Journal of Statistical Physics, Vol. 55, Nos. 3/4, 1989

An Invariance Principle for Reversible Markov Processes. Applications to Random Motions in Random Environments

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Received December 23, 1988

We present an invariance principle for antisymmetric functions of a reversible Markov process which immediately implies convergence to Brownian motion for a wide class of random motions in random environments. We apply it to establish convergence to Brownian motion (i) for a walker moving in the infinite cluster of the two-dimensional bond percolation model, (ii) for a d-dimensional walker moving in a symmetric random environment under very mild assumptions on the distribution of the environment, (iii) for a tagged particle in a d-dimensional symmetric lattice gas which allows interchanges, (iv) for a tagged particle in a d-dimensional system of interacting Brownian particles. Our formulation also leads naturally to bounds on the diffusion constant.

KEY WORDS: Symmetric random environment; random potential; reversible Markov process; central limit theorem; invariance principle; interacting Brownian particles.

1. INTRODUCTION

In this paper we present an invariance principle which holds for a large class of "random variables" which arise naturally in the general setting of time-reversible Markov processes. Our main theorems (Theorems 2.1 and 2.2 of Section 2) may be surprising at first glance since they apply to strongly dependent variables and yet contain no explicit assumptions on mixing or decay of correlations. Remarkably, in our context these are replaced by an assumption on the *symmetry properties* of the variables

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0022-4715/89/0500-0787$06.00/0

under time reversal. Roughly speaking, we prove that an antisymmetric function X_t (t = time) of a time-symmetric, stationary, ergodic Markov process converges to a Brownian motion (with a finite diffusion matrix $\mathbf{D}$) when appropriately rescaled. In typical applications, X_t is an increment of a component made in time t (or a current flowing during this time), and is obviously antisymmetric. Since there are no difficult-to-check mixing conditions in our theorems, they apply in these situations with ridiculous ease. This is especially useful in infinite-dimensional situations, where one cannot expect strong decay of correlations.

In addition to the invariance principle, we obtain a formula for the diffusion matrix $\mathbf{D}$. This is of a type known to physicists as an "Einstein–Green–Kubo formula": it expresses $\mathbf{D}$ in terms of an explicitly computable term and the time integral of the "velocity autocorrelation function."

The proofs of our main theorems employ central limit theorems obtained by several authors.[12,26] These authors assumed the square integrability of the velocity function of the process, as well as the integrability of the velocity autocorrelation function (equivalent to a condition on the spectral measure of the velocity function). We show that the first condition was unnecessary, thus generalizing a result of Kipnis and Varadhan.[26] For velocity functions arising from antisymmetric variables we show that the second condition is automatically satisfied. Thus the only hypothesis we need on the (antisymmetric) variable X_t is the existence of a velocity function (conditional drift), which is a weak requirement equivalent to asking that X_t be a semimartingale.

The abstract theorems of Section 2 yield an invariance principle, but not that the limiting Brownian motion is nonsingular (i.e., that $\mathbf{D}$ has strictly positive eigenvalues). However, our expression for $\mathbf{D}$ suggests a general scheme for finding lower bounds. Often these lower bounds have an interesting probabilistic meaning. We also obtain an (explicitly computable) upper bound. All this is discussed in Section 3.

In the subsequent three sections we reconsider some models in the light of the theorems in Section 2. These are: random walks in (symmetric) random environments, interacting random walks on the lattice (exclusion processes), and interacting diffusion in the continuum. In each case Theorem 2.1 or Theorem 2.2 applies as soon as the existence problem is settled. In several cases we obtain results more general than those that have appeared in the literature. For example, for the symmetric random walk in a random environment of Section 4, we require only that the first moment of the bond conductivities (jump rates) be finite, and obtain the invariance principle with a nonsingular diffusion matrix. This contrasts with the results in the literature, in which the bond conductivities were assumed to be bounded away from zero and infinity. We can even treat the case of bond conductivities which are zero with positive probability (above the critical probability for bond percolation), by conditioning on the event that the infinite cluster contains the random walker at the initial instant. Thus, we establish the asymptotic diffusion of a random walk on an infinite percolation cluster (see Section 4 for references).

An additional feature of our approach is that our formula for $\mathbf{D}$ can be used to relate diffusion constants for different processes. For instance, we prove that the motion of a tagged particle in a moving environment is generally faster than the motion of the "same" particle in a "frozen" environment. We also show that the diffusion matrix is typically an increasing function of the space dimension.

A preliminary report of our results was presented in ref. 10.

2. PRINCIPAL THEOREMS

Our invariance principle will apply to a family $X_{\mathbf{I}} \in \mathbb{R}^d$ indexed by intervals $\mathbf{I} = [a, b] \subset \mathbb{R}$ or $\mathbb{Z}$ of random variables enjoying certain symmetry properties. The $X_{\mathbf{I}}$ will be functionals of a time-reversible Markov process ξ_t. We treat both discrete time ($t \in \mathbb{Z}$) and continuous time ($t \in \mathbb{R}$), although the applications discussed in Sections 4–6 will usually involve continuous time. We begin by introducing the class of Markov processes.

Let ξ_t, $t \in \mathbb{Z}$ or $t \in \mathbb{R}$, be a reversible, ergodic Markov process with state space $\aleph$ (a measurable space) and invariant probability measure μ. Let Ω be the space of trajectories (paths) of the process, maps from $\mathbb{Z}$ or $\mathbb{R}$ into $\aleph$. Let $F_{\mathbf{I}}$, $\mathbf{I} \subset \mathbb{Z}$ or $\mathbf{I} \subset \mathbb{R}$, be the σ-algebra generated by ξ_t, $t \in \mathbf{I}$, with $F_t = F_{(-\infty, t]}$ and $F = \bigcup_t F_t$. Let θ_τ and $\mathbf{R}_\tau$ denote the time-translation operator and time-reflection operator (in τ), defined, respectively, by

$$(\theta_\tau \xi)(t) = \xi(t - \tau), \qquad (\mathbf{R}_\tau \xi)(t) = \xi(2\tau - t)$$

[More precisely, $\mathbf{R}_\tau \xi$ is a suitable modification of $t \to \xi(2\tau - t)$. For instance, in $\aleph$ is a topological space and the paths of Ω are in $D = D(\aleph)$, i.e., are right continuous with left limits, then $\mathbf{R}_\tau \xi$ is the modification in D of $t \to \xi(2\tau - t)$. Note that $\mathbf{R}_\tau \mathbf{R}_\sigma \xi$ is a version of $\theta_{2(\tau - \sigma)} \xi$. We assume that $\mathbf{R}_\tau \xi$ is so defined that $\mathbf{R}_\tau \mathbf{R}_\sigma = \theta_{2(\tau - \sigma)}$.]

P_μ (resp. P_ξ) will denote the distribution of our process on Ω with initial measure μ (resp. with $\xi_0 = \xi$ a.s.), with corresponding expectation E_μ (resp. E_ξ). We assume that ξ: $[0, \infty) \to \aleph$, $\xi = \xi(t, \omega)$, is jointly measurable. From this it follows that the probability semigroup T_t: $\mathbf{L}^p(\mu) \to \mathbf{L}^p(\mu)$ given by

$$E_\mu[f(\xi_t) \mid F_0] = T_t f \qquad \text{a.s.} \tag{2.1}$$

is strongly continuous (in t) in $\mathbf{L}^p(\mu)$, for both $p = 1$ and $p = 2$.

To say that ξ_t is *reversible* means that P_μ is invariant under $\mathbf{R}_\tau$ for all τ. In the discrete-time case, this is ensured if

$$\mu(d\xi)\, T(d\xi' \mid \xi) = \mu(d\xi')\, T(d\xi \mid \xi') \tag{2.2}$$

where $T(\cdot \mid \cdot)$ is the transition operator. In the continuous-time case, this is ensured if the Markov semigroup T_t is self-adjoint in $\mathbf{L}^2(\mu)$. *Stationarity* of the process, that P_μ is θ_τ invariant for all τ, is an immediate consequence of reversibility. Finally, to say that ξ_t is *ergodic* means that P_μ is ergodic under the time-translation group, or equivalently that if, for all $t \geqslant 0$, $T_t(A \mid \xi) = 1$ for μ-a.e. $\xi \in A$, then $\mu(A) = 1$ or 0 ($A \subset \aleph$ measurable) (see ref. 40, Corollary 5, p. 97).

We introduce next the notion of *antisymmetric random variables*, which will play a key role in what follows. Let $X_\mathbf{I}$ with values in $\mathbb{R}^d$ $(d \geqslant 1)$ be an $F_\mathbf{I}$-measurable random variable. We say that $X_\mathbf{I}$ is *antisymmetric* if

$$X_\mathbf{I} \circ \mathbf{R}_m = -X_\mathbf{I} \tag{2.3}$$

a.s., where m is the midpoint of $\mathbf{I}$. A *symmetric* random variable is defined in an analogous way. More generally, we shall say that a family $X_\mathbf{I}$, indexed by intervals $\mathbf{I} \subset \mathbb{R}$, is antisymmetric if each $X_\mathbf{I}$ is $F_\mathbf{I}$-measurable and antisymmetric.

Antisymmetric random variables arise naturally in several ways. The simplest case is if $X_\mathbf{I}$ is an increment of a component of the process, $X_{[a,b]} = X(\xi_b) - X(\xi_a)$ for X a function on $\aleph$. For an example not of this type, let ξ_t be a jump process on a finite state space $\aleph$ and define $X_\mathbf{I}$ to be (for fixed x, y in $\aleph$) the number of jumps from x to y minus the number of reverse jumps, made in the time interval $\mathbf{I}$. In other words,

$$X_\mathbf{I} = |\{t: \xi_{t-} = x, \xi_t = y; t \in \mathbf{I}\}| - |\{t: \xi_{t-} = y, \xi_t = x; t \in \mathbf{I}\}|$$

where $|A|$ is the cardinality of the set A. More generally, if $X_\mathbf{I}$ has an interpretation as a *current* flowing during the interval $\mathbf{I}$, it should be antisymmetric.

Our invariance principle will apply to an antisymmetric family enjoying certain additional properties, which we list below.

A.1. $X_\mathbf{I} \in \mathbf{L}^1(P_\mu)$ for each bounded interval $\mathbf{I}$.

A.2. The family $X_\mathbf{I}$ is *covariant*:

$$X_{[a,b]} \circ \theta_\tau = X_{[a+\tau, b+\tau]} \tag{2.4}$$

a.s. (whenever both sides are defined, see below).

A.3. The family $X_{\mathbf{I}}$ is *additive*:

$$X_{\mathbf{I}}+X_{\mathbf{I}'}=X_{\mathbf{I}\cup\mathbf{I}'} \tag{2.5}$$

a.s. when the intervals intersect in exactly one point.

A.4. In addition, in the continuous time case we assume that $X_t\equiv X_{[0,t]}$ has paths in $D([0,\infty);\mathbb{R}^d)$.

Concerning the index set $\{\mathbf{I}\}$, we wish to consider two cases: $X_{\mathbf{I}}$ is defined for all closed, bounded $\mathbf{I}\subset\mathbb{R}$ or only for intervals of the form $[k,h]$, $k<h$, k, $h\in\mathbb{Z}$. Note that in the latter case A.1–A.3 follow easily if

$$X_{[k,h]}=\sum_{j=k}^{h-1} X\circ\theta_j \tag{2.6}$$

where $X=X_{[0,1]}$ is an $F_{[0,1]}$-measurable, integrable, antisymmetric random variable. For discrete time this means simply that $X=\tilde{X}(\xi_0,\xi_1)$, where $\tilde{X}$ is an antisymmetric function on $\aleph\times\aleph$, such that X is integrable.

The variance of the limiting Brownian motion in our invariance principle will be expressed in terms of quadratic forms associated naturally with the process, which we now introduce.

Consider first a general nonnegative, self-adjoint operator B on a separable Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. The *quadratic form* associated with B is the form

$$\|\psi\|_1^2\equiv\|\psi\|_{1,B}^2\equiv\|B^{1/2}\psi\|^2=\int_0^\infty x\,v_\psi(dx) \tag{2.7}$$

where v_ψ is the spectral measure associated with $\psi\in H$ relative to the spectral decomposition of B. Note that $\mathbf{D}(B^{1/2})=\{\psi\in H\mid\|\psi\|_1<\infty\}$ [here $\mathbf{D}(\cdot)$ denotes the domain of the self-adjoint operator $\cdot$]. We denote by $H_1\equiv H_1(B)$ the completion of this domain in the $\|\cdot\|_1$ norm, taken modulo null vectors [equivalently, complete the space $\mathbf{D}(B^{1/2})\cap\operatorname{Ker}(B)^{\perp}$]. With a slight abuse of notation we write $H\cap H_1$ for $\mathbf{D}(B^{1/2})$.

We denote by $H_{-1}\equiv H_{-1}(B)$ the dual of H_1. One may identify H_{-1} (antilinearly) with the completion in the norm $\|\cdot\|_{-1}$

$$\|\varphi\|_{-1}^2\equiv\|\varphi\|_{-1,B}^2\equiv\|B^{-1/2}\varphi\|^2=\int_0^\infty\frac{v_\varphi(dx)}{x} \tag{2.8}$$

of the domain $\mathbf{D}(B^{-1/2})=\{\varphi\in H\mid\|\varphi\|_{-1}<\infty\}$, since (by the spectral theorem) for $\varphi\in H$ the linear functional $\psi\to(\varphi,\psi)$, $\psi\in H\cap H_1$, has norm $\|\varphi\|_{-1}$ under $\|\cdot\|_1$ and hence defines an element of H_{-1} of norm $\|\varphi\|_{-1}$ precisely if $\varphi\in\mathbf{D}(B^{-1/2})$. We often write $H\cap H_{-1}$ for $\mathbf{D}(B^{-1/2})$. Note that if $\varphi \in H \cap H_{-1}$, $\nu_\varphi(\{0\}) = 0$. The map B: $\mathbf{D}(B) \to \mathbf{D}(B^{-1})$ extends to a unitary from H_1 onto H_{-1}.

For $g \in H_{-1}$ and $\psi \in H_1$, we write (g, ψ) for the natural action of g on ψ. Note that if g and ψ belong to H, this agrees with the usual inner product on H. Note also that for $g \in H_{-1}$,

$$\|g\|_{-1} = \sup_{\psi \in H \cap H_1} \frac{|(g, \psi)|}{\|\psi\|_1} \tag{2.9}$$

We write $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_{-1}$ for the inner products on H_1 and H_{-1}, respectively. Note that since B: $H_1 \to H_{-1}$ (B^{-1}: $H_{-1} \to H_1$), $(\psi, B\psi) \equiv (B\psi, \psi)^*$ is well defined for $\psi \in H_1$ and in fact equal $\|\psi\|_1^2$ [$(\varphi, B^{-1}\varphi)$ is well defined for $\varphi \in H_{-1}$ and in fact equals $\|\varphi\|_{-1}^2$), in agreement with the obvious meaning of these expressions for $\psi \in \mathbf{D}(B)$ [$\varphi \in \mathbf{D}(B^{-1})$)].

Let $T_t \equiv e^{-Bt}$ be the contraction semigroup on H generated by B. The semigroup T_t extends to a contraction semigroup on $H_{\pm 1}$. Note that, by the spectral theorem, $T_t(H) \subset H_1$ for $t > 0$ and for $\psi \in H$

$$\|\psi\|_1^2 = \lim_{\delta \to 0} \left(\psi, \frac{1 - T_\delta}{\delta} \psi\right) \tag{2.10}$$

The spectral theorem may be extended to provide a simultaneous spectral representation of H, H_1, and H_{-1} relative to the spectral decomposition of B. We represent $H \cup H_1 \cup H_{-1}$ as a set of functions on a σ-finite measure space $\{\Omega, \nu\}$ in such a way that

$$\begin{aligned} B &\simeq \text{multiplication by the nonnegative function } \Lambda \text{ on } \{\Omega, \nu\} \\ H &\simeq \mathbf{L}^2(\Omega, \nu) \\ H_1 &\simeq \mathbf{L}^2(\Omega, \Lambda\nu) \\ H_{-1} &\simeq \mathbf{L}^2(\Omega, \Lambda^{-1}\nu) \end{aligned}$$

(here "$\simeq$" means "is represented by"). Moreover, the duality between H_1 and H_{-1} has the canonical representation: For $g \in H_{-1}$ and $\psi \in H_1$, $(g, \psi) = \int_\Omega \tilde{g}^* \tilde{\psi}\, d\nu$, where $g \simeq \tilde{g}$ and $\psi \simeq \tilde{\psi}$. Note that the semigroup $T_t \simeq$ multiplication by $e^{-\Lambda t}$, and is strongly continuous and self-adjoint on $H_{\pm 1}$ as well as on H. When we say "by the spectral theorem" or "by spectral theory" we often refer to this simultaneous spectral representation.

We now specialize to the case of interest, in which B is Markovian: $H = \mathbf{L}^2(\mu)$ and $B = -\mathbf{L}$, where $\mathbf{L}$ is the strong $\mathbf{L}^2$-generator of our reversible Markov process ξ_t for continuous time and is $T_1 - 1$ for discrete time. By reversibility, $-\mathbf{L}$ is a nonnegative self-adjoint operator on $\mathbf{L}^2(\mu)$. Note that for discrete time $H_{-1} \subset H \subset H_1$. We will often denote the integral in $\mathbf{L}^1(\mu)$ by $\langle \cdot \rangle$.

Note that in the case of discrete time we are faced with a minor notational inconsistency: the semigroup $T_t = \exp(-Bt) = \exp[(1-T_1)t]$ does not agree with T_1, the transition probability, for $t=1$. For the remainder of Section 2 we distinguish these two possibilities by writing T_1 or T_1^n whenever the original discrete-time transition probability is intended. Note that T_t is the transition probability for a continuous-time jump process also reversible with respect to μ.

In the Markovian case, $H \cap H_1 \equiv \mathbf{L}^2(\mu) \cap H_1$ has a natural probabilistic interpretation. For discrete time, if $\psi \in \mathbf{L}^2(\mu)$, then

$$E_\mu[(\psi(\xi_1) - \psi(\xi_0))^2] = 2(\psi, (1-T_1)\psi) = 2\|\psi\|_1^2 \tag{2.11}$$

For continuous time, if $\psi \in \mathbf{L}^2(\mu)$, then

$$\lim_{\delta \to 0} \delta^{-1} E_\mu[(\psi(\xi_\delta) - \psi(\xi_0))^2] = 2 \lim_{\delta \to 0} \delta^{-1}(\psi, (1-T_\delta)\psi) = 2\|\psi\|_1^2 \tag{2.12}$$

and, in particular, $\psi \in \mathbf{L}^2(\mu)$ belongs to H_1 precisely if the left-hand side of (2.12) is finite.

Some functions in $\mathbf{L}^1(\mu)$, not necessarily belonging to $\mathbf{L}^2(\mu)$, can be identified with elements of H_{-1}. The key ingredient for this identification is the following result.

Lemma 2.1. $\mathbf{L}^\infty(\mu) \cap H_1$ is dense in H_1 (under $\|\cdot\|_1$). ∎

Proof. Since $B = -\mathbf{L}$ is Markovian, $T_t(\mathbf{L}^\infty(\mu)) \subset \mathbf{L}^\infty(\mu)$. Therefore $T_t(\mathbf{L}^\infty(\mu) \cap H_1) \subset \mathbf{L}^\infty(\mu) \cap H_1$. Since, for $t > 0$, $T_t(\mathbf{L}^2(\mu)) \subset H_1$ and T_t is strongly continuous on $\mathbf{L}^2(\mu)$, $\mathbf{L}^\infty(\mu) \cap H_1$ is dense in $\mathbf{L}^2(\mu)$. The lemma thus follows from Lemma 3.2. (It also follows directly from the basic properties of Dirichlet forms.) ∎

Now every function φ in $\mathbf{L}^1(\mu)$ defines a linear functional g_φ on $\mathbf{L}^\infty(\mu) \cap H_1$:

$$(g_\varphi, \psi) \equiv \langle \varphi^* \psi \rangle \equiv (\varphi, \psi)$$

If g_φ is bounded, we say, with a slight abuse of terminology, that $\varphi \in \mathbf{L}^1(\mu) \cap H_{-1}$:

$$\mathbf{L}^1(\mu) \cap H_{-1} = \{\varphi \in \mathbf{L}^1(\mu) \mid \text{there exists a constant } C < \infty \text{ such that } |\langle \varphi^* \psi \rangle| \leqslant C\|\psi\|_1 \text{ for all } \psi \in \mathbf{L}^\infty(\mu) \cap H_1\}$$

Thus, using Lemma 2.1, if $\varphi \in \mathbf{L}^1(\mu) \cap H_{-1}$, g_φ defines an element of H_{-1}. [Note that $(\mathbf{L}^1(\mu) \cap H_{-1}) \cap \mathbf{L}^2(\mu) = \mathbf{L}^2(\mu) \cap H_{-1}$.] Moreover, we have the following result.

Lemma 2.2. The map $\mathbf{L}^1(\mu) \cap H_{-1} \to H_{-1}$, $\varphi \to g_\varphi$ is an injection. ▮

Proof. We must show that if $g_\varphi = 0$, then $\varphi = 0$. Since T_t is strongly continuous on $\mathbf{L}^1(\mu)$, it is weakly continuous on $\mathbf{L}^\infty(\mu)$. Therefore, since $T_t(\mathbf{L}^\infty(\mu)) \subset \mathbf{L}^\infty(\mu) \cap H_1$, it follows that $\mathbf{L}^\infty(\mu) \cap H_1$ is weakly dense in $\mathbf{L}^\infty(\mu)$. Thus, $\langle \varphi^* \psi \rangle = 0$ for all $\psi \in \mathbf{L}^\infty(\mu) \cap H_1$ implies that $\varphi = 0$. ▮

Note that T_t acts, naturally, on $\mathbf{L}^1(\mu)$ and, by extension from $\mathbf{L}^2(\mu)$, on H_{-1}. Nonetheless, T_t on $\mathbf{L}^1(\mu) \cap H_{-1}$ is unambiguous: Carefully distinguishing between $\varphi \in \mathbf{L}^1(\mu) \cap H_{-1}$ and $g_\varphi \in H_{-1}$, we have, by duality, that

$$T_t(\mathbf{L}^1(\mu) \cap H_{-1}) \subset \mathbf{L}^1(\mu) \cap H_{-1} \qquad \text{and} \qquad T_t g_\varphi = g_{T_t \varphi}$$

since

$$(T_t \varphi, \psi) = (\varphi, T_t \psi) \equiv (g_\varphi, T_t \psi) = (T_t g_\varphi, \psi) \qquad \text{for} \quad \psi \in \mathbf{L}^\infty(\mu) \cap H_1$$

By the preceding observation and Lemma 2.2, we obtain, using the spectral theorem, that for $t > 0$, $T_t(\mathbf{L}^1(\mu) \cap H_{-1}) \subset \mathbf{L}^2(\mu)$. Thus, since T_t is strongly continuous on H_{-1} and on $\mathbf{L}^1(\mu)$, we have proven the following useful result.

