The remaining integral does not contribute to the pole term.

Collecting together all the terms, we have

$$J_1=\frac{1}{\varepsilon}\int_0^1\frac{dx\,x(5x-3)}{\sqrt{1-x}}+\frac{4}{\varepsilon}+O(1)=\frac{16}{3}\frac{1}{\varepsilon}+O(1).$$

Calculating similarly the asymptotic behavior of J_2, we finally obtain $I_1 = 8\pi^3 m/\varepsilon$. For I_2, integrating by means of Eqs. (A.1) and (A.2), we obtain $I_2 = 2\pi^3/m\varepsilon$. Thus,

$$\text{P.P.}\,\tilde{\Gamma}_1^{(1)}=-\frac{m}{\pi^3\varepsilon}+\frac{A_2 m}{8\pi^2\varepsilon}.$$

Besides the graph of Fig. 4c, it is necessary to take into account the contribution to $\Gamma_1^{(1)}$ of the graphs of Figs. 4d and 4e, in which the cross denotes the vertices corresponding to P.P. $\Gamma_3^{(1)}$ and P.P. $\Gamma_2^{(1)}$. The calculation shows that their contribution to $\Gamma_1^{(1)}$ completely compensates the contribution of the graph of Fig. 4c and, therefore, $a_{01} = 0$.

LITERATURE CITED

1. H. E. Stanley, Introduction to Phase Transitions and Critical Phenomena, Oxford (1971).
2. S. L. Adler and R. F. Dashen, Current Algebras and Applications to Particle Physics, New York (1968).
3. A. A. Slavnov, Teor. Mat. Fiz., **10**, 305 (1972).
4. L. D. Faddeev, Dokl. Akad. Nauk SSSR, **210**, 807 (1973).
5. T. R. H. Skyrme, Proc. R. Soc., **A247**, 260 (1958); J. Math. Phys., **12**, 1735 (1971).
6. L. D. Faddeev, in: Nonlocal, Nonlinear, and Unrenormalizable Field Theories [in Russian], JINR, Dubna (1976).
7. A. Belavin, A. Polyakov, A. Schwartz, and Yu. Tyupkin, Phys. Lett., **B59**, 85 (1975).
8. K. Wilson, Phys. Rev., **D10**, 2445 (1974).
9. A. M. Polyakov, Phys. Lett., **B59**, 79 (1975); A. A. Migdal, Zh. Eksp. Teor. Fiz., **69**, 1457 (1975); E. Brézin, J. Zinn-Justin, and Y. G. Le Guillou, Preprint DPh-T/76.58, Saclay (1976).
10. E. Brézin and J. Zinn-Justin, Preprint DPh-T/76.27, Saclay (1976); J. Zinn-Justin, in: Nonlocal, Nonlinear, and Unrenormalizable Field Theories [in Russian], JINR, Dubna (1976).
11. I. Ya. Aref'eva, Teor. Mat. Fiz., **29**, 147 (1976).
12. S. Weinberg, Phys. Rev. Lett., **17**, 616 (1966).
13. D. Bessis and J. Zinn-Justin, Preprint DPh-T/71, Saclay (1971).
14. G. 't Hooft and M. Veltman, Preprint 73-9, CERN (1973).

GENERALLY COVARIANT THEORIES OF GAUGE FIELDS ON SUPERSPACE

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Different variants of a generally covariant theory of superfields with nonzero values of the curvature and torsion tensors are discussed from the point of view of the holonomy group. A study is made of the example of a Lagrangian that is quadratic in the torsion tensor in the linear approximation of weak fields with interaction switched off and includes free fields with spin 2 and Rarita–Schwinger fields with spin 3/2.

1. After the introduction of the concept of supersymmetry [1-3] and the construction of the simplest supersymmetric theories as field theories on superspace [4-7] the problem arose of finding gauge and generally covariant generalizations of these theories. The searches for such generalizations have been made and are currently being made in essentially two directions, depending on the type of projection operators that

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are used to separate the physical and unphysical states.

In one of these directions, one uses projection operators that directly select a value of the superspin of particles in such a way that superfunctions with other values of the superspin play the role of gauge transformations. In this direction, we have the important papers of Ferrara and Zumino [8] and Salam and Strathdee [9], in which a supersymmetric generalization of the Yang–Mills theory was obtained, and of Ogievetskii and Sokatchev [10], who use a vector superfield to construct a supersymmetric generalization of the theory of gravitation. The recent papers of Freedman, van Niewenhuizen, and Ferrara and Deser and Zumino [11], in which a unified supersymmetric description of the gravitational field and the Rarita–Schwinger field is proposed, belong, as regards their physical content, to this direction, although from the technical side the construction of gauge transformations with projection operators corresponding to a definite value of the superspin was not undertaken.

The other direction in the search for gauge and generally covariant generalizations of supersymmetric theories proceeds from attempts to construct various kinds of connection on superspaces in close analogy with the schemes that use connections to describe the Yang–Mills and Einstein fields in ordinary space. The projection operators that then arise are related to the operation of exterior differentiation. Such an approach was proposed for the first time in [12] with the only exceptional feature that the action integral was taken in the form of an invariant integral over a four-dimensional surface in superspace. It is obvious that the possibility is not precluded of taking the action integral in the form of an invariant integral over a six-dimensional surface in superspace.

The direct generalization of Einstein's equations to the case of an eight-dimensional Riemannian superspace was proposed by Arnowitt, Nath, and Zumino [13, 14] and somewhat later for the case of an arbitrary connection with nonzero curvature and torsion tensors by Zumino [15] and the present authors [16].

