QUASIPOTENTIAL EQUATION FOR A RELATIVISTIC HARMONIC OSCILLATOR

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In the framework of the quasipotential approach, a study is made of a relativistic generalization of the exactly solvable problem of an harmonic oscillator. Quasipotential wave equations are constructed in the form of expansions with respect to the wave functions of the corresponding nonrelativistic problem. Relativistic corrections to the energy levels are obtained.

1. Introduction

As the quasipotential approach [1-3] is further developed, problems which can be regarded as relativistic generalizations of well-known exactly solvable problems of quantum mechanics acquire considerable interest. Thus, the quasipotential formalism is used to study the relativistic Coulomb problem in [4-6] and the problem of a relativistic particle in a potential well [5-7]. It is then inevitable to ask how one must formulate the harmonic oscillator problem in the quasipotential theory.

If the answer to this question is known, one could develop, for example, a relativistic quasipotential version of the shell model, which could, in turn, perhaps be applied to the quark model.

In nonrelativistic quantum mechanics, the Hamiltonian of a three-dimensional isotropic oscillator is [8]

$$H=\frac{\mathbf{p}^2}{2m}+\frac{m\omega^2\mathbf{r}^2}{2}. \tag{1.1}$$

Because of the symmetry between the momentum and coordinate operators in this expression it is immaterial whether one seeks the energy levels and the wave functions of the oscillator in the **r** or **p** representation.

If we choose the p representation, then the first term in (1.1), the kinetic energy of a free nonrelativistic particle, is a c number and the second term, which is responsible for the interaction, is proportional to the differential Laplace operator in the **p** space:

$$\mathbf{r}^2=-\hbar^2\frac{\partial^2}{\partial\mathbf{p}^2}. \tag{1.2}$$

It should be noted that the expression (1.2) is the Casimir operator of the group of motions of the three-dimensional Euclidean p space. This circumstance plays a fundamental role in the relativistic generalization of the Hamiltonian (1.1) which we develop later.

Let us first recall that in the quasipotential approach the p space can be regarded as a Lobachevskii space [3]. The corresponding Laplace operator Δ_L is related to the square of the relativistic coordinate $\mathbf{r}^2$ by the equation (see [3])

$$\Delta_L=-\left(1+\frac{m^2c^2}{\hbar^2}\mathbf{r}^2\right). \tag{1.3}$$

Taking into account the relation for the energy of a free relativistic particle,

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$$E = \sqrt{m^2c^4 + \mathbf{p}^2c^2}, \tag{1.4}$$

we may surmise that the relativistic analog of (1.1) is

$$H = \sqrt{m^2c^4 + \mathbf{p}^2c^2} - \frac{\omega^2\hbar^2}{2mc^2}\Delta_L. \tag{1.5}$$

In the spherical coordinates

$$\begin{aligned} \frac{p_0}{c} &= mc\,\mathrm{ch}\,\chi, \\ \mathbf{p} &= mc\,\mathrm{sh}\,\chi\mathbf{n}, \\ \mathbf{n} &= (\sin\vartheta\cos\varphi,\ \sin\vartheta\sin\varphi,\ \cos\vartheta), \end{aligned} \tag{1.6}$$

the Laplace operator (1.3) is

$$\Delta_L = \frac{1}{\mathrm{sh}^2\chi}\frac{\partial}{\partial\chi}\left(\mathrm{sh}^2\chi\frac{\partial}{\partial\chi}\right) + \frac{\Delta_{\vartheta,\varphi}}{\mathrm{sh}^2\chi}, \tag{1.7}$$

where

$$\Delta_{\vartheta,\varphi} = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}.$$

The stationary quasipotential Schrödinger equation with the Hamiltonian (1.5) is therefore:

$$\left\{2mc^2\,\mathrm{ch}\,\chi - \frac{\omega^2\hbar^2}{mc^2}\left[\frac{1}{\mathrm{sh}^2\chi}\frac{\partial}{\partial\chi}\left(\mathrm{sh}^2\chi\frac{\partial}{\partial\chi}\right) + \frac{\Delta_{\vartheta,\varphi}}{\mathrm{sh}^2\chi}\right]\right\}\Psi(\chi,\vartheta,\varphi) = 2E\Psi(\chi,\vartheta,\varphi). \tag{1.8}$$

