and (36).

LITERATURE CITED

1. M. Duneau, D. Jagolnitzer, and B. Souillard, Commun. Math. Phys., **31**, 191 (1973).
2. M. Duneau, D. Jagolnitzer, and B. Souillard, Commun. Math. Phys., **35**, 307 (1974).
3. M. Duneau, D. Jagolnitzer, and B. Souillard, J. Math. Phys., **16**, 1662 (1975).
4. M. Duneau and B. Souillard, Commun. Math. Phys., **47**, 155 (1976).
5. D. Ruelle, Statistical Mechanics, New York (1969).
6. J. Riordan, An Introduction to Combinatorial Analysis, Wiley, New York (1958).
7. R. A. Minlos, Usp. Mat. Nauk, **23**, 139 (1968).

ANALYTIC STRUCTURE OF THE S MATRIX FOR SOME CLASSES OF POTENTIALS

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The properties of Jost functions and generalizations of them introduced in the paper are used to study the analytic properties in the complex plane of wave numbers of the S matrix of elastic scattering for local potentials with hard core, nonlocal separable potentials, complex local potentials, and nonlocal separable potentials with hard core.

Among the various representations of the S matrix $S_l(k)$, one frequently uses a representation in the form of an expansion in an infinite product with respect to pole terms; this enables one to express explicitly the dependence of the cross sections of the scattering processes on the positions and widths of the resonances [1-3]. In the present paper, on the basis of the generalization of the method proposed in [1], we obtain an analogous representation of $S_l(k)$ for the following classes of quasipotentials:

(I) local potentials $V(r)$ with hard core $(V(r)=\infty,\ 0\leqslant r\leqslant R_c)$;

(II) nonlocal potentials $q_l(r)\cdot q_l(r')$;

(III) complex local potentials $W(r) = V_1(r) + iV_2(r)$;

(IV) nonlocal potentials $p_l(r)\cdot p_l(r')$ with hard core $(p_l(r)=\infty,\ 0\leqslant r\leqslant R_c)$.

It is assumed that all these quasipotentials vanish when $r \geq R$.

Our result is of physical interest and can be used in investigations of the three-particle scattering problem and in problems using the ordinary and the generalized optical models of nuclear reactions.

We shall proceed from the following general scheme. We define Jost functions $f_l(k)$, in terms of which the function $S_l(k)$ is expressed. If the Jost functions that are obtained are not entire functions of k, we construct equivalent entire functions, which can be expanded in an infinite product in accordance with Hadamard's theorem, and the expansion then obtained is used in the expression for the S matrix.

In what follows, we shall use this notation. Expressions indicated by the Roman numerals I-IV correspond to the types of quasipotential listed above. All other expressions are common to all the quasipotentials.

By the usual method set forth, for example, in [1, 4] one can obtain the following expressions for the Jost functions:

$$f_{l+}^{(1)}(k)=\exp\left(-i\frac{l\pi}{2}\right)w_l^{(+)}(kR_c)-\int_{R_c}^{R}dr'g_l(k;R_c,r')V(r')f_{l+}(k,r'), \tag{1.I}$$

$$f_{l+}^{(2)}(k)=1+\frac{N_l^{(2)}(k)}{D_l^{(2)}(k)}k^{-1}\exp\left(-i\frac{l\pi}{2}\right)\int_0^R dr\,u_l(kr)q_l(r), \tag{1.II}$$

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$$f_{l+}^{(3)}(k)=1+\frac{k^l\exp(-il\pi)}{(2l+1)!!}\int_0^R dr\,w_l^{(+)}(kr)W(r)\varphi_l(k,r), \tag{1.III}$$

$$f_{l+}^{(4)}(k)=\exp\left(-i\frac{l\pi}{2}\right)w_l^{(+)}(kR_c)-\frac{N_l^{(4)}(k)}{D_l^{(4)}(k)}\int_{R_c}^R dr'\,g_l(k;R_c,r')p_l(r'), \tag{1.IV}$$

where

$$g_l(k;R_c,r)=\exp(-il\pi)k^{-1}[u_l(kR_c)w_l^{(+)}(kr)-u_l(kr)w_l^{(+)}(kR_c)],\quad N_l^{(2)}(k)=\exp\left(-i\frac{l\pi}{2}\right)\int_0^R dr\,q_l(r)w_l^{(+)}(kr),$$

$$N_l^{(4)}(k)=\exp\left(-i\frac{l\pi}{2}\right)\int_{R_c}^R dr\,p_l(r)w_l^{(+)}(kr),\quad D_l^{(2)}(k)=1-\int_0^R dr\,q_l(r)\int_0^r dr'\,g_l(k;r,r')q_l(r'),$$

$$D_l^{(4)}(k)=1-\int_{R_c}^R dr\,p_l(r)\int_{R_c}^r dr'g_l(k;r,r')p_l(r'),\quad D_l^{(2,4)}(k)=D_l^{(2,4)}(-k),$$