The basic ideas relating to the possibility of constructing the theory of a gauge superfield in closer analogy with the ordinary theory of Yang–Mills fields were recently formulated in the review [17] by Ogievetskii and Mesincescu, and a concrete variant of such a theory was proposed in [18] by Ogievetskii and Sokatchev. As is noted in [18], the proposed variant of the theory of the gauge superfield is clearly not free of difficulties associated with the appearance of redundant field components when an interaction is present.

As will be shown in the present paper, similar difficulties arise when one considers generally covariant theories, and the question of their consistent elimination is at the present open.

In this paper, in more detail than in [15, 16], we consider the formalism for introducing connections on superspace, emphasizing the importance for physical applications of the specification of a definite holonomy group.* We consider examples of generally covariant action integrals and some of their consequences.

2. An arbitrary superspace with coordinates $z^a=(x^\mu, \varphi^\alpha, \varphi^{\dot\alpha})$ is described by the Cartan structure equations [16]

$$d\wedge\omega^A(\delta)+\omega^B(\delta)\wedge\Gamma_B{}^A(d)={}^1\!/_2\,\omega^B(\delta)\wedge\omega^C(d)S_{CB}{}^A, \tag{1}$$

$$d\wedge\Gamma_A{}^B(\delta)+\Gamma_A{}^C(\delta)\wedge\Gamma_C{}^B(d)={}^1\!/_2\,\omega^C(\delta)\wedge\omega^D(d)R_{DC,\,A}{}^B, \tag{2}$$

where the differential forms $\omega^A(d)=dz^a\omega_a{}^A$ define the transition from the natural local coordinate system associated with the coordinates $z^a=(x^\mu, \varphi^\alpha, \varphi^{\dot\alpha})$ to an arbitrary local frame (we shall denote the components of tensors in an arbitrary local frame and in a natural local coordinate system by upper case and lower case letters, respectively), $\Gamma_A{}^B(d)$ are the differential forms of the connection, and $S_{CB}{}^A$ and $R_{DC,\,A}{}^B$ are the torsion and curvature tensors.† The differentiations and products of forms in the expressions (1) and (2) are exterior and are defined in the same way as for the case of ordinary spaces by alternation of the differentials d and δ.

The form of the torsion and curvature tensors follows from Eqs. (1) and (2):

$$S_{CB}{}^A=[(-)^{C(B+f)}X_B{}^fX_C{}^l\partial_l\omega_f{}^A+\Gamma_{CB}{}^A]-(-)^{CB}[B\leftrightarrow C], \tag{3}$$

$$R_{DC,\,B}{}^A=[(-)^{D(C+f)}X_C{}^fX_D{}^l\partial_l\Gamma_{fB}{}^A+(-)^{D(C+B+F)}\Gamma_{CB}{}^F\Gamma_{DF}{}^A]-(-)^{CD}[C\leftrightarrow D], \tag{4}$$

* The holonomy group is the group of transformations of the frame when it is taken in parallel transport around an infinitesimally small closed contour.

† The invariant contraction with respect to tensor indices is defined in the Appendix.

where the matrix X_A^a is the inverse of the matrix $\omega_a{}^A$: $\omega_a{}^A X_A{}^b = \delta_a{}^b$, and $\Gamma_{fB}{}^A$ and $\Gamma_{FB}{}^A$ are the coefficients of the decomposition of the differential forms of the connection with respect to dz' and $\omega^F(d)$, respectively,

$$\Gamma_B{}^A(d) = dz^f \Gamma_{fB}{}^A = \omega^F(d) \Gamma_{FB}{}^A.$$

The torsion and curvature tensors and the forms of the connection, as follows from the Cartan equations (1) and (2), have the following properties of symmetry with respect to permutation of indices:

$$S_{CB}{}^A = -(-)^{CB} S_{BC}{}^A, \quad R_{DC,\,B}{}^A = -(-)^{DC} R_{CD,\,B}{}^A,$$

and with respect to Hermitian conjugation:

$$(\Gamma_B{}^A(d))^+ = \Gamma_{\dot{B}}{}^{\dot{A}}(d)\,(-)^{B(A+B)}, \quad (S_{CB}{}^A)^+ = S_{\dot{C}\dot{B}}{}^{\dot{A}}(-)^{C(A+B+C)+B(A+B)}, \quad (R_{DC.\,B}{}^A)^+ = R_{\dot{D}\dot{C},\,\dot{B}}{}^{\dot{A}}(-)^{(D+C+B)(A+B)+(D+C)C+D},$$

where $\dot{A} = (\mu, \dot{\alpha}, \alpha)$, if $A = (\mu, \alpha, \dot{\alpha})$. Under spatial reflection, the spinor subscripts of $\Gamma_B{}^A(d)$, $S_{CB}{}^A$, and $R_{DC,\,B}{}^A$ transform with sign opposite to the ordinary transformation of spinors, whereas the spinor superscripts transform in accordance with the usual law.

We define the covariant differential of vectors in the form

$$DV^A = dz^f D_f V^A = dV^A + V^B \Gamma_B{}^A(d), \quad DV_A = dz^f D_f V_A = dV_A - \Gamma_A{}^B(d) V_B,$$

where $D_f V^A$ and $D_f V_A$ are covariant derivatives:

$$D_f V^A = \partial_f V^A + (-)^{fB} V^B \Gamma_{fB}{}^A, \quad D_f V_A = \partial_f V_A - \Gamma_{fA}{}^B V_B.$$

These formulas can be readily generalized to the case of covariant differentials and derivatives of products of vectors and a tensor and have the form, for example, in the case of the covariant derivative of the product of three vectors

$$D_f(V^A U_B W_C) = (D_f V^A) U_B W_C + (-)^{fA} V^A (D_f U_B) W_C + (-)^{f(A+B)} V^A U_B (D_f W_C)$$

or, in the case of the covariant derivative of a third-rank tensor,

$$D_f V^A{}_{BC} = \partial_f V^A{}_{BC} + (-)^{F(A+F)} \Gamma_{fF}{}^A V^F{}_{BC} - (-)^{A(B+F)} \Gamma_{fB}{}^F V^A{}_{FC} - (-)^{(A+B)(C+F)} \Gamma_{fC}{}^F V^A{}_{BF}.$$

