The authors express gratitude to P. Ya. Ufimtsev for helpful comments made during the review of the article.

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ACOUSTIC RADIATION AND RADIATION DRAG IN CONNECTION WITH THE MOTION OF A SOURCE IN A STRATIFIED MEDIUM

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UDC 551.596

The radiation of acoustic gravity waves is investigated. In contrast with previous papers, correct allowance is made for the fact that the expressions for the Fourier components characterizing the wave can contain singularities in the upper half-plane of the complex variable ω (the radiation frequency). The method used here makes it possible, in particular, to determine correctly the energy losses of the source in the fast acoustic wave in the case $M < 1$ (M is the Mach number), whereas previously losses were assumed to be absent. An important consideration is the possibility of the case in which the source absorbs energy developed in takeoff during the course of uniform rectilinear motion.

The radiation of acoustic gravity waves by moving sources has been discussed in a great many papers (see, e.g., [1, 2]). In these papers, however, a solution of the problem is sought by the Fourier method without regard for the fact that the derived expressions for the Fourier components of the quantities characterizing the radiated wave, i.e., the expressions for the pressure

$$p_{\omega k} = (2\pi)^{-4} \int p(t, \boldsymbol{R}) \exp[-i(\boldsymbol{kR} - \omega t)]\, dt\, d\boldsymbol{R},$$
$$p(t, \boldsymbol{R}) = \int p_{\omega k} \exp[i(\boldsymbol{kR} - \omega t)]\, d\omega\, d\boldsymbol{k}, \quad \boldsymbol{R} = \{x, y, z\}, \tag{1}$$
$$\boldsymbol{r} = \{x, y, 0\}, \quad r = \sqrt{x^2 + y^2}$$

Scientific-Research Institute for Radiophysics. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 27, No. 12, pp. 1545-1556, December, 1984. Original article submitted September 21, 1983; revision submitted May 7, 1984.

 0033-8443/84/2712-1076$08.50

and the analogous expressions for the velocity v and the density ρ can contain singularities in the upper half-plane of the complex variable ω. For example, in the case of a mass source ascending with a constant velocity V (V < c, where c is the sound velocity) in an isothermal atmosphere, such a singularity is the point $\omega = i\alpha = iV/H(1-M^2)$ (H is a scale of the height of a homogeneous atmosphere, and M = V/c). The disregard of this fact can lead to incorrect results, because for α > 0 the path of integration in the complex plane of ω in the causally stated problem (see below) must bypass the point ω = iα above, rather than merely enclose the real axis as has been done previously.

Attention must be called to another circumstance. The causal formulation of the problem makes it possible to explain the role of interaction between the source and the acoustic field attributable to variation of the properties of the medium (for a subsonic velocity of the source). It has been assumed earlier that energy losses in sound generation are absent for M < 1. In reality, it follows from the ensuing discussion that the situation is far more complicated here. In the ascent of a mass source, when $\boldsymbol{V}\uparrow\downarrow\boldsymbol{g}$ ($\boldsymbol{g}$ is the acceleration of gravity), transient acoustic losses exist for M < 1, and their energy is proportional to the large parameter $\ln(l/a)$ (l and a are the length and transverse width of the source, respectively). In the case of the motion of a source in a stratified medium in the direction of increasing density, when $\boldsymbol{V}\downarrow\downarrow\boldsymbol{g}$, $M<1$, the bulk of the acoustic energy developed during takeoff of the source is absorbed.

The given problem is also interesting from another standpoint isofar as it is possible, with a certain change of notation, to describe the energy losses by an exponentially varying [according to the law $y(t) = e^{\pm Vt/2H}$] source moving with a constant velocity in a homogeneous medium. An important aspect of this situation is the fact that, unlike a moving oscillatory source, an exponentially varying source moving with a constant velocity (either for M < 1 or for M > 1) by and large excites only disturbances with a positive projection of the wave vector onto the direction of motion. Associated with these considerations is an exponential growth of the disturbance in a certain time interval for α > 0, i.e., for M < 1 when $\boldsymbol{V}\downarrow\uparrow\boldsymbol{g}$, or for M > 1 when $\boldsymbol{V}\downarrow\downarrow\boldsymbol{g}$.

In the present article we use a method that permits the above-mentioned singularity to be taken into account correctly. Moreover, we take into account the finite width of the source, and this enables us, on the one hand, to obtain a finite expression for the energy losses by the source in the fast acoustic waves and, on the other, to include effects associated with energy losses in a medium with variable parameters. We proceed from the linear equations for the perturbed values of p, ρ, $\boldsymbol{v}$ in a gas situated in the field of gravity:

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p + \rho \boldsymbol{g}, \quad \frac{\partial \rho}{\partial t} - \frac{v_z}{H}\rho_0 + \rho_0 \operatorname{div} \boldsymbol{v} = q(t, \boldsymbol{R}),$$
$$\frac{\partial p}{\partial t} - \frac{v_z}{H} p_0 = c^2 \left(\frac{\partial \rho}{\partial t} - \frac{v_z}{H}\rho_0\right). \tag{2}$$

Here $p_0 = p_{00}e^{-z/H}$, $\rho_0 = \rho_{00}e^{-z/H}$ are the equilibrium values of the pressure and density, $p_0 = gH\rho_0$, i.e., the force of gravity is directed antiparallel to the z axis, and $q(t, \boldsymbol{R})$ is a mass source (other sources of disturbance are not considered in this article). Of course, the investigated system can only afford a very approximate model of a real medium. In particular, this model is invalid for large positive and negative values of z. In the ensuing discussion, therefore, we consider only disturbances localized in a finite region of space and time. Accordingly, the source can be started up only in a bounded time interval (in the case of a moving source its track must be of finite length).