Note that in the configuration representation Eq. (1.8) goes over into a difference – differential equation since the free Hamiltonian $H_0 = \sqrt{m^2c^4 + p^2c^2}$ in the $\mathbf{r}$ space is [3]

$$H_0 = \left[2\,\mathrm{ch}\,i\frac{d}{dr} + \frac{2i}{r}\,\mathrm{sh}\,i\frac{d}{dr} - \frac{\Delta_{\vartheta,\varphi}}{r^2}e^{i\frac{d}{dr}}\right]. \tag{1.9}$$

Clearly, it is preferable to work with a differential equation rather than a difference – differential equation since the techniques for solving the latter have been developed to a much lesser extent. In other words, it is more expedient to solve the problem of a relativistic oscillator described by the Hamiltonian (1.5) in the $\mathbf{p}$ representation. Separating the variables in Eq. (1.8) in the standard manner, we arrive at a one-dimensional equation for the partial wave function:

$$\left[\frac{1}{\mathrm{sh}^2\chi}\frac{d}{d\chi}\mathrm{sh}^2\chi\frac{d}{d\chi} - \frac{l(l+1)}{\mathrm{sh}^2\chi} - \frac{4m^2c^4}{\omega^2\hbar^2}\mathrm{sh}^2\frac{\chi}{2} + \frac{2mc^2}{\omega^2\hbar^2}(E - mc^2)\right]\Psi_l(\chi) = 0. \tag{1.10}$$

The investigation of Eq. (1.10) and its solutions is, in fact, the main burden of our paper. Only Sections 2 and 5 are not devoted to this problem. In Section 2 we expound a modified procedure for solving the nonrelativistic equation for the function $\Psi_l(p)$

$$\left[\frac{1}{p^2}\frac{d}{dp}p^2\frac{d}{dp} - \frac{l(l+1)}{p^2} + \frac{2}{m\omega^2\hbar^2}\left('E_{n,r} - \frac{p^2}{2m}\right)\right]\Psi_l(p) = 0 \tag{1.11}$$

with the boundary conditions

$$\Psi_l(0) < \infty, \tag{1.12a}$$

$$\Psi_l(\infty) = 0. \tag{1.12b}$$

This proves helpful for the subsequent analysis of the relativistic case. In Section 5 we shall consider a quasipotential equation with an oscillator interaction that has a slightly different form from (1.8).

In the present paper we shall not consider specific physical applications of relativistic oscillator equations (a separate paper is to be devoted to this question); rather, we shall concentrate entirely on the mathematical aspects.

2. Solution of the Nonrelativistic Problem

In Eq. (1.11) we go over to the dimensionless variables

$$\xi = p^2/m\omega\hbar, \qquad \lambda = E/\omega\hbar \tag{2.1}$$

and define a new unknown function:

$$\Psi_l(p) = u_l(\xi)/\xi^{1/4}. \tag{2.2}$$

As a result, the function $u_l(\xi)$ satisfies the equation ($s \equiv 2l + 1/4$)

$$u_l''(\xi) + \left[-\frac{1}{4} + \frac{\lambda}{\xi} + \frac{\frac{1}{4} - s^2}{\xi^2}\right] u_l(\xi) = 0 \tag{2.3}$$

with the boundary conditions

$$\xi^{-1/4} u_l(\xi)\big|_{\xi=0} = 0, \tag{2.4a}$$

$$u_l(\xi) < \infty \quad \text{for} \quad \xi \neq 0. \tag{2.4b}$$

Equation (2.3) is identical with the Whittaker equation [9, 10], i.e., its general solution can be written in the form

$$u_l(\xi) = C_1 M_{\lambda,s}(\xi) + C_2 M_{\lambda,-s}(\xi), \tag{2.5}$$

or

$$u_l(\xi) = B_1 W_{\lambda,s}(\xi) + B_2 W_{-\lambda,s}(-\xi), \tag{2.6}$$

where $M_{\lambda,s}(\xi)$ and $W_{\lambda,s}(\xi)$ are Whittaker functions.