The choice of the holonomy group of the space is the decisive question for the determination of the physical content of a theory that uses the idea of a generally covariant treatment. As an example, we can point out the treatment of Einstein's unified field theory with torsion, and also Kibble's theory [19]. The holonomy group of the space uniquely determines the geometry of the tangent spaces, for which it is the group of motions. In particular, it determines the metric properties of the tangent spaces. The holonomy group actually coincides with the gauge group of transformations, since in the cases when the gauge group is larger than the holonomy group the redundant transformations of the gauge group can be eliminated by the choice of the gauge by virtue of the holonomy theorem [20, 21]. All that we have said applies equally to the choice of the holonomy group of superspace when the generally covariant theory is generalized to the case of superspace. Considering the generally covariant generalizations of supersymmetric theories, in the choice of the holonomy group of superspace, it is obviously sensible to be guided by correspondence, on the one hand, with Einstein's theory, and, on the other, with the ordinary variants of supersymmetric theories. In the case of Einstein's theory, the holonomy group of Riemannian space is the Lorentz group, the invariant tensors of which include the Minkowski metric tensor $g_{\mu\nu}$ and the Dirac γ^μ matrices. The presence of the latter lead to the possibility of first-order equations for fermions.

In the generalization of the generally covariant theory to the case of superspace carried out in [13, 14], the holonomy group of the Riemannian superspace is the pseudo-orthogonal group on eight-dimensional superspace, which leaves invariant the metric tensor

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & \varepsilon_{\alpha\beta} & 0 \\ 0 & 0 & \varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix},$$

where $g_{\mu\nu}$ is the ordinary Minkowski metric tensor and $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$ are matrices that lower spinor indices. The detailed treatment leads to certain difficulties: first, second-order equations are obtained for bosons and for fermions, and, second, ordinary supersymmetric theories do not satisfy the resulting equations. The first of these difficulties is associated with the circumstance that the holonomy group of the Riemannian superspace chosen in [13, 14] does not have invariant tensors of the type of the Dirac γ^μ matrices, or at least not for the representations used in [13, 14]. The second difficulty indicates that such a holonomy group does not correspond to ordinary supersymmetric theories.

In our paper, we adopt as holonomy group of the considered superspace the Poincaré group augmented by translations of the spinor variables. The invariant tensors of this holonomy group are the Minkowski metric tensor $g_{\mu\nu}$, the tensors $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\dot\alpha\dot\beta}$, which serve to lower spinor indices, and the tensors $\sigma_{\alpha\dot\beta}^{\mu}=(1, -\boldsymbol{\sigma})_{\alpha\dot\beta}$, where $\boldsymbol{\sigma}$ are Pauli matrices. The metric tensors g_{AB} and g^{AB}, which raise and lower indices in accordance with the rule

$$V^A=g^{AB}V_B \text{ and } V_A=V^B g_{BA},$$

can be chosen in our case in the form

$$g_{AB}=\begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & -a\varepsilon_{\alpha\beta} & 0 \\ 0 & 0 & -b\varepsilon_{\dot\alpha\dot\beta}\end{pmatrix},\quad g^{AB}=\begin{pmatrix} g^{\mu\nu} & 0 & 0 \\ 0 & \varepsilon^{\alpha\beta}/a & 0 \\ 0 & 0 & \varepsilon^{\dot\alpha\dot\beta}/b\end{pmatrix},$$

where a and b are arbitrary constants. The metric tensor has the obvious symmetry properties

$$g_{AB}=(-)^{AB}g_{BA},$$

and satisfies the relations

$$g^{CA}g_{AB}=\delta^C{}_B \text{ and } g_{BA}g^{AC}(-)^A=\delta_B{}^C,$$

where

$$\delta_B{}^C=(-)^{CB}\delta_B{}^C=\begin{pmatrix}\delta_\mu{}^\nu & 0 & 0\\ 0 & \delta_\alpha{}^\beta & 0 \\ 0 & 0 & \delta_{\dot\alpha}{}^{\dot\beta}\end{pmatrix},$$

and $\delta_\mu{}^\nu$, $\delta_\alpha{}^\beta$, and $\delta_{\dot\alpha}{}^{\dot\beta}$ are ordinary Kronecker symbols.

Because the homogeneous holonomy group of our superspace is in our case the Lorentz group, only the components $\Gamma_\mu{}^\nu(d)$, $\Gamma_\alpha{}^\beta(d)$, and $\Gamma_{\dot\alpha}{}^{\dot\beta}(d)$ of the differential form of the connection are nonzero, and they satisfy relations that are a consequence of the constancy of the tensors g_{AB} and $\sigma_{\alpha\dot\beta}{}^\mu$ for the given group:

$$\Gamma_{AB}(d)+(-)^{AB}\Gamma_{BA}(d)=0, \tag{5}$$

$$\Gamma_{\mu\nu}(d)=(\sigma_{\mu\nu})_\alpha{}^\beta\Gamma_\beta{}^\alpha(d)+(\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}\Gamma_{\dot\beta}{}^{\dot\alpha}(d), \tag{6}$$

where $\Gamma_{AB}(d)=\Gamma_A{}^C(d)g_{CB}$, and $(\sigma_{\mu\nu})_\alpha{}^\beta$ and $(\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}$ are defined in the Appendix. The relations (24) and (25) lead to the following symmetry properties of the curvature tensor with respect to the second pair of indices: $R_{DC,BA}=-(-)^{AB}R_{DC,AB}$, and the relations

$$R_{DC,\mu\nu}=(\sigma_{\mu\nu})_\alpha{}^\beta R_{DC,\beta}{}^\alpha+(\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}R_{DC,\dot\beta}{}^{\dot\alpha},$$

where $R_{DC,BA}=R_{DC,B}{}^F g_{FA}$.

