SPECTRA OF CERTAIN RANDOM WAVES PROPAGATING IN NONLINEAR MEDIA

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The spatial spectrum of a random simple wave propagating in an ideal medium is obtained. The law of conservation of the spectrum width is found for such waves along with other conservation laws. A portion of these laws is generalized for the case when the wave satisfies the Korteweg–de Vries equation or the Bürgers equation. The spatial spectra are found for the density and flux of a beam of noninteracting particles treated in the hydrodynamic approximation. It is shown that due to hydrodynamic instability of the beam these spectra acquire a universal power-law form over a sufficiently long time.

INTRODUCTION

The statistical properties of random waves propagating in nonlinear media and satisfying nonlinear partial differential equations have heretofore been studied little. This is associated mainly with the fact that the nonlinearity of the original equations does not allow reasonably accurate closure of the equations for the moments of random waves. Therefore such random waves are usually investigated on the assumption of low nonlinearity [1-4] or by means of inadequately substantiated hypotheses concerning the character of the relationship between the higher elements and the lower ones [1, 5, 6].

Nevertheless, it turns out that from a nonlinear partial differential equation of the first order one may go over to a closed equation for the finite-dimensional probability density function (PDF) of a wave which satisfies the equation considered [7]; this allows analytic investigation of the statistical properties of waves which propagate in essentially nonlinear media.

In the present paper equations whose derivation has been given in [7] are used to find and analyze the spatial spectrum of a random wave which satisfies the equation of a simple wave. The applicability limits of the derived expressions are clarified. The conservation laws are derived for certain statistical characteristics of a random simple wave, such as the width of the spatial spectrum. The validity of a portion of them is proved for waves which propagate in nonlinear media having dispersion and satisfy the Korteweg –de Vries or Bürgers equations. In conclusion the spatial spectrum is found for the density and flux of a beam of noninteracting particles which is treated as the hydrodynamic approximation. It is demonstrated that over sufficiently long times these spectra acquire a universal power-law form which is produced by the multistream nature of the beam.

1. The Spectra of Simple Waves

1.1. One example of random waves propagating in nonlinear media which is among those which are simplest and of greatest practical interest, may consist of waves satisfying the equation of a simple wave:

$$\frac{\partial u}{\partial t} + \beta u \frac{\partial u}{\partial x} = 0 \tag{1.1}$$

and the random initial condition

$$u(x, 0) = u_0(x), \quad \langle u_0 \rangle = 0. \tag{1.2}$$

One arrives at this equation in describing a wave in acoustics or in a plasma, waves on the surface of a fluid, etc., in those cases when dispersion effects may be neglected (see, for example, [8, 9]). It is also

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of interest to investigate the statistical properties of the solution of the equation of a simple wave because it provides a qualitatively correct description of the evolution of more complex nonlinear waves and, specifically, the velocity fields of a turbulent fluid.

1.2. As was shown in [7], from (1.1) one may go over to the closed equation for the two-dimensional PDF $W_2(u_1, u_2; x_1, x_2, t)$ of the values of the wave $u(x, t)$ at two points x_1 and x_2 at one time t:

$$\frac{\partial W_2}{\partial t} + \beta \frac{\partial}{\partial u_1}\left(u_1 \int_{-\infty}^{u_1} \frac{\partial W_2}{\partial x_1} du_1\right) + \beta \frac{\partial}{\partial u_2}\left(u_2 \int_{-\infty}^{u_2} \frac{\partial W_2}{\partial x_2} du_2\right) = 0 \tag{1.3}$$

with the initial condition

$$W_2(u_1, u_2; x_1, x_2, 0) = W_0(u_1, u_2; x_1, x_2), \tag{1.4}$$

which is completely determined from the initial condition (1.2).