$f_{l+}(k,r)$ is the solution of the radial Schrödinger equation with the boundary condition

$$\lim_{r\to\infty}f_{l+}(k,r)\exp(-ikr)=1,$$

$\varphi_l(k,r)$ is the regular solution of the radial Schrödinger equation defined by the boundary condition

$$\lim_{r\to 0}r^{-l-1}\varphi_l(k,r)=1,$$

R_c is the radius of the hard core, R is the cutoff radius of the potentials, and u_l, v_l, $w_l^{(+)}$ are the Riccati–Bessel functions defined in [1].

It follows from the treatment given in [4] that the Jost function $f_{l+}^{(2)}(k)$ is analytic in the complete complex plane, excluding the infinitely distant point, if the following condition holds:

$$\int_{0^+} dr\,r^{-l}|q_l(r)|<\infty.$$

The corresponding analyticity of $f_{l+}^{(3)}(k)$ follows from the arguments in [5], and the analyticity of $f_{l+}^{(1,4)}(k)$ from the results of [6]. At the same time, the following conditions must hold:

$$\int_{R_c^+} dr|p_l(r)|<\infty,\quad \int_{R_c^+}|V(r)|dr<\infty,\quad \int_{0^+} dr\,r^2|W(r)|<\infty.$$

The S matrix is expressed in terms of the reduced Jost functions as follows:

$$S_l(k)=\exp(il\pi)\frac{f_{l+}^{(1,4)}(-k)}{f_{l+}^{(1,4)}(k)}=\frac{f_{l+}^{(2,3)}(-k)}{f_{l+}^{(2,3)}(k)}. \tag{2}$$

Note that the functions $f_{l+}^{(2,4)}(k)$ may have singularities at the points where $D_l^{(2,4)}(k)=0$. With allowance for this, and also for reasons of convenience when we study the asymptotic behavior of the functions $f_{l+}^{(1,3)}(k)$, we introduce the new functions

$$F_l^{(1)}(k)=\exp(-ikR_c)f_{l+}^{(1)}(k),\quad F_l^{(2)}(k)=f_{l+}^{(2)}(k)D_l^{(2)}(k),\quad F_l^{(3)}(k)=f_{l+}^{(3)}(k),\quad F_l^{(4)}(k)=\exp(-ikR_c)f_{l+}^{(4)}(k)D_l^{(4)}(k). \tag{3}$$

Taking into account the analyticity with respect to k of the functions $g_l(k;r,r')$, $\varphi_l(k,r)$, $f_{l+}(k,r)$, and $w_l^{(+)}(k,r)$, we can readily conclude that the function $F_l^{(1,2,3,4)}(k)$ is analytic in the open complex plane k. Considering the behavior of the functions $F_l^{(j)}(k)$ $(j=1,2,3,4)$ as $k\to\infty$ $(\mathrm{Im}\,k\geq 0)$ and using (1.I)-(1.IV), we readily conclude that

$$F_l^{(j)}(k)\underset{\substack{k\to\infty\\(\mathrm{Im}\,k\geq 0)}}{\longrightarrow}1+O(k),\quad j=1,2,3,4. \tag{4}$$

Similarly, as $k\to\infty$ $(\mathrm{Im}\,k<\infty)$, assuming that for r ≈ R the quasipotentials behave as $C(R-r)^{\alpha}$, where $\alpha>0$ and C is a constant, and, following the usual technique [1], we can show that

$$F_l^{(j)}(k)\underset{\substack{k\to\infty\\(\mathrm{Im}\,k<0)}}{\longrightarrow}Ck^{-\alpha-2}\exp(2ik\rho), \tag{5}$$

where

$$\rho=\begin{cases} R-R_c, & j=1,4,\\ R, & j=2,3.\end{cases}$$

Taking into account this asymptotic behavior of the functions F_l and using arguments similar to those employed in [1], we conclude that the functions $F_l^{(1,2,3,4)}(k)$ are entire functions of order 1 and normal type,* if the conditions on the behavior of the quasipotentials given above are satisfied.