The invariant action integral for the superfields that determine the differential forms $\omega^A(d)$ and $\Gamma_B^A(d)$ is constructed in the general case in the form

$$\int \mathscr{L}(R,S)\,W\prod_a dz^a, \tag{7}$$

where $\mathscr{L}(R,S)$ is an invariant function of the curvature and torsion tensors* and $W=\det\omega_a{}^A$.

In the case when $\mathscr{L}(R,S)$ is a linear function of the curvature and torsion tensors, the expressions for the variations of these last have the form

$$\widetilde{S_{CB}{}^A W}=\{[-\tilde\omega_C{}^F S_{FB}{}^A+{}^1\!/_2(-)^F\tilde\omega_F{}^F S_{CB}{}^A+{}^1\!/_2 S_{CB}{}^F\tilde\omega_F{}^A+(-)^{AB+F}\tilde\omega_C{}^A S_{BF}{}^F+\tilde\Gamma_{CB}{}^A]-(-)^{BC}[B\leftrightarrow C]\}W, \tag{8}$$

$$\widetilde{R_{DC,B}{}^A W}=\{[-\tilde\omega_D{}^F R_{FC,B}{}^A+{}^1\!/_2(-)^F\tilde\omega_F{}^F R_{DC,B}{}^A-(-)^F S_{DF}{}^F\tilde\Gamma_{CB}{}^A+{}^1\!/_2 S_{DC}{}^F\tilde\Gamma_{FB}{}^A]-(-)^{CD}[C\leftrightarrow D]\}W, \tag{9}$$

where on the right-hand sides of the expressions (8) and (9) we have omitted the terms that are total covariant derivatives. The tilde above quantities denotes their variation, and $\tilde\omega_A{}^B=X_A{}^b\tilde\omega_b{}^B$ and $\Gamma_{BC}{}^A=X_B{}^b\tilde\Gamma_{bC}{}^A$ are covariant variations. In the derivation of (8) and (9), we have used the following expressions for the variations of X_A^a and W:

$$\tilde X_A{}^a=-\tilde\omega_A{}^B X_B{}^a,\quad \tilde W=(-)^F\tilde\omega_F W, \tag{10}$$

and their covariant derivatives:

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* The properties of determinants of matrices that have anticommuting matrix elements is discussed in [13, 14] (see also the Appendix).

$$D_a X_A{}^b = -(-)^{a(A+f)} X_A{}^f (D_a \omega_f{}^F) X_F{}^b,\quad D_a W = (-)^{a(F+f)+F} X_F{}^f (D_a \omega_f{}^F) W.$$

In the linear approximation of weak fields, all quantities are expanded with respect to the ordinary supersymmetric superspace [2, 5]:

$$\omega_a{}^A = \overset{0}{\omega}_a{}^B (\delta_B{}^A + h_B{}^A(z)),\quad X_A{}^a = (\delta_A{}^B - h_A{}^B(z)) \overset{0}{X}_B{}^a,$$

where

$$\overset{0}{\omega}_a{}^A = \begin{pmatrix} \delta_\nu{}^\mu & 0 & 0 \\ ia(\varphi^+\sigma^\mu)_\beta & \delta_\beta{}^\alpha & 0 \\ ia(\sigma^\mu\varphi)_{\dot\beta} & 0 & \delta_{\dot\beta}{}^{\dot\alpha} \end{pmatrix},\quad \begin{matrix} A = (\mu, \alpha, \dot\alpha), \\ a = (\nu, \beta, \dot\beta), \end{matrix}\quad \overset{0}{X}_A{}^a = \begin{pmatrix} \delta_\mu{}^\nu & 0 & 0 \\ -ia(\varphi^+\sigma^\nu)_\alpha & \delta_\alpha{}^\beta & 0 \\ -ia(\sigma^\nu\varphi)_{\dot\alpha} & 0 & \delta_{\dot\alpha}{}^{\dot\beta} \end{pmatrix},\quad \begin{matrix} a = (\nu, \beta, \dot\beta), \\ A = (\mu, \alpha, \dot\alpha). \end{matrix}$$

The matrices $\overset{0}{\omega}_a{}^A$ and $\overset{0}{X}_A{}^a$ are the values of the matrices $\omega_a{}^A$ and $X_A{}^a$ in the case of the ordinary supersymmetric superspace, and the superfields $h_B^A(z)$ describe small deviations from the ordinary supersymmetric superspace. In this approximation, the torsion and curvature tensors (3) and (4) are represented in the form

$$S_{CB}{}^A = \{\overset{0}{D}_C h_B{}^A + (-)^{C(B+f)} [\overset{0}{X}_B{}^f (\overset{0}{D}_C \overset{0}{\omega}_f{}^F)(\delta_F{}^A + h_F{}^A) - (h_B{}^F \overset{0}{X}_F{}^f \overset{0}{D}_C + \overset{0}{X}_B{}^f h_C{}^F \overset{0}{D}_F) \overset{0}{\omega}_f{}^A] + \Gamma_{CB}{}^A\} - (-)^{CB} \{C \leftrightarrow B\},$$

$$R_{DC,B}{}^A = [\overset{0}{D}_D \Gamma_{CB}{}^A + (-)^{D(C+f)} \overset{0}{X}_C{}^f (\overset{0}{D}_D \overset{0}{\omega}_f{}^F) \Gamma_{FB}{}^A] - (-)^{DC} [C \leftrightarrow D],$$

where $\overset{0}{D}_A = \overset{0}{X}_A{}^a \partial_a$ are quantities known in supersymmetry theories as covariant derivatives [6-9], and they anticommute with the generators of the supersymmetry transformations. These quantities satisfy the following commutation relations:

$$\{\overset{0}{D}_\alpha, \overset{0}{D}_{\dot\beta}\} = -2ia\sigma_{\dot\beta\alpha}{}^\mu \partial_\mu.$$