At the initial time let $u_0(x)$ be a homogeneous function of x. Then (1.4) takes the form

$$W_0 = W_0(u_1, u_2; s) \quad (s = x_1 - x_2), \tag{1.5}$$

while (1.3) goes over into the equation

$$\frac{\partial W_2}{\partial t} + \beta \frac{\partial}{\partial u_1}\left(u_1 \int_{-\infty}^{u_1} \frac{\partial W_2}{\partial s} du_1\right) - \beta \frac{\partial}{\partial u_2}\left(u_2 \int_{-\infty}^{u_2} \frac{\partial W_2}{\partial s} du_2\right) = 0. \tag{1.6}$$

It is well known that the one-dimensional PDF of the wave $u(x, t)$ does not vary with time [7]. From Eq. (1.6) it is evident that one cannot say this of the two-dimensional PDF, since the nonlinearity of the original equation (1.1) leads to a substantial expansion of the spectrum of the wave with time and consequently to a pronounced change in the shape of the two-dimensional PDF.

1.3. Let us go over from the equation for the two-dimensional PDF (1.4) to the equation for the Fourier transform with respect to s of the two-dimensional characteristic function of a simple wave:

$$\Theta_2(\omega_1, \omega_2, \varkappa, t) = \int_{-\infty}^{\infty} \langle \exp[i\omega_1 u(x_1, t) + i\omega_2 u(x_2, t)] \rangle e^{i\varkappa s} ds. \tag{1.7}$$

The equation in $\Theta_2(\omega_1, \omega_2, \varkappa, t)$ can easily be derived from (1.6):

$$\frac{\partial \Theta_2}{\partial t} - \beta\omega_1 \frac{\partial}{\partial \omega_1}\left(\frac{\varkappa}{\omega_1}\Theta_2\right) + \beta\omega_2 \frac{\partial}{\partial \omega_2}\left(\frac{\varkappa}{\omega_2}\Theta_2\right) = 0. \tag{1.8}$$

Its initial condition, which derives from (1.5), has the form

$$\Theta_2(\omega_1, \omega_2, \varkappa, 0) = \Theta_0 = \iiint_{-\infty}^{\infty} \exp[i(\omega_1 u_1 + \omega_2 u_2 + \varkappa s)] W_0(u_1, u_2; s) du_1 du_2 ds. \tag{1.9}$$

Solving Eq. (1.8) by the method of characteristics, we obtain

$$\Theta_2 = \Theta_1(\omega_1)\Theta_1(\omega_2) 2\pi\delta(\varkappa) + \frac{g(\omega_1 + \beta\varkappa t, \omega_2 - \beta\varkappa t)\omega_1\omega_2}{(\omega_1 + \beta\varkappa t)(\omega_2 - \beta\varkappa t)}. \tag{1.10}$$

Here

$$g(\omega_1, \omega_2, \varkappa) = \Theta_0(\omega_1, \omega_2, \varkappa) - \Theta_1(\omega_1)\Theta_1(\omega_2) 2\pi\delta(\varkappa),$$

while

$$\Theta_1(\omega) = \langle e^{i\omega u(x, t)} \rangle$$

is the one-dimensional characteristic function of a homogeneous simple wave.

The solution (1.10) of Eq. (1.8) describes the evolution of the Fourier transform of the two-dimensional characteristic function of a simple wave accurately up to those t until which the ensemble of realizations which defines $\Theta_2(\omega_1, \omega_2, \varkappa, t)$ is an ensemble of unique functions. Otherwise Eq. (1.10) describes the statistical properties of a simple wave in that approximation in which one may neglect nonunique realizations.

1.4. Let us find the expression for the spatial spectrum $S(\varkappa, t)$ of a random simple wave:

$$S(\varkappa, t) = \int_{-\infty}^{\infty} \langle u(x, t) u(x + s, t) \rangle e^{i\varkappa s} ds. \tag{1.11}$$

The spectrum is associated with the characteristic function $\Theta_2(\omega_1, \omega_2, \varkappa, t)$ calculated above via the following equation:

$$S(\varkappa, t) = -\left.\frac{\partial^2 \Theta_2(\omega_1, \omega_2, \varkappa, t)}{\partial \omega_1 \partial \omega_2}\right|_{\omega_1 = \omega_2 = 0},$$

which can be rewritten in the following form with allowance for (1.10):

$$S(\varkappa, t) = \frac{1}{\beta^2 \varkappa^2 t^2} g(\beta\varkappa t, -\beta\varkappa t, \varkappa). \tag{1.12}$$