We use Eqs. (2) for an isothermal atmosphere. Expanding all variables in Fourier integrals, after suitable transformations we obtain

$$p(t, \boldsymbol{R}) = -ic^2 \int_0^{2\pi} d\Phi \int_0^{\infty} d\varkappa \int_{-\infty}^{\infty} \{dk_z d\omega \varkappa\omega(\omega_g^2 - \omega^2) q_{\omega\boldsymbol{k}} \exp[i(-\omega t + \boldsymbol{kR})]/\Delta(\omega, \boldsymbol{k})\}, \tag{3}$$

where ω_g is the Brunt–Väisälä frequency, $\omega_g^2 = (g/H)(1 - gH/c^2) = (\gamma-1)g^2/c^2$, $\gamma = c_p/c_V$ is the specific-heat ratio,

$$\Delta(\omega, \boldsymbol{k}) = \omega^4 - c^2\omega^2(k^2 - ik_z/H) + \omega_g^2 \varkappa^2 c^2,$$

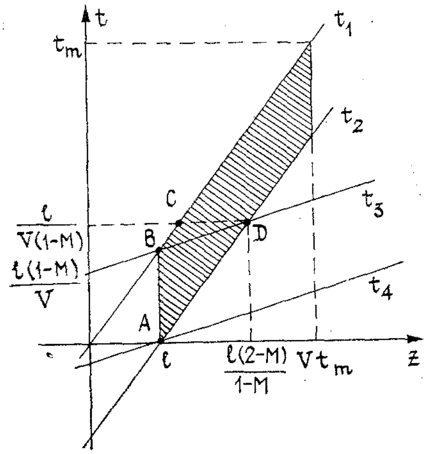

Fig. 1. $M < 1$, $t_1(z) = z/V$, $t_2(z) = (z-l)/V$, $t_3(z) = z/c + l(1-M)/V$, $t_4(z) = (z-l)/c$.

$$q_{\omega k} = (2\pi)^{-4} \int\limits_{-\infty}^{\infty} q(t, \boldsymbol{R}) \exp[-i(\boldsymbol{kR} - \omega t)]\, dt d\boldsymbol{R},$$

$$\boldsymbol{k} = \{k_x, k_y, k_z\}, \quad \boldsymbol{\varkappa} = \{k_x, k_y, 0\} = \varkappa\{\cos\Phi, \sin\Phi, 0\}.$$

Inasmuch as the integrand in (3) can contain singularities in the domain $\mathrm{Im}\,\omega = \omega_2 > 0$, the investigated system outwardly resembles a system with an instability. For this reason, the indicated facts must be included in the statement of the problem, i.e., the source q must be considered to be localized in a finite domain of space and time, $0 < t < t_m$. Prior to startup of the source, i.e., for $t < 0$, all disturbances are absent. [Consequently, the path of integration L_ω with respect to ω in (3) must be drawn above all singularities of the integrand (3) as a function of ω, i.e., it can be assumed that ω_2 is large along L_ω ($\omega = \omega_1 + i\omega_2$).] It must also be borne in mind that $q(t, \boldsymbol{R})$ determines $p(t, \boldsymbol{R})$ through expression (3) in the form of a convergent integral. In accordance with the foregoing discussion, we specify the expression for the source in the form

$$q(t, \boldsymbol{R}) = q_0 \Pi(t, z)\, \delta(r - a)/2\pi r, \; l \to 0, \quad a \to a, \quad a/l \to 0, \tag{4}$$

where $\delta(r-a)$ is a delta function, $\Pi(t, z) = 1$ for $l < z < Vt_m$, $(z-l)/V < t < z/V$ and $\Pi(t, z) = 0$ for all other values of t, z [the domain in which $\Pi(t, z) = 1$ is shown hatched in Fig. 1], i.e.,

$$q_{\omega k} = \frac{q_0 J_0(\varkappa a)(1 - \exp(-i\omega l/V))}{(2\pi)^4 \omega (k_z - \omega/V)} \{\exp[i(-k_z + \omega/V) Vt_m] - \exp[i(-k_z + \omega/V) l]\}, \tag{5}$$

where $J_0(\varkappa a)$ is a Bessel function. This form of the source expression allows for the fact, in particular, that the source is not activated to total discharge of its mass instantaneously, but in a time l/V. Extending the integration with respect to the variable $\varkappa$ in (3) in the usual way to the domain $-\infty < \varkappa < \infty$ and making use of Eq. (5) we find

$$p(r > a) = \frac{q_0 V}{16\pi^2} \int\limits_{L_\omega} \int\limits_{L_{k_z}} d\omega\, dk_z\, J_0(\mu a/c) H_0^{(1)}(\mu r/c)(1 - \exp(-i\omega l/V)) \times$$
$$\times \left\{\exp\left[iVt_m\left(-k_z + \frac{\omega}{V}\right)\right] - \exp\left[il\left(-k_z + \frac{\omega}{V}\right)\right]\right\} \frac{\exp[i(k_z z - \omega t)]}{k_z V - \omega}. \tag{6}$$