Noting also that [9]

$$M_{\lambda,-s}(\xi) = \xi^{(-2l+1)/4} e^{-\xi/2} \Phi\left(-\frac{2l-1}{4} - \lambda;\ \frac{1}{2} - l, \xi\right) \tag{2.7}$$

and taking into account the boundary condition (2.4a), we find that $C_2 = 0$. Consequently [10]

$$u_l(\xi) = C_1 M_{\lambda,s}(\xi) = C_1 \left[\frac{\Gamma\left(l + \frac{1}{2}\right) e^{-i\pi(\lambda - s - 1/2)}}{\Gamma\left(\lambda + s + \frac{1}{2}\right)} W_{\lambda,s}(\xi) + \frac{\Gamma\left(l + \frac{1}{2}\right) e^{-i\pi\lambda}}{\Gamma\left(\frac{1}{2} + s - \lambda\right)} W_{-\lambda,s}(\xi)\right]. \tag{2.8}$$

Now as $\xi \to \infty$

$$W_{-\lambda,s}(-\xi) \sim \xi^{\lambda} e^{\xi/2}(1 + O(\xi^{-1})), \tag{2.9}$$

and the boundary condition (2.4b) is therefore equivalent to the requirement

$$\frac{1}{\Gamma\left(\frac{1}{2} + s - \lambda\right)} = 0. \tag{2.10}$$

By virtue of the well-known property of the Γ function [9], Eq. (2.10) yields the quantization rule for the energy of a nonrelativistic oscillator:

$$E_n = \omega\hbar(2n + l + 3/2), \qquad n = 0, 1, 2, \ldots \tag{2.11}$$

The eigenfunctions corresponding to the spectrum (2.11) have the form

$$u_l(\xi) = C_{n,l} W_{n + 1/2(l + 3/2), \frac{2l+1}{4}}(\xi) = D_{n,l} M_{n + \frac{1}{2}\left(l + \frac{3}{2}\right), \frac{2l+1}{4}}(\xi), \tag{2.12}$$

where $C_{n,l}$ and $D_{n,l}$ are normalization constants.

If we had started from (2.6) and had first taken into account the boundary condition (2.4b), we would have obtained $B_2 = 0$. Then, using the representation

$$W_{\lambda,s}(\xi) = \frac{\Gamma(-2s)}{\Gamma\left(\frac{1}{2} - s - \lambda\right)} M_{\lambda,s}(\xi) + \frac{\Gamma(2s)}{\Gamma\left(\frac{1}{2} + s - \lambda\right)} M_{\lambda,s}(-\xi), \tag{2.13}$$

we would have obtained the energy spectrum (2.11) as a consequence of the boundary condition (2.4a). Thus, in the investigated representation of the oscillator solutions (2.5)-(2.6) in terms of Whittaker functions, the boundary conditions at the origin and at infinity enter symmetrically in the derivation of the expression for the energy levels. This contrasts with the usual approach [8], in which the boundary conditions at the origin and at infinity play a different role. The condition at infinity leads to two different energy quantization rules, of which one, namely,(2.11), is then selected by means of the condition at the origin.*

The solutions of Eq. (1.11) that satisfy the normalization conditions

$$\int_0 \Psi_{nl}(p)\Psi_{n'l}(p)p^2\,dp = \delta_{nn'}, \tag{2.14}$$

can be expressed in terms of Whittaker functions as follows:

$$\Psi_{nl}(p) = \frac{1}{(m\omega\hbar\xi)^{1/4}}\sqrt{\frac{2(-1)^{n+l+1}\Gamma\left(-n-l-\frac{1}{2}\right)}{\pi n!}}\; W_{n+\frac{1}{2}\left(l+\frac{3}{2}\right),\,\frac{2l+1}{4}}(\xi). \tag{2.15}$$

3. Relativistic Problem

Let us now turn to Eq. (1.10). Setting

$$\Psi_l(\chi) = \frac{\eta_l(\chi)}{(mc\,\mathrm{sh}\,\chi)^{1/2}}, \tag{3.1}$$

we obtain an equation for the new desired function $\eta_l(\chi)$

$$\eta_l''(\chi) - \mathrm{cth}\,\chi\,\eta_l'(\chi) + \left[2\frac{mc^2}{\omega^2\hbar^2}(E - mc^2) - \frac{3}{2} + \frac{3}{4}\mathrm{ch}^2\chi - \frac{4m^2c^4}{\omega^2\hbar^2}\mathrm{sh}^2\frac{\chi}{2} - \frac{l(l+1)}{\mathrm{sh}^2\chi}\right]\eta_l(\chi) = 0. \tag{3.2}$$

It is convenient to introduce the dimensionless variable

$$\xi = \frac{4mc^2}{\omega\hbar}\mathrm{sh}^2\frac{\chi}{2}, \tag{3.3}$$

which is proportional to the relativistic binding energy:

$$W = 2(E - mc^2) = \omega\hbar\xi. \tag{3.4}$$

In the nonrelativistic limit we obviously have $\xi \to p^2/m\omega\hbar$ [cf. (2.1)].