Note that the obtained spectrum has no singularities for $\varkappa = 0$, since from the definition of the function g it follows that

$$g(\beta\varkappa t, 0, \varkappa) = g(0, -\beta\varkappa t, \varkappa) = 0.$$

1.5. From (1.12) it is evident that the zero component of the spatial spectrum of the homogeneous simple wave is conserved: $S(0, t) = S_0$, or, what amounts to the same thing, the integral of the correlation function is conserved:

$$\int_{-\infty}^{\infty} \langle u(x, t) u(x+s, t) \rangle ds = \int_{-\infty}^{\infty} \langle u_0(x) u_0(x+s) \rangle ds = S_0.$$

Previously it was demonstrated in [7] that the one-point moments of a homogeneous simple wave are conserved – specifically, its mean-square: $\langle u^2(x, t) \rangle = \langle u_0^2 \rangle$. If the correlation length of the wave u(x, t) is introduced according to the equation

$$l(t) = \frac{1}{2} \frac{\int_{-\infty}^{\infty} \langle u(x, t) u(x+s, t) \rangle ds}{\langle u^2(x, t) \rangle},$$

it follows from what has been said above that the correlation length of a homogeneous simple wave $l = l_0 = S_0/2\langle u_0^2 \rangle$ is also conserved.

Thus, the nonlinearity of the propagation velocity of a homogeneous simple wave has a substantial effect on the form of its correlation function and spatial spectrum, and it does not lead to a change in the correlation length or the spectrum width $\varkappa_u = \varkappa_0 = \pi\langle u_0^2 \rangle / S_0$ if the latter is defined as

$$\varkappa_u(t) = \frac{\pi}{l(t)} = \frac{\int_{-\infty}^{\infty} S(\varkappa, t) d\varkappa}{S(0, t)}. \tag{1.13}$$

The result obtained becomes understandable if one takes into account the fact that the correlation length of the random wave u(x, t) is approximately equal to the mean distance between its neighboring zero values which is evidently conserved in the case of a simple wave.

It is easy to generalize the conservation laws derived above for other higher conservation moments of a homogeneous simple wave:

$$\int_{-\infty}^{\infty} \langle u^m(x, t) u^n(x+s, t) \rangle ds = \text{const}, \quad \langle u^n(x, t) \rangle = \text{const}.$$

1.6. Let us find the spatial spectrum of the wave u(x, t) in the case when $u_0(x)$ is a Gaussian homogeneous function. In this case the initial condition of Eq. (1.8) is the following:

$$\Theta_0 = \exp\left[-\frac{\sigma^2}{2}(\omega_1^2 + \omega_2^2)\right] \int_{-\infty}^{\infty} \exp(-K(s)\omega_1\omega_2 + i\varkappa s) ds,$$

$$\sigma^2 = K(0),$$

and the equation for the spatial spectrum (1.12) goes over into

$$S(\varkappa, t) = \exp(-\sigma^2 \beta^2 \varkappa^2 t^2) \int_{-\infty}^{\infty} \frac{\exp(\beta^2 \varkappa^2 t^2 K(s)) - 1}{\beta^2 \varkappa^2 t^2} e^{i\varkappa s} ds. \tag{1.14}$$

Let us clarify the asymptotics of this expression for $\varkappa \to \infty$. Having restricted ourselves to the first two terms of the expansion of K(s) into a Maclaurin series $K(s) \approx \sigma^2-(a/2)s^2$, we find that for such $\varkappa$ the spatial spectrum can be described by the equation

$$S(\varkappa, t) = \frac{1}{\varkappa^3}\sqrt{\frac{2\pi}{a}}\,\frac{1}{\beta^3 t^3}\exp\left(-\frac{1}{2\beta^2 t^2 a}\right). \tag{1.15}$$

The same kind of asymptotic expression was derived in [14] for the spectrum of a simple wave. Let us analyze the causes of the appearance of such asymptotics. It is clear that in order for the spectrum of the random wave u(x, t) to fall off according to the law $\varkappa^{-3}$ for $\varkappa \to \infty$ it is necessary for its Fourier spectrum to fall off at $\varkappa^{-3/2}$. But, as is known from the theory of Fourier integrals, for this it is required that u(x, t) be proportional to the function $\sqrt{x-x_0}$ in the neighborhood of at least one point x_0. However, if the evolution of a simple wave that is smooth for t = 0 is traced, then one can see that points in whose neighborhood u(x, t) behaves in such a way appear simultaneously with the appearance of nonuniqueness of the function u(x, t). Consequently, the asymptotic (1.15) of the spatial spectrum is associated with the appearance of nonunique realizations in the ensemble which defines the spectrum.