Here

$$\mu = \sqrt{\frac{\omega^2\{\omega^2 - c^2[(k_z - i/2H)^2 + 1/4H^2]\}}{\omega^2 - \omega_g^2}}, \tag{7}$$

$\mathrm{Im}\,\mu = \mu_2 > 0$, $H_0^{(1)}(\mu r/c)$ is a Hankel function, and the path of integration L_{k_z} runs along the real axis. We note first of all that the integral in (6) associated with the first term

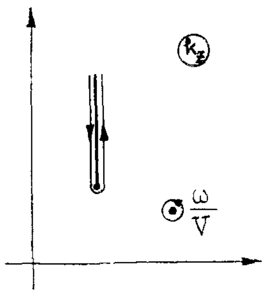

Fig. 2

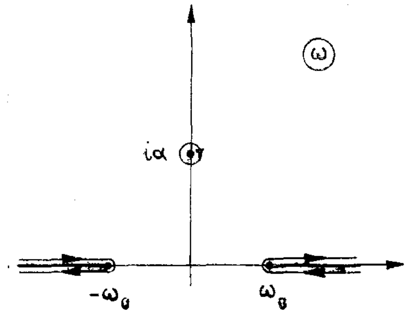

Fig. 3

in the braces has a nonzero value only for $t>t_m-l/V$. For the effects investigated here, we can restrict the problem to the time interval $t<t_m-l/V$, $l\to 0$, and, as a rule, this will be assumed below. Also, inasmuch as our primary concern is to determine the energy losses, it is mainly required to calculate the pressure in the vicinity of the track of the source $(r=a+0\to 0)$. We consider the integral (for $z_1>0$)

$$I=\int_{-\infty}^{\infty} dk_z J_0\left(\frac{\mu a}{c}\right) H_0^{(1)}\left(\frac{\mu r}{c}\right)\frac{\exp(ik_z z)}{k_z V-\omega}=I_1+I_2,\quad z_1=z-l. \tag{8}$$

Clearly, its computation is reducible to integration around the pole $k_z=\omega/V$ ($\omega_2>0$, ω_2 is large) and integration along both sheets of the cut of the logarithmic branch point

$$k_1=\frac{i}{2H}+\sqrt{\frac{\omega^2}{c^2}-\frac{1}{4H^2}}=\frac{i}{2H}+\eta\quad\left(\sqrt{\frac{\omega^2}{c^2}-\frac{1}{4H^2}}\to\frac{\omega}{c}\quad\text{as}\quad\omega\to\infty,\ \omega_2>0\right)$$

(Fig. 2), since

$$H_0^{(1)}(x)=(2i/\pi)\,J_0(x)\ln(x/2)+R_1(x),\quad |\arg x|<\pi, \tag{9}$$

where $R_1(x)$ is a function that is regular as $x\to 0$ (see, e.g., [3]). Integration along the sheets of the cut (for $z_1>0$, $r=a+0\to 0$) yields

$$I_1=\frac{2\exp(iz_1\omega/V)}{V}\int_{-\infty+i\zeta_2}^{\zeta_0}\frac{e^{\zeta}}{\zeta}\,d\zeta=\frac{2}{V}\exp(iz_1\omega/V)\,\mathrm{Ei}(\zeta_0), \tag{10}$$

where $\zeta_0=(iz_1/V)\,[-\omega+V(i/2H+\eta)]=\zeta_1+i\zeta_2$, $\eta\to\omega/c$ as $\omega\to\infty$, $\omega_2>0$, and $\mathrm{Ei}(\zeta_0)$ is the integral exponential function. We note that for $r\neq 0$

$$I_1=-2\int_{i\infty+\mathrm{Re}\,k_1}^{k_1} e^{ik_z z}J_0(\mu r/c)\,J_0(\mu a/c)(\omega-k_zV)^{-1}\,dk_z,\quad \mathrm{Im}\,\mu>0,\quad r>a.$$

Making use of (10), we obtain from (6) the corresponding expression for p_1, which takes the form

$$p_1=-\frac{q_0}{8\pi^2}\int_{-\infty+i\omega_2}^{\infty+i\omega_2} d\omega e^{i\xi\omega}(1-\exp(-i\omega l/V))\int_{-\infty+i\zeta_2}^{\zeta_0}\frac{e^{\zeta}}{\zeta}\,d\zeta, \tag{11}$$

or, after integration by parts ($\xi=-t+z/V$, $\xi_1=-t-l/V+z/V$),

$$p_1=\frac{q_0\exp(-z_1/2H)}{8\pi^2 i}\int_{L_\omega}\exp\,[i(-\omega t_1+\eta z_1)]\left(\frac{1}{\xi}-\frac{\exp(-i\omega l/V)}{\xi_1}\right)\times$$

$$\times\left(1-\frac{V\omega}{c^2\eta}\right)\left[\omega-V\left(\frac{i}{2H}+\eta\right)\right]^{-1}d\omega=\frac{q_0\exp(-z_1/2H)}{8\pi^2 i}\left(\frac{Q_1}{\xi}+\frac{Q_2}{\xi_1}\right), \tag{12}$$