When one considers the generally covariant theory in the tetrad formalism, there are two groups of gauge transformations: a) the group of generally covariant coordinate transformations $z^{a'} = f^{a'}(z)$, b) the group of frame transformations $\omega^{A'}(d) = \omega^B(d) L_B{}^{A'}(z)$. Since the transformations of both gauge groups leave the observable physical quantities unchanged, unimportant unphysical components can be eliminated by the choice of the gauge. In the linear approximation of weak fields, we shall consider infinitesimally small transformations of these groups so as not to destroy the approximation. Under intinitesimally small generally covariant transformations of the coordinates $z^{a'} = z^a + h^a(z)$ the transformation of the differential forms $\omega^A(d) = \overset{0}{\omega}{}^B(d)(\delta_B{}^A + h_B{}^A)$ in the linear approximation reduces simply to a transformation of the small superfields $h_B^A(z)$:

$$h'_B{}^A(z) = h_B{}^A(z) + \delta h_B{}^A(z).$$

The increments $\delta h_B^A(z)$ can be expressed in terms of infinitesimally small functions of the transformations of the coordinates $h^a(z)$ as follows:

$$\delta h_\nu{}^\mu = \partial_\nu H^\mu,\ \delta h_\alpha{}^\mu = \overset{0}{D}_\alpha H^\mu + 2ia\sigma_{\alpha\dot\beta}{}^\mu h^{\dot\beta},\ \delta h_{\dot\alpha}{}^\mu = \overset{0}{D}_{\dot\alpha} H^\mu + 2ia\sigma_{\dot\alpha\beta}{}^\mu h^\beta,\ \delta h_A{}^\alpha = \overset{0}{D}_A h^\alpha,\ \delta h_A{}^{\dot\alpha} = \overset{0}{D}_A h^{\dot\alpha}, \tag{11}$$

where $H^\mu = h^\mu + ia(h^+\sigma^\mu\varphi - \varphi^+\sigma^\mu h)$.

Infinitesimally small frame transformations $\omega^{A'}(d) = \omega^B(d)(\delta_B{}^{A'} + \xi_B{}^{A'}(z))$ lead to the following transformations of the small superfields $h_B^A(z)$:

$$h'_B{}^A(z) = h_B{}^A(z) + \xi_B{}^A(z),$$

where the matrices $\xi_B{}^A(z)$ satisfy the relations $\xi_{BA}(z) + (-)^{AB}\xi_{AB}(z) = 0$, where $\xi_{AB}(z) = \xi_A{}^C(z) g_{CB}$.

3. As an example, let us consider a Lagrangian in the linear approximation of weak fields proportional to the following quadratic combination of the torsion tensor:

$$T_\nu{}^\mu T_\mu{}^\nu - T_\mu{}^\mu T_\nu{}^\nu, \tag{12}$$

with the additional covariant gauge conditions

$$S_{AB}{}^\alpha = S_{AB}{}^{\dot\alpha} = S_{\alpha\beta}{}^\mu = S_{\dot\alpha\dot\beta}{}^\mu = S_{\nu\rho}{}^\mu = 0 \tag{13}$$

and

$$S_{\mu\dot\alpha}{}^\mu = S_{\mu\alpha}{}^\mu = S_{\nu\alpha}{}^\mu \sigma_\mu{}^{\alpha\dot\beta} = S_{\nu\dot\alpha}{}^\mu \sigma_\mu{}^{\dot\alpha\beta} = 0, \tag{14}$$

where $T_\nu{}^\mu = \sigma_\nu{}^{\alpha\dot\beta} T_{\alpha\dot\beta}{}^\mu$, and

$$T_{\alpha\dot\beta}{}^\mu = S_{\alpha\dot\beta}{}^\mu - 2ia\sigma_{\alpha\dot\beta}{}^\mu = 2ia[\sigma_{\alpha\dot\beta}{}^\nu h_\nu{}^\mu - (h_\alpha{}^\gamma \sigma_{\gamma\dot\beta}{}^\mu + h_{\dot\beta}{}^{\dot\gamma} \sigma_{\dot\gamma\alpha}{}^\mu)] + \overset{0}{D}_\alpha h_{\dot\beta}{}^\mu + \overset{0}{D}_{\dot\beta} h_\alpha{}^\mu$$

is the difference between the torsion of the considered superspace in the presence of gauge fields and the constant vector torsion $\overset{0}{S}_{\alpha\dot\beta}{}^{\mu}=2ia\sigma_{\alpha\dot\beta}{}^{\mu}$ of the ordinary supersymmetry superspace in the absence of gauge fields, i.e., $T_{\alpha\dot\beta}{}^{\mu}$ is the contribution of the gauge fields to the torsion $S_{\alpha\dot\beta}{}^{\mu}$.