Lemma 2.3. $\mathbf{L}^2(\mu) \cap H_{-1}$ is dense in $\mathbf{L}^1(\mu) \cap H_{-1}$ in the norm $\|\cdot\|_{-1} + \|\cdot\|_{\mathbf{L}^1(\mu)}$. ▮

The following two lemmas describe the probabilistic significance of H_{-1}.

Lemma 2.4. Let ξ_t be a continuous-time Markov process reversible with respect to the (stationary) probability measure μ. Then, for $\varphi \in \mathbf{L}^2(\mu)$

$$\lim_{t \to \infty} t^{-1} E_\mu \left[\left| \int_0^t \varphi(\xi_s)\, ds \right|^2 \right] < \infty \Leftrightarrow \varphi \in H_{-1} \tag{2.13}$$

Moreover, for any $\varphi \in \mathbf{L}^1(\mu) \cap H_{-1}$, $\int_0^t \varphi(\xi_s)\, ds \in \mathbf{L}^2(P_\mu)$ for all $t > 0$ and

$$\lim_{t \to \infty} t^{-1} E_\mu \left[\left| \int_0^t \varphi(\xi_s)\, ds \right|^2 \right] = 2\, \|\varphi\|_{-1}^2 = 2 \int_0^\infty (\varphi, T_s \varphi)\, ds \tag{2.14}$$

Conversely, suppose $\varphi \in \mathbf{L}^1(\mu)$, $\int_0^t \varphi(\xi_s)\, ds \in \mathbf{L}^2(P_\mu)$ for all $t > 0$ and

$$\liminf_{t \to \infty} t^{-1} E_\mu \left[\left| \int_0^t \varphi(\xi_s)\, ds \right|^2 \right] < \infty$$

Then $\varphi \in \mathbf{L}^1(\mu) \cap H_{-1}$. ▮

Proof. For $\varphi \in \mathbf{L}^2(\mu)$

$$E_\mu\left[\left|\int_0^t \varphi(\xi_s)\,ds\right|^2\right] = 2\int_0^t \int_0^s (\varphi, T_\tau \varphi)\,d\tau\,ds$$

$$= 2\int_0^t (\varphi, (1-T_s)\varphi)_{-1}\,ds \leqslant 2t\,\|\varphi\|_{-1}^2 \tag{2.15}$$

The statements concerning $\varphi \in \mathbf{L}^2(\mu)$ then follow from the spectral theorem. [Note that even if $\varphi \notin H_{-1}$, the third term in (2.15) is naturally defined via the spectral theorem].

Now consider the map $\mathbf{L}^2(\mu) \cap H_{-1} \to \mathbf{L}^2(P_\mu)$ given by $\varphi \to \int_0^t \varphi(\xi_s)\,ds$. By (2.15) this map extends, by continuity, to a bounded map on all of H_{-1}. This extension agrees with the usual integral on $\mathbf{L}^1(\mu) \cap H_{-1}$, since the usual integral is continuous as a map from $\mathbf{L}^1(\mu)$ to $\mathbf{L}^1(P_\mu)$, and $\mathbf{L}^2(\mu) \cap H_{-1}$ is dense in $\mathbf{L}^1(\mu) \cap H_{-1}$ in $\|\cdot\|_{-1} + \|\cdot\|_{\mathbf{L}^1(\mu)}$. Moreover, since the left-hand side and the third term of (2.15) are both continuous on H_{-1}, we have that

$$t^{-1}E_\mu\left[\left|\int_0^t \varphi(\xi_s)\,ds\right|^2\right] = 2t^{-1}\int_0^t (\varphi, (1-T_s)\,\varphi)_{-1}\,ds \tag{2.16}$$

for all $\varphi \in \mathbf{L}^1(\mu) \cap H_{-1}$. Passing to the limit $t \to \infty$, we obtain (2.14), using the spectral theorem.

Now suppose φ satisfies the specified conditions for the converse. Let $\delta > 0$, $t = N\delta$, and let

$$\varphi_\delta = \int_0^\delta (T_s \varphi)\,ds = E_\mu\left(\int_0^\delta \varphi(\xi_s)\,ds \,|\, F_0\right)$$

Then $\varphi_\delta \in \mathbf{L}^2(\mu)$ and

$$t^{-1}E_\mu\left[\left|\int_0^t \varphi(\xi_s)\,ds\right|^2\right]$$

$$= (N\delta)^{-1}E_\mu\left[\left|\sum_{k=1}^{N} \int_{(k-1)\delta}^{k\delta} \varphi(\xi_s)\,ds\right|^2\right]$$

$$= 2(N\delta)^{-1}\sum_{k=1}^{N-1}\sum_{i=0}^{k-1} (\varphi_\delta, T_{i\delta}\varphi_\delta) + \delta^{-1}E_\mu\left[\left|\int_0^\delta \varphi(\xi_s)\,ds\right|^2\right] \tag{2.17}$$

In this computation we have used the reversibility of the process and the "time reversal" symmetry of $\int_0^t \varphi(\xi_s)\,ds$. From (2.17) and the converse conditions we see, using the spectral theorem, that there is a constant $C<\infty$ such that for all $\delta>0$

$$\delta^{-1}\lim_{M\to\infty} 2M^{-1}\sum_{k=1}^{M}\sum_{i=0}^{k-1}(\varphi_\delta, T_{i\delta}\varphi_\delta)=2\delta^{-1}(\varphi_\delta, (1-T_\delta)^{-1}\varphi_s)\leqslant 2C^2 \tag{2.18}$$

where the middle term is defined, say, via the spectral theorem.

Let $\psi\in\mathbf{L}^\infty(\mu)\cap H_1$. From (2.18) and the Cauchy–Schwarz inequality

$$|(\varphi_\delta,\psi)|\leqslant(\varphi_\delta,(1-T_\delta)^{-1}\varphi_\delta)^{1/2}(\psi,(1-T_\delta)\psi)^{1/2}\leqslant\delta^{1/2}C(\psi,(1-T_\delta)\psi)^{1/2} \tag{2.19}$$

Since T_t is strongly continuous on $\mathbf{L}^1(\mu)$, dividing both sides of (2.19) by δ and letting $\delta\to 0$, we obtain, using (2.10),

$$|(\varphi,\psi)|\leqslant C\|\psi\|_1 \tag{2.20}$$

Therefore $\varphi\in\mathbf{L}^1(\mu)\cap H_{-1}$. This completes the proof of Lemma 2.4. ∎

We next consider the discrete time version of Lemma 2.4. The delicate point now is not that φ need not be in $\mathbf{L}^2(\mu)$, since $H_{-1}\subset\mathbf{L}^2(\mu)$ in this case. Rather, it is that T_1 need not be positive and in fact may have -1 as an eigenvalue.

Lemma 2.5. Let $\xi_0, \xi_1, \xi_2,\ldots$ be a Markov process reversible with respect to the (stationary) probability measure μ. Suppose $\varphi\in\mathbf{L}^2(\mu)$. Then

$$\liminf_{n\to\infty} n^{-1}E_\mu\left[\left|\sum_{i=0}^{n-1}\varphi(\xi_i)\right|^2\right]<\infty\Leftrightarrow\varphi\in H_{-1} \tag{2.21}$$

and for $\varphi\in H_{-1}$

$$\begin{aligned}\lim_{n\to\infty} n^{-1}E_\mu\left[\left|\sum_{i=0}^{n-1}\varphi(\xi_i)\right|^2\right]&=2\|\varphi\|_{-1}^2-(\varphi,\varphi)\equiv 2(\varphi,(1-T_1)^{-1}\varphi)-(\varphi,\varphi)\\&=\left(\varphi,\frac{1+T_1}{1-T_1}\varphi\right)\quad\blacksquare\end{aligned} \tag{2.22}$$

Proof. Observe that

$$n^{-1}E_\mu\left[\left|\sum_{i=0}^{n-1}\varphi(\xi_i)\right|^2\right]=2n^{-1}\sum_{k=0}^{n-1}\sum_{i=0}^{k}(\varphi, T_1^i\varphi)-(\varphi,\varphi) \tag{2.23}$$

If $T_1 \geqslant 0$, the limit $n \to \infty$ equals

$$\sum_{k=0}^{\infty} (\varphi, T_1^k \varphi) - (\varphi, \varphi) = (\varphi, (1-T_1)^{-1} \varphi) - (\varphi, \varphi)$$

and the result follows for this case. More generally, let λ be the spectral measure associated with φ relative to the spectral decomposition of T_1. Since $\|T_1\| = 1$, $\lambda(\mathbb{R} \backslash [-1, 1]) = 0$. In terms of λ, the right-hand side of (2.23) becomes

$$\begin{aligned}
&2n^{-1} \sum_{k=0}^{n-1} \int_{-1}^{1} \lambda(dx) \sum_{i=0}^{k} x^i - (\varphi, \varphi) \\
&= 2n^{-1} \sum_{k=0}^{n-1} \left[\int_{-1}^{1-} \frac{1-x^{k+1}}{1-x} \lambda(dx) + (k+1)\, \lambda(\{1\}) \right] - (\varphi, \varphi) \\
&= \lambda(\{-1\}) n^{-1} \sum_{k=0}^{n-1} [1 - (-1)^{k+1}] \\
&\quad + 2n^{-1} \sum_{k=0}^{n-1} \left[\int_{-1+}^{1-} \frac{1-x^{k+1}}{1-x} \lambda(dx) + (k+1)\, \lambda(\{1\}) \right] - (\varphi, \varphi) \\
&= \lambda(\{-1\}) n^{-1} \sum_{k=0}^{n-1} [1 - (-1)^{k+1}] + 2n^{-1} \sum_{k=0}^{n-1} \int_{-1+}^{0-} \frac{1-x^{k+1}}{1-x} \lambda(dx) \\
&\quad + 2n^{-1} \sum_{k=0}^{n-1} \left[\int_{0}^{1-} \frac{1-x^{k+1}}{1-x} \lambda(dx) + (k+1)\, \lambda(\{1\}) \right] - (\varphi, \varphi) \qquad (2.24)
\end{aligned}$$

Now, if $\varphi \in H_{-1}$, then $\lambda(\{1\}) = 0$,

$$\int_{-1}^{1} \frac{\lambda(dx)}{1-x} < \infty$$

and passing to the limit $n \to \infty$ on the third term of (2.24), using dominated convergence, we obtain

$$\begin{aligned}
&\lambda(\{-1\}) + 2 \int_{-1+}^{1} \frac{\lambda(dx)}{1-x} - (\varphi, \varphi) \\
&= 2 \int_{-1}^{1} \frac{\lambda(dx)}{1-x} - (\varphi, \varphi) \\
&= 2(\varphi, (1-T_1)^{-1} \varphi) - (\varphi, \varphi) = \left(\varphi, \frac{1+T_1}{1-T_1} \varphi \right)
\end{aligned}$$

Now suppose that

$$\liminf_{n \to \infty} n^{-1} E_\mu \left[\left| \sum_{i=0}^{n-1} \varphi(\xi_i) \right|^2 \right] < \infty$$

Then, since the first two terms on the right-hand side of (2.24) are bounded in n and the integrand in the third term of the right-hand side is monotone increasing, it follows that

$$\int_0^1 \frac{\lambda(dx)}{1-x} < \infty$$

(and, in particular, $\lambda(\{1\})=0$), so that $\varphi \in H_{-1}$. This completes the proof of Lemma 2.5. ∎

In applications, the state ξ of our reversible Markov process will represent the environment seen from a "tagged" particle. Since we wish to investigate asymptotic behavior for a fixed initial environment, as well as the behavior arising from averaging (with respect to μ) over the initial environment, we employ the following notion of convergence. Let $X^\varepsilon = (X_t^\varepsilon)_{t \geqslant 0}$, $\varepsilon > 0$, be a family of $\mathbb{R}^d$-valued processes defined on (Ω, P_μ). Then we say that X^ε converges *weakly in μ-measure* (or probability) to the $\mathbb{R}^d$-valued process Y, and we write $X^\varepsilon \to Y$, if for all bounded, continuous functions F on $D \equiv D([0, \infty), \mathbb{R}^d)$ (equipped with the Skorohod topology)

$$E_\mu[F(X^\varepsilon) \mid \xi_0 = \xi) \to E(F(Y))$$

as $\varepsilon \to 0$ in μ-probability. Note that $X^\varepsilon \to Y$ in μ-probability implies that X^ε tends to Y in distribution. We similarly define the notion of convergence of finite-dimensional distributions in μ-measure, for which we write

$$X^\varepsilon \xrightarrow{f} Y$$

We next state and prove two theorems. Theorem 2.1 applies to a discrete family, Theorem 2.2 to a continuous family. Although we shall rely primarily on Theorem 2.2 in the sequel, we state and prove Theorem 2.1 because of its simplicity and paucity of assumptions. Since we are emphasizing simplicity, we set $d=1$; the reader will easily supply the generalization to $d>1$.

Theorem 2.1. Let ξ_t be a (discrete- or continuous-time) Markov process which is reversible and ergodic with stationary probability measure

μ. Let X $(=X_{[0,1]})$ be an $F_{[0,1]}$-measurable, square-integrable, antisymmetric random variable. Define

$$X_n = X \circ \theta_{n-1}, \qquad n = 1, 2,... \tag{2.25}$$

$$X_t^\varepsilon = \varepsilon \sum_{n=1}^{[\varepsilon^{-2}t]} X_n \tag{2.26}$$

($[\cdot]$ = greatest integer function), and

$$\varphi = E_\mu[X \mid F_0] \tag{2.27}$$

Let D be given by

$$D = E_\mu(X^2) - 2(\varphi, (1 - T_1)^{-1}\varphi) \tag{2.28}$$

[In (2.28), T_1 means T for discrete time.] Then both terms are finite. (The second term is the dual quadratic form associated with the self-adjoint, nonnegative operator $1 - T_1$ and may be expressed in terms of a power series; see remark below.)

Let W_D be a Brownian motion with variance $Dt = E(W_D^2(t))$, starting from zero. Then $X^\varepsilon \to W_D$, weakly in μ-measure. Furthermore,

$$\lim_{\varepsilon \to 0} E_\mu[(X_t^\varepsilon)^2] = Dt \quad \blacksquare \tag{2.29}$$

Remarks. 1. The second term in (2.28) can be interpreted as the "integral" of the "velocity autocorrelation function" (φ should be understood as the "velocity" of the process). If T_1 is a positive operator in $\mathbf{L}^2(\mu)$—which is valid in continuous time or if the process can be imbedded in a continuous-time process—then the second term can be expressed as

$$(\varphi, (1 - T_1)^{-1}\varphi) = \sum_{n=0}^{\infty} (\varphi, T_1^n\varphi) \tag{2.30}$$

where the series converges absolutely. If T_1 is not positive (e.g., for a two-state process for a certain parameter range), (2.30) is still correct provided the sum is interpreted in the Cesaro sense. In fact, only the presence of a point mass at -1 in the spectral measure associated with φ relative to the spectral decomposition of T_1 can destroy the convergence of the series. (See the proof of Lemma 2.5.)

2. At this point there is nothing which guarantees that $D > 0$; i.e., exact cancellation is possible in (2.28) (the second term is negative). For example, let ξ_t be a two-state process, $\aleph = \{-1, 1\}$, and $X = \xi_1 - \xi_0$. Clearly, X_t is bounded, so $D = 0$. See Section 3 for conditions implying $D > 0$.

3. The extra assumption $X \in \mathbf{L}^2(\mu)$ is usually harmless when Theorem 2.1 is applied to discrete-time processes. However, for the con-

tinuous-time case it is often difficult to check directly that X_t is square-integrable. Therefore, in the next theorem (for continuous-time processes) we do not assume $X_t \in \mathbf{L}^2(\mu)$ at the outset.

4. The condition $X \in \mathbf{L}^2(\mu)$ implies that $\varphi \in \mathbf{L}^2(\mu)$. However, for continuous-time processes (cf. Theorem 2.2) the analogue of φ need not be, and in some interesting cases is not, in $\mathbf{L}^2(\mu)$.

Proof of Theorem 2.1. We first establish (2.29). Let

$$\varphi_{n-1} = E_\mu[X_n \mid F_{n-1}], \qquad n = 1, 2,..., \quad \varphi_0 = \varphi \tag{2.31}$$

$$M_n = X_n - \varphi_{n-1}, \qquad n = 1, 2,... \tag{2.32}$$

Then M_n, $n = 1, 2,...$, forms a stationary, square-integrable, martingale-difference sequence.[7] Summing in (2.32) and computing the variance of the martingale on the left, we get (using stationarity)

$$\begin{aligned} nE_\mu[M_1^2] &= E_\mu\left[\left(\sum_{k=1}^{n} X_k\right)^2\right] + E_\mu\left[\left(\sum_{k=0}^{n-1} \varphi_k\right)^2\right] \\ &\quad - 2E_\mu\left[\left(\sum_{k=1}^{n} X_k\right)\left(\sum_{k=0}^{n-1} \varphi_k\right)\right] \\ &= E_\mu\left[\left(\sum_{k=1}^{n} X_k\right)^2\right] + E_\mu\left[\left(\sum_{k=0}^{n-1} \varphi_k\right)^2\right] + 2E_\mu\left[\varphi_n \sum_{k=1}^{n} X_k\right] \end{aligned} \tag{2.33}$$

In this computation we used that

$$E_\mu\left[\left(\sum_{k=1}^{n} X_k\right)\left(\sum_{k=0}^{n} \varphi_k\right)\right] = 0 \tag{2.34}$$

which follows from (i) the invariance of P_μ under $\mathbf{R}_{n/2}$, and (ii) the symmetry properties of the two sums in the integrand.

Applying the Cauchy–Schwarz inequality to the third term in (2.33), we conclude that the first two terms are of order n and the third is of order $n^{1/2}$. It therefore follows from Lemma 2.5 that $\varphi \in H_{-1}$ and

$$\begin{aligned} &\lim_{n\to\infty} n^{-1} E_\mu\left[\left(\sum_{k=1}^{n} X_k\right)^2\right] \\ &\quad = E_\mu(M_1^2) - \lim_{n\to\infty} n^{-1} E_\mu\left[\left(\sum_{k=0}^{n-1} \varphi_k\right)^2\right] \\ &\quad = E_\mu(M_1^2) + E_\mu(\varphi_0^2) - 2(\varphi, (1-T_1)^{-1}\varphi) \\ &\quad = E_\mu(X^2) - 2(\varphi, (1-T_1)^{-1}\varphi) \end{aligned} \tag{2.35}$$

from which (2.29) directly follows. The last equality in (2.35) follows from (2.31).

Since $\varphi \in H_{-1}$, we may invoke Dürr and Goldstein[12] or Kipnis and Varadhan,[26] who show that under this hypothesis on φ one can write

$$\varphi_{n-1} = N_n + E_n - E_{n-1} \tag{2.36}$$

where N_n is another square-integrable martingale-difference sequence, $E_0 = 0$, and

$$n^{-1/2} E_n \to 0 \quad \text{in} \quad \mathbf{L}^2(\mu) \tag{2.37}$$

Since by (2.32) and (2.36)

$$X_t^\varepsilon = \varepsilon \sum_{n=1}^{[\varepsilon^{-2}t]} (M_n + \varphi_{n-1}) = \varepsilon \sum_{n=1}^{[\varepsilon^{-2}t]} (M_n + N_n) + \varepsilon E_{[\varepsilon^{-2}t]} \tag{2.38}$$

the central limit theorem for X_t^ε follows now from (2.37) and the central limit theorem for martingale-difference sequences.[7] That D is the diffusion constant follows now from (2.29), (2.37), (2.38), and the fact that the diffusion constant given by the martingale-difference central limit theorem is $E((M_1 + N_1)^2)$. Without further hypotheses the invariance principle follows as well.[26] We postpone further discussion of the invariance principle to the proof of the next theorem. ▮

Theorem 2.2. Let ξ_t, $t \in \mathbb{R}$, and $X_\mathbf{I}$ (with values in $\mathbb{R}^d$), indexed by the set of all closed bounded intervals $\mathbf{I} \subset \mathbb{R}$, be a reversible, ergodic Markov process and antisymmetric family as specified above, and let $X_\delta = X_{[0,\delta]}$. Assume that the mean forward velocity φ exists; that is,

$$\lim_{\delta \to 0} \frac{1}{\delta} E_\mu[X_\delta \mid F_0] = \varphi(\xi_0) \tag{2.39}$$

exists as a strong $\mathbf{L}^1$ limit ($\varphi \in \mathbf{L}^1(\mu)$ and has values in $\mathbb{R}^d$). In addition, assume that the martingale[46]

$$M_t = X_t - \int_0^t \varphi(\xi_\tau)\, d\tau \tag{2.40}$$

is square-integrable.

Then $\varphi_i \in H_{-1}$ and the following hold:

(i) Let the matrix $\mathbf{D}$ be given by

$$D_{ij} = C_{ij} + 2(\varphi_i, \mathbf{L}^{-1}\varphi_j) = C_{ij} - 2\int_0^\infty (\varphi_i, T_t \varphi_j)\, dt \tag{2.41}$$

$1 \leqslant i, j \leqslant d$, where $\mathbf{C}$ is the symmetric matrix determined by

$$e^T \cdot \mathbf{C} \cdot e = E_\mu[(e \cdot M_1)^2] \tag{2.42}$$

for all $e \in \mathbb{R}^d$ (e^T is the transpose of e).

Let $X_t^\varepsilon = \varepsilon X_{\varepsilon^{-2} t}$ and let $W_\mathbf{D}$ be a Brownian motion with diffusion matrix $\mathbf{D}$ $[D_{ij} t = E_\mathbf{D}(W_i(t)\, W_j(t))]$ starting at zero. Then $X^\varepsilon \xrightarrow{f} W_\mathbf{D}$ as $\varepsilon \to 0$, in μ-measure. Furthermore,

$$\lim_{\varepsilon \to 0} E_\mu[(e \cdot X_t^\varepsilon)^2] = (e^T \cdot \mathbf{D} \cdot e)\, t \tag{2.43}$$

(ii) If, in addition, $\varphi \in \mathbf{L}^2(\mu)$ or, more generally, for some $\bar{t} > 0$,

$$\sup_{0 \leqslant t \leqslant \bar{t}} |X_t| \in \mathbf{L}^2(P_\mu)$$

then $X^\varepsilon \to W_\mathbf{D}$ as $\varepsilon \to 0$, weakly in μ-measure. ▌

Remarks. 1. $\mathbf{D}$ is finite and nonnegative definite but may be degenerate. Strategies for proving $\mathbf{D}$ nondegenerate are discussed in Section 3.

2. Ergodicity is needed only for the application (in the proof) of the martingale version of the invariance principle. If ξ_t is not ergodic, one can decompose $\aleph$ into ergodic components and consider the restricted processes.