$$Q_1=\int_{L\omega}\exp\,[i(-\omega t_1+\eta z_1)]\left(1-\frac{V\omega}{c^2\eta}\right)\left[\omega-V\left(\frac{i}{2H}+\eta\right)\right]^{-1}d\omega,$$

$$t_1=t-l/V,\quad z_1>0.$$

We direct the cuts in the ω plane from the branch points $\omega=\pm\omega_0$, $\omega_0=c/2H$, as indicated in Fig. 3, where these cuts are represented by heavy lines. Then the integrand in Eq. (12) does not have any poles on the selected sheet of the Riemann surface for M > 1, and it has one pole in the case M < 1: $\omega=i\alpha$, $\alpha=V/H(1-M^2)$. Consequently,

$$p_1(M<1)=-\frac{q_0}{4\pi}\left[\frac{e^{-\alpha\xi}}{\xi}1\left(t_1-\frac{z_1}{c}\right)-\frac{e^{-\alpha\xi_1}}{\xi_1}1\left(t_1+\frac{l}{V}-\frac{z_1}{c}\right)\right]+p_{st1}1\left(t_1-\frac{z_1}{c}\right)-p_{st2}1\left(t_1+\frac{l}{V}-\frac{z_1}{c}\right)\quad(13)$$

where $|\xi|\gg a/V$, $|\xi_1|\gg a/V$, $1\left(t_1-\frac{z_1}{c}\right)=\begin{cases}1, & t_1>z_1/c\\ 0, & t_1<z_1/c\end{cases}$, and $p_{st\,1}\,1(t_1-z_1/c)$ − $p_{st\,2}\,1(t_1+l/V-z_1/c)$ are the integrals in (12) along the sheets of the cuts from the points $\omega=\pm\omega_0$ (Fig. 3);

$$p_1(M>1)=p_{st1}1(t_1-z_1/c)-p_{st2}(t_1+l/V-z_1/c),\quad t<t_m-l/V.\quad(14)$$

The expression for $p_{st\,1}$ is readily obtained for large values of t and z by the stationary-phase method. As a result, we have

$$p_{st1}=-\frac{q_0\exp(-z_1/2H)\pi^{1/2}cV}{4\pi^2H^{1/4}(t_1^2c^2-z_1^2)^{3/4}}\left\{\frac{\omega_0}{\sqrt{t_1^2c^2-z_1^2}}\times\right.$$

$$\left.\times(t_1c-Mz_1)\sin\left(\frac{\omega_0}{c}\sqrt{t_1^2c^2-z_1^2}\right)-\frac{V}{2H}\cos\left(\frac{\omega_0}{c}\sqrt{t_1^2c^2-z_1^2}\right)\right\}\times$$

$$\times\left\{1(t_1-z_1/c)\left[\frac{\omega_0^2}{t_1^2c^2-z_1^2}(t_1c-Mz_1)^2+V^2/4H^2\right]^{-1}\right\},\quad(15)$$

$$\frac{t_1^2c\omega_0}{z_1^2}\sqrt{t_1^2c^2-z_1^2}\gg1,\quad z\to\infty,\quad t\to\infty.$$

According to (15), for $\sqrt{t_1^2c^2-z_1^2}\simeq z_1$ we have $p_{st\,1}\simeq z_1^{-3/2}$. We cannot make the transition to the case of a homogeneous medium ($H\to\infty$) in (15) (this fact is evinced by the need to observe the condition $t_1^2c^2\sqrt{t_1^2c^2-z_1^2}/2Hz_1^2\gg1$), i.e., the method is invalid for a homogeneous medium. It also follows from (15) that $p_{st\,1}$ has the structure of a spherical disturbance diverging from the point $z_1 = 0$. To obtain the expression for $p_{st\,2}$ it is necessary to replace t by $t + l/V$ in the expression for $p_{st\,1}$ (see (6)].

We now determine the contribution to the pressure expression from the pole $k_z=\omega/V$ [I_2 in (8)]; this expression can be written in the form [see (6) and Fig. 2)]

$$p_2(t,r,z)=-\frac{iq_0}{8\pi}\int_{L_\omega}d\omega\,J_0\left(\frac{\mu_1a}{c}\right)H_0^{(1)}\left(\frac{\mu_1r}{c}\right)\times\quad(16)$$

$$\times(1-\exp(-i\omega l/V))e^{i\omega\xi},\quad t>t_m-l/V;$$

$$\mu_1=\mu\left(k_z=\frac{\omega}{V}\right)=\sqrt{\frac{\omega^3[\omega(1-M^{-2})+ic/MH]}{\omega^2-\omega_g^2}},\quad r\to a+0\to0.\quad(17)$$

We also analyze relation (16) only for the case $r\to a+0\to0$. The integrand in (16) has four singularities ($\omega=0$, $\pm\omega_g$, $i\alpha$) in the ω plane [see (8)].