Further, if we introduce the notation†

$$2\frac{mc^2}{\omega\hbar} \equiv k, \quad \frac{2l+1}{4} = s, \quad \frac{E - mc^2}{2\omega\hbar} - \frac{3}{8k} \equiv \lambda, \tag{3.5}$$

Eq. (3.2) is finally replaced by

$$\eta_l''(\xi) + \left[-\frac{1}{4(1+\xi/2k)} + \frac{\lambda}{\xi(1+\xi/2k)} + \frac{\frac{1}{4} - s^2}{\xi^2(1+\xi/2k)^2}\right]\eta_l(\xi) = 0, \tag{3.6}$$

the boundary conditions for $\eta_l(\xi)$ taking the form

$$\xi^{-1/4}\eta_l(\xi)\big|_{\xi=0} = 0, \tag{3.7a}$$

$$\eta_l(\xi) < \infty \quad \text{for} \quad \xi \neq 0. \tag{3.7b}$$

Equation (3.6) has three singular points: two regular points at $\xi = 0$ and $\xi = -2k$ and one irregular at $\xi = \infty$. It follows from the general theory of differential equations [11] that the solution of this type cannot be expressed in terms of elementary functions. If one seeks the solution in the form of expansions with respect to known special functions:

$$\eta(\xi) = \sum a_n(\lambda)\varphi_n(\xi), \tag{3.8}$$

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* By itself, fulfilment of the boundary condition at the origin does not yield any expression for the energy levels at all.

†In the nonrelativistic limit $\lambda \to 2E_{n.r}/\omega\hbar \quad \lambda_{n.r}$.

then recursion relations are obtained for the coefficients $\alpha_n(\lambda)$ that contain not less than three terms. Such a situation obtains, for example, in the case of the Mathieu equation and the equations for spheroidal functions.

The complexity of the resulting recursion relations depends largely on the extent to which a felicitous choice of the basis functions $\varphi_n(\xi)$ is made, the solutions $\alpha_n(\lambda)$ of these relations being such that the expansion (3.8) converges only for certain (proper!) values of the parameter λ.

In the nonrelativistic limit ($k \to \infty$) the boundary-value problem (3.6)-(3.7) goes over into the previously considered problem (2.3)-(2.4). In this limit the points $\xi = 0$ and $\xi = \infty$ remain singular for the "degenerate" equation. Taking into account this circumstance in solving the relativistic problem, we choose the Whittaker functions, i.e., the solutions of the nonrelativistic problem, as the basis $\varphi_n(\xi)$. In other words, the expansion (3.8) takes the form

$$\eta_l(\xi) = \sum_{r=0}^{\infty} a_r W_{r+\frac{2l+3}{4},\ \frac{2l+1}{4}}(\xi). \tag{3.9}$$

Substituting (3.9) into (3.6) and taking into account (2.4), we obtain

$$\sum_{r=0}^{\infty} a_r \Big[(\lambda - \varkappa) W_{\varkappa,s}(\xi) + \frac{\lambda}{2k} \xi W_{\varkappa,s}(\xi) - \frac{1}{8k} \xi^2 W_{\varkappa,s}(\xi) + \frac{1}{k} \xi^2 W_{\varkappa,s}(\xi) + \frac{\xi^3}{4k^2} W''_{\varkappa,s}(\xi) \Big] = 0, \tag{3.10}$$

where

$$\varkappa = r + \frac{2l+3}{4} = r + s + \frac{1}{2}. \tag{3.11}$$

Now the Whittaker functions satisfy the recursion relations [9, 10]

$$\xi W_{\varkappa,s}(\xi) = 2\varkappa W_{\varkappa,s}(\xi) + W_{\varkappa+1,s}(\xi) + \left[\left(\varkappa - \frac{1}{2} \right)^2 - s^2 \right] W_{\varkappa-1,s}(\xi), \tag{3.12a}$$

