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Generalized One-Sided Laws of the Iterated Logarithm for Random Variables Barely with or without Finite Mean

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The almost sure limiting behavior of weighted sums of independent and identically distributed random variables barely with or without finite mean are established. Results for these partial sums,

$$\sum_{k=1}^{n} k^{\alpha} X_k, \qquad \alpha \in R$$

have been studied, but only when $\alpha = -1$ or $\alpha = 0$. As it turns out, the two cases of major interest are $\alpha = -1$ and $\alpha > -1$. The purpose of this article is to examine the latter.

KEY WORDS: Law of the iterated logarithm; strong law of large numbers; slow variation; weak law of large numbers.

1. INTRODUCTION

Throughout, $\{X, X_n, n \geqslant 1\}$ will be i.i.d. unbounded asymmetrical random variables with larger right tail than left. We will explore the almost sure limiting behavior of weighted sums of these random variables when strong laws fail. In Adler and Rosalsky[1] the authors showed that if $\{X, X_n, n \geqslant 1\}$ are i.i.d. nonintegrable random variables and $\{a_n, n \geqslant 1\}$ are constants satisfying

$$\sum_{k=1}^{n} |a_k| = O(n\,|a_n|) \text{ and } n\,|a_n| \uparrow \tag{1.1}$$

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then

$$P\left\{\lim_{n\to\infty}\sum_{k=1}^{n} a_k X_k/b_n = 1\right\} = 0$$

for *all* sequences $\{b_n, n \geq 1\}$.

On the other hand if (1.1) fails, then a strong law can exist; see Adler.[2] The important point here is that those random variables are barely with or without finite means. Here we unite these two ideas. This is accomplished by examining weights where (1.1) holds and random variables of the type that can be found in Adler[2]; see Sec. 4 of this paper.

These interesting random variables were also studied by Klass and Teicher.[3] Many of the results obtained here owe much to the work of these two men. Furthermore, some of the properties used freely here can be found in Klass and Teicher.[3] This allows us to omit details at times.

A few remarks about notation are needed. The symbol C will denote a generic finite constant which is not necessarily the same in each appearance. Also, let $\log x = \log_e(\max\{e, x\})$ and $\log_2 x = \log\log x$.

2. PRELIMINARIES

As in Klass and Teicher[3] let

$$\tilde{\mu}(x) = \int_x^\infty P\{|X| > t\}\,dt \qquad \text{provided} \quad E|X| < \infty$$

and

$$\mu(x) = \int_0^x P\{|X| > t\}\,dt \qquad \text{when} \quad E|X| = \infty$$

Also, set

$$c_x = \left(\frac{x}{\tilde{\mu}(x)}\right)^{-1} \qquad \text{when} \quad E|X| < \infty$$

and

$$c_x = \left(\frac{x}{\mu(x)}\right)^{-1} \qquad \text{when} \quad E|X| = \infty$$

Thus, $c_n = n\tilde{\mu}(c_n)$ if $E|X| < \infty$ and $c_n = n\mu(c_n)$ if $E|X| = \infty$.

We consider weights of the form $a_n = n^\alpha$, $n \geq 1$. In Adler[2] we studied the behavior of our partial sum, $S_n = \sum_{k=1}^n a_k X_k$, $n \geq 1$, when $\alpha = -1$. The question at hand is what happens when $\alpha \neq -1$ and either $\mu(x)$ or $\tilde{\mu}(x)$ is slowly varying at infinity. In Theorem 1 it is shown that if $\alpha < -1$, then $S_n/b_n \to 0$ a.s. for all sequences $\{b_n, n \geqslant 1\}$ with the property that $|b_n| \to \infty$. Owing to the fact that the random variables under consideration possess all moments less than unity, we present the following theorem.

Theorem 1. Let $\{X, X_n, n \geqslant 1\}$ be i.i.d. random variables with $E|X|^p < \infty$ for all $p < 1$. If $\alpha < -1$, then S_n converges almost surely.