Under the conditions $r\to a+0\to0$, $l/a\to\infty$, we restrict the computation of (16) to the determination of the principal part of the excited disturbance, which contains the large parameter $\ln(l/a)$ in the expression for the radiation drag (see below). The occurrence of this parameter, as in the case of a homogeneous medium, is associated with the fact that if the dimensions of the source are small, the drag is rendered large by a high-frequency sound field. With the indicated reservations and with allowance for (9), we readily carry out the integration in (16). Assuming that $r\to a+0\to0$, we obtain

$$p_2(t,z)=\frac{q_0}{4\pi}\left[\frac{1(-\xi)}{\xi}\left(2\sin^2\frac{\omega_g\xi}{2}+1+e^{-\alpha\xi}\right)-\frac{1(-\xi_1)}{\xi_1}\left(2\sin^2\frac{\omega_g\xi_1}{2}+1+e^{-\alpha\xi_1}\right)\right],\quad(18)$$

$$|\xi|\gg a/V,\quad|\xi_1|\gg a/V,\quad t<t_m-l/V.$$

Equation (18) is applicable either for the case M > 1 or for the case M < 1. The total pressure is $p = p_1 + p_2$ [see Eqs. (13), (14), and (18)]. For the ensuing discussion it is convenient to distinguish the expression for the pressure exclusive of $p_{st\,1}$ and $p_{st\,2}$, i.e.,

$$p_k(M<1) = p - p_{st1} - p_{st2} = -\frac{q_0}{4\pi}\left[4\frac{1(-\xi)}{\xi}\sin^2\frac{\omega_g\xi}{2} + 4\frac{1(-\xi_1)}{\xi_1}\sin^2\frac{\omega_g\xi_1}{2} + \right. \tag{19}$$

$$\left.\frac{e^{-\alpha\xi}}{\xi}1\left(t_1-\frac{z_1}{c}\right) - \frac{e^{-\alpha\xi_1}}{\xi_1}1\left(t+\frac{l}{c}-\frac{z}{c}\right) - (1+e^{-\alpha\xi})\frac{1(-\xi)}{\xi} + (1+e^{-\alpha\xi_1})\frac{1(-\xi_1)}{\xi_1}\right],$$

$$z>l,\quad l/V(1-M)<t<t_m-l/V;$$

$$p_k(M>1) = -\frac{q_0}{4\pi}\left[-4\frac{1(-\xi)}{\xi}\sin^2\frac{\omega_g\xi}{2} + 4\frac{1(-\xi_1)}{\xi_1}\sin^2\frac{\omega_g\xi_1}{2} - (1+e^{-\alpha\xi})\frac{1(-\xi)}{\xi} + (1+e^{-\alpha\xi_1})\frac{1(-\xi_1)}{\xi_1}\right],$$

$$z>l,\quad l/V<t<t_m-l/V. \tag{20}$$

It follows from Eqs. (19) and (20) that p_k comprises two distinct parts. The first part (which depends on ω_g and does not have a singularity as $\xi\to 0, \xi_1\to 0$) is associated with the radiation of slow internal waves, which occurs both for M > 1 and for M < 1. The second part of (19)-(20) (which does not depend on ω_g) is associated with energy losses of the acoustic field and for M > 1 is similar in some measure to acoustic radiation in a homogeneous medium, except that important differences from the case of a homogeneous medium exist here. First, the indicated losses in a stratified medium also exist for M < 1 (see below). Moreover, the expressions (19) for the pressure in the fast acoustic wave contain terms with c $e^{-\alpha\xi}$, $e^{-\alpha\xi_1}$.

For M < 1, after passage of the front z = ct, the disturbance at a given point begins to grow exponentially with time. This growth continues as long as z > Vt (we neglect small quantities, $l \to 0$). For $z\simeq Vt$ the growth of the disturbance with time ceases at the given point.

We now consider the stated fact somewhat more in detail. If we set $\omega_g = 0$, Eqs. (12) and (18) will then correspond to the solution of the equation

$$\frac{1}{c^2}\frac{\partial^2 p^{(1)}}{\partial t^2} - \Delta p^{(1)} + \frac{p^{(1)}}{4H^2} = \exp(z/2H)\frac{\partial q(t,R)}{\partial t} \tag{21}$$

as $r \to 0$. Here $p^{(1)}=p\exp(z/2H)$ and $q(t,R)$ is given by Eq. (4). We note that the left-hand side of (21) is analogous to the equation for the magnetic field in an isotropic plasma (with Langmuir frequency $\omega_e = c/2H$ and dielectric constant $\varepsilon=1-c^2/4H^2\omega^2$). Making use of Eq. (21), we can readily explain the exponential growth of p with time for $M<1$, $z/c<t<z/V$. Thus, we infer from (6) and a subsequent analysis that the disturbance from the source ($z>0$, $t<t_m$) in the investigated system propagates in the positive z direction, since the only contribution to p_k is from values of $k_z=\omega/V$. The right-hand side of (21) increases exponentially with t. Therefore, disturbances emitted by the source at an earlier time will contribute less to the wave pressure at a given point than disturbances emitted later. For M < 1, disturbances emitted at earlier times by the source arrive earlier at a given point. For this reason, the pressure p_k at the given point grows exponentially with time for M < 1. For M > 1, on the other hand, waves emitted by the source at later times $\tilde{t}$ (but $\tilde{t} < z/V$) arrive earlier at a given point z. As a result, p_k decays with time ($z/V < t < z/c$) at the given point for M > 1.