For any t > 0 the coefficient of $\varkappa^{-3}$ in (1.15) is not equal to zero. Thus, for arbitrarily small t nonunique realizations appear in the ensemble of realizations of a wave that is originally Gaussian. This can be explained by the fact that the Gaussian function du_0/dx exceeds an arbitrarily large stipulated value with a probability that is not equal to zero, while it is known that the simple wave u(x, t) is unique at the given time t only if the inequality $du_0/dx > 1/\beta t$ is satisfied (see, for example, [10]). With a growth of t to $t^* = 1/\beta\sqrt{3a}$ the coefficient of $\varkappa^{-3}$ in (1.15) increases, which indicates a growth of the contribution to the ensemble from the nonunique realization. It is obvious that the spectrum (1.14) may be considered sufficiently accurate only for $t < t^*$.

2. The Conservation Laws of Random Waves in Nonlinear Media Having Dispersion

2.1. Above, several conservation laws were derived for the spectrum and other spatial characteristics of a homogeneous simple wave. It turns out that certain of these characteristics are also valid when a random wave propagates in a nonlinear medium having a low dispersion and can be described by the Korteweg–de Vries equation or by the Bürgers equation.

2.2. We shall begin by considering a wave which propagates in a weakly dispersive medium having random sources and is governed by the Korteweg–de Vries equation:

$$\frac{\partial u}{\partial t} + \beta u\frac{\partial u}{\partial x} + \mu\frac{\partial^3 u}{\partial x^3} = \eta(x, t). \tag{2.1}$$

Henceforth we shall assume that the random sources $\eta(x, t)$ are Gaussian and delta-correlated in time:

$$\langle \eta(x, t)\eta(x+s, t+\tau)\rangle = D(s)\delta(\tau).$$

Moreover, we assume, as previously, that $u(x, 0) = u_0(x)$ is a random homogeneous function having known statistical properties ($\langle u_0\rangle = 0$).

We multiply (2.1) out by u(x, t) and average the result. Calculating the average which depends implicitly on $\eta(x, t)$ by means of the local method, as in [13], or by means of the Furutsu–Novikov formula [11, 12], we obtain the result

$$\frac{\partial\langle u^2\rangle}{\partial t} + \frac{\beta}{3}\frac{\partial\langle u^3\rangle}{\partial x}\left[\frac{\partial^2\langle u^2\rangle}{\partial x^2} - \frac{3}{2}\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle\right] = D(0). \tag{2.2}$$

From the homogeneity of $u_0(x)$ and $\eta(x, t)$ along x it follows that the wave u(x, t) is homogeneous for all t > 0. Therefore all averages in Eq. (2.2) are independent of x, and this equation goes over into

$$\frac{d\langle u^2(x, t)\rangle}{dt} = D(0).$$

Thus,

$$\langle u^2(x, t)\rangle = D(0)t + \langle u_0^2\rangle, \tag{2.3}$$

i.e., a nonlinear wave propagating in a weakly dispersive medium having sources that are delta-correlated in time behaves like a Brownian particle. However, if there are no random sources, then $\langle u^2(x, t)\rangle = \langle u_0^2\rangle$, which is the mean-square and at the same time the variance of the wave, is conserved. The latter fact is evidently a corollary of the energy conservation law of a wave which satisfies the Korteweg–de Vries equation [8].