Proof. Note that

$$\sum_{k=1}^{\infty} |k^\alpha X_k| = \sum_{k=1}^{\infty} k^\alpha |X_k| \, I(|X_k| \leqslant k^2) + \sum_{k=1}^{\infty} k^\alpha |X_k| \, I(|X_k| > k^2)$$

The second term is almost surely finite, via the Borel–Cantelli Lemma, since $E|X|^{1/2} < \infty$. On the other hand, the first sum is finite (a.s.) since $\alpha < -1$ and

$$\begin{aligned} E \sum_{k=1}^{\infty} k^\alpha |X_k| \, I(|X_k| \leqslant k^2) &= \sum_{k=1}^{\infty} \sum_{j=1}^{k} k^\alpha E|X| \, I((j-1)^2 < |X| \leqslant j^2) \\ &= \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} k^\alpha E|X| \, I((j-1)^2 < |X| \leqslant j^2) \\ &\leqslant C \sum_{j=1}^{\infty} j^{\alpha+1} E|X| \, I((j-1)^2 < |X| \leqslant j^2) \\ &\leqslant CE|X|^{[(\alpha+3)/2]^+} < \infty \end{aligned}$$ □

Next, we turn our attention to the situation of $\alpha > -1$. We choose as our norming sequence, $b_n = n^\alpha c_n$, $n \geqslant 1$. We need not only $b_n \uparrow \infty$, but also $b_n(\log_2 n)^{-2} \uparrow \infty$. In the non-$L_1$ case, since $c_n/n \uparrow \infty$, this is quite clear. However, this is not so apparent in the other situation.

Lemma 1. $n^\rho c_n$ is eventually increasing (to ∞) for all $\rho > -1$.

Proof. The only case of interest is $\rho \in (-1, 0)$ and $E|X| < \infty$.

Due to the fact that the slow variation of $\tilde{\mu}$ implies that $\tilde{\mu}(c_n)$ is slowly varying it follows that $n^\rho c_n = n^{\rho+1} \tilde{\mu}(c_n) \to \infty$.

Noting that $xP\{|X| > c_x\} = o(1)$, we choose $x < y < x+1$ sufficiently large, so that

$$\frac{(y/x)^\rho}{1 + yP\{|X| > c_x\}} > -\rho$$

By virtue of the mean value theorem, there is a $z \in (x, y)$ such that $(y^\rho - x^\rho)(y-x)^{-1} = \rho z^{\rho-1}$, whence

$$\frac{y^\rho - x^\rho}{y-x} \geqslant \rho x^{\rho-1} \tag{2.1}$$

Since $0 < c_x \uparrow \infty$

$$\begin{aligned}\frac{\tilde{\mu}(c_y) - \tilde{\mu}(c_x)}{c_y - c_x} &= \frac{\int_{c_y}^\infty P\{|X| > t\}\,dt - \int_{c_x}^\infty P\{|X| > t\}\,dt}{c_y - c_x} \\ &= \frac{-\int_{c_x}^{c_y} P\{|X| > t\}\,dt}{c_y - c_x} \geqslant -P\{|X| > c_x\}\end{aligned} \tag{2.2}$$

Using (2.2) and the fact that $c_x = x\tilde{\mu}(c_x)$ we observe that

$$\begin{aligned}\frac{c_y - c_x}{y-x} &= \frac{y\tilde{\mu}(c_y) - x\tilde{\mu}(c_x)}{y-x} \\ &= y\left[\frac{\tilde{\mu}(c_y) - \tilde{\mu}(c_x)}{y-x}\right] + \tilde{\mu}(c_x) \\ &= y\left[\frac{\tilde{\mu}(c_y) - \tilde{\mu}(c_x)}{c_y - c_x}\right]\left[\frac{c_y - c_x}{y-x}\right] + x^{-1}c_x \\ &\geqslant -yP\{|X| > c_x\}\left(\frac{c_y - c_x}{y-x}\right) + x^{-1}c_x\end{aligned}$$

Therefore

$$\frac{c_y - c_x}{y-x} \geqslant \frac{x^{-1}c_x}{1 + yP\{|X| > c_x\}} \tag{2.3}$$

Combining (2.1) and (2.3) we conclude that

$$\begin{aligned}\frac{y^\rho c_y - x^\rho c_x}{y-x} &= y^\rho\left(\frac{c_y - c_x}{y-x}\right) + \left(\frac{y^\rho - x^\rho}{y-x}\right)c_x \\ &\geqslant \frac{y^\rho x^{-1}c_x}{1 + yP\{|X| > c_x\}} + \rho x^{\rho-1}c_x \\ &= x^{\rho-1}c_x\left[\frac{(y/x)^\rho}{1 + yP\{|X| > c_x\}} + \rho\right] > 0\end{aligned}$$

This shows that for all large values of x and y, $y^\rho c_y > x^\rho c_x$. □

As in Klass and Teicher[3] we first establish weak laws, which allow us to make conclusions about the almost sure limiting behavior of particular subsequences of S_n/b_n.