3. Since $\varphi \in H_{-1}$, the time integral in (2.41) converges absolutely for all i, j.

4. Since $\varphi \in H_{-1}$, it follows from Eq. (2.16) that

$$\lim_{t \to 0} t^{-1} E_\mu \left[\left(\int_0^t e \cdot \varphi(\xi_s)\, ds \right)^2 \right] = 0$$

Therefore

$$e^T \cdot \mathbf{C} \cdot e = E_\mu[(e \cdot M_1)^2] = \lim_{t \to 0} t^{-1} E_\mu[(e \cdot M_t)^2] = \lim_{t \to 0} t^{-1} E_\mu[(e \cdot X_t)^2] \tag{2.44}$$

Now consider the weak $\mathbf{L}^1(P_\mu)$ limit

$$\lim_{t \to 0} t^{-1} E_\mu[(e \cdot X_t)^2 \mid F_0] \equiv e^T \cdot \boldsymbol{\psi} \cdot e$$

where $\boldsymbol{\psi}$ is symmetric. It follows from (2.44) that

$$C_{ij} = \langle \psi_{ij} \rangle$$

which in practice is usually very easy to compute.

5. Note that

$$\sup_{0 \leqslant t \leqslant \bar{t}} \left| \int_0^t \varphi(\xi_\tau)\, d\tau \right| \leqslant \int_0^{\bar{t}} |\varphi(\xi_\tau)|\, d\tau \in \mathbf{L}^2(P_\mu)$$

if $\varphi \in \mathbf{L}^2(\mu)$, and since M_t is a square-integrable martingale,[46]

$$\sup_{0 \leqslant t \leqslant \bar{t}} |M_t| \in \mathbf{L}^2(P_\mu)$$

It therefore follows from (2.40) that if $\varphi \in \mathbf{L}^2(\mu)$,

$$\sup_{0 \leqslant t \leqslant \bar{t}} |X_t| \in \mathbf{L}^2(P_\mu)$$

We also note that many of the conclusions of Theorem 2.2 follow from Theorem 2.1. However, we prefer to give a direct proof of Theorem 2.2, especially since the formula (2.41) for the diffusion matrix, which plays a crucial role in obtaining the estimates discussed in later sections, does not directly follow from Theorem 2.1.

Proof. We begin by establishing that X_t and the integrated drift are actually square-integrable. For this we exploit the symmetry properties of the variables. Writing

$$M_t = X_t - \int_0^t \varphi(\xi_\tau)\, d\tau \equiv A_t - S_t \tag{2.45}$$

we note that A_t is antisymmetric, S_t is symmetric, and so

$$\begin{aligned} A_t &= (1/2)(M_t - M_t \circ \mathbf{R}_{t/2}) \\ S_t &= -(1/2)(M_t + M_t \circ \mathbf{R}_{t/2}) \end{aligned} \tag{2.46}$$

Thus (since P_μ is invariant under $\mathbf{R}_{t/2}$), both A_t and S_t are in $\mathbf{L}^2(P_\mu)$.

It thus follows from (2.45), again using the symmetry properties, that

$$E_\mu(S_t^2) + E_\mu(X_t^2) = E_\mu(M_t^2) = tE_\mu(M_1^2) \tag{2.47}$$

The last equality follows since M_t is a square-integrable martingale having stationary increments. We conclude from (2.47) that both $E(S_t^2)$ and $E(X_t^2)$ are $O(t)$. In particular, by Lemma 2.4, $\varphi \in H_{-1}$.

It similarly follows from (2.45) that

$$E_\mu[(e \cdot X_t)^2] = tE_\mu[(e \cdot M_1)^2] - E_\mu\left[\left(\int_0^t e \cdot \varphi(\xi_s)\, ds\right)^2\right] \tag{2.48}$$

Thus, again using Lemma 2.4,

$$\lim_{t \to \infty} t^{-1} E_\mu[(e \cdot X_t)^2] = e^T \cdot \mathbf{C} \cdot e + 2(e \cdot \varphi, \mathbf{L}^{-1} e \cdot \varphi)$$

which is equivalent to (2.43).

We now begin the proof of the convergence of X^ε to the Brownian motion $W_{\mathbf{D}}$. The idea is to replace the drift term S_t by another square-integrable martingale (plus a negligible error term) following a line of argument given in ref. 26. We require a generalization of a result of Kipnis and Varadhan,[26] which we interrupt the proof of Theorem 2.2 to state.

Theorem 2.3. Let ξ_t, $t \in \mathbb{R}$, be a Markov process reversible and ergodic with respect to the (stationary) probability measure μ and let $\varphi \in \mathbf{L}^1(\mu) \cap H_{-1}$ with values in $\mathbb{R}^d$. Then

$$\varepsilon \int_0^{\varepsilon^{-2} t} \varphi(\xi_\tau)\, d\tau = \varepsilon N_{t\varepsilon^{-2}} + Q^\varepsilon(t) \tag{2.49}$$

where N_t is a square-integrable, $\mathbb{R}^d$-valued martingale (with respect to the filtration F_t) with stationary increments and (i)

$$\lim_{\varepsilon \to 0} E_\mu[Q^\varepsilon(t)^2] = 0 \tag{2.50}$$

and (ii) if $\varphi \in \mathbf{L}^2(\mu)$ or, more generally, if for some $\bar{t} > 0$,

$$\sup_{0 \leqslant t \leqslant \bar{t}} \left| \int_0^t \varphi(\xi_\tau)\, d\tau \right| \in \mathbf{L}^2(P_\mu)$$

then

$$\lim_{\varepsilon \to 0} P_\mu[\sup_{0 \leqslant t \leqslant \tau} |Q^\varepsilon(t)| > \eta] = 0 \tag{2.51}$$

for all τ and $\eta > 0$; (iii) furthermore,

$$\lim_{\varepsilon \to 0} E_\mu\left[\left(\varepsilon \int_0^{\varepsilon^{-2} t} e \cdot \varphi(\xi_\tau)\, d\tau\right)^2\right] = t E_\mu[(e \cdot N_1)^2] = t 2 \|e \cdot \varphi\|_{-1}^2 \quad \blacksquare \tag{2.52}$$

Remark. Without loss of generality we may assume that the martingale N_t appearing in Theorem 2.3 has paths in $D([0, \infty), \mathbb{R}^d)$; see, e.g., ref. 47.

To complete the proof of Theorem 2.2, we write, using Theorem 2.3,

$$X_t^\varepsilon = \varepsilon \int_0^{\varepsilon^{-2} t} \varphi(\xi_\tau)\, d\tau + \varepsilon M_{\varepsilon^{-2} t} = \varepsilon \hat{M}_{\varepsilon^{-2} t} + Q^\varepsilon(t) \tag{2.53}$$

where $\hat{M}_t = N_t + M_t$ is a square-integrable martingale (with respect to F_t) with stationary increments and paths in $D([0, \infty), \mathbb{R}^d)$ and $Q_t^\varepsilon \to 0$ in the appropriate sense. Note that it follows from the condition

$$\sup_{0 \leqslant t \leqslant \bar{t}} |X_t| \in \mathbf{L}^2(P_\mu)$$

in Theorem 2.2(ii) that the condition

$$\sup_{0 \leqslant t \leqslant \bar{t}} \left| \int_0^t \varphi(\xi_\tau)\, d\tau \right| \in \mathbf{L}^2(P_\mu)$$

in Theorem 2.3(ii) is also satisfied; see Remark 5 following the statement of Theorem 2.2.

Since ξ_t is ergodic, we may invoke the invariance principle for d-dimensional square-integrable martingales $\hat{M}_t$ with stationary increments,[7,22]

$$\hat{M}_t^\varepsilon \equiv \varepsilon \hat{M}_{\varepsilon^{-2}t} \to W_{\mathbf{D}}(t) \tag{2.54}$$

in the sense of standard weak convergence for processes with paths in $D([0, \infty), \mathbb{R}^d)$, where $e^T \cdot \mathbf{D} \cdot e = E_\mu[(e \cdot \hat{M}_1)^2]$. By (2.50), (2.53), and (2.43), $\mathbf{D}$ in Eq. (2.54) is given by (2.41). The convergence $X^\varepsilon \to W_{\mathbf{D}}$ in the standard sense follows easily—convergence in finite-dimensional distribution from (2.50) and weak convergence from (2.51). Moreover, this convergence can be strengthened to the convergence in μ-measure of Theorem 2.2 by observing that in the martingale case there is an "a.s." form of the invariance principle: For any "ergodic" square-integrable martingale $\hat{M}_t$ (with respect to F_t) with stationary increments and paths in $D([0, \infty), \mathbb{R}^d)$, a.e. martingale arising from $\hat{M}_t$ via formation of regular conditional probabilities given F_0 converges weakly to $W_{\mathbf{D}}$ under the usual ε-scaling $\hat{M}_t^\varepsilon \equiv \varepsilon \hat{M}_{\varepsilon^{-2}t}$ as $\varepsilon \to 0$, with $e^T \cdot \mathbf{D} \cdot e = E[(e \cdot \hat{M}_1)^2]$. This follows from Theorem 5.4 of Helland[22] and the ergodic theorem. (Because of the form of the ε-scaling, the usual condition on the size of the jumps[22] is easily verified using the ergodic theorem.) In particular, for every bounded, continuous function G on $D([0, \infty), \mathbb{R}^d)$

$$\lim_{\varepsilon \to 0} E[G(\hat{M}^\varepsilon) \mid F_0] = E[G(W_{\mathbf{D}})], \quad \text{a.s.} \tag{2.55}$$

The convergence of X_t^ε to $W_{\mathbf{D}}$ in μ-measure follows easily. This completes the proof of Theorem 2.2. ∎

Proof of Theorem 2.3. It will suffice to consider the one-dimensional case, $d = 1$, since the martingales which arise are in fact all martingales with respect to the same filtration F_t. In particular, if φ has values in $\mathbb{R}^d$, $d>1$, the martingales arising from each of the components φ_i of φ together form an $\mathbb{R}^d$-valued martingale.

For $\lambda>0$ define ψ_λ by

$$\psi_\lambda = \int_0^\infty e^{-\lambda t} T_t \varphi \, dt$$

$\psi_\lambda \in \mathbf{L}^1(\mu)$ and satisfies

$$\lambda\psi_\lambda - \tilde{\mathbf{L}}\psi_\lambda = \varphi \tag{2.56}$$

where $\tilde{\mathbf{L}}$ is the strong generator of T_t acting on $\mathbf{L}^1(\mu)$. We claim that

$$\psi_\lambda \equiv (\lambda - \tilde{\mathbf{L}})^{-1}\varphi = (\lambda - \mathbf{L})^{-1}\varphi \tag{2.57}$$

i.e., that $g_{\psi_\lambda} = (\lambda - \mathbf{L})^{-1} g_\varphi \equiv (\lambda - \mathbf{L})^{-1}\varphi$ [where $\mathbf{L}$ is the canonical map $H_1 \to H_{-1}$ arising from the $\mathbf{L}^2(\mu)$ generator], so that the spectral theorem may be used to analyze ψ_λ. Note that by the spectral theorem $(\lambda - \mathbf{L})^{-1}\varphi \in H_{-1}$. The claim follows by observing that for $\psi \in \mathbf{L}^\infty(\mu) \cap H_1$, using the spectral theorem,

$$\begin{aligned}
(\psi_\lambda, \psi) &= \left(\int_0^\infty e^{-\lambda t} T_t \varphi \, dt, \psi \right) \\
&= \int_0^\infty e^{-\lambda t} (T_t \varphi, \psi) \, dt \\
&= \int_0^\infty e^{-\lambda t} (\varphi, T_t \psi) \, dt \\
&= \left(\varphi, \int_0^\infty e^{-\lambda t} T_t \psi \, dt \right) \\
&= (\varphi, (\lambda - \mathbf{L})^{-1}\psi) = ((\lambda - \mathbf{L})^{-1}\varphi, \psi)
\end{aligned}$$

Since $\varphi \in H_{-1}$, we have that $\psi_0 \equiv (-\mathbf{L})^{-1}\varphi \in H_1$ and, by the spectral theorem, we may associate with φ a spectral measure v_φ relative to the spectral decomposition of $-\mathbf{L}$ which satisfies $\int_0^\infty v_\varphi(dx)/x < \infty$. It follows easily that $\psi_\lambda \in \mathbf{L}^2(\mu)$,

$$\lim_{\lambda \to 0} \|\psi_\lambda - \psi_0\|_1 = 0 \tag{2.58}$$

and

$$\lim_{\lambda \to 0} \lambda(\psi_\lambda, \psi_\lambda) = 0 \tag{2.59}$$

We now obtain the decomposition (2.49). We have for every $\lambda > 0$

$$\int_0^t \varphi(\xi_\tau)\, d\tau = N_t^\lambda + R_t^\lambda \tag{2.60}$$

where

$$N_t^\lambda \equiv \psi_\lambda(\xi_t) - \psi_\lambda(\xi_0) - \lambda \int_0^t \psi_\lambda(\xi_\tau)\, d\tau + \int_0^t \varphi(\xi_\tau)\, d\tau \tag{2.61}$$

and

$$R_t^\lambda \equiv -\psi_\lambda(\xi_t) + \psi_\lambda(\xi_0) + \lambda \int_0^t \psi_\lambda(\xi_\tau)\, d\tau \tag{2.62}$$

By (2.56), N_t^λ is a martingale, and it follows from Lemma 2.4 and the fact that $\psi_\lambda \in \mathbf{L}^2(\mu)$ that it is square-integrable. Moreover, N^λ is Cauchy in the space of $\mathbf{L}^2$-martingales as $\lambda \to 0$: Since $\psi_\lambda \in \mathbf{L}^2(\mu)$, it follows from stationarity, (2.58), and (2.12) that

$$\begin{aligned} E_\mu[(N_t^\lambda - N_t^{\lambda'})^2] &= t \lim_{\tau \to 0} \frac{1}{\tau} E_\mu[(N_\tau^\lambda - N_\tau^{\lambda'})^2] \\ &= 2t\, \|\psi_\lambda - \psi_{\lambda'}\|_1^2 \xrightarrow[\lambda, \lambda' \to 0]{} 0 \end{aligned} \tag{2.63}$$

Thus there exists a square-integrable martingale N_t, with respect to the filtration F_t, such that

$$N_\cdot^\lambda \to N_\cdot \tag{2.64}$$

in the space of $\mathbf{L}^2$-martingales. Thus, for any $\lambda > 0$,

$$\int_0^t \varphi(\xi_\tau)\, d\tau = N_t + (N_t^\lambda - N_t) + R_t^\lambda \equiv N_t + Q_{\lambda, t} \tag{2.65}$$

and introducing the scaling parameter $\varepsilon > 0$, putting $\lambda = \varepsilon^2$ and

$$Q^\varepsilon(t) = \varepsilon Q_{\varepsilon^2, \varepsilon^{-2} t} \tag{2.66}$$

we arrive at the decomposition (2.49).

We now verify the properties (i)–(iii) of this decomposition. Since

$$|Q_{\lambda, t}| \leqslant |(N_t^\lambda - N_t)| + |\psi_\lambda(\xi_0)| + |\psi_\lambda(\xi_t)| + \lambda \int_0^t |\psi_\lambda(\xi_\tau)|\, d\tau \tag{2.67}$$

(2.50) follows easily from (2.64) and (2.59). Then (2.52) follows from (2.50) and Lemma 2.4.

Finally, to verify (2.51), we first note that since the left-hand side of (2.65) has continuous paths and N_t has a modification with paths in $D([0, \infty), \mathbb{R})$ (see, e.g., ref. 47), it is sufficient to verify (2.51) with the supremum taken over a dense set. Thus, we may conclude the proof by establishing that for all $\eta > 0$

$$\lim_{t \to \infty} P_\mu[\sup_{\substack{0 \leq s \leq t \\ s \text{ rational}}} Q_s^{(i)}/\sqrt{t} > \eta] = 0 \tag{2.68}$$

where $Q_s^{(i)}$, $i = 1, 2, 3, 4$, are the terms on the right-hand side of (2.67) with $\lambda = t^{-1}$.

Verification of (2.68) for $i = 2$ and $i = 4$ just involves more or less the same $\mathbf{L}^2$ estimates of $Q_t^{(2)}$ and $Q_t^{(4)}$ required for (2.50), while (2.68) for $i = 1$ follows from (2.64) and the Doob–Kolmogorov inequality.[46,47] The analysis of (2.68) for $i = 3$ is more delicate. Here, as in ref. 26, we appeal to the following result.

Lemma 2.6 (Kipnis and Varadhan[26]). Let ξ_t, $t \in \mathbb{R}$, be a Markov process reversible with respect to the probability measure μ, and let $\psi \in \mathbf{L}^2(\mu) \cap H_1$. Then

$$P_\mu[\sup_{\substack{0 \leq s \leq t \\ s \text{ rational}}} |\psi(\xi_s)| > \eta] \leq 3\eta^{-1}\{(\psi, \psi) + t\|\psi\|_1^2\}^{1/2} \quad \blacksquare$$

This lemma is exploited as follows. We write, for $\lambda_0 > 0$,

$$|\psi_{t^{-1}}(\xi_t)| \leq |\psi_{t^{-1}}(\xi_t) - \psi_{\lambda_0}(\xi_t)| + |\psi_{\lambda_0}(\xi_t)| \equiv Q_t^{(5)} + Q_t^{(6)}$$

and apply the lemma with $\psi = \psi_{t^{-1}} - \psi_{\lambda_0}$ and η replaced by $\eta\sqrt{t}$ to obtain that

$$\limsup_{t \to \infty} P_\mu[\sup_{\substack{0 \leq s \leq t \\ s \text{ rational}}} Q_s^{(5)}/\sqrt{t} > \eta] \leq \|\psi_0 - \psi_{\lambda_0}\|_1$$

By (2.58), the proof may be completed by taking $\lambda_0 \to 0$ provided we can establish (2.68) with $i = 6$ for any fixed $\lambda_0 > 0$. To do this, we note that

$$\sup_{\substack{0 \leq s \leq t \\ s \text{ rational}}} |\psi_{\lambda_0}(\xi_s)| \leq \sup_{n = 0, 1, \dots, [t/\bar{t}]} \hat{\psi}^{(n)} \tag{2.69}$$

where

$$\hat{\psi}^{(n)} = \sup_{\substack{n\bar{t} \leq s \leq (n+1)\bar{t} \\ s \text{ rational}}} |\psi_{\lambda_0}(\xi_s)| \tag{2.70}$$

By (2.61),

$$\hat{\psi}^{(0)} \leqslant |\psi_{\lambda_0}(\xi_s)| + \lambda_0 \int_0^{\bar{t}} |\psi_{\lambda_0}(\xi_\tau)|\, d\tau$$

$$+ \sup_{\substack{0 \leqslant s \leqslant \bar{t} \\ s \text{ rational}}} |N_s^{\lambda_0}| + \sup_{\substack{0 \leqslant s \leqslant \bar{t} \\ s \text{ rational}}} \left| \int_0^s \varphi(\xi_\tau)\, d\tau \right| \tag{2.71}$$

so that choosing $\bar{t}$ as in Theorem 2.3(ii), we have that $\hat{\psi}^{(0)} \in \mathrm{L}^2(P_\mu)$. It then folows, using stationarity and (2.69), that

$$P_\mu[\sup_{\substack{0 \leqslant s \leqslant t \\ s \text{ rational}}} |\psi_{\lambda_0}(\xi_s)|/t^{1/2} > \eta]$$

$$\leqslant ([t/\bar{t}] + 1)\, P_\mu[\hat{\psi}^{(0)} > t^{1/2}\eta]$$

$$\leqslant [([t|\bar{t}] + 1)/(t\eta^2)] \int_{\{\hat{\psi}^{(0)} > t^{1/2}\eta\}} dP_\mu(\hat{\psi}^{(0)})^2 \to 0 \tag{2.72}$$

as $t \to \infty$. This completes the proof of Theorem 2.3. ∎

Remarks. 1. Theorem 2.3 is established in ref. 26 under the additional hypothesis that $\varphi \in \mathrm{L}^2(\mu)$, which immediately implies the square-integrability of the time integral of φ and thus of N_t. However (see Lemma 2.4), the condition that $\varphi \in H_{-1}$ itself guarantees this square-integrability.

2. One might expect the integrated drift $S_t \equiv \int_0^t \varphi(\xi_\tau)\, d\tau$ to provide a good indication of the behavior of X_t as $t \to \infty$. However, these processes are asymptotically independent. In fact, by (2.40) and Theorem 2.3,

$$(X_t^\varepsilon, S_t^\varepsilon) = (N_t^\varepsilon + M_t^\varepsilon, N_t^\varepsilon) + (Q^\varepsilon(t), Q^\varepsilon(t)) \equiv \tilde{N}_t^\varepsilon + \tilde{Q}^\varepsilon(t)$$

where $S_t^\varepsilon = \varepsilon S_{\varepsilon^{-2}t}$, $N_t^\varepsilon = \varepsilon N_{\varepsilon^{-2}t}$, and $M_t^\varepsilon = \varepsilon M_{\varepsilon^{-2}t}$. Since $\tilde{N}_t^\varepsilon = \varepsilon \tilde{N}_{\varepsilon^{-2}t}$, where $\tilde{N}_t \equiv (N_t + M_t, N_t)$ is a square-integrable $\mathbb{R}^{2d}$-valued martingale with stationary increments, $\tilde{Q}^\varepsilon(t)$ is asymptotically negligible in the sense of (2.50) or (2.51), and $E_\mu[(e \cdot X_t^\varepsilon)(\hat{e} \cdot S_t^\varepsilon)] = 0$ by symmetry, we have that the processes X_t^ε and S_t^ε are asymptotically independent Brownian motions.

Theorem 2.4. With the hypotheses and notation of Theorem 2.2, let $S_t = \int_0^t \varphi(\xi_\tau)\, d\tau$ and $S_t^\varepsilon = \varepsilon S_{\varepsilon^{-2}t}$. Then $(X^\varepsilon, S^\varepsilon) \xrightarrow{f} (W_{\mathbf{D}}, W_{\mathbf{D}^*}^*)$ as $\varepsilon \to 0$, in μ-measure, where $W_{\mathbf{D}}$ and $W_{\mathbf{D}^*}^*$ are independent Brownian motions with $\mathbf{D}$ given by (2.41) and (2.42) and $\mathbf{D}^*$ by $D_{ij}^* = -2(\varphi_i, \mathrm{L}^{-1}\varphi_j)$. If the hypothesis in Theorem 2.2(ii) is satisfied, the convergence "$\xrightarrow{f}$" can be strengthened to "$\to$," weak convergence in μ-measure.

3. BOUNDS ON THE DIFFUSION MATRIX

In the models treated in subsequent sections we will be interested in bounds on the matrix **D**, the diffusion matrix of the limiting Brownian motion. Typically we cannot compute **D** explicitly (exceptions are random walk and diffusion in a random environment in one dimension), so we resort to finding upper and lower bounds. Of course, a lower bound is more interesting theoretically than an upper bound—we already know that **D** is finite, but not that it is nondegenerate.

A simple upper bound always follows by dropping the (negative-definite) second term in (2.28) or (2.41): as a matrix, $\mathbf{D} \leqslant \mathbf{C}$. In examples one can usually compute **C** explicitly, as the definition of the model and the choice of X_t usually specify the quadratic variation of X_t as well; see Remark 4 after Theorem 2.2. Note that unless $\varphi \equiv 0$ (i.e., X_t is already a martingale), the second term in (2.28) or (2.41) is not zero, so the inequality is strict.

