 1 [since $\|S(X:L^{(n)})\|_1 \to \infty$ by (3.11) and (3.27)]. This completes the proof of Proposition 1 and Theorem 5. □

Sketch of Proof of Theorem 4. There are positive constants c_1 and c_2 such that $\forall L := (L_1,..., L_d) \in \mathbf{N}^d$; one has that

$$c_1 \cdot (L_1 \cdot \cdots \cdot L_d) \cdot EX_0^2 \leqslant E\,|S(X:L)|^2 \leqslant c_2 \cdot (L_1 \cdot \cdots \cdot L_d) \cdot EX_0^2 \tag{3.28}$$

The existence of c_2 comes from Lemma 2, and the existence of (positive) c_1 is an elementary consequence of the assumption that the (continuous) spectral density f on T^d satisfies $f(1,..., 1) > 0$. (We leave its proof to the reader.) As a consequence of Lemma 2 and (3.28), by truncation one can reduce the proof of Theorem 4 to the case where the X_k's are bounded. Then using (3.28), one can easily carry out the argument of Proposition 1 (appropriately modified) even if $\rho^*(1) = 1$. The details are left to the reader. □

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