$$\xi^2 W''_{\varkappa,s}(\xi) - \frac{\xi^2 W_{\varkappa,s}(\xi)}{8} = \frac{1}{8} \left[\left(\varkappa - \frac{1}{2} \right)^2 - s^2 \right] \left[\left(\varkappa - \frac{3}{2} \right)^2 - s^2 \right] W_{\varkappa-2,s}(\xi)$$
$$- \frac{\varkappa}{2} \left[\left(\varkappa - \frac{1}{2} \right)^2 - s^2 \right] W_{\varkappa-1,s}(\xi) + \frac{1}{4} \left(3s^2 - 5\varkappa^2 - \frac{3}{4} \right) W_{\varkappa,s}(\xi) - \frac{1}{2} W_{\varkappa+1,s}(\xi) + \frac{1}{8} W_{\varkappa+2,s}(\xi), \tag{3.12b}$$

$$\xi^3 W''_{\varkappa,s}(\xi) = \frac{1}{4} \left[\left(\varkappa - \frac{1}{2} \right)^2 - s^2 \right] \left[\left(\varkappa - \frac{3}{2} \right)^2 - s^2 \right] \left[\left(\varkappa - \frac{5}{2} \right)^2 - s^2 \right] W_{\varkappa-3,s}(\xi)$$
$$+ \frac{1}{4} \left[\left(\varkappa - \frac{1}{2} \right)^2 - s^2 \right] \left[\left(\varkappa - \frac{3}{2} \right)^2 - s^2 \right] (2\varkappa - 5) W_{\varkappa-2,s}(\xi)$$
$$+ \frac{1}{4} \left[\left(\varkappa - \frac{1}{2} \right)^2 - s^2 \right] \left[s^2 - \varkappa^2 - 5\varkappa + \frac{15}{4} \right] W_{\varkappa-1,s}(\xi)$$
$$- \frac{1}{4} \left[4\varkappa(\varkappa^2 - s^2) - \varkappa \right] W_{\varkappa,s}(\xi) + \frac{1}{4} \left[s^2 - \varkappa^2 + 5\varkappa + \frac{15}{4} \right] W_{\varkappa+1,s}(\xi)$$
$$+ \frac{1}{4} (2\varkappa + 5) W_{\varkappa+2,s}(\xi) + \frac{1}{4} W_{\varkappa+3,s}(\xi). \tag{3.12c}$$

Taking into account (3.12), we find the desired recursion relations for the coefficients α_r:

$$a_3^{(s)} a_3 + b_2^{(s)} a_2 + c_1^{(s)} a_1 + d_0^{(s)} a_0 = 0,$$

$$a_4^{(s)} a_4 + b_3^{(s)} a_3 + c_2^{(s)} a_2 + d_1^{(s)} a_1 + e_0^{(s)} a_0 = 0,$$

$$a_5^{(s)} a_5 + b_4^{(s)} a_4 + c_3^{(s)} a_3 + d_2^{(s)} a_2 + e_1^{(s)} a_1 + f_0^{(s)} a_0 = 0,$$

$$a_{r+3}^{(s)} a_{r+3} + b_{r+2}^{(s)} a_{r+2} + c_{r+1}^{(s)} a_{r+1} + d_r^{(s)} a_r + e_{r-1}^{(s)} a_{r-1} + f_{r-2}^{(s)} a_{r-2} + \frac{1}{16k^2} a_{r-3} = 0 \quad (r \geqslant 3), \tag{3.13}$$

where

$$a_r^{(s)} = \frac{1}{16k^2}(r-2)(r-1)r(r-2+2s)(r-1+2s)(r+2s),$$

$$b_r^{(s)} = \frac{(r-1)r(r-1+2s)(r+2s)}{8k}\left(1-\frac{r-2+s}{k}\right)$$

$$c_r^{(s)} = \frac{r(r+2s)}{2k}\left(\lambda - r - s - \frac{1}{2} - \frac{r^2+2rs+6r+6s-1}{8k}\right),$$

$$d_r^{(s)} = \lambda - \left(r+s+\frac{1}{2}\right) + \frac{\lambda}{k}\left(r+s+\frac{1}{2}\right) - \frac{1}{4k}(5r^2+2s^2$$

$$+ 10rs + 5r + 5s + 2) - \frac{r+s+1/2}{4k^2}(r^2+2sr+r+s),$$

$$e_r^{(s)} = \frac{1}{2k}\left(\lambda - 1 + \frac{6-r^2-2rs+4r+4s}{8k}\right),$$

$$f_r^{(s)} = \frac{1}{8k}\left(1+\frac{r+s+3}{k}\right) \quad \left(s=\frac{2l+1}{4}\right) \tag{3.14}$$