Lemma 2. Let $\{X, X_n, n \geqslant 1\}$ be unbounded i.i.d. random variables with $\tilde{\mu}$ or μ slowly varying according as $E|X|$ is finite or infinite. Suppose further that either $E(X^-/\tilde{\mu}(X^-)) < \infty$ or $E(X^-/\mu(X^-)) < \infty$ holds in the respective cases. If $\alpha > -1/2$, then

$$\frac{S_n}{b_n} \xrightarrow{P} \begin{cases} -(\alpha+1)^{-1} & \text{when } EX=0 \\ (\alpha+1)^{-1} & \text{when } E|X| = \infty \end{cases}$$

Proof. In either case c_n/n is slowly varying and $nP\{|X| > c_n\} = o(1)$, whence we conclude, via Adler and Rosalsky,[4] that

$$\frac{\sum_{k=1}^{n} k^\alpha [X_k - EXI(|X| \leqslant c_n)]}{b_n} \xrightarrow{P} 0$$

Note, in the mean zero situation, that $EXI(|X| \leqslant c_n) = -EXI(|X| > c_n) = -[1+o(1)]\,\tilde{\mu}(c_n)$, whence

$$b_n^{-1} \sum_{k=1}^{n} k^\alpha EXI(|X| \leqslant c_n)$$
$$= -[1+o(1)]\,\tilde{\mu}(c_n)(n^\alpha c_n)^{-1} \sum_{k=1}^{n} k^\alpha \to -(\alpha+1)^{-1}$$

In the other case, the conclusion obtains in similar fashion due to the fact that $EXI(|X| \leqslant c_n) \sim \mu(c_n)$. □

We are now ready to state and prove our main results.

3. MAIN RESULTS

Theorem 2. Let $\{X, X_n, n \geqslant 1\}$ be i.i.d. random variables with $E|X| = \infty$ and $E(X^-/\mu(X^-)) < \infty$. If $\mu(x) \sim \mu(x \log_2 x)$, then

$$\liminf_{n \to \infty} S_n/b_n = (\alpha+1)^{-1} \quad \text{a.s.} \tag{3.1}$$

and

$$\limsup_{n \to \infty} S_n/b_n = \infty \quad \text{a.s.} \tag{3.2}$$

whenever $\alpha > -1$.

Proof. From Lemma 2 we observe that $\liminf_{n\to\infty} S_n/b_n \leqslant (\alpha+1)^{-1}$ a.s., if $\alpha > -1/2$.

Let $\alpha \in (-1, -1/2]$ and $\delta > 0$. Note that for all $\varepsilon > 0$ if n is sufficiently large, then $\mu(\varepsilon b_n/k^\alpha) \leqslant 2\mu(c_n)$ for all $k = 1,..., n$. Also, the slow variation of μ implies that $2xP\{|X| > x\} \leqslant \delta\varepsilon(\alpha+1)\,\mu(x)$ for all large x. Then for all large n

$$\sum_{k=1}^{n} P\{|X| > \varepsilon b_n/k^\alpha\} \leqslant \delta(\alpha+1)\,\mu(c_n)\,b_n^{-1} \sum_{k=1}^{n} k^\alpha \to \delta$$

Using the fact that $EX^2I(|X| \leqslant x) = o(x\mu(x))$ we have for all large x, $EX^2I(|X| \leqslant x) \leqslant \delta(\alpha+1)\,x\mu(x)$. Hence for all large n

$$\begin{aligned} b_n^{-2} \sum_{k=1}^{n} k^{2\alpha} EX^2 I(|X| \leqslant b_n/k^\alpha) &\leqslant \delta(\alpha+1)\,b_n^{-1} \sum_{k=1}^{n} k^\alpha \mu(b_n/k^\alpha) \\ &\leqslant \delta(\alpha+1)\,\mu(c_n)\,b_n^{-1} \sum_{k=1}^{n} k^\alpha \to \delta \end{aligned}$$

Therefore, via the degenerate convergence criterion (Chow and Teicher,[5] p. 338), $S_n/b_n - A_n \to^P 0$, where $A_n = b_n^{-1} \sum_{k=1}^n k^\alpha EXI(|X| \leqslant b_n/k^\alpha) + o(1)$. However, since

$$\begin{aligned} &\limsup_{n\to\infty} b_n^{-1} \sum_{k=1}^{n} k^\alpha EXI(|X| \leqslant b_n/k^\alpha) \\ &\qquad \leqslant \limsup_{n\to\infty} b_n^{-1} \sum_{k=1}^{n} k^\alpha E\,|X|\, I(|X| \leqslant c_n) = (\alpha+1)^{-1} \end{aligned}$$

we have $\liminf_{n\to\infty} S_n/b_n \leqslant (\alpha+1)^{-1}$ a.s. for all $\alpha > -1$.