2.3. Let us derive still another conservation law for the wave u(x, t) which satisfies Eq. (2.1). For this purpose we differentiate the average $\langle u(x_1, t)u(x_2, t)\rangle$ with respect to time and write the following equation, after making use of (2.1):

$$\frac{\partial \langle u(x_1, t) u(x_2, t)\rangle}{\partial t} + \frac{\beta}{2}\frac{\partial}{\partial x_1}\langle u^2(x_1, t) u(x_2, t)\rangle$$

$$+\frac{\beta}{2}\frac{\partial}{\partial x_2}\langle u(x_1, t) u^2(x_2, t)\rangle + \mu\left(\frac{\partial^3}{\partial x_1^3} + \frac{\partial^3}{\partial x_2^3}\right)\langle u(x_1, t) u(x_2, t)\rangle = \langle \eta(x_1, t) u(x_2, t)\rangle + \langle \eta(x_2, t) u(x_1, t)\rangle. \quad (2.4)$$

Taking account of the homogeneity of u(x, t) and η(x, t) along x and calculating the average in the right side by analogy with the calculation of the similar average in the derivation of (2.2), we rewrite (2.4) thus:

$$\frac{\partial K(s, t)}{\partial t} + \frac{\beta}{2}\frac{\partial}{\partial s}[B(s, t) - B(-s, t)] = D(s). \quad (2.5)$$

Here we have used the notation $s = x_1 = x_2$, $K = \langle u(x_1, t)u(x_2, t)\rangle$, $B(s) = \langle u^2(x_1, t)u(x_2, t)\rangle$. Integrating (2.5) with respect to s and assuming that K(s, t) and B(s, t) tend to zero for $|s| \to \infty$, we find that

$$\frac{dS(0, t)}{dt} = D,$$

where $D = \int_{-\infty}^{\infty} D(s)ds$, while $S(0, t) = \int_{-\infty}^{\infty} K(s, t)ds$ is the spatial spectrum of the wave u(x, t) for $\varkappa = 0$. Thus,

$$S(0, t) = Dt + S_0. \quad (2.6)$$

Consequently, the zero component of the spectrum of a wave which propagates in a weakly dispersive medium having random sources and satisfies Eq. (2.1) increases linearly with time as does its mean-square.

Having determined the width of the spectrum of the wave u(x, t) according to Eq. (1.13), we obtain the following expression for it with allowance for (2.3) and (2.6):

$$\varkappa_u(t) = \pi\frac{\langle u_0^2\rangle + D(0)t}{S_0 + Dt}. \quad (2.7)$$

Over times $t \ll D(0)/\langle u_0^2\rangle$, D/S_0, the effect of the random sources on the spatial spectrum is insubstantial, and $\varkappa_u = \pi\langle u_0^2\rangle/S_0$ coincides with the width of the wave spectrum at the initial time. Thus, for a wave propagating in an ideal medium having weak dispersion, the width of the spectrum is conserved, although the spectrum itself varies greatly with time.

Over times $t \ll D(0)/\langle u_0^2\rangle$, D/S_0, the width of the spatial spectrum (2.7) is entirely determined by the statistical properties of the sources and is independent of $u_0(x)$.

2.4. Let the homogeneous wave u(x, t) propagate in a dissipative medium and satisfy the Bürgers equation:

$$\frac{\partial u}{\partial t} + \beta u\frac{\partial u}{\partial x} = \gamma\frac{\partial^2 u}{\partial x^2}.$$

The energy of this wave decreases with time, and consequently $\langle u^2(x, t)\rangle$ decreases also. Nevertheless, it may be shown by analogy with the procedure used above that the value of the spatial spectrum at zero is conserved. However, the width of the spectrum decreases with increasing t, since all of the remaining components of the spectrum attenuate due to high-frequency dissipation.

3. Fluctuations of the Density and Flux of a Beam of Noninteracting Particles in the Hydrodynamic Approximation

3.1. We consider fluctuations of the density and flux near the beam of noninteracting particles. Usually only fluctuations associated with the discreteness of such beams (shot noise) are considered.

Sometimes, however, specifically in investigating a cold plasma (see, for example, [15]), it is more convenient to treat them as continuous fields which are characterized by two smooth functions, $\mathbf{u}(\mathbf{x}, t)$ (the velocity of a physically infinitely small volume) and $\rho(\mathbf{x}, t)$ (the density of particles in the volume) which satisfy the equation

$$\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} = F_\alpha(\boldsymbol{x}, \boldsymbol{u}, t),$$
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\alpha}(u_\alpha \rho) = 0 \qquad (\alpha, \beta = 1, 2, 3), \tag{3.1}$$

where $\mathbf{F}$ is the force acting on a particle of the stream. In a plasma this is the force created by the electric and magnetic fields.