Equations (3.13) in conjunction with the normalization condition for the wave function enable one, in principle, to determine the proper values of λ and the coefficients $\alpha_n(\lambda)$ in the expansion (3.9). Let us calculate λ to terms of order $1/k$. Setting

$$\lambda = \lambda_0 + \lambda'/k, \tag{3.15}$$

substituting (3.15) into (3.13), and comparing the coefficients of the same powers of k, we obtain

$$\lambda_{nl}^0 = n + s + \frac{1}{2} = n + \frac{1}{2}\left(l+\frac{3}{2}\right), \tag{3.16}$$

$$\lambda_{nl}' = \frac{1}{4}\left[\left(n+s+\frac{1}{2}\right)^2 - 3s^2 + \frac{3}{4}\right] = \frac{1}{32}[8n^2 - 4l^2 + 8nl + 12n + 9]. \tag{3.17}$$

On the other hand, it follows from (3.5) in the limit $k \to \infty$ that

$$\lambda_{nl} = \frac{E_{nl} - mc^2}{2\omega\hbar} - \frac{3}{2k} \xrightarrow[k\to\infty]{} \frac{E_{nl}^{\text{n.r}} + \Delta E'}{2\omega\hbar} - \frac{3}{8k} = \lambda_{nl}^0 + \frac{\lambda_{nl}'}{k}, \tag{3.18}$$

from which we obtain an expression for the energy levels of the relativistic oscillator to terms $\sim 1/c^2$

$$E_{nl} = mc^2 + E_{nl}^{\text{n.r}} + \Delta E_{nl}', \tag{3.19}$$

where

$$E_{nl}^{\text{n.r}} = \omega\hbar\left(2n + l + \frac{3}{2}\right),$$

$$\Delta E_{nl}' = \frac{\omega^2\hbar^2}{16mc^2}\left[\left(2n+l+\frac{3}{2}\right)^2 - 3l(l+1) + \frac{33}{4}\right]. \tag{3.20}$$

As Eq. (3.20) shows, the relativistic correction lifts the fortuitous degeneracy of the energy levels in the orbital angular momentum. Equations (3.19)-(3.20) can also be deduced in another manner. Namely, we substitute (3.12) into (3.10) and retain only the terms $W_{r+1/2+s,s}(\xi)$ in the expansion (the remaining terms can be neglected in the limit $k \to \infty$). Then, taking into account (3.17), we obtain

$$\sum_{r=0}^{\infty} a_r\left\{\lambda^0 - \varkappa + \frac{1}{k}\left[\lambda^0\varkappa + \lambda' - \frac{1}{4}\left(5\varkappa^2 - 3s^2 + \frac{3}{4}\right)\right]\right\} W_{\varkappa,s}(\xi) = 0. \tag{3.21}$$

Since the functions $W_{\varkappa,s}(\xi)$ are independent, each of the coefficients in the expansion (3.21) vanishes, which again leads to Eqs. (3.19)-(3.20). This method of calculating the proper values is equivalent to perturbation theory.

4. Equation for the s Wave

It is of interest to consider Eq. (1.10) for $l = 0$ (s wave) separately, since the mathematically well-known modified Mathieu functions [12] are its solutions. We may mention that the solutions of Eq. (12) for arbitrary l can be regarded as associated modified Mathieu functions.*

*In [13], the associated Mathieu functions are introduced on the basis of the equation for spheroidal functions.

In Eq. (1.10) for $l = 0$, we introduce a new desired function:

$$y(\chi) = \Psi_0(\chi)\operatorname{sh}\chi. \tag{4.1}$$

Then

$$\frac{d^2y(\chi)}{d\chi^2} + \left[\frac{2mc^2}{\omega^2\hbar^2}E - 1 - \frac{2m^2c^4}{\omega^2\hbar^2}\operatorname{ch}\chi\right]y(\chi) = 0. \tag{4.2}$$

The nonrelativistic equation corresponding to (4.2) is

$$\frac{d^2y(p)}{dp^2} + \left[\frac{2E}{m\omega^2\hbar^2} - \frac{p^2}{m^2\omega^2\hbar^2}\right]y(p) = 0, \tag{4.3}$$