Applying Hadamard's theorem, we can expand the function $F_l^{(j)}(k)$ in the infinite product

$$F_l^{(j)}(k)=F_l^{(j)}(0)\exp(ick)\prod_{n=1}^{\infty}(1-k/k_n)\exp(k/k_n), \tag{6}$$

where c is a complex constant that must be found and k_n are the zeros of $F_l^{(j)}(k)$, which in the case of a complex or a nonlocal quasipotential may be multiple [7]. If this is so, the multiplicity of the zeros is completely due to the zeros of the functions $D_l^{(2,4)}(k)$. Without loss of generality, we can assume that $F_l^{(j)}(0)\neq 0$.

We turn to the determination of the constant c, for which we require the following definition [8]. Functions F(k) satisfying the condition

$$\int_{-\infty}^{\infty} dk\frac{\ln^+|F(k)|}{1+k^2}<\infty,$$

where $\ln^+(k)=\ln k,\ k\geqslant 1,\ \ln^+(k)=0,\ 0\leqslant k<1$, belong to the class C.

Obviously $F_l^{(j)}(k)$ belongs to the class C. Then [8]

THEOREM. The set of roots $\{k_n\}$, $k_n\neq 0$, of the entire function F(k) of the class C satisfies the conditions:

1) $\sum\limits_{-\infty}^{\infty}|\mathrm{Im}(1/k_n)|<\infty$;

2) for any φ, $0<\varphi\leqslant\pi/2, \lim\limits_{r\to\infty} n_+(r,\varphi)/r=\lim\limits_{r\to\infty} n_-(r,\varphi)/r$, where both limits exist; here $n_+(r,\varphi)$ is the number of roots of F(k) within the sector $|k|<r$, $|\arg k|<\varphi$, and $n_-(r,\varphi)$ is the number of roots of F(k) in the sector $|k|<r$, $|\arg k-\pi|<\varphi$;

3) there exists the limit

$$\lim_{K\to\infty}\sum_{|k_n|<K}(1/k_n).$$

In accordance with Pfluger's theorem [9], if the conditions 1-3 are satisfied, the entire function of exponential type $F_l^{(j)}(k)$ has the asymptotic behavior

$$|k|^{-1}\ln\left|\frac{F_l^{(j)}(k)}{\exp(ick)F_l^{(j)}(0)}\right| \underset{k\to\infty}{\cong} \sum_n \mathrm{Re}(k_n^{-1})\cos\varphi-\sum_n \mathrm{Im}(k_n^{-1})\sin\varphi+A|\sin\varphi|+O(1), \tag{7}$$

where $k=(k)\exp(i\varphi)$,

$$\lim_{r\to\infty} n_+(r,\pi/2)/r=\lim_{r\to\infty} n_-(r,\pi/2)/r=A/\pi.$$

From (7), taking into account the asymptotic behavior of $F_l^{(j)}(k)$ in the upper and lower half-planes of k, we obtain

$$0=-\sum_n \mathrm{Im}(k_n^{-1})+A-\mathrm{Re}\,c+O(1),\ \varphi=\pi/2,\ k=i|k|,\ |k|\to\infty;$$

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* We recall that by definition the type of an entire function F(k) is equal to

$$\sigma_F=\lim_{r\to\infty}\frac{\ln M(r)}{r^\nu},\quad M(r)=\max_{|k|=r}|F(k)|.$$

If for $\nu>0$ the inequality $0<\sigma_F<\infty$ holds then F(k) is of normal type. The order ν of an entire function is by definition equal to

$$\nu=\limsup_{r\to\infty}\frac{\ln\ln M(r)}{\ln(r)}.$$

$$2\rho=\sum_n \mathrm{Im}(k_n^{-1})+A+\mathrm{Re}\,c+O(1),\ \varphi=-\pi/2,\ k=-i|k|,\ |k|\to\infty;$$

whence

$$\rho=\sum_n \mathrm{Im}(k_n^{-1})+\mathrm{Re}\,c. \tag{8}$$

Similarly

$$\mathrm{Im}\,c=\sum_n \mathrm{Re}(k_n^{-1}),\quad \varphi=0,\quad k=|k|,\quad |k|\to\infty.$$