We discuss lower bounds on **D** in the continuous-time case. We have written our formula for **D** in terms of a quadratic form, and in fact, as we have seen, quadratic forms (Dirichlet forms and their duals) are natural for the study of reversible Markov processes. Now it often happens that the quadratic form

$$Q(\psi) \equiv \|\psi\|_1^2$$

associated with our process has a decomposition

$$Q = Q_1 + Q_2 \tag{3.1}$$

where Q_1 and Q_2 are closed, symmetric, nonnegative quadratic forms. Such a decomposition corresponds to a sum (let $A = -\mathbf{L}$)

$$A = A_1 + A_2 \tag{3.2}$$

where A_1 and A_2 are self-adjoint, nonnegative operators on $\mathbf{L}^2(\mu)$, and (3.2) is understood "in the form sense," i.e., (3.1) holds on

$$\mathbf{D}(Q) = \mathbf{D}(Q_1) \cap \mathbf{D}(Q_2)$$

where Q_i and $\mathbf{D}(Q_i) \equiv H_1(A_i)$ are the quadratic form and form domain of A_i, respectively. [Sometimes we know (3.1) only for a subdomain of $\mathbf{D}(Q)$ consisting of "nice" functions on which the quadratic forms Q_i are defined.] In fact, one can always define the "form sum" (3.2), provided $\mathbf{D}(Q_1) \cap \mathbf{D}(Q_2)$ is dense in the Hilbert space; $-\mathbf{L}$ is the operator associated with Q. (Q is automatically a closed, symmetric quadratic form, and furthermore is Dirichlet if the Q_i are Dirichlet, implying that $\mathbf{L}$ is a Markovian generator.)

In typical applications the Markov process is a composite of different motions, each reversible with respect to the same measure μ, and this is reflected by a corresponding decomposition of the quadratic form. For example, our process might have jumps or Brownian motions in different directions. The "environment process observed from the tagged particle" considered in subsequent sections often has a decomposition into the "motion of the environment" and the "motion of the particle." Yet another decomposition is used in Section 5.

If in such a decomposition of the process into several motions, one of these motions yields the same φ and $\boldsymbol{\psi}$ (or $\mathbf{C}$) as the full process, we obtain inequalities (lower bounds) on the diffusion matrix of the limiting Brownian motion using Eq. (2.41) and Lemma 3.1 below. Often these inequalities have interesting probabilistic meanings, relating diffusion constants for different processes. For example, $\mathbf{D}$ is typically increasing with the dimension of the motion. For the motion of a tagged particle, we have the following quasitheorem: The diffusion matrix for a particle moving in a random, dynamic environment is greater than that for the "same" particle moving in the "frozen" environment. These statements are made precise in Sections 5 and 6.

We give here several lemmas of a technical nature which are useful for obtaining the inequalities. The basic idea is that when one can establish (3.1) on a suitably large class of functions (a "form core" of the generator), the inequality $Q \geqslant Q_1$ implies the reversed inequality for the dual quadratic forms.

For self-adjoint operators $0 \leqslant A \leqslant B$ on a separable Hilbert space H (with norm $\|\cdot\|$), it follows directly from the definition of $\|\cdot\|_{-1}$ as a dual norm [see Eq. (2.9)] that $B^{-1} \leqslant A^{-1}$. More precisely, for any self-adjoint operator $B \geqslant 0$ on H, we may write $(\psi, B\psi)$ for $\|\psi\|_{1,B}^2$ [the square of the norm on $H_1(B)$], $(\varphi, B^{-1}\varphi)$ for $\|\varphi\|_{-1,B}^2$ [the square of the norm on $H_{-1}(B)$], and for $\psi \in H$ set $\|\psi\|_{\pm 1,B} = \infty$ when it is not otherwise defined [so that $(\psi, B\psi)$ and $(\psi, B^{-1}\psi)$ are defined for all $\psi \in H$]. We then have, by (2.9), that for $\varphi \in H$

$$(\varphi, B^{-1}\varphi) = \sup_{\psi \in H} \frac{|(\varphi, \psi)|^2}{(\psi, B\psi)} \tag{3.3}$$

Here, and in similar formulas, we set $0/0 = 0$, $a/0 = \infty$ for $a > 0$, and $a/\infty = 0$.

It follows immediately from (3.3) that if $A \geqslant 0$ and $B \geqslant 0$ are self-adjoint operators on H satisfying

$$(\psi, A\psi) \leqslant (\psi, B\psi) \qquad \text{for all} \quad \psi \in H$$

then for all $\varphi \in H$

$$(\varphi, B^{-1}\varphi) \leqslant (\varphi, A^{-1}\varphi) \tag{3.4}$$

Now suppose a subspace $\mathbf{D} \subset H \cap H_1(B)$ is dense in $H_1(B)$ under $\|\cdot\|_{1,B}$. Then, since for $\varphi \in H \cap H_{-1}(B)$

$$\sup_{\psi \in \mathbf{D}} \frac{|(\varphi, \psi)|^2}{(\psi, B\psi)} = \sup_{\psi \in H \cap H_1(B)} \frac{|(\varphi, \psi)|^2}{(\psi, B\psi)} \left(= \sup_{\psi \in H} \frac{|(\varphi, \psi)|^2}{(\psi, B\psi)} \right)$$

we have that for $\varphi \in H \cap H_{-1}(B)$

$$(\varphi, B^{-1}\varphi) = \sup_{\psi \in \mathbf{D}} \frac{|(\varphi, \psi)|^2}{(\psi, B\psi)} \tag{3.5}$$

Thus, if for all $\psi \in \mathbf{D}$

$$(\psi, A\psi) \leqslant (\psi, B\psi) \tag{3.6}$$

then for all $\varphi \in H \cap H_{-1}(B)$

$$(\varphi, B^{-1}\varphi) \leqslant \sup_{\psi \in \mathbf{D}} \frac{|(\varphi, \psi)|^2}{(\psi, A\psi)} \leqslant \sup_{\psi \in H} \frac{|(\varphi, \psi)|^2}{(\psi, A\psi)} = (\varphi, A^{-1}\varphi)$$

We call a subspace $\mathbf{D} \subset H \cap H_1(B)$ dense in $H_1(B)$ under $\|\cdot\|_{1,B}$ a *form core* for B. This terminology does not quite agree with the *conventional*, which refers to density in $H \cap H_1$ in $\|\cdot\| + \|\cdot\|_1$, or, equivalently, to density in $H_1(I+B)$ (under $\|\cdot\|_{1,I+B}$). We remark that if $\mathbf{D}$ is a form core *in the conventional sense*, then if (3.6) holds for all $\psi \in \mathbf{D}$, it holds, in fact, for all $\psi \in H$.

Now suppose further that A and B are Markovian (i.e., are the negative $\mathbf{L}^2$ generators of a probability semigroup). Then, as explained in Section 2, we have a canonical embedding $\mathbf{L}^1(\mu) \cap H_{-1}(B) \subset H_{-1}(B)$ and $(\varphi, B^{-1}\varphi)$ is defined for all $\varphi \in \mathbf{L}^1$. [It is ∞ for $\varphi \in \mathbf{L}^1 \backslash H_{-1}(B)$.] Moreover, as we shall now explain, the conclusion (3.4) can be extended to $\varphi \in \mathbf{L}^1$.

First, observe that if (3.6) holds for all $\psi \in H$, then $H \cap H_1(B) \subset H \cap H_1(A)$ and with any linear functional $g \in H_{-1}(A)$ we may naturally *associate* a linear functional $\hat{g} \in H_{-1}(B)$; merely let $\hat{g}$ be the continuous extension to $H_1(B)$ of g restricted to $H \cap H_1(B)$. It then follows trivially that

$$(\hat{g}, B^{-1}\hat{g}) \leqslant (g, A^{-1}g) \tag{3.7}$$

for all $g \in H_{-1}(A)$. In particular, for $\varphi \in \mathbf{L}^1$, we have that

$$(\varphi, B^{-1}\varphi) \leqslant (\varphi, A^{-1}\varphi)$$

since $(\varphi, A^{-1}\varphi) < \infty$ implies that φ defines a linear functional $g = g_\varphi \in H_{-1}(A)$ which is associated with the linear functional $\hat{g} \in H_{-1}(B)$ also defined by φ.

Now suppose that (3.6) holds for all ψ in a form core $\mathbf{D}$ for B. Then with any $g \in H_{-1}(A)$ we may still associate $\hat{g} \in H_{-1}(B)$, by extension to $H_1(B)$ of the restriction of g to $\mathbf{D}$. Moreover,

$$(\hat{g}, B^{-1}\hat{g}) = \sup_{\psi \in H_1(B)} \frac{|(\hat{g}, \psi)|^2}{(\psi, B\psi)} = \sup_{\psi \in \mathbf{D}} \frac{|(\hat{g}, \psi)|^2}{(\psi, B\psi)} \leqslant \sup_{\psi \in \mathbf{D}} \frac{|(\hat{g}, \psi)|^2}{(\psi, A\psi)}$$

$$\leqslant \sup_{\psi \in H_1(A)} \frac{|(\hat{g}, \psi)|^2}{(\psi, A\psi)} = (g, A^{-1}g)$$

Suppose that $\varphi \in \mathbf{L}^1$, $(\varphi, A^{-1}\varphi) < \infty$, and $\mathbf{D} \subset \mathbf{L}^\infty$. Let $g = g_\varphi \in H_{-1}(A)$ be the linear functional defined by φ and let $\hat{g} \in H_{-1}(B)$ be the linear functional associated with g. Then, since $(\hat{g}, \psi) = (\varphi, \psi)$ for all $\psi \in \mathbf{D}$, it follows that if $\varphi \in \mathbf{L}^1 \cap H_{-1}(B)$,

$$(\hat{g}, \psi) = (\varphi, \psi) \qquad \text{for all} \quad \psi \in \mathbf{L}^\infty \cap H_1(B)$$

Thus, if $\varphi \in \mathbf{L}^1 \cap H_{-1}(B)$, $\hat{g}$ is defined by φ and we may conclude that

$$(\varphi, B^{-1}\varphi) \leqslant (\varphi, A^{-1}\varphi)$$

We summarize the preceding discussion in the following result (see also Faris,[13] Proposition 6.1).

Lemma 3.1. Suppose A and B are nonnegative, self-adjoint operators on a Hilbert space H, and let $\mathbf{D}$ be a form core for B. Then:

(i) If $(\psi, A\psi) \leqslant (\psi, B\psi)$ for all $\psi \in H$, it follows that $(\varphi, B^{-1}\varphi) \leqslant (\varphi, A^{-1}\varphi)$ for all $\varphi \in H$.

(ii) If $(\psi, A\psi) \leqslant (\psi, B\psi)$ for all $\psi \in \mathbf{D}$, it follows that $(\varphi, B^{-1}\varphi) \leqslant (\varphi, A^{-1}\varphi)$ for all $\varphi \in H \cap H_{-1}(B)$.

If, in addition, A and B are Markovian, (i) and (ii) can be strengthened to:

(iii) If $(\psi, A\psi) \leqslant (\psi, B\psi)$ for all $\psi \in H$, if follows that $(\varphi, B^{-1}\varphi) \leqslant (\varphi, A^{-1}\varphi)$ for all $\varphi \in \mathbf{L}^1$.

(iv) If $\mathbf{D} \subset \mathbf{L}^\infty$ and $(\psi, A\psi) \leqslant (\psi, B\psi)$ for all $\psi \in \mathbf{D}$, it follows that $(\varphi, B^{-1}\varphi) \leqslant (\varphi, A^{-1}\varphi)$ for all $\varphi \in \mathbf{L}^1 \cap H_{-1}(B)$.

Finally, if $\mathbf{D}$ is a *conventional* form core for B, then (i) and (iii) still hold when "$\psi \in H$" is replaced by "$\psi \in \mathbf{D}$." ∎

Note that whenever B is Markovian, $\mathbf{L}^\infty \cap H_1(B)$ is a form core for B by Lemma 2.1. Since $\mathbf{L}^\infty$ is invariant under the semigroup $T_t \equiv e^{-Bt}$ generated by B when B is Markovian, the preceding observation is, in fact, a special case of the following lemma, whose operator version is widely employed. This lemma usually provides the easiest way to check that a set is a form core.

Lemma 3.2. Let B be a nonnegative, self-adjoint operator on the Hilbert space H and let $\mathbf{D} \subset H \cap H_1(B)$ be a subspace dense under $\|\cdot\|$ in $H^\perp \equiv (\operatorname{Ker} B)^\perp$, the orthogonal complement in H of Ker B. Suppose $\mathbf{D}$ is invariant under the semigroup $T_t \equiv e^{-Bt}$ generated by B, $T_t\mathbf{D} \subset \mathbf{D}$. Then $\mathbf{D}$ is a form core for B. If, in addition, $\mathbf{D}$ is dense in H under $\|\cdot\|$, it is a conventional form core for B. ▌

Proof. $\bigcup_{s>0} T_s H^\perp$ is dense in $H_1(B)$ under $\|\cdot\|_1 \equiv \|\cdot\|_{1,B}$. Since $Be^{-Bt} \leqslant (e^{-1}/t)I$, we have that $T_s\mathbf{D}$ is a dense subset of $T_s H^\perp$ under $\|\cdot\|_1$. Therefore, $\bigcup_{s>0} T_s\mathbf{D}$ is dense in $H_1(B)$ under $\|\cdot\|_1$. Hence, by invariance, so also is $\mathbf{D}$. The last statement of the lemma follows by applying the preceding part of the lemma to $I+B$. ▌

4. RANDOM WALK IN RANDOM ENVIRONMENT

Random walks in a random environment have been widely studied; see, for instance, refs. 3–5, 23, 25, and 44. For review papers we refer to Papanicolaou.[37,38]

In this section we apply the general theorems of Section 2 to establish diffusive behavior for a random walk on an inhomogeneous lattice. To be definite, the model is the following. In the d-dimensional lattice $\mathbb{Z}^d$, to each bond $(x, x+e)$, $x \in \mathbb{Z}^d$, $|e| = 1$, associate a random rate $a_e(x) = a_{-e}(x+e) \geqslant 0$ with some distribution μ. We denote by $\aleph$ the space of the environments:

$$\aleph = \aleph(\mathbb{Z}^d) = \{a_e : \mathbb{Z}^d \to [0, \infty],\ |e| = 1\} \tag{4.1}$$

Sometimes we will write $a_i(x) = a_{e_i}(x)$, where e_i is the unit vector in the positive i direction. Let μ be a probability measure on $\aleph$ satisfying:

H1. μ is translation invariant; i.e., for any measurable set $A \subset \aleph$, $\mu(S_x A) = \mu(A)$, where S_x denotes the translation by $x \in \mathbb{Z}^d$ [see (4.8c) for the formal definition]. Furthermore, μ is ergodic.

Given a configuration $a \in \aleph$ with distribution μ, put a particle at the origin and let it move to its nearest neighbors with rates a. That is, define the jump process $X_t(a)$, $t \in \mathbb{R}$, as the Markov process with state space $\mathbb{Z}^d$ and transition probabilities defined by

$$P[X_{t+h}(a) = x + e \mid X_t(a) = x] = a_e(x)h + o(h) \tag{4.2a}$$

$$P[X_{t+h}(a) = x \mid X_t(a) = x] = 1 - \sum_{|e|=1} a_e(x)h + o(h) \tag{4.2b}$$

and

$$X_0(a) = 0 \tag{4.2c}$$

The problem we are interested in is the convergence to a Brownian motion of the rescaled process $\varepsilon X_{\varepsilon^{-2}t}(a)$ as $\varepsilon \to 0$.

The case in which μ has support on the set of configurations

$$\aleph([b_1, b_2]) = \{a \in \aleph: a_e(x) \in [b_1, b_2], \forall x \in \mathbb{Z}^d, |e| = 1\}, \qquad 0 < b_1 < b_2 < \infty$$

has been widely studied; convergence to Brownian motion has been proven in refs. 5, 15, and 29.

One natural question is whether it is possible to relax this condition, allowing both unbounded and zero rates. This, as we shall see, involves two different kinds of difficulties: the unboundedness is somehow a technical problem (see below), while zero rates will imply a nontrivial problem in bond percolation.

If the rates are unbounded, one even has some difficulty in proving the existence of the process, that is, in excluding explosion. We assume

$$\langle a_e(0) \rangle \equiv \mu(a_e(0)) < \infty \tag{4.3}$$

Condition (4.3) turns out to be the "minimal" requirement for existence. In fact, one can given examples in which the first moment does not exist and explosion does occur. Even assuming (4.3), it is not straightforward to see that the mean squared displacement of the particle is finite, i.e., $E(X_t(a)^2) < \infty$, though it is true. Using the general theorems of Section 2, we establish convergence to a Brownian motion under hypothesis (4.3).

The case in which the distribution μ gives positive probability for the rates to be zero is a deeper problem. The general theorems will give convergence to Brownian motion, but one has to prove that the diffusion matrix is not degenerate, and this will be true only under certain hypotheses and will require a nontrivial argument. Let us for the moment assume that

$$p \equiv \mu(\{a_e(0) > 0\}) < 1 \tag{4.4a}$$

and that

$$\mu \text{ is a product measure} \tag{4.4b}$$

Given $a \in \aleph$, define

$$x \sim y \text{ if } \exists x = x_1, x_2, \ldots, x_n = y \text{ with } |x_i - x_{i-1}| = e_{i-1} \text{ and } a_{e_{i-1}}(x_{i-1}) > 0 \tag{4.5a}$$

and

$$W(x, a) \equiv \{ y \in \mathbb{Z}^d \mid y \sim x \} \tag{4.5b}$$

We call $W(0, a)$ the cluster containing the origin. It is then clear that for the configurations $a \in \aleph$ for which $|W(0, a)| < \infty$ (here $|A|$ denotes the cardinality of the set A), the position $X_t(a)$ of the particle will be bounded for any t. So the minimal condition one has to require is that $|W(0, a)| = \infty$. It is well known[24] that there exists a critical probability $p_c(d)$, depending on the dimension d, such that

$$p_c(1) = 1 \tag{4.6a}$$

$$\theta_p \equiv \mu(\{a: |W(0, a)| = \infty\}) > 0 \qquad \text{iff} \quad p > p_c(d) \tag{4.6b}$$

So one has to require that $d \geqslant 2$ and μ is such that $p > p_c(d)$. Even under these hypotheses it is still not so obvious that the particle walking on the infinite cluster will converge under the scaling limit to a Brownian motion. This seems to depend very much on the geometry of the cluster; but computer simulations, and physical considerations,[35] predict a diffusive behavior no matter what the geometry of the cluster is.

In fact, we prove this result in dimension $d = 2$ for $p > 1/2$, $1/2$ being the critical probability in two dimensions. In $d > 2$ dimensions the results we have are less satisfactory; see Remark 4.16.

To separate the difficulties, we proceed as follows. In Section 4.1 we consider the unbounded case, while in Section 4.2 we prove the diffusive behavior for the random walk on the infinite cluster of the bond percolation problem in two dimensions.

4.1. Case of Unbounded Rates

In this subsection we consider the case in which μ (i.e., the distribution of the rates) satisfies the following conditions:

C1. μ is translation invariant and ergodic.

C2. $\langle a_e(0) \rangle_\mu < \infty$, where $\langle \cdot \rangle_\mu$ denotes expectation with respect to μ (sometimes we simply write $\langle \cdot \rangle$).

C3. $p \equiv \mu(\{a_e(0) > 0\}) = 1$.

In most of the statements condition C3 is not really needed, but, as we pointed out before, we will always assume it in this subsection to avoid confusion.

To use the general theorems of Section 2, consider the process describing the environment as seen by the traveling particle (see Definition 4.1 below). The generator $\mathbf{L}$ of this process is formally given by

$$\mathbf{L}f(a) = \sum_{|e|=1} a_e(0)[f(S_{-e}a) - f(a)] \tag{4.7}$$

and the position $X_t(a)$ of the traveling particle is given by the algebraic number of shifts of the environment during the time interval $[0, t]$ [see Eq. (4.12) for the formal definition of $X_t(a)$]. Using the translation invariance of the measure μ, one checks, formally, that

$$\mu(g\mathbf{L}f) = \mu(f\mathbf{L}g)$$

Without further assumptions it is not, however, directly clear that the generator $\mathbf{L}$ is, in fact, self-adjoint on $\mathbf{L}^2(\mu)$, from which reversibility would follow. Rather, we directly establish the existence of this process and reversibility and ergodicity with respect to the measure μ (obtaining the self-adjointness of $\mathbf{L}$ as a consequence). Afterward the invariance principle as well as bounds on the diffusion matrix are established.

The basic process in our analysis, which plays the role of ξ_t (in Theorem 2.2), describes the environment seen by the traveling particle.

Definition 4.1. The Environment Process

(i) *The Discrete Process.* We define a discrete Markov process $a(k) = \{a_e(x, k),\ x \in \mathbb{Z}^d,\ |e| = 1\}$, $k \in \mathbb{N}$, with state space $\aleph$, by giving the transition probabilities $p(a, a')$, $a, a' \in \aleph$, in the following way:

$$p(a, a') = \begin{cases} \alpha(a)^{-1} a_e(0) & \text{iff} \quad a' = S_{-e}a \text{ for some } e,\ |e| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.8a}$$

where

$$\alpha(a) = \sum_{|e|=1} a_e(0) \tag{4.8b}$$

and $S_{-e}a$ is defined by

$$S_{-e}a = a^*, \qquad a^*_{e'}(x) = a_{e'}(x+e) \qquad \forall x \in \mathbb{Z}^d \qquad \forall e', \quad |e'| = 1 \tag{4.8c}$$

We will write

$$pf(a) = \sum_{|e|=1} p(a, S_{-e}a) f(S_{-e}a) \tag{4.8d}$$

Furthermore, we define the discrete process $X(k)$, $k \in \mathbb{N}$, with state space $\mathbb{Z}^d$, by setting

$$X(1) = x \qquad \text{iff} \quad a(1) = S_{-x} a(0) \tag{4.8e}$$

$$X(0) = 0 \tag{4.8f}$$

and

$$X(k) = \sum_{i=1}^{k} X(1) \cdot \theta_{i-1} \tag{4.8g}$$

If we wish to focus on the process starting from the environment a, we write $X(a, k)$.

(ii) *The continuous process.* The Markov jump process $X_t(a)$ we are interested in, with transition probabilities given by Eq. (4.2), is defined via a random time change of the discrete process. We follow the standard construction (ref. 8, Chapter 15, Section 6). For $n \in \mathbb{N}$, $a \in \aleph$ construct independent exponential random variables $\tau_n(a)$ with mean $\alpha(a)$ [see Eq. (4.8b)]. Let these exponential variables be independent of the process $a(k)$, $k \in \mathbb{N}$, constructed so far. $\tau_n(a)$ will serve as the waiting time in the state a after the nth jump. Then, we define random variables as follows:

$$R_0 = 0, \qquad R_n = \tau_{n-1}(a(n-1)) + \cdots + \tau_0(a(0)) \tag{4.9a}$$

$$n^*(t) = n \qquad \text{iff} \quad R_n \leqslant t < R_{n+1} \tag{4.9b}$$

The continuous-time Markov process $a(t)$, $t \in \mathbb{R}$, is defined by setting

$$a(t) = a(n^*(t)) \tag{4.10}$$

Note that the generator $\mathbf{L}$ of this process is formally given by

$$\mathbf{L} f(a) = -\alpha(a)(I - p) f(a) = \sum_{|e|=1} a_e(0)[f(S_{-e} a) - f(a)] \tag{4.11}$$

where I is the identity and p is defined in (4.8).