In the Lagrangian (12) with the additional conditions (13) and (14) we in addition set equal to zero the connections $\Gamma_{AB}{}^{C}$, which corresponds to switching off the interaction and considering only the kinetic parts of the fields in the Lagrangian, i.e., to considering the spectrum of the fields that are present. It is a consequence of the conditions (13) that the superfields $h^A_B(z)$ can be represented in the form

$$h_A{}^{\alpha}=\overset{0}{D}_A\bar h^{\alpha},\ h_A{}^{\dot\alpha}=\overset{0}{D}_A\bar h^{\dot\alpha},\ h_\nu{}^{\mu}=\partial_\nu\bar h^{\mu},\ h_\alpha{}^{\mu}=\overset{0}{D}_\alpha\bar H^{\mu}+2ia\bar h^{\dot\gamma}\sigma_{\dot\gamma\alpha}{}^{\mu},\ h_{\dot\alpha}{}^{\mu}=\overset{0}{D}_{\dot\alpha}(\bar H^{\mu})^{+}+2ia\bar h^{\gamma}\sigma_{\gamma\dot\alpha}{}^{\mu},$$

where $\bar h^{\alpha}, \bar h^{\dot\alpha}=(\bar h^{\alpha})^{+}, \bar h^{\mu}$, and $\bar H^{\mu}$ are certain new superfields.

We use the freedom associated with the gauge transformations (11) and choose a gauge in which $\bar h^{\mu}(z)=0$. Then the vector torsion $T_{\alpha\dot\beta}{}^{\mu}$ in this gauge can be expressed in terms of the new superfields as follows

$$T_{\alpha\dot\beta}{}^{\mu}=\{\overset{0}{D}_\alpha,\ \overset{0}{D}_{\dot\beta}\}X^{\mu}+i[\overset{0}{D}_{\dot\beta},\ \overset{0}{D}_\alpha]Y^{\mu}, \tag{15}$$

where X^{μ} and Y^{μ} are the Hermitian components of the superfield $\bar H^{\mu}=X^{\mu}+iY^{\mu}$. Substituting into (13) the expression (15) for $T_{\alpha\dot\beta}{}^{\mu}$ and using the additional conditions (14), we find that the Lagrangian* can be expressed solely in terms of the single Hermitian superfield $Y^{\mu}(z)$:

$$T_\nu{}^{\mu}T_\mu{}^{\nu}-T_\mu{}^{\mu}T_\nu{}^{\nu}\sim Y^{\mu}\varepsilon_{\mu\nu\rho\lambda}\partial^{\rho}\overset{0}{D}{}^{\alpha}\sigma_{\alpha\dot\beta}{}^{\lambda}\overset{0}{D}{}^{\dot\beta}Y^{\nu}. \tag{16}$$

In deriving the expression (16) for the Lagrangian, we have used the following relation between the σ^{μ} matrices

$$\varepsilon_{\mu\nu\rho\lambda}\bar\sigma^{\lambda}=i(\bar\sigma_\mu\sigma_\rho\bar\sigma_\nu-\bar\sigma_\mu g_{\nu\rho}-\bar\sigma_\nu g_{\mu\rho}+\bar\sigma_\rho g_{\mu\nu}).$$

The action (7) with the Lagrangian (16) can be expressed in terms of ordinary fields by substituting into (16) the decomposition of the superfield $Y^{\mu}(z)$ with respect to the ordinary fields:

$$Y^{\mu}(z)=A^{\mu}(x)+X_\alpha{}^{\mu}(x)\varphi^{\alpha}+\varphi^{\dot\alpha}\chi_{\dot\alpha}{}^{\mu}(x)+B^{\mu}(x)\varphi_\alpha\varphi^{\alpha}+(B^{\mu}(x))^{+}\varphi^{\dot\alpha}\varphi_{\dot\alpha}+a_\nu{}^{\mu}(x)\varphi^{+}\sigma^{\nu}\varphi+\left[\theta_\alpha{}^{\mu}(x)-\frac{ia}{2}(\partial_\nu X^{\mu}(x)^{+}\sigma^{\nu})_\alpha\right]\times$$
$$\varphi^{\alpha}\varphi_{\dot\alpha}\varphi^{\dot\alpha}+\left[\theta_{\dot\alpha}{}^{\mu}(x)+\frac{ia}{2}(\sigma^{\nu}\partial_\nu X^{\mu}(x))_{\dot\alpha}\right]\varphi^{\dot\alpha}\varphi_\alpha\varphi^{\alpha}+{}^{1}/_{4}\,[C^{\mu}(x)+a^{2}\Box A^{\mu}(x)]\,\varphi_\alpha\varphi^{\alpha}\varphi_{\dot\alpha}\varphi^{\dot\alpha}$$

and integrating in (7) with respect to the anticommuting variables in accordance with the rules

$$\int\varphi^{\alpha}\,d\varphi^{\beta}=\varepsilon^{\alpha\beta},\qquad \int d\varphi^{\alpha}=0.$$

In the linear approximation, it is necessary to replace $\det\omega^A_a$ by $\det\overset{0}{\omega}_a{}^{A}=1$. As a result, we obtain

$$\int Y^{\mu}\varepsilon_{\mu\nu\rho\lambda}\partial^{\rho}\overset{0}{D}{}^{\alpha}\sigma_{\alpha\dot\beta}{}^{\lambda}\overset{0}{D}{}^{\dot\beta}Y^{\nu}d^4x\,d^4\varphi=\int(L_2+L_{3/2}+L_1)\,d^4x,$$

where

$$L_2=-a(a^{(\mu\nu)}\Box a_{(\nu\mu)}-2a^{(\mu\rho)}\partial_\rho\partial^{\nu}a_{(\nu\mu)}+2a_\mu{}^{\mu}\partial^{\rho}\partial^{\nu}a_{(\nu\rho)}-a_\mu{}^{\mu}\Box a_\nu{}^{\nu})$$

is the Lagrangian that gives the equations for a free field with spin 2,

$$L_{3/2}=\varepsilon_{\mu\nu\rho\lambda}(\theta^{\mu})^{+}\sigma^{\lambda}\partial^{\rho}\theta^{\nu}$$