[it can be obtained from Eq. (1.11) for $l = 0$ by the substitution $y(p) = p\Psi_0(p)$]. Introducing the new variable $x = \chi/2$ into (4.2) and setting

$$a(k) = 4\left(\frac{2mc^2}{\omega^2\hbar^2}E - 1\right) = 4\left(\frac{E}{\omega\hbar}k - 1\right), \quad k = 2\frac{mc^2}{\omega\hbar}, \tag{4.4}$$

we obtain the equation

$$\frac{d^2y(x)}{dx^2} + (a - 2k^2\operatorname{ch}2x)y(x) = 0. \tag{4.5}$$

The boundary conditions imposed on $y(x)$ are

$$y(0) = 0, \tag{4.6}$$

$$y(x) < \infty \quad \text{for } x \neq 0. \tag{4.7}$$

Study of the boundary-value problem (4.5)-(4.7) showed that a solution exists only for the values $a = a_{2n+1}(k^2)$ that satisfy the transcendental equation

$$a - 1 - q - \cfrac{q^2}{a - 9 - \cfrac{q^2}{a - 25 - \ldots \atop \cdots\cdots\cdots \atop a - (2r+1) - \cfrac{q^2}{\cdots}}} = 0. \tag{4.8}$$

The left side of (4.8) is an infinite continued fraction. For $a = a_{2n+1}(k^2)$ the function $y(x)$ can be expressed in terms of one of the modified Mathieu functions [9, 12]

$$y(x) = Gek_{2n+1}(x, -k^2) \equiv \frac{ce_{2n+1}(0, k^2)}{\pi k A_1^{2n+1}(k^2)}\operatorname{th}\chi\sum_{r=0}^{\infty}(-1)^r(2r+1)A_{2r+1}^{2n+1}(k^2)K_{2r+1}(2k\operatorname{ch}x), \tag{4.9}$$

where $ce_{2n+1}(x, k^2)$ is a Mathieu function of the second kind and $K_{2r+1}(z)$ is a MacDonald function. The coefficients of the expansion (4.9) are determined by the recursion relations

$$(a - k^2 - 1)A_1^{2n+1} - k^2A_3 = 0,$$

$$[a - (2r+1)^2]A_{2r+1}^{2n+1} - k^2(A_{2r-1}^{2n+1} + A_{2r+3}^{2n+1}) = 0 \quad (r \geqslant 1). \tag{4.10}$$

To calculate the relativistic corrections to the energy levels, it is necessary to find asymptotic expressions for the parameters (4.10) as $k \to \infty$. It should be noted that the asymptotic expansions given in the literature [9, 11, 12] for the eigenvalues of the modified Mathieu equation (4.5) are incorrect. This error is due to the fact that in the derivation of the expansions use is made of symmetry properties of the eigenvalues that do not hold for large k^2. Applying to Eq. (4.5) the method developed in [14], we obtain

$$a_n(k) \xrightarrow[k\to\infty]{} 2k^2 + 2(2n+1)k + \frac{2n^2 + 2n + 1}{4}. \tag{4.11}$$

Noting also that, in accordance with (4.4),

$$a(k) \xrightarrow[k\to\infty]{} \frac{8mc^2}{\omega^2\hbar^2}(mc^2 + E^{n.r} + \Delta E), \tag{4.12}$$

where ΔE is the correction of order $1/k$, and equating (4.11) and (4.12), we find

$$E^{n,l}=\omega\hbar\left(2n+\frac{3}{2}\right),$$
$$\Delta E=\frac{\omega^2\hbar^2}{16mc^2}\left[\left(2n+\frac{3}{2}\right)^2+\frac{33}{4}\right]. \tag{4.13}$$

Obviously, the relations (4.13) are a special case of (3.19) for $l = 0$.

5. Exactly Solvable Relativistic Problem

It is well known that the Hamiltonian (1.1) of the nonrelativistic oscillator is invariant under transformations of the group U(3). If one requires that the relativistic Hamiltonian be U(3)-symmetric, then one can show that the interaction potential must have the form (in units for which $\hbar = 2m = c = 1$)

$$V=\omega^2\left(\Delta_{\vartheta,\varphi}+r^{(2)}\right)e^{i\frac{d}{dr}}, \tag{5.1}$$

where $\Delta_{\vartheta,\varphi}$ is the angular part of the Laplace operator, and $r^{(2)}$ is the generalized degree calculated in accordance with the formula [3]

$$r^{(\lambda)}=i^{\lambda}\frac{\Gamma(-ir+\lambda)}{\Gamma(-ir)}. \tag{5.2}$$

One can readily show that in the nonrelativistic limit

$$V(r)\rightarrow\omega^2r^2. \tag{5.3}$$

The radial part of the relativistic Schrödinger equation with the potential (5.1) can be expressed as follows:

$$\left[2\operatorname{ch}i\frac{d}{dr}+\frac{l(l+1)}{r^{(2)}}e^{i\frac{d}{dr}}+\omega^2\left(l(l+1)+r^{(2)}\right)e^{i\frac{d}{dr}}-2E_q\right]\Psi_{ql}(r)=0. \tag{5.4}$$