The variable X_t we are interested in, defined by

$$X_t \equiv X(n^*(t)) \tag{4.12}$$

is the algebraic number of shifts of the environment during the time interval $[0, t]$. Thus, $X_t(a) = X(a, n^*(t))$.

Note that the function $t \to n^*(t)$ is nondecreasing; following Billingsley[7] (Chapter 3, Section 3), we call $n^*(t)$ the *random time change function.* We denote by P_μ (resp. P_a) the law of the process $a(t)$, $t \in \mathbb{R}$, with starting measure μ [resp. with $a(0)=a$ a.s.], and by E_μ (resp. E_a) the expectation with respect to P_μ (resp. P_a).

Remark 4.2

(i) Note that (4.8e) is meaningless if the rates a are periodic with period less than or equal to two in a coordinate direction. Our theorems apply also to this case, provided we suitably enlarge the state space. We avoid giving more details here.

(ii) The process $a(t)$ and the process X_t are well defined if and only if the random time change function is well defined, or, equivalently, if explosion is excluded. In Lemma 4.3 below we prove absence of explosion as well as reversibility and ergodicity of both the discrete and the continuous processes defined so far.

Lemma 4.3. Let μ^* be the measure on $\aleph$ defined by

$$\mu^*(da) = \langle \alpha(a) \rangle^{-1} \alpha(a)\, \mu(da) \tag{4.13}$$

Then the following hold:

(i) The discrete process with starting measure μ^* is reversible and ergodic.

(ii) The following relation holds:

$$\lim_{t \to \infty} \frac{1}{t} n^*(t) = \langle \alpha(a) \rangle \qquad \text{a.s.} \tag{4.14}$$

(iii) For any $t > 0$ fixed

$$P_a[\{ \sup_{0 \leqslant \tau \leqslant t} |X_\tau(a)| < \infty \}] = 1 \qquad \text{a.s.} \tag{4.15}$$

(iv) The continuous process with starting measure μ is reversible and ergodic. ∎

Proof. (i) Reversibility follows from the following easy calculation:

$$\mu^*(da)\, p(a, a') = \mu^*(da')\, p(a', a)$$

To prove ergodicity, let $A \subset \aleph$ be an invariant set, i.e.,

$$p(a, A) = 1, \qquad \forall a \in A$$

Then, by definition of p [see Eq. (4.8)], A is translation invariant. Since μ is ergodic with respect to space translations (see condition C1), $\mu(A)$ is zero or one, and therefore as a consequence $\mu^*(A)=0$ or 1, μ^* being absolutely continuous with respect to μ.

(ii) By part (i), the process $\{a(k),\ \tau_k(a(k));\ k\in\mathbb{Z}\}$ is ergodic; therefore

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\tau_k(a(k))=\lim_{n\to\infty}\frac{1}{n}R_n=E_{\mu^*}[\tau_0(a)]$$

$$=\int d\mu^*(a)\,\alpha(a)^{-1}=\frac{1}{\langle\alpha(a)\rangle}\qquad \mu\text{-a.s.}\qquad (4.16)$$

Equation (4.16) proves Eq. (4.14), since by definition [see Eq. (4.9)],

$$n^*(t)/R_{n^*(t)+1}\leqslant n^*(t)/t\leqslant n^*(t)/R_{n^*(t)}$$

(iii) Equation (4.15) follows from (ii): in fact,

$$\{|X_t|>\mathbf{C}\}\subset\{n^*(t)>\mathbf{C}\}$$

(iv) Since, by (4.3), the environment process has rates $r(a, db)$ satisfying $\int_{\aleph\times\aleph}\mu(da)\,r(a, db)<\infty$ and since by the translation invariance of μ and the bond form of the rates we have the evident symmetry $\mu(da)\,r(a, db)=\mu(db)\,r(b, da)$, reversibility follows easily.

We also give a more concrete verification of reversibility: We prove, in fact, that the continuous process $a(t)$ is "reversible for any fixed initial configuration." More explicitly, we prove that for any $a, b\in\aleph$ such that b is a translate of a,

$$P_a[\{a(t)=b\}]=P_b[\{a(t)=a\}]\qquad (4.17)$$

Let $\tau_0=0$, τ_1, τ_2,... be the times of jump of the process $a(t)$ and let $n^*(t)$ be the number of jumps up to time t [see Eq. (4.9b)]. Let $a=b_0, b_1$,... be a fixed trajectory for the process $a(t)$. Then

$$P_a[\{a(\tau_0)=b_0,..., a(\tau_n)=b_n;\, n^*(t)=n\}]$$

$$=\int\cdots\int 1\left\{\sum_{i=1}^{n} t_i'<t\right\}\frac{a_0}{\alpha_0}\cdots\frac{a_{n-1}}{\alpha_{n-1}}$$

$$\times\,\alpha_0\exp(-\alpha_0 t_1')\cdots\alpha_{n-1}\exp(-\alpha_{n-1}t_n')\,dt_1'\cdots dt_n'$$

$$=P_b[\{a(\tau_0)=b_n,..., a(\tau_n)=b_0;\, n^*(t)=n\}]$$

where

$$t_n'=t-\sum_{i=1}^{n-1}t_i'$$

and the a_i are the rates associated with the bonds involved in the transition at time t_i' and $\alpha_i = \alpha(b_i)$, where $\alpha(b)$ is defined in Eq. (4.8b).

Equation (4.17) follows by summing first over all trajectories with $b_0 = a$ and $b_n = b$ and then over n. By translation invariance of the measure μ, this suffices to establish reversibility, which in turn implies time invariance. Ergodicity follows from the fact that μ is ergodic with respect to translations. ▮

Remark 4.4. To prove the invariance principle for X_t, we first establish convergence to Brownian motion for the discrete process and then we use a random time change argument to conclude the proof.

Theorem 4.5. The following hold:

(i) The rescaled discrete process $X_t^{(n)} \equiv n^{-1/2} X([nt])$, with $[nt]$ the integer part of nt, converges weakly in μ^*-measure to a Brownian motion $W_{\mathbf{D}^*}(t)$ with finite diffusion matrix $\mathbf{D}^*$. Furthermore, $\mathbf{D}^* = (D_{ij},\ i,\ j = 1,..., d)$ is given by

$$\frac{1}{2} D_{ij}^* = \left\langle \frac{a_i(0)}{\alpha(a)} \right\rangle_{\mu^*} \delta_{ij} - (\varphi_i^*(a), (I-p)^{-1} \varphi_j^*(a))_{\mu^*} \tag{4.18}$$

where

$$\varphi_i^*(a) = \alpha(a)^{-1} [a_i(0) - a_i(-e_i)] \in H_{-1}(I-p; \mu^*) \tag{4.19}$$

I is the identity, and p is defined in Eq. (4.8).

(ii) The rescaled continuous process $\varepsilon X_{\varepsilon^{-2} t}$ converges weakly in μ-measure to a Brownian motion $W_{\mathbf{D}}(t)$ with finite diffusion matrix $\mathbf{D}$. Furthermore, $\mathbf{D}$ is given by

$$\mathbf{D} = \langle \alpha(a) \rangle \mathbf{D}^* \tag{4.20}$$

where $\mathbf{D}^*$ is defined in (4.18).

(iii) Setting $\varphi_i(a) = [a_i(0) - a_i(-e_i)]$, we have that $\varphi_i \in H_{-1} \equiv H_{-1}(-\mathbf{L}; \mu)$, where $\mathbf{L}$ is the strong $\mathbf{L}^2(\mu)$ generator of the environment process $a(t)$, and

$$\tfrac{1}{2} D_{ij} = \langle a_i(0) \rangle_\mu \delta_{ij} + (\varphi_i(a), \mathbf{L}^{-1} \varphi_j(a))_\mu \quad ▮ \tag{4.21}$$

Proof. (i) We use Theorem 2.1. By Definition 4.1, we have that $|X(1)| = 1$, $X(1)$ is antisymmetric, and

$$E_{\mu^*}(X_i(1) \mid F_0) = \varphi_i^*(a(0))$$

Furthermore, by Lemma 4.3(i), the discrete environment process is reversible and ergodic; therefore, from Theorem 2.1, part (i) of Theorem 4.5 follows.

(ii) The continuous-time process X_t is obtained from the discrete one via the random time change function $n^*(t)$. By Theorem 17.1 of ref. 7, $\varepsilon X_{\varepsilon^{-2}t}$ converges weakly in μ-measure to a Brownian motion provided that the random time change function behaves like t in probability. Therefore, the invariance principle and Eq. (4.20) follow from the fact that [see Lemma 4.3(ii)]

$$\lim_{t\to\infty} \frac{1}{t} n^*(t) = \langle \alpha(a)\rangle \qquad \text{in measure}$$

Finally, (4.21) follows from (4.18) and (4.20): in fact,

$$\begin{aligned}\frac{1}{2} D_{ij}^* &= \frac{1}{\langle \alpha(a)\rangle}\left\{\langle a_i(0)\rangle_{\mu^*}\delta_{ij} - \left(\varphi_i^*, \frac{1}{\alpha(I-p)}\alpha\varphi_j^*\right)_{\mu^*}\right\}\\ &= \frac{1}{\langle \alpha(a)\rangle}\left\{\langle a_i(0)\rangle_{\mu}\delta_{ij} + \left(\alpha\frac{\varphi_i}{\alpha}, L^{-1}\alpha\frac{\varphi_j}{\alpha}\right)_{\mu}\right\}\end{aligned}$$

where the preceding formal calculation is justified using Eq. (3.3). ∎

Remark. One can also obtain parts (ii) (except "tightness") and (iii) of Theorem 4.5 from Theorem 2.2. One need only check that (2.39) holds and that M_t as defined in (2.40) is square-integrable. This can be easily done by considering a sequence of processes with cutoff (bounded) rates and observing that the corresponding sequence of martingales is Cauchy in $\mathbf{L}^2$, so that upon the removal of the cutoffs, (2.40) is obtained with $\varphi \in \mathbf{L}^1(\mu)$ and M_t square-integrable. [Note that our hypothesis (4.3) on the moments of $a_i(0)$ guarantees only that $\varphi \in \mathbf{L}^1(\mu)$, but not necessarily $\mathbf{L}^2(\mu)$, so the Kipnis–Varadhan theorem is not applicable as originally stated. Note also that without further hypotheses on the moments of $a_i(0)$ it is by no means obvious that X_t, let alone M_t, is square-integrable.] The usual asymptotics for the mean-squared displacement for the continuous-time model, which do not directly follow from the random time change analysis presented above, also follow easily upon application of Theorem 2.2.

In the next theorem we will establish bounds on the diffusion matrix $\mathbf{D}$. First of all we observe that $\mathbf{D}$ is explicitly computable in dimension $d=1$. This is a well-known result.[3,23,36] We have that for $d=1$

$$D = 2\left\langle \frac{1}{a_e(0)}\right\rangle^{-1} \tag{4.22}$$

provided the rhs is finite. We prove (4.22) at the end of this section.

Theorem 4.6. Let $\mu = \mu_d$ be a measure on $\aleph(\mathbb{Z}^d)$ satisfying C1–C3. Then the following hold:

(i) Let μ_{d-1} be the projection of μ_d on the hyperplane $\{x = (x_1, ..., x_d): x_d = 0\}$. Then we have the matrix inequality

$$\mathbf{D}(\mu_{d-1}) \leqslant \mathbf{D}(\mu_d) \tag{4.23a}$$

and, in particular,

$$D_{ii}(\mu_{d-1}) \leqslant D_{ii}(\mu_d) \tag{4.23b}$$

Furthermore,

$$\left\langle \frac{1}{a_i(0)} \right\rangle^{-1} \leqslant \frac{1}{2} D_{ii}(\mu_d) \leqslant \langle a_i(0) \rangle \tag{4.23c}$$

(ii) The lower bound in Eq. (4.23c) is attained if μ has support on the set

$$\{a: a_i(x) = a_i(x+z) \text{ if } z \text{ is orthogonal to } e_i\} \tag{4.24a}$$

The upper bound in Eq. (4.23c) is attained if and only if μ has support on the set

$$\{a: \ a_i(x) = a_i(x + e_i) \ \ \forall x \in \mathbb{Z}^d\} \tag{4.24b}$$

(iii) If μ is either reflection invariant or isotropic (i.e., invariant with respect to $\mathbb{Z}^d$ rotations by $\pi/2$), then $\mathbf{D} = \mathbf{D}(\mu)$ is a diagonal matrix. In the isotropic case $\mathbf{D}$ is a multiple of the identity matrix. ∎

Proof. (i) To establish (4.23a), we use Eq. (4.21) and Lemma 3.1. In fact, the generator $\mathbf{L}$ of the process is a sum of generators corresponding to the various directions in $\mathbb{Z}^d$. The lower bound in (4.23c) then follows from Eq. (4.22), while the upper bound is the first term in the formula (4.21) for $\mathbf{D}$ (the second one being nonpositive). In order to check the hypothesis of Lemma 3.1, we need only check that $\mathbf{L}$ is given by the expected formula, i.e., the one in Eq. (4.7), on $\mathbf{D} \equiv \mathbf{L}^\infty(\mu)$, and note that $\mathbf{D}$ is a form core by Lemma 3.2 (or by Lemma 2.1).

(ii) If μ has support in the set defined in (4.24a), the ith marginal of the process X_t makes a one-dimensional random walk in the environment given by $\{a_i(0, ..., 0, x_i, 0, ..., 0), x_i \in \mathbb{Z}\}$ (x_i is the ith coordinate) distributed according to μ. Thus, by (4.22), $D_{ii} = 2\langle 1/a_i(0)\rangle_\mu^{-1}$. If μ concentrates on the set given by (4.24b), $X_{i,t}$ is a martingale. Thus,

$$D_{ii} = \lim_{t \to 0} \frac{1}{t} E_\mu[X_{i,t}^2] = 2\langle a_i(0) \rangle_\mu$$

If $D_{ii} = 2\langle a_i(0)\rangle$, then by (4.21), $(\varphi_i, \mathbf{L}^{-1}\varphi_j) = 0$, so in particular $(\varphi_i, \mathbf{L}^{-1}\varphi_i) \equiv \langle(\mathbf{L}^{-1/2}\varphi_i)^2\rangle = 0$. This implies that $\varphi_i = 0$ μ-a.s. Therefore, since μ is translation invariant, it must concentrate on the set given by (4.24b).

(iii) Suppose μ is isotropic. Then $\mathbf{D}$ is invariant under all rotations $\mathbf{R} = \mathbf{R}^{ij}$ by $\pi/2$ in a coordinate 2-plane. Thus, if v is an eigenvector for $\mathbf{D}$, $\mathbf{R}v$ is an eigenvector belonging to the same eigenvalue. Since $\{\mathbf{R}^{ij}v \mid i, j = 1,..., d\}$ spans, it follows that $\mathbf{D}$ is a multiple of the identity.

Now suppose μ is reflection invariant. Then $\mathbf{D}$ is reflection invariant, and if $v = (v_1,..., v_i,..., v_d)$ is a eigenvector for $\mathbf{D}$, then $\mathbf{R}_i v \equiv (v_1,..., -v_i,..., v_d)$ is an eigenvector belonging to the same eigenvalue. It follows that every coordinate vector e_i is an eigenvector: Let v be an eigenvector such that $v_i \neq 0$. Then $e_i = (v - \mathbf{R}_i v)/(2v_i)$. Thus, $\mathbf{D}$ is diagonal. ∎

4.2. Diffusion in Percolation Regime

In this subsection we will establish diffusive behavior for a random walk in the infinite cluster of the bond percolation model. So, we consider the model described so far, but we will restrict our considerations to the two-dimensional lattice.

The main result of this subsection is the following theorem.

Theorem 4.7. Let μ be a measure on $\aleph(\mathbb{Z}^2)$ satisfying:

(i) Under μ, $\{a_e(x): |e| = 1, x \in \mathbb{Z}^2\}$ is a family of independent identically distributed random variables.

(ii) $p \equiv \mu(\{a_e(0) > 0\}) < 1$ and $p > 1/2$.

(iii) $\mu(\{\sup_{x\in\mathbb{Z}^2} a_e(x) \leqslant b\}) = 1$ for some $b > 0$.

Let [see Eqs. (4.5) and (4.6)]

$$\aleph^* = \{a \in \aleph(\mathbb{Z}^2): \ |W(0, a)| = \infty\} \tag{4.25a}$$

$$\theta_p \equiv \mu(\aleph^*) \tag{4.25b}$$

(in ref. 24 it is proven that $\theta_p > 0$), and

$$\mu^*(\cdot) \equiv \mu(\cdot \mid \aleph^*) \tag{4.25c}$$

For $a \in \aleph$, let $X_t(a)$, $t \in \mathbb{R}$, be the jump Markov process with state space $\mathbb{Z}^2$ and transition probabilities defined in Eq. (4.2). Then the following hold:

1. The following limits exist and are finite, for all $i, j = 1, 2$:

$$\lim_{t\to\infty} \frac{1}{t} E_\mu(X_{i,t} X_{j,t}) \equiv D_{ij}, \qquad \lim_{t\to\infty} \frac{1}{t} E_{\mu^*}(X_{i,t} X_{j,t}) \equiv D^*_{ij}$$

2. $\varepsilon X_{\varepsilon^{-2}t} \to W_{\mathbf{D}^*}(t)$ weakly in μ^*-measure. Here $W_{\mathbf{D}^*}(t)$ is the two-dimensional Brownian motion with diffusion matrix $\mathbf{D}^* = \{D^*_{ij}\}$.

3. $\mathbf{D} = \{D_{ij}\}$ and $\mathbf{D}^*$ are diagonal, i.e., $\mathbf{D} = D\mathbf{I}$ and $\mathbf{D}^* = D^*\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Furthermore,

$$D^* < \infty, \qquad \theta_p \mathbf{D}^* = \mathbf{D} \tag{4.26a}$$

and D can be written as follows:

$$\tfrac{1}{2}D = \langle a_{e_1}(0)\rangle - \int_0^\infty dt\, \langle [a_{e_1}(0) - a_{e_1}(-e_1)]\, E_a[a_{e_1}(X_t) - a_{e_1}(X_t - e_1)]\rangle \tag{4.26b}$$

4. $D^* > 0$ ∎ (4.26c)

Remark 4.8. We prove statements 1–3 using the general Theorem 2.2. For this part of the proof only ergodicity is not an obvious property. We prove ergodicity in Lemma 4.9 below. The difficult part is the proof of statement 4: we prove $D^* > 0$ by relating D^* to the limit of the "effective current" in a finite box. Then the result follows from Kesten's proof of the fact that the limit of the effective current is strictly positive.

Lemma 4.9. The measure μ^* [see (4.25c)] is ergodic with respect to the process $a(t)$, $t \in \mathbb{R}$ (see Definition 4.1), i.e., with respect to the process of the environment seen from the traveling particle. ∎

Proof. The proof is based on the fact[2] that there exists only one infinite cluster, μ-a.s., and on the ergodicity of the random environment under translations.

The first observation is that the set $\aleph^*$ is invariant with respect to the time evolution; i.e., if $a \in \aleph^*$, then obviously $a(t) \in \aleph^*$, $\forall t > 0$. Let $A \subset \aleph^*$ be a nontrivial time-invariant set; i.e., assume

$$a \in A \Rightarrow a(t) \in A \qquad \forall t > 0 \quad \text{and} \quad \mu(A) > 0 \tag{4.27}$$

Ergodicity will follow upon showing that $\mu(A) = \mu(\aleph^*) = \theta_p$.

From (4.27) it follows that

$$\forall a \in A \qquad S_{-x} a \in A \qquad \text{if} \quad x \in W(0, a) \tag{4.28}$$

In fact, if there exists an $x \in W(0, a)$ for which $S_{-x} a \notin A$, then for those trajectories for which there exists a $t > 0$ such that $X_t(a) = x$ we would have $a(t) \notin A$. Note that the set of these trajectories has positive probability.

Define

$$B = \{a \in \aleph\colon \exists x \in \mathbb{Z}^2 \quad S_{-x} a \in A\} \tag{4.29}$$

Clearly, B contains the set A. Since μ is ergodic with respect to the translations, B is invariant under translations, and $\mu(B) \geqslant \mu(A) > 0$, we have that

$$\mu(B) = 1 \tag{4.30}$$

We will show that

$$B \cap \aleph^* \subset A \tag{4.31}$$

Observe that since $A \subset B \cap \aleph^*$, by (4.31) and (4.30), $\mu(A) = \mu(\aleph^*)$, concluding the proof.

To prove (4.31), we observe that $a \in B \cap \aleph^* \Rightarrow \exists x \in \mathbb{Z}^2$ such that (i) $S_{-x} a \in A$ and (ii) $S_x(S_{-x} a) = a \in \aleph^*$.

Since there is only one infinite cluster,[2] (i) + (ii) imply that $x \in W(0, a)$. By Eq. (4.28), $S_x(S_{-x} a) = a \in A$. ∎

Proof of Theorem 4.7: Statements 1–3. Statements 1 and 2 in Theorem 4.7 are an easy consequence of Theorem 2.2. In fact, first of all we observe that the operator $\mathbf{L}$ given by

$$\mathbf{L} = \sum_i \mathbf{L}_i \tag{4.32a}$$

$$\mathbf{L}_i f(a) = a_i(0)(\nabla_i^+ f)(a) - a_i(-e_i)(\nabla_i^- f)(a) \tag{4.32b}$$

$$(\nabla_i^+ f)(a) \equiv f(S_{-e_i} a) - f(a) \tag{4.32c}$$

$$(\nabla_i^- f)(a) \equiv f(a) - f(S_{e_i} a) \tag{4.32d}$$

is the generator of the process $a(t)$ (the environment as seen from the traveling particle).

The four hypotheses of Theorem 2.2 are satisfied:

1. *Reversibility and ergodicity.* Ergodicity has been proven in Lemma 4.9, while reversibility follows from the fact that the operator $\mathbf{L}$ is symmetric on $\mathbf{L}^2(\mu^*)$, i.e.,

$$(f, \mathbf{L} g)_{\mu^*} = (g, \mathbf{L} f)_{\mu^*} \qquad \forall f,\ g \in \mathbf{L}^2(\mu^*) \tag{4.33}$$

Equation (4.33) follows from the translation invariance of the measure μ and the identity

$$a_i(0)\, 1\{a \in \aleph^*\} = a_i(0)\, 1\{a \in \aleph^*,\ S_{-e_i} a \in \aleph^*\}$$

2. *Antisymmetry.* The variable X_t = the algebraic number of shifts during the time interval $[0, t]$ is obviously antisymmetric with respect to time reversal.