Setting $d_n = c_n(\log_2 n)^{-2}$, $n \geqslant 1$, we note that $n^\alpha d_n \uparrow \infty$, via Lemma 1. Observe that

$$\begin{aligned} b_n^{-1} \sum_{k=1}^{n} k^\alpha X_k \geqslant\; & b_n^{-1} \sum_{k=1}^{n} k^\alpha [X_k I(-c_k < X_k \leqslant 0) - EXI(-c_k < X \leqslant 0)] \\ &+ b_n^{-1} \sum_{k=1}^{n} k^\alpha [X_k I(0 < X_k \leqslant d_k) - EXI(0 < X \leqslant d_k)] \\ &+ b_n^{-1} \sum_{k=1}^{n} k^\alpha X_k I(X_k \leqslant -c_k) \\ &+ b_n^{-1} \sum_{k=1}^{n} k^\alpha EXI(-c_k < X \leqslant d_k) \end{aligned}$$

The first normalized partial sum converges almost surely to zero via the usual Khintchine–Kolmogorov–Kronecker argument. Utilizing the Borel–Cantelli lemma, the third term is $o(1)$ a.s.

As for the second term, since its variance does not exceed

$$\begin{aligned} b_n^{-2} \sum_{k=1}^{n} k^{2\alpha} EX^2 I(0<X\leqslant d_k) &\leqslant Cb_n^{-2} \sum_{k=1}^{n} k^{2\alpha} d_k \mu(d_k) \\ &\leqslant Cb_n^{-2} \sum_{k=1}^{n} (\log_2 k)^{-2} k^{2\alpha} c_k \mu(c_k) \\ &= O((\log_2 n)^{-2}) \end{aligned}$$

then, by virtue of Lemma 1 of Klass and Teicher[3] with $C_n = n^\alpha d_n$, and the Borel–Cantelli lemma it follows that

$$\liminf_{n\to\infty} b_n^{-1} \sum_{k=1}^{n} k^\alpha [X_k I(0<X_k\leqslant d_k) - EXI(0<X\leqslant d_k)] \geqslant 0 \qquad \text{a.s.}$$

It remains to show that

$$\liminf_{n\to\infty} b_n^{-1} \sum_{k=1}^{n} k^\alpha EXI(-c_k<X\leqslant d_k) \geqslant (\alpha+1)^{-1}$$

However, this follows immediately from $EXI(-c_n<X\leqslant d_n)\geqslant [1+o(1)]\,\mu(c_n)$. Hence (3.1) obtains. This in turn, with the aid of Theorem 3 of Adler and Rosalsky,[1] implies (3.2). □

Next, we obtain a similar result in the slightly more difficult L_1 situation.

Theorem 3. Let $\{X, X_n,\ n\geqslant 1\}$ be unbounded i.i.d. mean zero random variables with $E(X^-/\tilde{\mu}(X^-))<\infty$. If $\tilde{\mu}(x)\sim\tilde{\mu}(x\log_2 x)$ and

$$\tilde{\mu}(b_n) = O(\tilde{\mu}(c_n)) \qquad \text{when} \quad \alpha\in(-1, -1/2] \tag{3.3}$$

then

$$\liminf_{n\to\infty} S_n/b_n = -(\alpha+1)^{-1} \qquad \text{a.s.}$$

and

$$\limsup_{n\to\infty} S_n/b_n = \infty \qquad \text{a.s.}$$

whenever $\alpha > -1$.

Proof. Via Lemma 2 we realize that $\liminf_{n\to\infty} S_n/b_n \leqslant -(\alpha+1)^{-1}$ a.s. whenever $\alpha > -1/2$.