By analogy with the problem of the nonrelativistic oscillator, we shall seek the solution of Eq. (5.4) in the form

$$\Psi_{ql}(r)=C(-r)^{(l+1)}M(r)\Omega_{nl}(r^2). \tag{5.5}$$

The factors $(-r)^{(l+1)}$ and

$$M(r)=\omega^{ir}\Gamma\left(ir+\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4}{\omega^2}}\right) \tag{5.6}$$

are related to the behavior of the solution $\Psi_{ql}(r)$ at the points $r = 0$ and $r = \infty$, respectively, and $\Omega_{nl}(r^2)$ is a polynomial of n-th degree (n is the radial quantum number).

In the nonrelativistic limit we obviously have

$$(r)^{(l+1)}\rightarrow(-r)^{l+1},$$
$$M(r)\rightarrow e^{-\omega r^2/2}. \tag{5.7}$$

Substituting (5.5) into Eq. (5.4), we obtain an equation for the polynomials $\Omega_{nl}(r^2)$:

$$\left\{A(v)e^{-i\frac{d}{dr}}+B(r)e^{i\frac{d}{dr}}-C(r)\,2E\right\}\Omega_{ln}(r^2)=0, \tag{5.8}$$

where the coefficients A, B, and C are given by the expressions

$$A(r)=\omega^2r^4-ir^3\omega^2\left[2l+1+\sqrt{1+\frac{4}{\omega^2}}\right]-r^2\left[\omega^2l(l+1)+\omega^2(2l+1)\sqrt{1+\frac{4}{\omega^2}}+1\right]$$
$$+ir\left[\omega^2l(l+1)\sqrt{1+\frac{4}{\omega^2}}+(2l+1)+l(l+1)\right],$$
$$B(r)=-\omega^2r^4-2i\omega^2r^3+r^2[\omega^2-\omega^2l(l+1)-1]-ir[1+\omega^2l(l+1)]-l(l+1),$$
$$C(r)=-ir^3\omega-\omega r^2\left[l+\frac{1}{2}\sqrt{1+\frac{4}{\omega^2}}-\frac{1}{2}\right]+\frac{irl\omega}{2}\left[\sqrt{1+\frac{4}{\omega^2}}-1\right]. \tag{5.9}$$

Simple, but rather lengthy calculations lead to the following exact formula for the energy levels of the relativistic oscillator:

$$E_{ln} = \omega\left(2n + l + \frac{3}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{\omega^2}}\right). \tag{5.10}$$

It can be seen from (5.10) that the levels we have found differ from the corresponding nonrelativistic levels only by the shift of the "zero-point" oscillations by $(\omega/2)\sqrt{1 + (4/\omega^2)}$. The degeneracy of the levels in Eq. (5.10) is the same as that of the levels (2.11) of the nonrelativistic oscillator. This fact indicates that the relativistic oscillator described by Eq. (5.4) possesses a hidden "dynamical" U(3) symmetry. We note in this connection that it would be interesting to go over, on the basis of Eq. (5.4), to the formalism of infinite-component fields, in the same way as is done in [6] for the Coulomb interaction.

We shall now write down explicitly the polynomials $\Omega_{nl}(r^2)$ for $l = 0$ and a few of the first values of n:

$$\Omega_{00}(r^2) = 1, \quad \Omega_{10}(r^2) = 1 - \frac{4}{5 + 3\sqrt{1 + 4/\omega^2}}r^2,$$

$$\Omega_{20}(r^2) = 1 - \frac{40(3 + \sqrt{1 + 4/\omega^2})r^2}{2(7 + 5\sqrt{1 + 4/\omega^2}) + 5(5 + 3\sqrt{1 + 4/\omega^2})(3 + \sqrt{1 + 4/\omega^2})} + \frac{16r^4}{2(7 + 5\sqrt{1 + 4/\omega^2}) + 5(5 + 3\sqrt{1 + 4/\omega^2})(3 + \sqrt{1 + 4/\omega^2})}. \tag{5.11}$$

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