Since the rates are bounded, one easily checks that the other two hypotheses of Theorem 2.2 are satisfied, i.e., that the following hold:

3. *The mean forward velocity exists.* The following limit exists in $\mathbf{L}^2(P_{\mu^*})$:

$$\lim_{\delta \to 0} \frac{1}{\delta} E_{\mu^*}[X_{i,\delta} \mid F_0] = a_i(0, 0) - a_i(-e_i, 0) \equiv \varphi_i(a(0)) \tag{4.34}$$

4. X_δ is *square-integrable* and

$$\lim_{\delta \to 0} \frac{1}{\delta} E_{\mu^*}[X_{i,\delta}^2] = 2\langle a_i(0)\rangle_{\mu^*} \tag{4.35}$$

Therefore, from Theorem 2.2 we can conclude that the process $\varepsilon X_{\varepsilon^{-2}t} \to W_{\mathbf{D}^*}(t)$ weakly in μ^*-measure and the matrix $\mathbf{D}^*$ is given by

$$\begin{aligned} D_{ij}^* &= \lim_{t \to \infty} \frac{1}{t} E_{\mu^*}[X_{i,t} X_{j,t}] \\ &= 2\langle a_i(0)\rangle_{\mu^*} \delta_{ij} - 2 \int_0^\infty dt \langle \varphi_i(a)\, E_a[\varphi_j(a(t))]\rangle_{\mu^*} \end{aligned} \tag{4.36}$$

where $\varphi_i(a)$ is defined in (4.34).

To prove (4.26b), we show that

$$\lim_{t \to \infty} \frac{1}{t} E_\mu[X_{i,t} X_{j,t}] = \theta_p \lim_{t \to \infty} \frac{1}{t} E_{\mu^*}[X_{i,t} X_{j,t}] \tag{4.37}$$

and

$$D_{ij} = \lim_{t \to \infty} \frac{1}{t} E_\mu[X_{i,t} X_{j,t}] = 2\langle a_i(0)\rangle - 2 \int_0^\infty dt\, \langle \varphi_i(a)\, E_a[\varphi_j(a(t))] \rangle_\mu \tag{4.38}$$

To prove (4.37), we observe that

$$\begin{aligned} \frac{1}{t} E_\mu[X_{i,t} X_{j,t}] &= \frac{1}{t} \int d\mu(a)\, 1\{a \in \aleph^*\}\, E_a[X_{i,t}(a)\, X_{j,t}(a)] \\ &\quad + \frac{1}{t} \int d\mu(a)\, 1\{a \notin \aleph^*\}\, E_a[X_{i,t}(a)\, X_{j,t}(a)] \end{aligned} \tag{4.39}$$

The limit as $t \to \infty$ of the first term on the rhs of Eq. (4.39) is equal to the rhs of Eq. (4.37). On the other hand, if $a \notin \aleph^*$, then

$$|X_{i,t}(a)\, X_{j,t}(a)| < h(a)^2 \qquad \text{for any} \quad t \geqslant 0$$

where $h(a)$ is the diameter of $W(0, a)$, so that $h(a) < \infty$ if $a \notin \aleph^*$.

It is known (ref. 24, Section 5, Theorem 5.1) that

$$\mu(\{a: a \notin \aleph^*, h(a) > N\}) \leqslant c_1 \exp(-c_2 N)$$

for some positive constants c_1, c_2. Hence, in particular,

$$\int d\mu(a)\, 1\{a \notin \aleph^*\}\, h(a)^2 < C$$

for some positive constant C. Therefore, the second term on the rhs of (4.39) goes to zero as $t \to \infty$.

To prove (4.38), we observe that the mean forward velocity for the process X_t with starting measure μ (instead of μ^*) still exists in $\mathbf{L}^2(\mu)$ and it is equal to φ, i.e.,

$$\lim_{\delta \to 0} \frac{1}{\delta} E_\mu[X_{i,\delta} \mid F_0] = \varphi_i(a(0)) \qquad \text{in} \quad \mathbf{L}^2(P_\mu) \tag{4.40a}$$

and

$$\lim_{\delta \to 0} \frac{1}{\delta} E_\mu[X_{i,\delta}^2] = 2\langle a_i(0)\rangle_\mu \tag{4.40b}$$

Furthermore, μ is reversible but not ergodic. Since in the proof of Theorem 2.2 only reversibility, antisymmetry, and Eqs. (4.40) were used for the computation of $\lim_{t\to\infty} E(X_{i,t}^2)$, Eq. (4.38) can be derived as in Theorem 2.2.

By hypothesis, μ is invariant under rotations of $\pi/2$. Thus, by Theorem 4.6(iv), $\mathbf{D}$ is a multiple of the identity. As a consequence of Eq. (4.26), $\mathbf{D}^*$ is also a multiple of the identity. ∎

We now start the proof that $D^* > 0$. We prove instead the positivity of

$$\tfrac{1}{2} D = \langle a_1(0)\rangle_\mu - \int_0^\infty dt\, \langle \varphi_1(a)\, E_a[\varphi_1(a(t))]\rangle_\mu \tag{4.41a}$$

$$\varphi_1(a) = a_1(0) - a_1(-e_1) \tag{4.41b}$$

Since D^* is proportional to D by Eq. (4.26), this will conclude the proof.

Sketch of the Proof that $D > 0$. The proof that $D > 0$ is given also in ref. 10; we report it here for the sake of completeness; we use the same notation and (sometimes) even the same words used in ref. 10.

We consider periodic configurations with period $2N$ and call D_N the diffusion coefficient [i.e., lim $E(X_{1,t}^2/t)$] for the corresponding random walk. Since with positive probability all the rates are strictly positive, one expects that $D_N > 0$, which can, in fact, be easily shown. The problem left is to prove that (a) as N increases, D_N remains bounded away from 0, and (b) D_N and D are suitably related.

The relationship between D_N and D is easily established by looking at the "explicit" formula for D in Eq. (4.41). In that formula the integrand of the second term is positive and involves $E[\varphi(X_t)]$. For each fixed t the expectation $E_N[\varphi(X_t)]$ related to the periodic configuration converges to $E[\varphi(X_t)]$. On the other hand, the first terms for both D and D_N are identical. So we can use Fatou's lemma to obtain the inequality

$$D \geqslant \limsup D_N \tag{4.42}$$

The positive of $\limsup D_N$ is not clear; here deep percolation problems enter. We prove that

$$\tfrac{1}{2} D_N = \sigma_N \tag{4.43}$$

where σ_N is the current flowing in a box of size N when a unit potential difference is established, which is better thought of as the effective conductivity. Then we use the result of Kesten,[24] who proved that $\liminf \sigma_N > 0$.

Thus, to conclude the proof, we define the periodic random walk, with diffusion coefficient D_N; we establish that $D \geqslant \limsup D_N$; then we define the conductivity σ_N and determine its relationship to D_N.

Definition 4.11. *The periodic random walk.* Let $B_N \subset \mathbb{Z}^2$ be the box

$$B_N = \{x = (x_1, x_2) \in \mathbb{Z}^2\colon -N \leqslant x_1 \leqslant N,\ -N \leqslant x_2 < N\} \tag{4.44}$$

Given a configuration $a \in \aleph$ of rates, let a^* be the periodic rate configuration, with period $2N$ (on bonds linking nearest neighbor points) in the strip

$$\Sigma_N = \{x = (x_1, x_2) \in \mathbb{Z}^2\colon -N \leqslant x_2 < N\} \tag{4.45}$$

which agrees with a on the box B_N, except that the vertical bonds on $x_1 = \pm N$ are given infinite rates. (This is required to obtain a simple relation between D_N and σ_N.)

Since bonds leading out of Σ_N on $\{x_2 = -N\}$ and $\{x_2 = N - 1\}$ do not exist for a^*, Σ_N has "reflecting boundary conditions" on top and bottom. In order to construct a random walk on Σ_N with rates a^*, it is necessary to identify the points with $x_1 = -N$, and similarly for the points with $x_1 = N$, $3N$, $-3N$,..., but as long as we consider only functions which agree on equivalent points, this may safely be ignored, provided we bear in mind that from *all* points on, e.g., $x_1 = N$ the rates a^*

to the point with coordinates $(N+1, x_2)$, $[(N-1, x_2)]$ are now $(2N)^{-1}a_1(-N, x_2)[(2N)^{-1}a_1(N-1, x_2)]$.

Denote by $X_N^*(x, t; a)$ [or simply $X_N^*(t)$ if no confusion is likely] the Markov process on Σ_N defined by the rates a^* starting from x, and let $X_N(t)$ be the x_1 component of $X_N^*(t)$. Denote by $Y_N(x, t; a)$ [or simply $Y_N(t)$] the Markov process $X_N^*(t)/\approx$, where $(x_1, x_2) \approx (x_1', x_2') \Leftrightarrow x_1' - x_1 = 0 \bmod 2N$ and $x_2' = x_2$. One may regard $Y_N(t)$ as a Markov process with state space B_N^*, the box B_N with the left side identified with the right [i.e., $(-N, x_2)$ identified with (N, x_2)]. Let m_N be the uniform distribution on B_N^*, i.e.,

$$m_N(f) = \frac{1}{(2N)^2} \sum_{x \in B_N^*} f(x) \tag{4.46}$$

Then (for fixed a) with respect to m_N, $Y_N(t)$ is clearly a reversible Markov process, in terms of which $X_N^*(t)$, and hence $X_N(t)$, can be realized. Under this realization

$X_N(t) =$ the number of jumps of $Y_N(t)$ in the positive horizontal direction minus the number of jumps of $Y_N(t)$ in the negative horizontal direction during the time interval $[0, t]$

[In the above definition the jump from $N-1$ to $-N \equiv N$ (respectively from $N \equiv -N$ to $N-1$) is considered as a unit positive (respectively negative) jump.] Note that this is well defined if $N \geqslant 2$, which we assume.

We now need some notation. The rates are most naturally regarded as bond functions: $(x, y) \equiv b \to a^*(b)$, the rate for going from x to y. [Though the original $a(b)$ was symmetric, this may no longer be the case for a^*, because of the redefinition of the rates for leaving $x_1 = \cdots, N, 3N, -3N,\ldots$, necessitated by granting infinite "conductivities," i.e. rates, to the vertical bonds on these lines; e.g., $a^*((N, x_2), (N+1, x_2')) = a^*((N+1, x_2'), (N, x_2))$ may fail.] For any oriented bond $b = (x, y)$, let

$$\nabla_b f = f(y) - f(x) \tag{4.47a}$$

and let $(\nabla_i^\pm f)(x) = \pm \nabla_{(x, x \pm e_i)} f$. Let

$$E_x \equiv \text{the set of bonds emanating from } x \tag{4.47b}$$

Then the generator $\mathbf{L}_a$ of the process $X_N^*(x, t; a)$ is given by

$$\mathbf{L}_a f(x) = \sum_{b \in E_x} a^*(b) \nabla_b f(x) \tag{4.48}$$

while the generator for the process $Y_N(x, t; a)$ may be identified with the restriction of $\mathbf{L}_a$ to periodic functions. Here, and from now on, "periodic" means periodic with period $2N$ (in the x_1 direction).

Let

$$\varphi_a^N = \mathbf{L}_a x_1 \tag{4.49a}$$

and note that for $x \in B_N$, $|x_1| \neq N$,

$$\varphi_a^N(x) = (\nabla_1^- a_1)(x) \equiv a_1(x) - a_1(x - e_1) \tag{4.49b}$$

$$\varphi_a^N(-N, x_2) = \frac{1}{2N} \sum_{x_2 = -N}^{N} a_1(-N, x_2) - a_1(N-1, x_2) \tag{4.49c}$$

Denote by P_a^N [resp. $P_{a,x}^N$] the law of the process $Y_N(t)$ with starting measure m_N [resp. δ_x]; E_a^N [$E_{a,x}^N$] denotes the expectation with respect to P_a^N [resp. $P_{a,x}^N$].

Lemma 4.12. For any $a \in \aleph$ fixed the following limit exists and is finite:

$$\lim_{t \to \infty} \frac{1}{t} E_a^N[X_N(t)^2] \equiv D_N(a) \tag{4.50a}$$

Furthermore,

$$D_N(a) = m_N(\psi_a^N) - 2 \int_0^\infty m_N(\varphi_a^N E_{a,x}^N[\varphi_a^N(Y_N(s))])\, ds \tag{4.50b}$$

where φ_N^a is given in (4.49) and

$$\psi_a^N((x_1, x_2)) = a_1(x_1, x_2) + a(x_1 - 1, x_2) \qquad \text{for} \quad x_1 \neq -N \tag{4.50c}$$

$$\psi_a^N((-N, x_2)) = \frac{1}{2N} \sum_{x_2 = -N}^{N-1} [a_1(-N, x_2) + a_1(N-1, x_2)] \quad \blacksquare \tag{4.50d}$$

Proof. We have that

$$\lim_{\delta \to 0} \frac{1}{\delta} E_a^N[X_N(\delta) \mid F_0] = \varphi_a^N(X_N(0)) \tag{4.51a}$$

and

$$\lim_{\delta \to 0} \frac{1}{\delta} E_a^N[X_N(\delta)^2 \mid F_0] = \psi_a^N(X_N(0)) \tag{4.51b}$$

Therefore, using reversibility, Eqs. (4.51), and the fact that the variable $X_N(t)$ is antisymmetric, Eqs. (4.49) and (4.50) can be proven as in Theorem 2.2. ▮

Remark. A fixed configuration a^* induces a partition of B_N into disjoint clusters. Each cluster determines an ergodic component of the process $Y_N(t)$.

It follows easily from Theorem 2.2, with $Y_N(t)$ playing the role of the Markov process ξ_t, that the process $X_N(t)$ converges to a mixture of Brownian motions with (average) diffusion constant $D_N(a)$. Since the rates are infinite in the vertical strip $\{x_1 = N\}$, there is at most one cluster with an interior path connecting $\{x_1 = -N\}$ to $\{x_1 = N\}$. For this component the diffusion coefficient is positive, and for the others it is zero.

Now we are ready to prove the inequality (4.42) between D and D_N.

Proposition 4.13. The following hold:

(i) For any $t > 0$ fixed

$$\lim_{N \to \infty} \langle m_N(\varphi_a^N E_{a,x}^N[\varphi_a^N(Y_N(t))] \rangle_\mu = \langle \varphi_1(a)\, E_a[\varphi_1(a(t))] \rangle_\mu \tag{4.52}$$

where φ^N is defined in Eq. (4.49) and φ_1 is defined in Eq. (4.41b).

(ii) The following inequality holds:

$$D \geqslant \limsup_{N \to \infty} \langle D_N(a) \rangle_\mu \tag{4.53}$$

where D is defined in Eq. (4.41a) and $D_N(a)$ in Eq. (4.50). ▮

Proof. (ii) Part (ii) follows from (i). In fact, if (i) holds, we have

$$\begin{aligned}
&\int_0^\infty dt \langle \varphi_1(a)\, E_a[\varphi_1(a(t))] \rangle_\mu \\
&\quad = \int_0^\infty dt \lim_{N \to \infty} \langle m_N(\varphi_a^N E_{a,x}^N[\varphi_a^N(Y_N(t))]) \rangle_\mu \\
&\quad \leqslant \liminf_{N \to \infty} \int_0^\infty dt\, \langle m_N(\varphi_a^N E_{a,x}^N[\varphi_a^N]) \rangle_\mu \\
&\quad = \liminf_{N \to \infty} \left\langle \int_0^\infty dt\, m_N(\varphi_a^N E_{a,x}^N[\varphi_a^N(Y_N(t))]) \right\rangle_\mu
\end{aligned} \tag{4.54a}$$

where the positivity of the integrands and Fatou's lemma have been used. Furthermore, by the translation invariance of μ,

$$\langle m_N(\psi_a) \rangle_\mu = 2 \langle a_1(0) \rangle_\mu \tag{4.54b}$$

Therefore from the formulas for D [see (4.41)] and D_N [see (4.50)], and from (4.54) it follows that

$$\begin{aligned} D &= 2\langle a_1(0)\rangle - 2\int_0^\infty dt\,\langle \varphi_1(a)\,E_a[\varphi_1(a(t))]\rangle_\mu \\ &\geqslant 2\langle a_1(0)\rangle - 2\liminf_{N\to\infty}\left\langle \int_0^\infty dt\, m_N(\varphi_a^N E_{a,x}^N[\varphi_a^N(Y_N(t))])\right\rangle_\mu \\ &= \limsup_{N\to\infty}\langle D_N(a)\rangle \end{aligned}$$

(i) To prove (4.52), we first note that

$$E_a[\varphi_1(a(t))] = E_{a,0}[a_1(X(t)) - a_1(X(t)-e_1)]$$

and also that

$$\begin{aligned} &E_{S_{-x}a,0}[a_1(X(t)+x) - a_1(X(t)+x-e_1)] \\ &\quad = E_{a,x}[a_1(X(t)) - a_1(X(t)-e_1)] \\ &\quad \equiv E_{a,x}[\varphi_1(X(t))] \end{aligned}$$

Using the abuse of notation $\varphi_i(x) = \varphi_i(S_{-x}a)$, from the translation invariance of the measure μ we have that for any $N > 1$,

$$\langle \varphi_1(a)\,E_a\varphi_1(a(t))\rangle = \frac{1}{(2N)^2}\sum{}^*\,\langle \varphi_1(x)\,E_{a,x}\,\varphi_1(X(t))\rangle \qquad (4.55a)$$

where $\sum^*$ is the sum over $\{x \in B_N\colon |x_1| < N\}$. Furthermore, by definition [see Eq. (4.49)] we have that

$$\varphi^N(x) \equiv \varphi_1(x) \qquad \forall x \in B_N \backslash \{|x_1| = N\} \qquad (4.55b)$$

Let $\tau_N = \tau(N, x, a)$ be the following (possibly infinite) stopping time:

$$\tau_N = \inf\{t\colon |X(t)| = N\} \qquad (4.56a)$$

For $0 < \alpha < 1$ let $B_N(\alpha)$ be the set

$$B_N(\alpha) = \{x \in B_N\colon |x| \leqslant N - N^\alpha\} \qquad (4.57)$$

Then the rhs of Eq. (4.55a) can be written as

$$\frac{1}{(2N)^2}\sum{}^*\,\langle \varphi_1(x)\,E_{a,x}[\varphi_1(X(t))]\rangle_\mu = G_1 + G_2 + G_3$$

where

$$G_1 = \frac{1}{(2N)^2} \sum_{x \in B_N \setminus B_N(\alpha)} \langle \varphi_1(x) \, E_{a,x}[\varphi_1(X(t))] \rangle_\mu$$

$$G_2 = \frac{1}{(2N)^2} \sum_{x \in B_N(\alpha)} \langle \varphi_1(x) \, E_{a,x}[\varphi_1(X(t)) \, 1\{\tau_N \leqslant t\}] \rangle_\mu \tag{4.58a}$$

$$G_3 = \frac{1}{(2N)^2} \sum_{x \in B_N(\alpha)} \langle \varphi_1(x) \, E_{a,x}[\varphi_1(X(t)) \, 1\{\tau_N > t\}] \rangle_\mu$$

The first two terms are bounded as follows:

$$G_1 \leqslant (2b)^2 \, [2N^{1+\alpha}/(2N)^2] \tag{4.58b}$$

$$G_2 \leqslant (2b)^2 ([N - N^\alpha]^2 (2N)^{-2}) \sup_{x \in B_N(\alpha)} P_x(\tau_N \leqslant t) \tag{4.58c}$$

where $\sup_x a_e(x) < b$ has been used.

If $x \in B_N(\alpha)$,

$$P_x[\{\tau_N \leqslant t\}] \leqslant P^*[\{n^* > N - x\}]$$

where n^* is a Poisson-distribution random variable with parameter bt and P^* is its law. Therefore, we have for $x \in B_N(\alpha)$

$$P_x[\{\tau_N \leqslant t\}] \leqslant P^*[\{n^* > N^\alpha\}] \leqslant C_1 \exp(-C_2 N^\alpha) \tag{4.59}$$

for some positive constants C_1 and C_2 depending on t. Therefore, $\alpha < 1$ and (4.59) imply that G_2 goes to zero as $N \to \infty$. Since also G_1 goes to zero, we have that

$$\langle \varphi_1(a) \, E_a[\varphi_1(a(t))] \rangle = \lim_{N \to \infty} \frac{1}{(2N)^2} \sum_{x \in B_N(\alpha)} \langle \varphi_1(x) \, E_{a,x}[\varphi_1(X(t)) \, 1\{\tau_N > t\}] \rangle_\mu \tag{4.60}$$

It is possible to construct a joint representation of the processes $X(t)$ and $Y_N(t)$ in such a way that

$$X(0) = x = Y_N(0), \qquad X(t) = Y_N(t) \qquad \forall t \leqslant \tau_N \tag{4.61}$$

The same computations as in (4.57) show that

$$\lim_{N \to \infty} \langle m_N(\varphi^N{}_a E^N_{a,x}[\varphi^N_a(Y_N(t))]) \rangle_\mu = \lim_{N \to \infty} \left(\frac{1}{2N}\right)^2 \sum_{x \in B_N(\alpha)} \langle \varphi_1(x) \, E_{a,x}[\varphi_1(X(t)) \, 1\{\tau_N > t\}] \rangle_\mu \tag{4.62}$$

where (4.55b) has been used. Now (4.52) follows from Eqs. (4.60)–(4.62). ∎

Definition 4.14. *The effective conductivity.* Let $a \in \aleph$ be fixed. The effective conductivity $\sigma_N(a)$ is the current established across B_N by a unit potential difference. Let V be the corresponding potential. It satisfies

$$\mathbf{L}_a V(x) = 0, \qquad -N < x_1 < N \tag{4.63a}$$

$$V(-N, x_2) = 0, \qquad V(N, x_2) = 1, \qquad -N \leqslant x_2 < N \tag{4.63b}$$

Due to the possible existence of clusters not connected to $\{|x_1| = N\}$, the solution of (4.63) may not be unique. But, since in this clusters V must be constant, all solutions give rise to the same current

$$\sigma_N(a) \equiv \sum_{x_2 = -N}^{N-1} a_1(x_1, x_2)(\nabla_1^+ V)(x_1, x_2) \tag{4.64}$$

where we recall that

$$(\nabla_1^+ V)(x) = V(x + e_1) - V(x)$$

The expression in (4.64) is independent of x_1 provided (4.63) is satisfied, and thus we have that

$$\sigma_N(a) = 2Nm_N(a_1 \nabla_1^+ V) \tag{4.65}$$

Let V^* be the natural extension of V to all of Σ_N, not the periodic extension but one producing periodic current:

$$V^*(x) = V(x) \quad \text{for} \quad x \in B_N, \qquad V^*(x_1 + 2N, x_2) = V(x_1, x_2) + 1$$

Because the left and right boundaries of B_N have been made "superconducting," V^* satisfies

$$\mathbf{L}_a V^*(x) = 0 \qquad \text{for all} \quad x \in \Sigma_N$$

since all that is now required at, e.g., $x_1 = -N$ is that the current into this line, from the left, equals the current out, into the right, which is already guaranteed by the fact that the expression (4.64) for $\sigma_N(a)$ is independent of x_1.