3.2. Equations (3.1) are the exact corollary of the Liouville equation for $f_{6N}(\mathbf{x}^1, \ldots, \mathbf{x}^N, \mathbf{u}^1, \ldots, \mathbf{u}^N, t)$ which is the PDF of the coordinates and velocities of N particles of the beam:

$$\frac{\partial f_{6N}}{\partial t} + \sum_{n=1}^{N} u_\alpha^n \frac{\partial f_{6N}}{\partial x_\alpha^n} + \sum_{n=1}^{N} \frac{\partial}{\partial u_\alpha^n}\{F_\alpha(\boldsymbol{x}^n, \boldsymbol{u}^n; t) f_{6N}\} = 0. \tag{3.2}$$

In the hydrodynamic approximation its solution may be represented as follows:

$$f_{6N} = \int_{-\infty}^{\infty} \overset{N}{\cdots} \int_{-\infty}^{\infty} \prod_{n=1}^{N} \rho_n \, d\rho_n \, W_{4N}(\rho_1, \ldots, \rho_N, \boldsymbol{u}^1, \ldots, \boldsymbol{u}^N, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^N; t), \tag{3.3}$$

where W_{4N} is the 4N-dimensional PDF of the values of the density ρ and velocity $\mathbf{u}$ of the beam at N points at one time t. The coordinates $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N$ are parameters here, rather than variables, as in f_{6N}.

Thus, the solution of Eq. (3.2) allows the correlation function to be calculated and this means that the spectrum of the fluctuations of the density, the flux, and the other macroscopic characteristics of the particle beam treated in the hydrodynamic approximation may also be calculated.

3.3. Let us find the spatial spectra of the fluctuations of the density and flux in a hydrodynamic beam of particles. Both of them are fully defined by f_{12}, for which the equation (as is evident from (3.2)) is the following:

$$\frac{\partial f_{12}}{\partial t} + u_\alpha^1 \frac{\partial f_{12}}{\partial x_\alpha^1} + u_\alpha^2 \frac{\partial f_{12}}{\partial x_\alpha^2} = 0. \tag{3.4}$$

Here and throughout the subsequent analysis we shall assume for simplicity that $\mathbf{F} \equiv 0$.

If at the initial instant $\rho = \rho_0$ is constant while the random velocity field is uniform, then the initial condition of Eq. (3.4) takes the form

$$f_{12}(\boldsymbol{u}^1, \boldsymbol{u}^2, \boldsymbol{x}^1, \boldsymbol{x}^2, 0) = \rho_0^2 W_6(\boldsymbol{u}^1, \boldsymbol{u}^2; \boldsymbol{s}). \tag{3.5}$$

Here W_6 is the probability distribution of the velocity field of the beam at the initial time, while $\mathbf{s} = \mathbf{x}^1 - \mathbf{x}^2$. It is obvious that Eq. (3.4) proper may be rewritten as follows in this case:

$$\frac{\partial f_9}{\partial t} + v_\alpha \frac{\partial f_9}{\partial s_\alpha} = 0, \tag{3.6}$$

where $\mathbf{v} = \mathbf{u}^1 - \mathbf{u}^2$. Its solution with the initial condition (3.5) is:

$$f_9(\boldsymbol{u}^1, \boldsymbol{u}^2, \boldsymbol{s}; t) = \rho_0^2 W_6(\boldsymbol{u}^1, \boldsymbol{u}^2; \boldsymbol{s} - \boldsymbol{v}t).$$