Suppose $\alpha \in (-1, -1/2]$ and let $\delta > 0$. For all $\varepsilon > 0$, the slow variation of $\tilde{\mu}$ and (3.3) imply that $\tilde{\mu}(\varepsilon b_n/k^\alpha) \leqslant C\tilde{\mu}(c_n)$, $n \geqslant 1$, $k = 1,..., n$. Furthermore, for all large x

$$xP\{|X| > x\} \leqslant \delta\varepsilon(\alpha+1)\, C^{-1}\tilde{\mu}(x)$$

Thus, for all large n, we have

$$\sum_{k=1}^{n} P\{|X| > \varepsilon b_n/k^\alpha\} \leqslant \delta(\alpha+1)\, \tilde{\mu}(c_n)\, b_n^{-1} \sum_{k=1}^{n} k^\alpha \to \delta$$

Next, utilizing the fact that $EX^2 I(|X| \leqslant x) = o(x\tilde{\mu}(x))$ and (3.3) it follows that for all large n

$$\begin{aligned} EX^2 I(|X| \leqslant b_n/k^\alpha) &\leqslant \delta(\alpha+1)\, C^{-1} k^{-\alpha} b_n \tilde{\mu}(b_n/k^\alpha) \\ &\leqslant \delta(\alpha+1)\, k^{-\alpha} b_n \tilde{\mu}(c_n), \qquad k = 1,..., n \end{aligned}$$

Therefore, if n is sufficiently large, then

$$b_n^{-2} \sum_{k=1}^{n} k^{2\alpha} EX^2 I(|X| \leqslant b_n/k^\alpha) \leqslant \delta(\alpha+1)\, \tilde{\mu}(c_n)\, b_n^{-1} \sum_{k=1}^{n} k^\alpha \to \delta$$

Hence, we conclude that $S_n/b_n - A_n \to^P 0$, where

$$A_n = b_n^{-1} \sum_{k=1}^{n} k^\alpha EXI(|X| \leqslant b_n/k^\alpha) + o(1)$$

Next, we show that

$$\limsup_{n\to\infty} b_n^{-1} \sum_{k=1}^{n} k^\alpha EXI(|X| \leqslant b_n/k^\alpha) \leqslant -(\alpha+1)^{-1} \tag{3.4}$$

In view of $EX = 0$, $EXI(|X| \leqslant x) = -EX^+ I(X^+ > x) + EX^- I(X^- > x)$, for all $x > 0$. The slow variation of $\tilde{\mu}$ plus the fact that $E(X^-/\tilde{\mu}(X^-)) < \infty$ ensures that $EX^- I(X^- > x) = o(\tilde{\mu}(x))$ and $EX^+ I(X^+ > x) \sim \tilde{\mu}(x)$, whence it follows that

$$b_n^{-1} \sum_{k=1}^{n} k^\alpha EX^- I(X^- > b_n/k^\alpha) = o(1)$$

and

$$\begin{aligned} &b_n^{-1} \sum_{k=1}^{n} k^\alpha EX^+ I(X^+ > b_n/k^\alpha) \\ &\qquad \geqslant [1 + o(1)]\, b_n^{-1} \tilde{\mu}(c_n) \sum_{k=1}^{n} k^\alpha \to (\alpha+1)^{-1} \end{aligned}$$

Thus (3.4) obtains, and so for all $\alpha > -1$

$$\liminf_{n\to\infty} S_n/b_n \leqslant -(\alpha+1)^{-1} \qquad \text{a.s.}$$

As in the previous proof, set $d_n = c_n(\log_2 n)^{-2}$, $n \geqslant 1$, and note that

$$\begin{aligned} b_n^{-1}\sum_{k=1}^n k^\alpha X_k \geqslant b_n^{-1}&\sum_{k=1}^n k^\alpha[X_k I(-c_k < X_k \leqslant 0) - EXI(-c_k < X \leqslant 0)] \\ &+ b_n^{-1}\sum_{k=1}^n k^\alpha[X_k I(0 < X_k \leqslant d_k) - EXI(0 < X \leqslant d_k)] \\ &+ b_n^{-1}\sum_{k=1}^n k^\alpha X_k I(X_k \leqslant -c_k) \\ &- b_n^{-1}\sum_{k=1}^n k^\alpha EXI(X > d_k) \\ &- b_n^{-1}\sum_{k=1}^n k^\alpha EXI(X \leqslant -c_k) \end{aligned}$$

As in the proof of Theorem 2, classical arguments show that the first and third terms converge to zero almost surely.

By virtue of $EX^2I(0 < X \leqslant x) = o(x\tilde{\mu}(x))$ it follows that the variance of the second term is $O((\log_2 n)^{-2})$, hence

$$\liminf_{n\to\infty} b_n^{-1}\sum_{k=1}^n k^\alpha[X_k I(0 < X_k \leqslant d_k) - EXI(0 < X \leqslant d_k)] \geqslant 0 \qquad \text{a.s.}$$

The last term is $o(1)$, since $EX^- I(X^- > x) = o(\tilde{\mu}(x))$. While, owing to the fact that $EX^+ I(X^+ > x) \sim \tilde{\mu}(x)$

$$b_n^{-1}\sum_{k=1}^n k^\alpha EXI(X > d_k) \to (\alpha+1)^{-1}$$

whence $\liminf_{n\to\infty} S_n/b_n = -(\alpha+1)^{-1}$ a.s.