The relationship between $D_N(a)$ and $\sigma_N(a)$ now follows easily:

Proposition 4.15. For any $N \geqslant 2$ and $a \in \aleph$

$$D_N(a) = 2\sigma_N(a) \quad \blacksquare \tag{4.66}$$

Proof. Consider the function

$$f = x_1 - 2NV^* + N \tag{4.67}$$

on Σ_N. The constant N is added so that, for convenience, $f=0$ on $x_1=\pm N$. Since

$$\varphi_a^N = \mathbf{L}_a x_1$$

it follows that f is a periodic function satisfying

$$\mathbf{L}_a f = \varphi_a^N \tag{4.68}$$

By the definition of D_N in Eq. (4.50),

$$D_N(a) = m_N(\psi_a^N) - 2m_N(\varphi_a^N \mathbf{L}_a^{-1} \varphi_a^N) \tag{4.69}$$

The solution of $\mathbf{L}_a g = \varphi_a^N$ is not unique, all solutions being of the form $g = f + h$, where f is given in (4.67) and h is a periodic, $\mathbf{L}_a$-harmonic function, i.e., $\mathbf{L}_a h(x) = 0$ for $x \in B_N^*$.

Observe that [see (4.49)]

$$\begin{aligned}
m_N(\varphi_a^N f) &= \frac{1}{(2N)^2} \sum_{x \in B_N^*} [\varphi_a^N(x) f(x)] \\
&= \frac{1}{(2N)^2} \left\{ \sum_{|x_1| \neq N} [a_1(x+e_1) - a_1(x)] f(x) \right. \\
&\quad \left. + \sum_{|x_1| = N} \frac{1}{2N} \sum_{|y_1| = N} [a_1(y+e_1) + a_1(y)] f(x) \right\} \\
&= \frac{1}{(2N)^2} \sum_{x \in B_N^*} [a_1(x+e_1) - a_1(x)] f(x) \\
&= m_N((\nabla_1^- a_1) f)
\end{aligned} \tag{4.70a}$$

The second equality follows from the fact that $f(x)$ is constant in $\{|x_1| = N\}$. Similar computations show that

$$m_N(\varphi_a^N(f+h)) = m_N((\nabla_1^- a_1)(f+h)) \tag{4.70b}$$

On the other hand, since h is harmonic and periodic, it must be constant in connected clusters. Observe that in clusters connected to $\{|x_1| = N\}$, because of the periodic boundary conditions, we have that

$$m_N((\nabla_1^- a_1) h) = 0 \tag{4.71}$$

On the other hand, in clusters nonconnected to $\{|x_1| = N\}$ we have zero a's on the boundaries and therefore (4.71) also follows.

From (4.71) we have that for all solutions g $(=f+h)$ of $\mathbf{L}_a g = \varphi_a^N$,

$$m_N(\varphi_a^N \mathbf{L}_a^{-1} \varphi_a^N) = m_N((\nabla_1^- a_1) f) \tag{4.72}$$

A computation as in (4.70a) shows that $m_N(\psi_a^N) = 2m_N(a_1)$; thus, (4.70) and (4.72) imply that (4.69) can be written as follows:

$$D_N(a) = 2m_N(a_1) + 2m_N((\nabla_1^- a_1) f) \tag{4.73}$$

"Integrating" by parts and using the translation invariance of m_N (i.e., periodicity), we obtain that

$$\begin{aligned} D_N(a) &= 2m_N(a_1) - 2m_N(a_1 \nabla_1^+ f) \\ &= 2m_N(a_1) - 2m_N(a_1(1 - 2N\nabla_1^+ V)) \\ &= 4Nm_N(a_1 \nabla_1^+ V) \\ &= 2\sigma_N(a) \quad \blacksquare \end{aligned}$$

Proof That $D > 0$. It follows from Proposition 4.13(ii) and Proposition 4.15 that

$$D \geqslant \limsup_{N\to\infty} \langle D_N(a) \rangle_\mu = 2 \limsup_{N\to\infty} \langle \sigma_N(a) \rangle_\mu \geqslant 2 \langle \liminf_{N\to\infty} \sigma_N(a) \rangle_\mu \tag{4.74}$$

Moreover, it has been proven in refs. 18 and 24 that there exist constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 \leqslant \liminf \sigma_N(a) \leqslant \limsup \sigma_N(a) \leqslant c_2, \qquad \mu\text{-a.s.} \tag{4.75}$$

Thus, $D \geqslant 2c_1 > 0$. $\blacksquare$

Remark 4.16. 1. Independence is required only for (4.75).

2. For boxes other than squares, and for higher dimensions, it remains true that

$$D_N(a) = 2\sigma_N(a) = m_N(a_1 \nabla_1^+ V)$$

where $V(x_-) = 2N$, $V(x_+) = 0$, and the box extends from $x_1 = x_-$ to $x_1 = x_+$ and $x_+ - x_- = 2N$. Thus, $\sigma_N(a)$ is the effective local conductivity, the current per unit cross section per unit average electric field.

3. In more than two dimensions, the invariance principle still holds, by Theorem 2.2. The proof of the positivity of the diffusion coefficient D works only for $p > 1/2$, i.e., above the two-dimensional critical probability. In fact, as in Theorem 4.6, it is possible to show that $D(d) > D(2)$ (where d is the dimension) and we know that $D(2) > 0$ if $p > 1/2$. Furthermore,

since Kesten proved that $\liminf \sigma_N(a) > c_1$ in any dimension if $p > 1/2$, we can also repeat the same argument as above to conclude that $D(d) > c_1$.

Proof of (4.22). In $d=1$ dimension, suppose that $\langle 1/a_e(0)\rangle_\mu < \infty$; then

$$D = 2\langle 1/a_e(0)\rangle^{-1} \quad \blacksquare \tag{4.76}$$

Proof. See ref. 23 and references therein. We give here a different, easy approach.

(i) Writing $a(x)$ for $a_{e_1}(x)$, define

$$\begin{aligned} h(x) &= \sum_{y=0}^{x-1} \frac{1}{a(y)} \qquad \text{for} \quad x>0 \\ h(0) &= 0 \\ h(x) &= -\sum_{y=x}^{-1} \frac{1}{a(y)} \qquad \text{for} \quad x<0 \end{aligned} \tag{4.77}$$

and check that $h(X_t)$ is a martingale [$h(x)$ is harmonic] for (almost) any fixed configuration a with $a(x)>0$, $\forall x$. The invariance principle for martingales then gives that $\varepsilon h(X_{\varepsilon^{-2}t})$ converges to a Brownian motion with diffusion constant

$$D_h \equiv \lim_{t\to 0} \frac{1}{t} E[h(X_t)^2] = 2\left\langle \frac{1}{a(0)} \right\rangle \tag{4.78}$$

where the stationarity and the ergodicity of μ have been used.

Now by the ergodicity of μ under translations we have that

$$\frac{h(x)}{x} \xrightarrow[|x|\to\infty]{} \left\langle \frac{1}{a(0)} \right\rangle, \qquad \mu\text{-a.s.}$$

Thus

$$\begin{aligned} X_t &= \frac{X_t}{h(X_t)} h(X_t) \\ &= \left\langle \frac{1}{a(0)} \right\rangle^{-1} h(X_t) + \left(\frac{X_t}{h(X_t)} - \left\langle \frac{1}{a(0)} \right\rangle^{-1} \right) h(X_t) \\ &\equiv \left\langle \frac{1}{a(0)} \right\rangle^{-1} h(X_t) + \varepsilon(t) \end{aligned}$$

where $\varepsilon(t)/\sqrt{t} \to_{t\to\infty} 0$ in probability, and X_t behaves asymptotically like $\langle 1/a(0)\rangle^{-1} h(X_t)$. In particular,

$$D = \langle 1/a(0)\rangle^{-2} D_h = 2\langle 1/a(0)\rangle^{-1}$$

[Note that since $\sup_{0 \leqslant s \leqslant t} \varepsilon(s)/\sqrt{t} \to_{t \to \infty} 0$ in probability, $h(X_t)/\sqrt{t}$ being small whenever $|X_t|$ is not large, the invariance principle for $d=1$ also directly follows.]

There is a second argument leading directly from the expression given in Theorem 2.2 for $\mathbf{D}$ to (4.22); the reader will infer this argument from the final part of Section 6, so we omit it here. ∎

5. SYSTEM OF INFINITELY MANY PARTICLES

In this section we discuss some applications of Theorem 2.2 to jump processes on the lattice. We consider two models: "the exclusion process" and the "stirring-exclusion dynamics." These models are examples of random walks in a random *moving* environment. We show that the "diffusion coefficient" in the case of a random moving environment is not less than the one in the case of the "same" frozen environment.

5.1. Exclusion Process

The state space is $\aleph^* = \{0, 1\}^{\mathbb{Z}^d}$; $\eta \in \aleph^*$ stands for a configuration of particles, $\eta(x)$ denoting the occupation variable for lattice site $x \in \mathbb{Z}^d$ with $\eta(x) = 1$ [0] corresponding to x occupied [empty]. The particles move by random jumps respecting a hard-core exclusion; see, in particular, Eq. (5.6c). The jump rates are functions $c(x, y; \eta) \geqslant 0$ which are translation invariant, uniformly (in x, y, and η) bounded above, short range [i.e., for fixed x and y they depend on only finitely many coordinates of η and $c(0, y; \cdot) \equiv 0$ for all but finitely many y's], and they are not identically zero. We distinguish one of the particles, which we call the tagged particle, and study its motion.

More precisely, the generator of the process described above is given by

$$\mathbf{L}^* f(\eta) = \tfrac{1}{2} \sum_{x, y \in \mathbb{Z}^d} c(x, y; \eta)[f(\eta^{x, y}) - f(\eta)] \tag{5.1}$$

where $\eta^{x, y}$ denotes the configuration η with $\eta(x)$ and $\eta(y)$ interchanged. We refer to Liggett[33] for the existence and the ergodic properties of this process. In order that the process be reversible, we require that the rates $c(\cdot, \cdot)$ satisfy the detailed balance condition

$$c(x, y; \eta) = c(x, y; \eta^{x, y}) \exp\{-[H(\eta^{x, y}) - H(\eta)]\} \tag{5.2a}$$

where, formally,

$$H(\eta) = \sum_{A \subset \mathbb{Z}^d, |A| < \infty} \Phi(A) \prod_{x \in A} \eta(x) \tag{5.2b}$$

with $\{\Phi(A) \mid A \subset \mathbb{Z}^d,\ |A| < \infty\}$ a family of translation-invariant, short-range potentials. We consider the extremal reversible translation-invariant Gibbs measures μ_ρ^*, $\rho \in (0, 1)$, associated with the Hamiltonian H, with the one-body potential chosen in such a way that

$$\mu_\rho^*(\eta(0)) = \rho$$

Let $Y(t)$ be the position of the tagged particle at time t. The role of the reversible Markov process of Theorem 2.2 is played, not by η_t, but by the process ξ_t defined by

$$\xi_t(x) \equiv \eta_t(Y(t) + x) \qquad \forall t \geqslant 0, \quad \forall x \in \mathbb{Z}^d \tag{5.3}$$

(the process as seen from the tagged particle). By definition

$$\xi_t(0) \equiv 1 \qquad \forall t \geqslant 0 \tag{5.4}$$

Furthermore, it is easy to see that ξ_t is the Markov process with state space

$$\aleph = \{\xi \in \aleph^*;\ \xi(0) = 1\} \tag{5.5}$$

and generator $\mathbf{L}$ given by

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 \tag{5.6a}$$

$$\mathbf{L}_1 f(\xi) = \frac{1}{2} \sum_{x, y \neq 0} c(x, y; \xi)[f(\xi^{x,y}) - f(\xi)] \tag{5.6b}$$

$$\mathbf{L}_2 f(\xi) = \sum_x c(0, x; \xi)[1 - \xi(x)][f(S_{-x}\xi^{0,x}) - f(\xi)] \tag{5.6c}$$

where S_x is the shift by $x \in \mathbb{Z}^d$.

Thus the generator $\mathbf{L}$ has a decomposition into a sum of two generators $\mathbf{L}_1$ and $\mathbf{L}_2$ as discussed in Section 3. $\mathbf{L}_2$ generates the shifts of the environment caused by the jump of the tagged particle and $\mathbf{L}_1$ generates the "motion" of the environment.

It is not difficult to see that the Gibbs measures

$$\mu_\rho(\cdot) \equiv \mu_\rho^*(\cdot \mid \xi(0) = 1) \tag{5.7}$$

are reversible and ergodic for ξ_t. Let $X(t)$ be the (vectorial) sum of shifts of the system during the time interval $[0, t]$. Clearly, $X(t) = Y(t)$ in distribution.

Theorem 5.1. The process $\varepsilon X(\varepsilon^{-2}t)$, $t \geqslant 0$, converges as $\varepsilon \to 0$ weakly in μ_ρ-measure to a Brownian motion with finite diffusion matrix $\mathbf{D} = (D_{ij}\colon i, j \in \{1, \ldots, d\})$ given by

$$D_{ij} = \delta_{ij} \sum_x x_i^2 \mu_\rho(c(0, x; \xi)(1 - \xi(x))$$
$$- 2 \int_0^\infty ds \sum_{x, y} x_i y_j \mu_\rho(c(0, x; \xi)(1 - \xi(x)) E_\xi[c(0, y; \xi_s)(1 - \xi_s(y))]) \tag{5.8}$$

where E_ξ denotes the expectation with respect to the process with starting configuration ξ.

Furthermore, if the functions $c(x, y; \xi)$ and $H(\xi)$ and the measure μ_ρ are invariant under reflection in the coordinate hyperplanes, then $\mathbf{D}$ is diagonal; if they are rotation (by $\pi/2$) invariant, $\mathbf{D}$ is a multiple of the identity. ▮

Theorem 5.1 does not preclude the possibility that $\mathbf{D}$ is degenerate.

Proof. We apply Theorem 2.2. Reversibility follows from the condition (5.2). In order to establish ergodicity it is sufficient to show that if $(f, \mathbf{L}f) = 0$, then f is constant μ-a.s.[26] Using reversibility, we have that

$$(f, (-\mathbf{L})f) = \mu\left(\sum_{x, y} c(x, y; \xi) \, |f(\xi^{x, y}) - f(\xi)|^2\right) + (f, (-\mathbf{L}_2)f)$$

Hence $(f, \mathbf{L}f) = 0$ implies that f is invariant under the permutations $\xi \to \xi^{x, y}$ for all x, y. Hence, f is measurable with respect to the tail field at infinity, which is trivial. Hence f is constant. The other hypotheses are easily verified. Finally, we have

$$X_i(t) = \int_0^t \varphi_i(\xi_\tau) \, d\tau + M_i(t)$$
$$\varphi_i(\xi) = \sum_x x_i c(0, x; \xi)[1 - \xi(x)] \tag{5.9}$$
$$E[M_i(t) \, M_k(t)] = \delta_{i,k} t \sum_x x_i^2 \mu_\rho(c(0, x; \xi)[1 - \xi(x)])$$

from which (5.8) follows. [(5.9) follows from standard computations for Markov processes; see, e.g., ref. 46.]

The same argument as for Theorem 4.6(iii) gives the properties of $\mathbf{D}$ in the case of reflection or rotation invariance. ▮

Remark. As far as we know, it is not known, in general, whether or not $\mathbf{D}$ is degenerate. In ref. 26 some results are obtained for the case in which the rates $c(x, y; \xi)$ are constant in ξ, i.e., $c(x, y; \xi) = p(x, y)$ with $p(x, y) = p(y, x)$. The resulting model is called the simple exclusion process.[33] For this model it is proven in ref. 24 that $\mathbf{D}$ is not degenerate if either the dimension is larger than one or $d=1$ and $p(x, y)$ does not vanish for at least four values of y. In $d=1$ in the nearest neighbor case, the behavior is quite different. Arratia[6] proved that $t^{-1/4}X(t)$ converges as $t \to \infty$ in distribution to a Gaussian random variable with mean zero and variance $(2/\pi)(1-\rho)\rho$. (See also ref. 48.)

5.2. The "Stirring-Exclusion" Process

In this subsection we consider the lattice analog of a model introduced in ref. 31 (see also ref. 26) for the ideal gas.

The intuitive description of the model is as follows: Given an initial configuration of particles in $\mathbb{Z}^d$ (the state space is $\{0, 1\}^{\mathbb{Z}^d}$), each particle performs a simple symmetric (continuous-time) random walk on the lattice obeying the exclusion condition: jumps to occupied sites are suppressed. Furthermore, when two particles are at nearest neighbor sites they interchange their positions (like "stirring" particles) with rate r. Just as in the previous subsection, we put a particle at the origin and study its motion. As before, $Y(t)$ is the position of this tagged particle at time t. Since the particles (other than the tagged one) are indistinguishable (i.e., we do not distinguish whether they interchange positions or not) we regard only the tagged particle as interacting via stirring-exclusion as described above.

To apply Theorem 2.2, we represent as usual $Y(t)$ in terms of the process ξ_t as seen from the tagged particle. The state space of this process is the set $\aleph$ defined in (5.5), and the generator is given by

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 \tag{5.10a}$$

$$\mathbf{L}_1 f(\xi) = \tfrac{1}{2} \sum_{\substack{x \neq 0,\, y \neq 0 \\ |x-y|=1}} [f(\xi^{x,y}) - f(\xi)] \tag{5.10b}$$

$$\mathbf{L}_2 f(\xi) = \sum_{|y|=1} \{[1-\xi(y)] + r\xi(y)\}[f(S_{-y}\xi^{0,y}) - f(\xi)] \tag{5.10c}$$

We denote by T_t $[T_{i,t}]$ the semigroup generated by $\mathbf{L}[\mathbf{L}_i]$ on $\aleph$. The Bernoulli measures μ_ρ of parameter ρ conditioned to have a particle at the origin (Palm measures) are reversible for this process.

Denote by $X(t)$ the (vectorial) sum of shifts of the environment up to time t. Clearly $X(t) = Y(t)$ in distribution.

Theorem 5.2. The process $\varepsilon X(\varepsilon^{-2} t)$, $t \geqslant 0$, converges as $\varepsilon \to 0$ weakly in μ_ρ-measure to a Brownian motion with finite diffusion matrix $\mathbf{D} = (D_{ij}\colon i, j \in \{1,..., d\})$ given by Eq. (5.11) below.

If $r > 0$, then $\mathbf{D}$ is not degenerate. $\mathbf{D} = D\mathbf{I}$($\mathbf{I}$ is the identity), where D is given by

$$D \equiv D(\rho, r, d) = 2[1 - \rho(1-r)] - 2\int_0^\infty ds\, \langle \varphi(\xi)\, E_\xi[\varphi(\xi_s)]\rangle \tag{5.11}$$

where

$$\varphi(\xi) = (1-r)[\xi(-e_1) - \xi(e_1)] \tag{5.12}$$

$\langle \cdot \rangle$ denotes expectation with respect to μ_ρ, and e_1 is the unit vector in the positive 1-direction.

Furthermore,

$$D(\rho, r, d) \quad \text{is decreasing in } d \tag{5.13}$$

$$D(\rho, r, d) \geqslant D_{\text{frozen}} \tag{5.14}$$

(the diffusion coefficient when only the tagged particle moves) and

$$2[\rho r^{-1} + (1-\rho)]^{-1} \leqslant D(\rho, r, d) \leqslant 2[1 - \rho(1-r)] \quad \blacksquare \tag{5.15}$$

Proof. The hypotheses of Theorem 2.2 are satisfied. One easily verifies that the drift is given by (5.12) and the average quadratic variation of the martingale in the decomposition of $X(t)$ is $(1 - \rho + \rho r)t$. This gives (5.11).

To prove Eq. (5.14), we observe that, by definition [see Eq. (5.10)], the generator $\mathbf{L}$ splits into the sum of two generators $\mathbf{L}_1$ and $\mathbf{L}_2$. The process with generator $\mathbf{L}_2$ is the "environment process" for the tagged particle moving in a "frozen" environment. It is easy to check that the same measure μ_ρ is reversible and ergodic for this process. We are thus in the framework of Theorem 2.2; hence, we can conclude that $X^*(t)$, the (vectorial) sum of shifts of the environment up to time t for this process, converges under the usual scaling to a Brownian motion with diffusion matrix $\mathbf{D}_{\text{frozen}} = D_{\text{frozen}}\mathbf{I}$, where

$$D_{\text{frozen}} \equiv 2[1 - \rho(1-r)] + 2(\varphi, \mathbf{L}_2^{-1}\varphi)_{\mu_\rho} \tag{5.16}$$

with φ given by (5.12).

Since the hypotheses of Lemma 3.1 are satisfied, (5.14) follows. Furthermore, it is not hard to realize that the process of a particle starting at the origin and moving in a frozen configuration η is distributed identically to a random walk in a random environment a, as in Section 4; for $d = 1$, a is given by

$$a_{x-1,x} = \begin{cases} r & \text{if} \quad x \geqslant 1 \text{ and } \eta(x) = 1 \text{ or } x \leqslant 0 \text{ and } \eta(x-1) = 1 \\ 1 & \text{otherwise} \end{cases}$$

With this in mind, we compute D_{frozen} for $d=1$ using (4.22):

$$D_{\text{frozen}} = 2[\rho r^{-1} + (1-\rho)]^{-1} \tag{5.17}$$

Equation (5.17) gives the lower bound in (5.15), once (5.13) has been established.

The proof of (5.13) is similar to the one for (5.14): Observe that

$$\mathbf{L} = \sum_{i=1}^{d} \mathbf{L}_i$$

where now $\mathbf{L}_i$ is the generator of the jumps in the i direction, and apply Lemma 3.1.

Finally, we observe that the upper bound in (5.15) is the first term in the expression (5.11) for D, the second one being negative. ∎

6. DIFFUSION IN A RANDOM POTENTIAL AND SELF-DIFFUSION FOR INTERACTING BROWNIAN PARTICLES

In this section we consider two models in the continuum similar to the lattice models discussed in previous sections. These are: diffusion of a Brownian particle in a random potential, and the diffusion of a tagged particle in an infinite system of interacting Brownian particles (self-diffusion). The first case has been treated in Papanicolaou and Varadhan,[36] the second case in Guo.[19] We are able to obtain the invariance principle with minimal hypotheses (essentially just those necessary to make the process in question well defined). After stating and proving the invariance principle for each model, we discuss various inequalities between the diffusion constants of the models, leading to the conclusion that they are all positive. The proof of this fact requires that some technical points concerning quadratic forms be cleared up; we state what we need without giving a full proof.

By a diffusion in a random potential we mean a particle moving in $\mathbb{R}^d$ according to the equations

$$dX(t) = -\nabla V(X(t))\, dt + dW(t) \tag{6.1a}$$

$$X(0) = 0 \tag{6.1b}$$

In (6.1), $W(t)$ is a standard Brownian motion (with diffusion matrix the identity matrix), starting at zero, and $V(x) \equiv V(x, \omega)$ is a *random potential.* By this we mean that $V(x, \omega)$ is a (C^1) function of x and a measurable function of ω, ω in some probability space (Ω, Σ, μ). We assume that this process is translation invariant and ergodic under translation. Translation acts naturally on V by

$$(S_y V)(x, \omega) = V(x - y, \omega)$$

If Σ is generated by $V(x, \cdot)$, $x \in \mathbb{R}^d$, we can assume S_y is a measurable and measure-preserving transformation on Ω. We assume that the V process is jointly measurable.