Using Eqs. (19) and (20), we can find the drag acting on the source. It is determined from the expression for the work done by the source on the disturbance generated by it, i.e., from

$$\dot{A} = \int qp\,d\mathbf{R}/\rho_0. \tag{22}$$

We note the following in connection with Eq. (22). From (2) we obtain

$$\rho_0\partial v_r/\partial t = -\partial p/\partial r. \tag{23}$$

Accordingly, for $r \to a+0$ in the given approximation

$$v_r \simeq -\frac{q_0}{2\pi\rho_0 r}\left[1\left(t-\frac{z}{V}\right)-1\left(t+\frac{l}{V}-\frac{z}{V}\right)\right], \quad t \gg l/V. \tag{24}$$

Consequently, the work described by Eq. (22) is readily perceived to be equal to the energy flux S_r across a cylindrical surface of small radius $a+0, a \to 0$, the axis of which coincides with the z axis:

$$S_r = 2\pi r \int_{Vt}^{Vt+l} p v_r dz, \quad r = a+0. \tag{25}$$

This result appears to be very significant, because it implies that the variation of the acoustic energy in the above-indicated cylinder of small radius can be neglected under the conditions described.

For the case of supersonic motion we obtain the following by means of Eq. (22) after suitable calculations:

$$\begin{gathered}\dot{A}(M>1) \simeq \frac{q_0^2 V(1+l/H)}{2\pi\rho_0(Vt)} \ln\frac{l}{a} + \frac{q_0^2 \omega_g^2 l^2}{8\pi\rho_0(Vt)V}, \\ 0 < t < t_m - l/V, \quad z \gg l, \quad l/a \gg 1, \quad \omega_g l/c \ll 1, \\ l/H \ll 1, \quad |\alpha| l/V \ll 1, \quad \ln l/a \gg |\alpha|(H/V),\end{gathered} \tag{26}$$

where $\rho_0(Vt)$ is the value of the unperturbed density at the point z = Vt, i.e., $\rho_0(Vt) = \rho_{00}\exp(-Vt/H)$.

The first term in Eq. (26) describes the radiation of acoustic waves, and the second term characterizes internal waves (the corresponding expression for a point source appears to have been first derived in [1]).

Inasmuch as Eq. (20) is valid for $|\xi| \gg a/V$, $|\xi_1| \gg a/V$, Eq. (26) is therefore approximately derived. On the other hand, this approximation is entirely satisfactory, because it merely disregards quantities that are small in comparison with $\ln l/a \gg 1$. Of course, this is true only of the expression for acoustic radiation. Equation (26) contains an increment of the order of l/H, which is associated with acoustic radiation in a medium having variable parameters. This increment could obviously be obtained only by taking into consideration the finite dimensions of the source. The presence of the large parameter $\ln(l/a)$ indicates that radiation at high frequencies provides the main contribution to the radiation of fast acoustic waves by sources of small dimensions.

Using Eqs. (19) and (22), we obtain the following for a subsonic source:

$$\begin{gathered}\dot{A}(M<1) \simeq \frac{q_0^2 V l \ln l/a}{4\pi\rho_0(Vt)H} + \frac{q_0^2 \omega_g^2 l^2}{8\pi\rho_0(Vt)V}, \\ \ln l/a \gg |\alpha|(H/V), \quad \ln l/a \gg 1.\end{gathered} \tag{27}$$

It is seen that the expression for the work done by drag on internal waves for M < 1 has the same form as for M > 1. This fact is a consequence of the existence of the resonance frequency $\omega = \omega_g$ in the investigated system [see Eqs. (7) and (17)].

Equations (26) and (27) involve expressions associated with the energy losses of the source in its motion through the stratified medium, which vanish in a homogeneous medium ($H \to \infty$). It is important to note that these losses also exist for the case of subsonic source velocities. In this case, Eq. (27) describes the principal losses associated with restructuring of the static field [see (19)], whence it follows that only the static field, which does not depend on the time in a coordinate system moving with the source (M < 1, $Vt \gg l$), is taken into account near the source for sound. Restructing takes place as a result of the slow variation of the velocity field described by Eq. (24), corresponding to a loss of energy by the source due to the variation of the quantity

$$W_k = \frac{1}{2}\int v_z^2 \rho_0 dR \simeq \frac{q_0^2}{4\pi}\frac{l}{\rho_0(Vt)}\ln\frac{l}{a}.$$

Since $l\to 0$, $l/a\to\infty$, the energy losses in sound generation are dominant both for M > 1 and for M < 1 (for internal waves $\dot{A}\sim l^2$).

It has been assumed above that $t < t_m$, i.e., radiation associated with shutdown of the source has been disregarded. It is difficult to carry out a corresponding analysis analogous to the preceding. In particular, an equation similar to (15) is obtained. We shall not pursue this problem in the present article, however,