Henceforth it is not this equation but the expression which is equivalent to it for

$$\Theta(\boldsymbol{\omega}, \boldsymbol{k}, \boldsymbol{\varkappa}, t) = \int_{-\infty}^{\infty} \overset{9}{\cdots} \int_{-\infty}^{\infty} f_9(\boldsymbol{u}^1, \boldsymbol{u}^2, \boldsymbol{s}; t) \exp(i\boldsymbol{\varkappa}\boldsymbol{s} + i\boldsymbol{\omega}\boldsymbol{u}^1 + i\boldsymbol{k}\boldsymbol{u}^2)\, d\boldsymbol{s}\, d\boldsymbol{u}^1 d\boldsymbol{u}^2 = \rho_0^2 \Theta_0(\boldsymbol{\omega} + \boldsymbol{\varkappa} t, \boldsymbol{k} - \boldsymbol{\varkappa} t, \boldsymbol{\varkappa}). \tag{3.7}$$

which is most convenient for us. Here

$$\Theta_0(\boldsymbol{\omega}, \boldsymbol{k}, \boldsymbol{\varkappa}) = \int_{-\infty}^{\infty} \overset{9}{\cdots} \int_{-\infty}^{\infty} W_6(\boldsymbol{u}^1, \boldsymbol{u}^2; \boldsymbol{s}) \exp(i\boldsymbol{\varkappa}\boldsymbol{s} + i\boldsymbol{\omega}\boldsymbol{u}^1 + i\boldsymbol{k}\boldsymbol{u}^2)\, d\boldsymbol{s}\, d\boldsymbol{u}^1 d\boldsymbol{u}^2.$$

The spatial spectrum of the fluctuations of ρ is determined by the following equation, with allowance for (3.3), (3.7):

$$S_\rho(\varkappa, t) = \rho_0^2 \Theta_0(\varkappa t, -\varkappa t, \varkappa), \tag{3.8}$$

while the spectral tensor of the fluctuations of the flux $\rho\mathbf{u}$ is determined by the equation

$$S_{\alpha\beta}(\varkappa, t) = -\rho_0^2 \frac{\partial^2}{\partial\omega_\alpha \partial k_\beta} \Theta_0(\omega + \varkappa t, k - \varkappa t, \varkappa)\Big|_{\omega=k=0}. \tag{3.9}$$

3.4. If at the initial time the uniform velocity field, which we shall assume to be one-dimensional for simplicity, is Guassian, it follows that (3.8) goes over into

$$S_\rho(\varkappa, t) = \rho_0^2 \int_{-\infty}^{\infty} [\exp(-\varkappa^2 t^2 D(s)) - \exp(-\varkappa^2 t^2 \sigma^2)]\, e^{i\varkappa s} ds + 2\pi\rho_0^2 \delta(\varkappa), \tag{3.10}$$

where $D(s) = \sigma^2 - K(s)$, while the spectrum of the flux (3.9) takes the form

$$S_{\rho u}(\varkappa, t) = \rho_0^2 \int_{-\infty}^{\infty} K(s) \exp(-\varkappa^2 t^2 D(s) + i\varkappa s)\, ds. \tag{3.11}$$

Let the random velocity field be a smooth function of coordinates for $t = 0$; then for $s \ll s_0$ (the characteristic scale of the initial velocity fluctuations) we have $D(s) = (a/2)s^2$, and at sufficiently large $\varkappa$ Eq. (3.10) goes over into

$$S_\rho(\varkappa, t) = \frac{\rho_0^2}{|\varkappa| t} \sqrt{\frac{2\pi}{a}} \exp\left(-\frac{1}{2t^2 a}\right). \tag{3.12}$$

The spectrum of the flux (3.11) has analogous asymptotics. Such a slow drop-off of the "tails" of the spectra, which leads to infinity of the mean-square density and flux of the beam, can be explained by the fact that the density of a multistream beam treated in the hydrodynamic approximation has singularities (see, for example, [8]).

Over sufficiently long times (i.e., in the region of developed multistream motion) Eq. (3.12) is simplified:

$$S_\rho(\varkappa, t) = \frac{\rho_0^2}{|\varkappa| t} \sqrt{\frac{2\pi}{a}},$$

and no longer describes the "tails" of the density and flux spectra but essentially describes the spectra themselves for all $\varkappa$ except a small domain near $\varkappa = 0$ having the width $\varkappa \sim (ts_0\sqrt{a})^{-1}$, which decreases with time.

It is clear that for $\varkappa \gtrsim 1/l_0$, where l_0 is the mean distance between stream particles, the hydrodynamic approximation is violated and the formulas derived here cease to describe the behavior of the beam correctly.

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