the Rarita–Schwinger Lagrangian for a field of spin $^{3}/_{2}$, and

$$L_1=a(a^{[\mu\nu]}\Box a_{[\mu\nu]}+2a^{[\mu\rho]}\partial_\rho\partial^{\nu}a_{[\nu\mu]})+C^{\mu}\varepsilon_{\mu\nu\rho\lambda}\partial^{\rho}a^{[\nu\lambda]}$$

a Lagrangian whose part within the brackets would describe the so-called notoph, i.e., a vector particle with helicity equal to zero [23]. However, the presence of the last term in the Lagrangian L_1 means that $a_{[\mu\nu]}$ has the structure

$$a_{[\mu\nu]}=\partial_\mu b_\nu-\partial_\nu b_\mu. \tag{17}$$

And since the Lagrangian of the notoph is invariant under gauge transformations:

$$a_{[\mu\nu]}=a_{[\mu\nu]}+\partial_\mu b_\nu-\partial_\nu b_\mu,$$

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* That a Lagrangian of the form of the expression on the right-hand side of (16) contains ordinary fields with spins 2 and 3/2 was pointed out to us by V. I. Ogievetskii and E. Sokatchev (see also [10, 22]).

the structure of (17) for $a_{[\mu\nu]}$ leads to the actual absence of the notoph. Thus, this Lagrangian in the linear approximation effectively contains the free Lagrangians for the gravitational field $a_{\{\mu\nu\}}$ and the field θ^{μ}_{α} with spin $^3/_2$. We have not examined the interaction of these fields. However, it should be noted that for the interaction it is not in advance obvious what are the gauge groups that ensure the absence of the redundant components in the case of the free fields.

We are very grateful to V. I. Ogievetskii and E. Sokatchev for a helpful discussion and a number of valuable tips.

Appendix

The invariant contraction over tensor indices in eight-dimensional superspace can be defined in two different ways, which in the simplest case of the product of two vectors have the form

$$V^A U_A = V^\mu U_\mu + V^\alpha U_\alpha + V^{\dot\alpha} U_{\dot\alpha} \text{ (first method)}, \quad U_A V^A = V^A U_A (-)^A = V^\mu U_\mu - V^\alpha U_\alpha - V^{\dot\alpha} U_{\dot\alpha} \text{ (second method)}.$$

We adopt the first method of defining the invariant contraction.

Note a simple but very useful rule concerning grading factors. All the calculations can be made without bothering about the grading factors, which can be readily reinstated by following this rule. We illustrate it by the example of the expression for the variation of the torsion, $\widetilde{S_{CB}{}^A W}$. The order of the tensor indices in this case is $\dot{}\dot{}^{A}_{CB\cdot}$, and the grading factors of the different terms on the right-hand side of (8) take into account what permutations of the indices must be made in each term in order to establish the order of the indices $\dot{}\dot{}^{A}_{CB\cdot}$, and to each transposed pair of indices, say E and F, there corresponds the factor $(-)^{EF}$, where the indices in the exponents take the values 0 for boson values and 1 for fermion values. The indices over which the summation is performed must be placed next to one another and in the order corresponding to the adopted definition for the invariant contraction. This rule works in all expressions, including the expressions for the volume and its variation discussed in [13], in which the determinant of the matrix M_A^B, which contains anticommuting matrix elements, is defined as

$$\det M = \exp \operatorname{Tr} (\ln M),$$

and the trace of these matrices in the form $\operatorname{Tr} M = (-)^A M_A{}^A$. In particular, one understands the grading factors $(-)^A$ in the definition of the trace of the matrix or a product of matrices, for example, $M_A{}^B N_B{}^C L_C{}^A (-)^A$, and in the expression (10) for the variation of the volume $W = \det \omega_a{}^A$, $\widetilde{W} = (-)^A X_A{}^a \widetilde{\omega}_a{}^A W$.

We should point out a fundamentally different possibility of generalizing the volume for the case of superspace. For this we note that in the case of ordinary space the volume can be expressed in the form of the exterior product of Cartan forms:

$$\omega^0(d_0) \wedge \omega^1(d_1) \wedge \omega^2(d_2) \wedge \omega^3(d_3). \tag{A.1}$$

If generalizations are made in accordance with (A.1) and the volume in superspace is expressed as an exterior product of eight Cartan ω forms, then because of the equation

$$\omega^A(d) \wedge \omega^B(\delta) = -(-)^{AB} \omega^B(d) \wedge \omega^A(\delta)$$

it can be seen that the exterior product of the spinor forms is a symmetric tensor, in contrast to the spatial forms, which are antisymmetric tensors. Therefore, this volume in superspace is a tensor, in contrast to the case of an ordinary space, in which it is an invariant. Such tensor volumes in superspace can obviously be used when one is considering generalizations of generally covariant theories to the case of superspace.

The entities $(\sigma_{\mu\nu})^\alpha{}_\beta$ and $(\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}$ are defined as follows:

$$(\sigma_{\mu\nu})^\alpha{}_\beta = \frac{1}{4} (\widetilde{\sigma}_\mu \sigma_\nu - \widetilde{\sigma}_\nu \sigma_\mu)^\alpha{}_\beta, \quad (\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta} = -\frac{1}{4} (\sigma_\mu \widetilde{\sigma}_\nu - \sigma_\nu \widetilde{\sigma}_\mu)_{\dot\alpha}{}^{\dot\beta},$$

where

$$(\sigma_\mu)_{\dot\alpha\beta} = (1, \sigma)_{\dot\alpha\beta} = \sigma_{\mu\dot\alpha\beta}, \quad (\widetilde{\sigma}_\mu)^{\alpha\dot\beta} = (1, -\sigma)_{\alpha\beta} = \sigma_\mu{}^{\alpha\dot\beta}$$

and σ are the ordinary Pauli matrices. The transition from the vector representation to the spinor representation in the case of the antisymmetric tensor $A_{\mu\nu} = -A_{\nu\mu}$ is made in accordance with the formulas

$$A_{\mu\nu} = (\sigma_{\mu\nu})_\alpha{}^\beta A_\beta{}^\alpha + (\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta} A_{\dot\beta}{}^{\dot\alpha}, \quad A_\alpha{}^\beta = \frac{1}{2} (\sigma_{\mu\nu})_\alpha{}^\beta A^{\nu\mu}, \quad A_{\dot\alpha}{}^{\dot\beta} = \frac{1}{2} (\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta} A^{\nu\mu}.$$