The foregoing discussion applies to the case in which the density of the medium decreases in the direction of motion of the source. We now look briefly at the description of a system in which the density of the medium increases in the direction of the source velocity. To derive the corresponding expressions it is necessary, as a rule, to replace g by −g and to replace H by −H in the equations used above (H_1 = −H for H_1 < 0, and H_1 = H for H_1 > 0). Here, by analogy with the preceding situation, we have in place of (19) and (20)

$$p_k(M<1) = -\frac{q_0}{4\pi}\left[-\frac{4}{\xi}\sin^2\frac{\omega_g\xi}{2}1(-\xi)+\frac{4}{\xi_1}\sin^2\frac{\omega_g\xi_1}{2}1(-\xi_1)-\right.$$
$$-\frac{1(-\xi)}{\xi}(1+e^{-\alpha\xi})+\frac{1(-\xi_1)}{\xi_1}(1+e^{-\alpha\xi_1})+ \qquad (28)$$
$$\left.+\frac{1(t_1-z_1/c)}{\xi}-\frac{1(t+(l-z)/c)}{\xi_1}\right],$$
$$z>l,\quad \frac{l}{V(1-M)}<t<t_m-\frac{l}{V},\quad |\xi|\gg\frac{a}{V},\quad |\xi_1|\gg\frac{a}{V};$$

$$p_k(M>1) = -\frac{q_0}{4\pi}\left\{-\frac{4}{\xi}\sin^2\frac{\omega_g\xi}{2}1(-\xi)+\frac{4}{\xi_1}\sin^2\frac{\omega_g\xi_1}{2}1(-\xi_1)+\right.$$
$$+\frac{1+e^{-\alpha\xi}}{\xi}\left[1\left(t_1-\frac{z_1}{c}\right)-1(-\xi)\right]-\frac{1+e^{-\alpha\xi_1}}{\xi_1}\times \qquad (29)$$
$$\left.\times\left[1\left(t+\frac{l-z}{c}\right)-1(-\xi_1)\right]\right\},$$
$$z>l,\quad \frac{l}{V(1-M^{-1})}<t<t_m-\frac{l}{V},\quad |\xi|\gg\frac{a}{V},\quad |\xi_1|\gg\frac{a}{V}.$$

It must be noted in connection with Eqs. (28) and (29) that after the replacement of H by −H the integrand in (12) has one pole (ω = 0) for M < 1 and two poles [$\omega=0$, $\omega=i\alpha=iV/H(M^2-1)$] for M > 1; the positions of the cuts in the complex plane of ω are the same as for H_1 > 0 (Fig. 3). In contrast with the case of entry of the source into a medium with a density decreasing in the direction of the velocity (H_1 > 0), it follows from Eq. (28) that for H_1 < 0 the pressure grows exponentially with time in the interval $z/V<t<z/c$, $l\to 0$ for a supersonic source. This result is associated with the fact that the right-hand side of (21) decreases exponentially with time for H_1 < 0 and waves radiated later contribute less to the pressure at a given point than waves radiated earlier [see (21) and the accompanying discussion].

It is instructive to calculate the total takeoff energy losses of sound generated by the source in the small time interval $0<t<l/V(1-M)$ $(M<1)$ or $0<t<l/V$ $(M>1)$ (see Fig. 1). In these intervals the intensity of the losses varies mainly as a result of the power variation of the source itself [$(\dot{A}\exp(Vt/H_1)$ depends on t]. Setting $z\simeq Vt$ in (12), we find

$$p_1\simeq\frac{q_0}{4\pi}\left[-\frac{1}{\xi}1\left(t_1-\frac{z_1}{c}\right)+\frac{1}{\xi_1}1\left(t+\frac{l-z}{c}\right)\right]1(1-M), \qquad (30)$$
$$1(1-M)=\begin{cases}1, & M<1\\ 0, & M>1\end{cases},$$

whereupon we obtain the following for acoustic waves [see (28) and (29)]:

$$\frac{q_0}{2\pi}\left\{-\frac{1(-\xi)}{\xi}+\frac{1(-\xi_1)}{\xi_1}+\left[\frac{1}{2\xi}\,1\left(t-\frac{z}{c}-\frac{l(1-M)}{V}\right)-\frac{1}{2\xi_1}\,1\left(t+\frac{l-z}{c}\right)\right]1(1-M)\right\},\quad z\simeq Vt. \tag{31}$$

This formula is valid either for $H_1 < 0$ or for $H_1 > 0$. Accordingly, the total takeoff (startup) energy losses [see (22)] are equal to ($M < 1$)

$$A_{st}=q_0\int\limits_{ABCD} p\,dz\,dt/\rho_0, \tag{32}$$

where the integration is carried out over the trapezoid ABCD (Fig. 1), i.e., over a domain comprising two subdomains $(z-l)/V<t<z/c+l(1-M)/V$, $l<z<l(2-M)/(1-M)$ and $ct-l(M^{-1}-1)<z<Vt$, $l/V<t<l/V(1-M)$. Consequently, $[(l/H)\ln l/a\ll 1]$,

$$A_{st}(M<1)\simeq\frac{q_0^2 l}{4\pi\rho_{00}}\ln\frac{l}{a},\qquad \ln\frac{l}{a}\gg\frac{|2M-1|}{1-M}. \tag{33}$$

The analogous A_{st} is obtained for $M > 1$:

$$A_{st}(M>1)\simeq\frac{q_0^2 l}{2\pi\rho_{00}}\ln\frac{l}{a}\left(1-\frac{3}{2}\frac{l}{H}\right). \tag{34}$$

Only quantities of the order of $\ln(l/a)$ are included in (33) and (34), in accordance with the order of accuracy of the analysis.