It remains to show that

$$\limsup_{n\to\infty} \sum_{k=1}^n k^\alpha X_k/b_n = \infty \qquad \text{a.s.}$$

Suppose this is false. Then $\limsup_{n\to\infty} |X_n/c_n| < \infty$ a.s., which in turn implies that $\sum_{n=1}^\infty P\{|X| > Mc_n\} < \infty$ for all large M, which is false. □

4. APPLICATIONS

In this section we present a few interesting examples. The first three should be compared with those in Adler.[2]

Here, let $\{X, X_n, n \geqslant 1\}$ be i.i.d. random variables with $f(x) = \sigma_\lambda x^{-2}(\log x)^{-\lambda} I_{(e,\infty)}(x)$, $-\infty < x < \infty$, where σ_λ is a suitably chosen constant.

Example 1. If $\lambda < 1$, then

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^n k^\alpha X_k}{n^{\alpha+1}(\log n)^{1-\lambda}} = \frac{\sigma_\lambda}{(1-\lambda)(\alpha+1)} \qquad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n k^\alpha X_k}{n^{\alpha+1}(\log n)^{1-\lambda}} = \infty \qquad \text{a.s.}$$

whenever $\alpha > -1$.

Proof. Since $\mu(x) \sim \sigma_\lambda(1-\lambda)^{-1}(\log x)^{1-\lambda}$, we have $c_n \sim \sigma_\lambda(1-\lambda)^{-1} n(\log n)^{1-\lambda}$. Thus all the hypotheses of Theorem 2 obtain. □

Example 2. If $\lambda = 1$, then

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^n k^\alpha X_k}{n^{\alpha+1}\log_2 n} = \frac{\sigma_1}{\alpha+1} \qquad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n k^\alpha X_k}{n^{\alpha+1}\log_2 n} = \infty \qquad \text{a.s.}$$

whenever $\alpha > -1$.

Proof. Here, $\mu(x) \sim \sigma_1 \log_2 x$ and $c_n \sim \sigma_1 n \log_2 n$. Hence, via Theorem 2, the conclusion holds. □

Example 3. If $\lambda > 1$ and $\alpha > -1$, then

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^n k^\alpha (X_k - EX)}{n^{\alpha+1}(\log n)^{1-\lambda}} = \frac{-\sigma_\lambda}{(\lambda-1)(\alpha+1)} \qquad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n k^\alpha (X_k - EX)}{n^{\alpha+1}(\log n)^{1-\lambda}} = \infty \qquad \text{a.s.}$$

Proof. Due to the fact that $\tilde{\mu}(x) \sim \sigma_\lambda(\lambda-1)^{-1}(\log x)^{1-\lambda}$ and $c_n \sim \sigma_\lambda(\lambda-1)^{-1} n(\log n)^{1-\lambda}$ the results follow directly from Theorem 3. □

We conclude with the St. Petersburg game. This game has been the motivation for many results in probability theory. For a more in-depth discussion of generalized St. Petersburg games see Adler and Rosalsky.[1]

Example 4. Let $\{Y, Y_n, \ n \geq 1\}$ be i.i.d. random variables with $P\{Y = q^{-k}\} = pq^{k-1}$, $k \geq 1$, where $0 < p < 1$ and $q = 1 - p$. Then, for all $\alpha > -1$

$$\liminf_{n \to \infty} \frac{\sum_{k=1}^{n} k^\alpha Y_k}{n^{\alpha+1} \log_{q^{-1}} n} = \frac{p}{q(\alpha+1)} \qquad \text{a.s.}$$

and

$$\limsup_{n \to \infty} \frac{\sum_{k=1}^{n} k^\alpha Y_k}{n^{\alpha+1} \log_{q^{-1}} n} = \infty \qquad \text{a.s.}$$

where $\log_{q^{-1}}$ denotes the logarithm to the base q^{-1}.

Proof. From properties that can be found in Adler and Rosalsky[1] we observe that $\mu(x) \sim pq^{-1} \log_{q^{-1}} x$ and $c_n \sim pq^{-1} n \log_{q^{-1}} n$. □

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