It is helpful to note the following properties of $(\sigma_{\mu\nu})^{\alpha\beta}$ and $(\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}$ with respect to permutation of indices:

$$(\sigma_{\mu\nu})_{\alpha}{}^{\beta}=(\sigma_{\mu\nu})^{\beta}{}_{\alpha},\ (\sigma_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}=(\sigma_{\mu\nu})_{\dot{\beta}}{}^{\dot{\alpha}},$$

where

$$(\sigma_{\mu\nu})_{\alpha}{}^{\beta}=\varepsilon_{\alpha\alpha'}\varepsilon^{\beta\beta'}(\sigma_{\mu\nu})^{\alpha'}{}_{\beta'},\ (\sigma_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}=\varepsilon^{\dot{\alpha}\dot{\alpha}'}\varepsilon_{\dot{\beta}\dot{\beta}'}(\sigma_{\mu\nu})_{\dot{\alpha}'}{}^{\dot{\beta}'},$$

with respect to complex conjugation:

$$((\sigma_{\mu\nu})^{\alpha}{}_{\beta})^{*}=(\sigma_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}$$

and spatial reflection:

$$(\sigma_{\mu\nu})^{\alpha}{}_{\beta}\rightarrow-(\sigma_{\tilde{\mu}\tilde{\nu}})_{\dot{\alpha}}{}^{\dot{\beta}},\ (\sigma_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}}\rightarrow-(\sigma_{\tilde{\mu}\tilde{\nu}})^{\alpha}{}_{\beta},$$

where the subscript $\tilde{\mu}$ means that the spatial components of the quantity with subscript $\tilde{\mu}$ have signs opposite to the spatial components of the quantity with subscript μ, i.e., $V^{\tilde{\mu}}=(V^0,-\mathbf{V})$, if $V^{\mu}=(V^0,\mathbf{V})$.

LITERATURE CITED

1. Yu. A. Gol'fand and E. P. Likhtman, Pis'ma Zh. Eksp. Teor. Fiz., **13**, 452 (1971); Problems of Theoretical Physics (collection dedicated to the memory of I. E. Tamm) [in Russian], Nauka (1972).
2. D. V. Volkov and V. P. Akulov, Pis'ma Zh. Eksp. Teor. Fiz., **16**, 621 (1972); Teor. Mat. Fiz., **18**, 39 (1974).
3. J. Wess and B. Zumino, Nucl. Phys., **B70**, 39 (1974).
4. J. Wess and B. Zumino, Phys. Lett., **B49**, 52 (1974).
5. A. Salam and J. Strathdee, Nucl. Phys., **B76**, 477 (1974).
6. S. Ferrara, J. Wess, and B. Zumino, Phys. Lett., **B51**, 239 (1974).
7. A. Salam and J. Strathdee, Phys. Rev., **D11**, 1521 (1975).
8. S. Ferrara and B. Zumino, Nucl. Phys., **B79**, 413 (1974).
9. A. Salam and J. Strathdee, Phys. Lett., **B51**, 353 (1974).
10. V. I. Ogievetskii and E. Sokatchev, Proc. of the Fourth International Symposium on Nonlocal Field Field Theories, Alushta, USSR (1976); JINR Preprint E2-9985, Dubna (1976).
11. D. Z. Freedman, P. van Niewenhuizen, and S. Ferrara, Phys. Rev., **D13**, 3214 (1976); S. Deser and B. Zumino, Phys. Lett., **B62**, 335 (1976).
12. D. V. Volkov and V. A. Soroka, Pis'ma Zh. Eksp. Teor. Fiz., **18**, 529 (1973); Teor. Mat. Fiz., **20**, 291 (1974).
13. R. Arnowitt, P. Nath, and B. Zumino, Phys. Lett., **56B**, 81 (1975).
14. R. Arnowitt and P. Nath, Phys. Lett., **56B**, 177 (1975).
15. B. Zumino, CERN Preprint TH.2120 (1975).
16. V. P. Akulov, D. V. Volkov, and V. A. Soroka, Pis'ma Zh. Eksp. Teor. Fiz., **22**, 396 (1975).
17. V. I. Ogievetskii and L. Mezincescu, Usp. Fiz. Nauk, **177**, 637 (1975).
18. V. I. Ogievetskii and E. Sokatchev, Pis'ma Zh. Eksp. Teor. Fiz., **23**, 66 (1976).
19. T. W. B. Kibble, J. Math. Phys., **2**, 212 (1961).
20. K. Nomizu, Lie Groups and Differential Geometry, Math. Soc. of Japan, Tokyo (1956).
21. A. Lichnerowicz, Théorie Globale des Connexions et des Groupes d'Holonomiè, Rome (1955).
22. E. Sokatchev, Nucl. Phys., **B99**, 96 (1975).
23. V. I. Ogievetskii and I. V. Polubarinov, Yad. Fiz., **4**, 216 (1966).
24. V. I. Pakhomov, Mat. Zam., **16**, 65 (1974).