The takeoff radiation energy for an internal wave is determined [see Eqs. (28), (29), and Fig. 1] by the intervals $0<t<l/V$, $l<z<Vt+l$ and is equal to (for $M > 1$ and $M < 1$)

$$A_{st.sw}=q_0^2\omega_g^2 l^3/24\pi V^2\rho_{00}. \tag{35}$$

It is interesting to note that the time interval in which the takeoff disturbance is generated for $M < 1$ differs for the slow and fast waves. In the case of the slow wave, as mentioned, this interval is $0<t<l/V$, and for the fast wave in the case $M < 1$ it is $0<t<l/V(1-M)$.

We can use Eqs. (28) and (29) to determine the interaction of the acoustic wave $[t_m>t>l/H(1-M),\ M<1]$ and the slow wave $(t_m>t>l/V)$ with the moving source. As a result, we obtain ($H_1 < 0$)

$$\dot{A}_{t.ac}(M<1)\simeq-\frac{q_0^2 Vl\exp(-Vt/H)}{4\pi\rho_{00}H}\ln\frac{l}{a},$$

$$\dot{A}_{t.sw}(M<1)=\frac{q_0^2\omega_g^2 l^2\exp(-Vt/H)}{8\pi\rho_{00}V}; \tag{36}$$

$$A_t(M<1)=A_{t.ac}+A_{t.sw}=\int\limits_{l/V(1-M)}^{t_m}\dot{A}_{t.ac}\,dt+\int\limits_{l/V}^{t_m}\dot{A}_{t.sw}\,dt=$$
$$=\frac{q_0^2 l}{4\pi\rho_{00}}\left(-\ln\frac{l}{a}+\frac{\omega_g^2 lH}{2V^2}\right),\quad Vt_m/H\gg 1,\quad l/H\ln l/a\ll 1. \tag{37}$$

It follows from Eqs. (36) and (37) that for $H_1 < 0$, $M < 1$ the work $A_{t.ac}$ corresponds to absorption by the source rather than to the loss of energy from it. This means that along the given path the source absorbs the energy developed in takeoff [see (33)]. The total energy losses are

$$A(M<1)=A_{st}(M<1)+A_t(M<1)\simeq q_0^2\omega_g^2 l^2H/8\pi\rho_{00}V^2. \tag{38}$$

Equation (38) does not contain an expression for the acoustic radiation energy. This fact indicates that the energy losses in acoustic radiation are relatively small (the large parameter $\ln(l/a)$ does not enter into the expression for these losses).

For $M > 1$, by analogy with (36) and (37), we have ($H_1 < 0$)

$$\dot{A}_{t.ac}(M > 1) = \frac{q_0^2 V (1 - l/H) \exp(-Vt/H)}{2\pi\rho_{00}} \ln\frac{l}{a},$$

$$\dot{A}_{t.sw}(M > 1) = \frac{q_0^2 \omega_g^2 l^2 \exp(-Vt/H)}{8\pi\rho_{00} V}; \quad (39)$$

$$A_t(M > 1) = \int_{l/V}^{t_m} \dot{A}_{t.ac}\,dt + \int_{l/V}^{t_m} \dot{A}_{t.sw}\,dt = \frac{q_0^2 H}{2\pi\rho_{00}} \left[\left(1 - \frac{l}{H}\right)\ln\frac{l}{a} + \frac{\omega_g^2 l^2}{4V^2}\right], \quad Vt_m/H \gg 1. \quad (40)$$

It follows from a comparison of Eqs. (26), (27) with (36)-(40) that when the source moves in the direction of increasing density of the medium, the energy losses are greatly diminished in comparison with the case of the source moving in the direction of decreasing density of the medium.

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ANALYSIS AND SYNTHESIS OF A MODULATED RODDED STRUCTURE EXCITED BY ELECTRIC AND MAGNETIC CURRENTS

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UDC 621.396.67

An infinite modulated rodded structure under both periodic and aperiodic excitation is analyzed. An expression is obtained for a partial beam pattern of the structure; by selecting the positions of the short-circuiting devices a synthesis is effected of the partial beam pattern which is close to tabular. The results of the synthesis show the possibility of the use of rodded structures in phased antenna arrays for finite scanning.

Currently, in uhf electronics and antenna engineering rodded structures are finding a number of important practical applications, among them reflective arrays, impedance antennas, decoupling devices, and others.

It is of interest to consider the possibility of using rodded structures in phased antenna arrays (PAA), in particular in PAA used for finite scanning [1]. As is well known, the latter are traditionally constructed so that between the control elements and the radiating array a passive multipole is placed, the purpose of which is the formation of a tabular partial beam pattern (BP) of a width equal (in the ideal case) to the width of a given scanning sector.

A disadvantage of the traditional method of constructing PAA for finite scanning is the presence of a complex feeder power supply system, which increases the weight and dimensions of PAA and decreases their reliability. However, the use of an appropriate rodded structure with specially selected parameters and excited by the field of a radiator array with a simple power supply system would eliminate this disadvantage to a significant degree.

The present article examines the possibility of using rodded structures in large finite-scanning PAA based on the example of a two-dimensional model: an infinite modulated rodded structure excited by a given distribution of electrical and magnetic currents. The Gaussian system of units and a time function of the form $\exp(-i\omega t)$ are used.

Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 27, No. 12, pp. 1557-1566, December, 1984. Original article submitted November 4, 1983.

0033-8443/84/2712-1085$08.50