If, in addition, there is a function $C(\omega)$, positive and finite a.e. on Ω, for which

$$|x \cdot \nabla V(x, \omega)| \leqslant C(\omega)(1 + |x|^2) \tag{6.2}$$

a.e., which we assume, then solutions $X(t)$ of (6.1) exist for a.e. V, and $X(t)$ is a diffusion.

An example (which plays the leading role in our discussion of the inequalities on diffusion constants) is the following. Let $U(x)$ be a symmetric, positive, compactly-supported C^∞, superstable[41] pair potential in $\mathbb{R}^d$. Define

$$V(x) = \sum_i U(x - y_i) \tag{6.3}$$

where $(y_i)_{i=1,\ldots,\infty}$ is a (locally-finite) point process on $\mathbb{R}^d$ distributed according to an (extremal translation-invariant) Gibbs state of the potential U. Using Ruelle's estimates, one sees that for some function $C_1(\omega)$ finite a.e.

$$|\nabla V(x, \omega)| \leqslant C_1(\omega)[1 + \log(1 + |x|)^{1/d}] \tag{6.4}$$

a.e., so (6.2) holds.

Following the by now standard procedure, we introduce the environment as seen from the moving particle. This is the process

$$V(t) \equiv S_{-X(V,t)} V \tag{6.5}$$

where $X(V, t)$ $[= X(t)]$ is the solution of (6.1). The corresponding semigroup acting on bounded measurable functions of V is

$$T_t F(V) = E_V^0[F(S_{-X(t)} V)] \tag{6.6}$$

[E_V^0 is the expectation with respect to the solution of (6.1)]. As usual, we have to prove that $V(t)$ is reversible and ergodic.

To prove reversibility, assume that

$$Z \equiv \langle \exp\{-2V(0)\} \rangle_\mu \equiv \int \mu(dV) \exp\{-2V(0)\} < \infty \tag{6.7}$$

and let

$$\mu^*(dV) = Z^{-1} \exp\{-2V(0)\}\, \mu(dV) \tag{6.8}$$

We prove that μ^* is a reversible measure. In order to do this, let $p_t(x \mid y; V)$ be the density of the transition function of the process governed by (6.1). The reversibility with respect to the measure $\exp\{-2V(x)\}\, dx$ is expressed by

$$\exp\{-2V(x)\}\, p_t(y \mid x; V) = \exp\{-2V(y)\}\, p_t(x \mid y; V) \tag{6.9}$$

which holds for all x, y. Let F and G be bounded measurable on Ω; we have

$$\begin{aligned} \langle GT_t F \rangle &\equiv \int \mu^*(dV)\, G(V)\, E_V^0[F(S_{-X(t)} V)] \\ &= Z^{-1} \int \mu(dV) \int dy \exp\{-2V(0)\}\, G(V)\, p_t(y \mid 0; V)\, F(S_{-y} V) \end{aligned} \tag{6.10}$$

Using Fubini's theorem and making the change of variables $V \to S_{-y} V$, we obtain

$$= Z^{-1} \int dy \int \mu(dV) \exp\{-2V(-y)\}\, G(S_y V)\, p_t(y \mid 0; S_y V)\, F(V) \tag{6.11}$$

Next, using the invariance property

$$p_t(y \mid x; V) = p_t(y + z \mid x + z; S_z V) \tag{6.12}$$

we obtain in (6.11)

$$\begin{aligned} &= Z^{-1} \int dy \int \mu(dV) \exp\{-2V(-y)\}\, G(S_y V)\, p_t(0 \mid -y; V)\, F(V) \\ &= \langle FT_t G \rangle_{\mu^*} \end{aligned} \tag{6.13}$$

In the last equality, (6.9) has been used.

Ergodicity of the process with invariant measure μ^* now follows from the full support of the process governed by (6.1) in $\mathbb{R}^d$, for μ-a.e. V, and from the ergodicity of μ under translations (note that $\mu^* \ll \mu$).

We next have to recover the motion of the particle from the process $V(t)$. Rewriting (6.1) in terms of the process $V(t)$, we have

$$dX(t) = -\nabla V(0; t)\, dt + dW(t) \tag{6.14}$$

or in integrated form

$$X(t) = \int_0^t \nabla V(0; s)\, ds + W(s) \tag{6.15}$$

There remains only the question whether $X(t)$ is suitably measurable (see Section 2) with respect to the process $V(t)$ (and antisymmetric). Clearly, the former is false if $\mu(\{V\equiv\text{const}\})=1$, but if μ is supported by non-periodic configurations, this is more or less clear. In any case one can always consider the process $(V(t), y(t))$ in place of $V(t)$, where $y(t)$ moves identically to $X(t)$ but is taken mod 1, so $y(t)$ remains in the unit cube. This process is reversible (with invariant measure μ^* times the Lebesgue measure on the unit cube), and clearly $X(t)$ is $\sigma(y(s), 0\leqslant s\leqslant t)$-measurable and antisymmetric.

We are ready to state the following theorem.

Theorem 6.1. Let $V(x)$ be a translation-invariant and ergodic random potential with $\exp\{-2V(0)\}\in \mathbf{L}^1(\mu)$ and $\varphi\equiv\nabla V(0)$ in $\mathbf{L}^1(\mu^*)$, μ^* given by (6.8). Then $\varepsilon X(\varepsilon^{-2}t)$ [$X(t)$ given by (6.15)] converges (weakly in μ-measure) to a Brownian motion with diffusion matrix given by

$$D_{ij}=\delta_{ij}+2(\varphi_i, \mathbf{L}^{-1}\varphi_j)_{\mu^*} \quad \blacksquare \tag{6.16}$$

Proof. The theorem follows from Theorem 2.2. Note that $\mu\ll\mu^*$ and convergence in measure is preserved by replacing the measure by one absolutely continuous with respect to it. ▌

Remarks. (i) Without further assumptions it is not immediately clear that the generator of the environment process is a self-adjoint extension of

$$\mathbf{L}F=[-\nabla V(0)]\cdot\nabla F+(1/2)\,\nabla^2 F \tag{6.17}$$

[where ∇ is the generator of translations on $\mathbf{L}^2(\mu)$] on some suitable domain, or that the quadratic form associated with the process is given by

$$\|F\|_1^2=(1/2)\int \mu^*(dV)\,|\nabla F|^2 \tag{6.18}$$

on $\mathbf{D}(\nabla)$ [as one would conjecture from (6.17)], but we expect that this is true. With some further assumptions on V these facts can be established. For instance, if V is bounded below by a constant μ-a.s. and C^2, one can differentiate in (6.6) with respect to time on the domain $\{F\colon F, \nabla F$, and $\nabla^2 F$ are in $\mathbf{L}^2(\mu)\}$, use the fact that $p_t(x\,|\,0; V\}$ solves the backward equation, and integrate by parts to establish (6.17) [and similarly on the domain $\{F\colon F$ and ∇F are in $\mathbf{L}^2(\mu)\}$ to establish (6.18)]. One can then use (6.6) again to show that these domains are invariant under the transition semigroup of the process. Lemma 3.2 then implies that these domains determine the generator and quadratic form, respectively.

(ii) One can treat just as easily the motion governed by

$$dX(t)=\nabla V(X(t))\,dt+[2V(X(t)]^{1/2}\,dW(t) \tag{6.19}$$

for a random potential $V(x, \omega)$ with $V \geqslant 0$. In this case the translation-invariant measure on environments V is itself the reversible measure, and $\mathbf{L}$ is a self-adjoint extension of $\nabla \cdot V(0)\nabla$. One assumes that $V(0)$ and $\nabla V(0)$ are in $\mathbf{L}^1(\mu)$, and Theorem 2.2 applies verbatim. (This is the continuum analog of the model treated in Section 4.)

We consider next the self-diffusion of a tagged particle in a system of interacting Brownian particles. The evolution of this system—with interaction given by the gradient of a pair potential U—is governed by the system of equations $(i = 1, 2,...)$

$$dX_i(t) = -\sum_{j \neq i} \nabla U(X_i(t) - X_j(t))\, dt + dW_i(t) \tag{6.20}$$

In (6.20), $X_i(t) \in \mathbb{R}^d$, U is a compactly-supported, C^2, positive, superstable (even) pair potential (as in refs. 30 and 31), and the $W_i(t)$ are independent standard Brownian motions. If (6.20) defines an evolution (in the space of locally-finite infinite-particle configurations), then the translation-invariant Gibbs states of the potential U with inverse temperature $\beta = 2$ and arbitrary finite density should be reversible, and among these the extremal ones will be ergodic.

Lang[30,31] and Shiga[43] have given a proof of the existence of the "equilibrium dynamics." This means that (strong, unique) solutions exist for (6.20) for a set of initial configurations of full measure with respect to any $\beta = 2$ Gibbs state (for a given density). Fritz[16] has shown the existence of nonequilibrium dynamics [that is, he described explicitly a "large" set of initial configurations for which (6.20) can be solved] for $d \leqslant 4$. Previous constructions for $d = 1$ were given by Rost[42] and Lippner.[34]

There are several approaches to constructing the environment process seen from the tagged particle. If we have available solutions of (6.20) for an explicitly described set of initial conditions (of labeled particles, 0 labeling the tagged particle), we can proceed by change of variables, as follows. We solve (6.20) for a solution $(X_0(t), X_1(t),...)$ and define $Y_i(t) = X_i(t) - X_0(t)$, $i \geqslant 1$. Then (6.20) becomes $(i = 1, 2,...)$

$$dX_0(t) = \sum_{j \geqslant 1} \nabla U(Y_j(t))\, dt + dW_0(t) \tag{6.21}$$

$$\begin{aligned} dY_i(t) = &\left\{ -\sum_{j \geqslant 1, j \neq i} \nabla U(Y_i(t) - Y_j(t)) - \nabla U(Y_i(t)) - \sum_{j \geqslant 1} \nabla U(Y_j(t)) \right\} dt \\ &+ d[W_i(t) - W_0(t)] \end{aligned} \tag{6.22}$$

Note that in (6.22) $Y_i(t)$, $i \geqslant 1$, has an autonomous Markovian evolution. This evolution is invariant under permutation of the labels, and solutions of (6.22) should exist for a set $\mathbf{X}^*$ of locally finite initial configurations. Furthermore, if μ^* is a $\beta=2$ Gibbs state of the (formal) Hamiltonian

$$\tfrac{1}{2}\sum_{j\neq i} U(y_i-y_j)+\sum_i U(y_i) \tag{6.23}$$

$\mu^*(\mathbf{X}^*)=1$ and μ^* should be reversible. Then (6.21) with $X_0(0)=0$ defines the motion $X(t)$ of the tagged particle, which is suitably determined by the Y motion: Note that by (6.22), $W_0(t)$ can be expressed as the limit of a sequence of averages involving only Y's. We are thus in the familiar framework of Theorem 2.2.

There are other constructions of this process—directly from the Lang process—which are perhaps more "intrinsic." The Palm process construction (see, e.g., Harris[20]) should lead directly to the Y-process with Palm measure μ^* as reversible measure. One would have then only the additional technical question of whether the tagged particle's motion can be recovered from the environment process, i.e., whether $X(t)$ is measurable with respect to $Y(s)=\{Y_i(s)\}$, $0\leqslant s\leqslant t$.

We remark after stating Theorem 6.2 on another construction beginning directly with quadratic forms. Since it is not the purpose of our paper, we do not present more details of any of these constructions or prove that they all lead to the "same" process.

Theorem 6.2. Let $X(t)$ be the position of a tagged particle in the system of interacting Brownian particles, as defined above. Let

$$\varphi=\sum_{i\geqslant 1}\nabla U(Y_i) \tag{6.24}$$

be in $\mathbf{L}^1(\mu^*)$, μ^* as above. Then $\varepsilon X(\varepsilon^{-2}t)$ tends weakly in μ^*-measure to a Brownian motion with diffusion matrix

$$D_{ij}=\delta_{ij}+2(\varphi_i,\mathbf{L}^{-1}\varphi_j)_{\mu^*} \tag{6.25}$$

where $\mathbf{L}$ is the strong $\mathbf{L}^2(\mu^*)$ generator of the environment (Y) process.

Remarks. 1. From (6.22) and Ito's lemma the (formal) generator of the environment process is easily calculated; one finds that

$$\mathbf{L}=\mathbf{L}_1+\mathbf{L}_2 \tag{6.26}$$

where

$$\mathbf{L}_1 f(y)=-\sum_{j\geqslant 1}\sum_{i\geqslant 1}\nabla U(y_j)\cdot\nabla_i f+\tfrac{1}{2}\sum_i \Delta_i f+\sum_{i\neq j}\nabla_i\cdot\nabla_j f \tag{6.27a}$$

($\nabla_i \equiv \nabla_{y_j}$; $\Delta_i = \nabla_i \cdot \nabla_i$) and

$$\mathbf{L}_2 f(y) = -\sum_{i \geqslant 1} \sum_{j \neq i} \nabla U(y_i - y_j) \cdot \nabla_i f - \sum_i \nabla U(y_i) \cdot \nabla_i f + \tfrac{1}{2} \sum_i \Delta_i f \tag{6.27b}$$

Techniques similar to those of Fritz[16] should allow one to show at least that $\mathbf{L}$ is an extension of the formal generator acting on local functions. We conjecture that the quadratic form of the environment process is given by

$$\|f\|_1^2 = \langle |\mathbf{D}f|^2 \rangle_{\mu^*} + \left\langle \sum_{i \geqslant 1} |\nabla_i f|^2 \right\rangle_{\mu^*} \tag{6.28}$$

where $\mathbf{D}$ is the generator of an overall translation ($\mathbf{D} = \sum_{i \geqslant 1} \nabla_i$, formally), and the quadratic form domain consists exactly of those f for which both terms in (6.28) are finite. One could prove this conjecture using Fritz's techniques by proving that a suitable subdomain is invariant under the semigroup (using Lemma 3.2), but we have not carried out the details. Of course if one can show that $\mathbf{L}$ is the Friedrich extension of the formal generator (or better, that it is self-adjoint on local functions), then (6.28) follows easily by integrating by parts. For the Lang process, Rost[42] has shown how to prove that the second term in (6.28) is the correct quadratic form: use the fact that Lang's process is a limit of finite-volume processes reversible for μ^* whose quadratic forms are increasing, and apply a theorem of Faris.[13]

2. One can simply *define* a quadratic form by the sum (6.28) on the intersection of the domains. Clearly, the form defined in this way is Dirichlet and so is associated with a Markovian semigroup on $\mathbf{L}^2(\mu^*)$. This semigroup is given by the Trotter product formula for the semigroups associated with $\mathbf{L}_1$ and $\mathbf{L}_2$, making clear that the associated process has the correct probabilistic interpretation. One can take this process as the underlying environment process. We have not investigated in detail the connection with the previous constructions.

Proof of Theorem 6.2. The theorem follows from Theorem 2.2. ∎

We now discuss inequalities between the various diffusion constants, using ideas already explained in Section 3. The only lack of rigor in our discussion concerns the exact quadratic form domains of the various processes. Sufficient for the conclusions to hold would be, e.g., that C^2-local functions are form cores of all the quadratic forms in the discussion. We expect that for the interacting case this will be settled in the near future.

As explained in Section 3, upper bounds are immediate. For lower bounds, we first note that $\mathbf{D}_{\mathrm{RP}}(d)$, the diffusion matrix for the diffusion in a random potential in dimension d, is a lower bound for the corresponding matrix $\mathbf{D}_{\mathrm{IB}}(d)$ for the interacting Brownian particles, provided $d\mu^* = Z^{-1} \exp[-2 \sum_i U(y_i)]\, d\mu$, where μ is translation invariant and we take the potential in the first model to be the "frozen environment" of the second model. That is, the random potential is given by (6.3) with (y_i) distributed according to μ and with the pair potential U the same as for the interacting case. The inequality follows from the decomposition (6.28) and Lemma 3.1.

Further inequalities between diffusion matrices for different dimensions follow. For example, in the random potential case fix $d \geqslant 1$ and a measure μ_d on potentials in $\mathbb{R}^d$. Let V_{d-1} denote the restriction of a potential to the hyperplane $\{x_d = 0\}$, and let μ_{d-1} be the induced measure on environments in $\mathbb{R}^{d-1}$. Then for $\alpha = 1,..., d-1$

$$\mathbf{D}_{\mathrm{RP}}(\mu_d)_{\alpha,\alpha} \equiv (\mathbf{D}_{\mathrm{RP}}(d))_{\alpha,\alpha} \geqslant (\mathbf{D}_{\mathrm{RP}}(d-1))_{\alpha,\alpha} \equiv \mathbf{D}_{\mathrm{RP}}(\mu_{d-1})_{\alpha,\alpha} \tag{6.29}$$

Inequality (6.29) follows from the decomposition

$$Q_d = Q_d^* + Q_{d-1}. \tag{6.30}$$

of the quadratic form of the d-dimensional generator, reflecting the decomposition of the process into motion in the x_d direction and the motions in the other directions. Now Lemma 3.1 applies.

In the case of a one-dimensional random potential, one can compute D explicitly; the result is well known.[37] There are two arguments leading to this result. First, define for each environment V

$$\begin{aligned} h_V(x) &= \int_0^x \exp\{2V(y)\}\, dy \qquad \text{for} \quad x \geqslant 0 \\ h_V(x) &= -\int_x^0 \exp\{2V(y)\}\, dy \qquad \text{for} \quad x < 0 \end{aligned} \tag{6.31}$$

Assume that

$$\langle \exp\{2V(0)\} \rangle_\mu < \infty \tag{6.32}$$

One checks easily that

$$\mathbf{L}_V h_V \equiv 0 \tag{6.33}$$

where $\mathbf{L}_V$ is the generator of the motion $X_V(t)$. Furthermore, by Ito's lemma

$$E_V^0[h_V^2[X_V(t)] = E_V^0\left[\int_0^t \exp\{4V(X_V(t'))\}\, dt'\right] \tag{6.34}$$

Let $h(t) = h_V(X_V(t))$ with V distributed according to μ^*. From (6.33) it is easy to see that $h(t)$ is a martingale with stationary increments, and from (6.34)

$$\begin{aligned} E_{\mu^*}[h(t)^2] &= t\langle \exp\{4V(0)\}\rangle_{\mu^*} \\ &= t\langle \exp\{2V(0)\}\rangle_\mu \langle \exp\{-2V(0)\}\rangle_\mu^{-1} \end{aligned} \tag{6.35}$$

Since, by the ergodic theorem, for μ-a.e. V

$$x^{-1} h_V(x) \to \langle \exp\{2V(0)\}\rangle_\mu \qquad \text{as} \quad |x| \to \infty \tag{6.36}$$

an argument similar to that used in Section 4 [proof of Eq. (4.22)] gives

$$D = [\langle \exp\{2V(0)\}\rangle_\mu \langle \exp\{-2V(0)\}\rangle_\mu]^{-1} \tag{6.37}$$

which is the formula we wished to establish.

Formula (6.37) for D follows also in one dimension directly from the general expression given in (6.16) if (6.32) holds and if we can establish that the Dirichlet form is given by (6.18) on a domain $\mathbf{D} \subset \mathbf{D}(\nabla)$ [∇ considered as an operator on $\mathbf{L}^2(\mu)$] on which this form is finite, and which is furthermore a core of $\mathbf{D}(\nabla)$ and invariant under T_t (and therefore a core of the Dirichlet form by Lemma 3.2). As remarked before, this can be checked using formula (6.6) for T_t if V is C^2 and b ounded below.

Given (6.18), we can compute an explicit expression for $\|\varphi\|_{-1}$, where $\varphi = -\nabla V(0)$. Note that, since $\mathbf{D}$ is a form core, from its definition

$$\begin{aligned} \|\varphi\|_{-1} &= \sup \frac{|Z^{-1} \int d\mu \exp[-2V(0)]\, \varphi F|}{\{(2Z)^{-1} \int d\mu \exp[-2V(0)]\, |\nabla F|^2\}^{1/2}} \\ &= (2Z)^{-1/2} \sup \frac{|\int d\mu \exp[-2V(0)]\, \nabla F|}{\{\int d\mu \exp[-2V(0)]\, |\nabla F|^2\}^{1/2}} \end{aligned} \tag{6.38}$$

where the supremum is over F in $\mathbf{D}$ with nonzero Dirichlet norm. Since ∇ generates a unitary semigroup in $\mathbf{L}^2(\mu)$, and $\mathbf{D}$ is a core of $\mathbf{D}(\nabla)$, $\{\nabla F\colon F \in \mathbf{D}\}$ is dense in the orthogonal complement of 1 in $\mathbf{L}^2(\mu)$. Therefore $\{\exp[-V(0)]\, \nabla F\}$ is dense in the orthogonal complement of $\exp[V(0)]$, so by (6.40),

$$\begin{aligned} \|\varphi\|_{-1}^2 &= (1/2) Z^{-1} \|(1 - P_{\exp[V(0)]}) \exp[-V(0)]\|_{2,\mu}^2 \\ &= (1/2)\{1 - \langle \exp[2V(0)]\rangle_\mu^{-1} \langle \exp[-2V(0)]\rangle_\mu^{-1}\} \end{aligned}$$

and again we obtain (6.37). ∎

We summarize the preceding discussion in the following theorem.

Theorem 6.3. Assume that

$$\langle \exp\{2V(0)\}\rangle_\mu < \infty$$

For $\alpha = 1,..., d$ we have that

$$\begin{aligned}(\mathbf{D}_{\rm IB}(d))_{\alpha,\alpha} &\geqslant (\mathbf{D}_{\rm RP}(d))_{\alpha,\alpha} \geqslant \cdots \geqslant (\mathbf{D}_{\rm RP}(1))_{\alpha,\alpha}\\ &= [\langle \exp\{2V_\alpha(0)\}\rangle_{\mu(\alpha,1)} \langle \exp\{-2V_\alpha(0)\}\rangle_{\mu(\alpha,1)}]^{-1} > 0 \quad (6.39)\end{aligned}$$

where V_α is the environment induced on the space $\{x_\beta = 0,\ \beta \neq \alpha\}$ with induced measure $\mu(\alpha, 1)$.

ACKNOWLEDGMENTS

We wish to thank E. Presutti and H. Spohn: without their support and ideas the present work would not have materialized. We thank J. L. Lebowitz and H. Rost for many helpful discussions and suggestions, and a referee for criticism and suggestions which led to an improved version of this paper. Three of us (A.D.M., P.A.F., and W.D.W.) thank the Mathematics Department of Rutgers and J. L. Lebowitz for friendly hospitality while this work was being done. Finally, one of us (P.A.F.) thanks the Dipartimento di Matematica delle Universita di L'Aquila e Roma I for their hospitality while this was being written. This work was partially supported by CNR grant CNR PS AITM, CNPq grant 311074-84 MA, and NSF grant DMS 85-12505.

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