Thus, $\rho=-i\sum_n k_n^{-1}+c$. Therefore

$$F_l^{(j)}(k)=F_l^{(j)}(0)\exp(ik\rho)\prod_{n=1}^{\infty}(1-k/k_n). \tag{9}$$

Now, proceeding from (9) and (2), we can in the same way as in the case of real local potentials [1, 4, 10] obtain the expansion

$$S_l^{(j)}(k)=\exp(-2ik\rho)\prod_{\alpha=1}^{N_1}\frac{\varkappa_\alpha+ik}{\varkappa_\alpha-ik}\prod_{\alpha=1}^{N_2}\frac{K_\alpha-ik}{K_\alpha+ik}\prod_{\alpha=1}^{\infty}\frac{|k_\alpha|^2-k^2-2ik\,\mathrm{Im}\,k_\alpha}{|k_\alpha|^2-k^2+2ik\,\mathrm{Im}\,k_\alpha} \tag{10}$$

where we have separated N_1 virtual states, N_2 bound states, and an infinite number of resonance states corresponding to zeros of the functions $F_l^{(j)}(k)$, $j=1, 2, 4$. The expression (10) generalizes the well-known result for local potentials [1].

In the case of complex potentials, the situation is much more complicated. Let us consider briefly the position of the zeros of the Jost functions $f_{l+}^{(3)}(k)$; these are simultaneously poles of $S_{l+}^{(3)}(k)$. As was shown in [5], if the imaginary part of the potential $W(r)$ is negative, $f_{l+}^{(3)}(k)$ cannot have zeros on the imaginary k axis, and on the positive real half-axis of k these zeros cannot be multiple. By virtue of the asymptotic behavior of $f_{l+}^{(3)}(k)$, the sequence of zeros of the entire function $f_{l+}^{(3)}(k)$ cannot have points of accumulation in the half-plane $\mathrm{Im}\,k \geq 0$. Therefore, the number of zeros is there finite.

The zeros of $f_{l+}^{(3)}(k)$ in the upper half-plane of k characterize metastable states with exponential damping in time of the flux of outgoing waves [5]. The zeros of $f_{l+}^{(3)}(k)$ on the positive real half-axis, or the so-called spectral points [11], describe maximal absorption when $S_l^{(3)}(k)=0$. The zeros in the lower half-plane correspond to decaying quasistationary states (fourth quadrant) and quasistationary states describing absorption (third quadrant) [5]. Accordingly, we separate N zeros $\varkappa_q$ of multiplicity α_q ($\mathrm{Im}\,\varkappa_q > 0$), N′ zeros k_r of multiplicity β_r ($k_r = \mathrm{Re}\,k_r < 0$), and an infinite (countable) number of zeros k_s of multiplicity γ_s ($\mathrm{Im}\,k_s < 0$). We obtain the expansion

$$S_l^{(3)}(k)=\exp(-2ikR)\prod_{q=1}^{N}\left(\frac{\varkappa_q+k}{\varkappa_q-k}\right)^{\alpha_q}\prod_{r=1}^{N'}\left(\frac{k_r+k}{k_r-k}\right)^{\beta_r}\prod_{s=1}^{\infty}\left(\frac{k_s+k}{k_s-k}\right)^{\gamma_s}. \tag{11}$$

The expansion (11) is obtained here for the first time. In the absence of spectral points and multiple zeros k_s it coincides with the result of [3] obtained on the basis of general physical principles.

LITERATURE CITED

1. R. G. Newton, Scattering Theory of Waves and Particles, McGraw-Hill, London (1966).
2. N. G. Van Kampen, Phys. Rev., **91**, 1267 (1953).
3. V. S. Ol'khovskii, Teor. Mat. Fiz., **20**, 211 (1974).
4. M. Coz, L. G. Arnold, and A. D. Mackellar, Ann. Phys., **59**, 219 (1970).
5. M. V. Nikolayev and V. S. Olkhovsky, Lett. Nuovo Cimento, 8, 703 (1973).
6. V. A. Plyuiko, V. S. Ol'khovskii, and M. V. Nikolaev, Visn. Kiiv. Derzh. Univ., Ser. Fiz., **15**, 42 (1974).
7. M. Bertero and D. Dillon, Nuovo Cimento, **2A**, 1024 (1971).
8. B. Ya. Levin, Entire Functions. Course of Lectures [in Russian], MGU (1971).
9. A. Pfluger, Commun. Math. Helv., **16**, 1 (1943).
10. M. Bertero, L. Talenti, and G. A. Viano, Nuovo Cimento, **46A**, 337 (1966).
11. M. A. Naimark, Tr. Mosk. Matem. Ob-va, **3**, 181 (1954).