FRACTIONAL INTEGRAL AND ITS PHYSICAL INTERPRETATION

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A relationship is established between Cantor's fractal set (Cantor's bars) and a fractional integral. The fractal dimension of the Cantor set is equal to the fractional exponent of the integral. It follows from analysis of the results that equations in fractional derivatives describe the evolution of physical systems with loss, the fractional exponent of the derivative being a measure of the fraction of the states of the system that are preserved during evolution time t. Such systems can be classified as systems with "residual" memory, and they occupy an intermediate position between systems with complete memory, on the one hand, and Markov systems, on the other. The use of such equations to describe transport and relaxation processes is discussed. Some generalizations that extend the domain of applicability of the fractional derivative concept are obtained.

INTRODUCTION

In connection with the introduction of the ideas of fractal geometry [1] into modern theoretical physics, many active attempts are being made to explain dependences of the type

$$\Phi(z)=A_\nu z^{-\nu}, \tag{1}$$

which are encountered in different fields of natural science. In Eq. (1), ν is some fractional exponent, A_ν is a constant that depends on ν, and z is some intensive variable (time, frequency, temperature, etc.). Analysis of the literature on fractal geometry reveals that the theoretical methods which lead to the dependence (1) can be divided roughly into three groups. The first involves the "transfer," usually by means of integration, of the fractional dimension of a fractal object to other physical quantities. The second group of methods reduces the problem to consideration of a functional equation of the type

$$\Phi\left(\frac{z}{b}\right)=\xi\Phi(z), \tag{2}$$

which has a unique solution of the form (1) with arbitrary constant A and exponent $\nu=\ln\xi/\ln b$. The third group of methods is based on numerical methods, in which the fractal geometry of a surface is specified numerically. Numerous examples can be found in the conference proceedings [2], and also in [3—5].

Besides these methods, attempts were made in [6—10] to relate dependences of the type (1) to solutions of equations in fractional derivatives. Although the mathematical formalism of fractional calculus has by now been well developed [11,12] and there even exists a field-free method of calculating boundary fluxes based on it [13], wide use of fractional integrals and derivatives is hindered for one simple reason — there is no clear physical interpretation of them. If we could find for them a clear physical interpretation, as exists for ordinary integrals and derivatives, their domain of application in physics would undoubtedly be enlarged. It should be mentioned that a physical realization of a derivative of half order was given in electrochemistry [14]. Some theoretical models show that ultraslow transport processes may be realized in branching fractal structures [15]. They are described by diffusion equations with fractional time derivative with exponent in the interval $0<\alpha<1$.

The main aim of this paper is to show that there is a direct relationship between fractional integrals and Cantor's fractal set. If the total number of remaining states in each stage of the division of this set is normalized to unity, then the fraction ν of remaining states, which occurs in the exponent of the fractional integral, is exactly equal to the fractal dimension of Cantor's set, and $0<\nu<1$. An intermediate asymptotic behavior in time in which manifestation of such an operator is to be expected is found. We discuss the application of equations in fractional derivatives to transport processes in porous and percolation systems, and also for systems in which the interaction is collisional in nature. We discuss "ultraslow" relaxation processes, for which a physical quantity varies more slowly than the first derivative. Generalizations of the approach proposed in the paper are given for arbitrary and random partitions of the Cantor set.

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1. FRACTIONAL INTEGRAL WITH SELF-SIMILAR EVOLUTION PROCESS

In order to understand clearly the physical interpretation of fractional integrals, it is helpful to recall two limiting cases widely used in physics.

We consider the evolution of a physical system in which some quantity $J(t)$ is related to another quantity $f(t)$ through a memory function $K(t)$:

$$J(t)=\int_0^t K(t-\tau)f(\tau)d\tau. \tag{3}$$

Let $K(t)$ be the step function

$$K(t-\tau)=\begin{cases}1/t, & 0<\tau<t,\\ 0, & \tau>t,\end{cases} \tag{4}$$

in which the factor $1/t$ is chosen to achieve normalization of the memory function to unity:

$$\int_0^t K(\tau)d\tau=1. \tag{5}$$

Then in the evolution process the system passes through all states continuously without any loss. In this case

$$J(t)=\frac{1}{t}\int_0^t f(\tau)d\tau \doteqdot \frac{1-e^{-pt}}{pt}F(p)\overset{pt\gg1}{=\!=}\frac{F(p)}{pt}. \tag{6}$$

and this corresponds to complete memory. Here and in what follows, $F(p)$ is the Laplace transform of the function $f(t)$, and p is the parameter of the Laplace transformation.

Another limiting case occurs when the system loses all its states except for one with infinitely high density. In this case, we have

$$J(t)=\int_0^t \delta(t-\tau)f(\tau)d\tau=f(t)\doteqdot e^{-pt}F(p). \tag{7}$$

The expression (7) corresponds to the well-known Markov process with complete absence of memory. This process relates all subsequent states to previous states through the single current state at each time t.

These two limiting cases, well known in physics, enable us to pose the following question. Do there exist physical systems that in the process of evolution occupy an intermediate position between a "line," i.e., when the system does not lose a single state during the process of evolution during the complete time t, and a "point," i.e., when the considered system loses all its states except for one, which is concentrated at the time t with infinitely high density? Ordinary geometry does not give an affirmative answer to this question, since it "does not know" an intermediate geometrical object between a line and a point. Fractal geometry answers the question in the affirmative, since such an object exists and is known as the Cantor set, or Cantor bars [1]. The problem can be formulated as follows. Suppose that in a system with given spatial geometry only some of the states "survive" during the process of evolution, the remainder being irreversibly lost during the process of evolution. Loss of some of the states is understood in the sense that they are lost irreversibly and are no longer accessible to the system. The Cantor set is constructed in such a way that it takes into account the inaccessibility of some of the states automatically (see Fig. 1, where $\xi=0.35$, $\nu=\ln 2/\ln(1/\xi)=0{,}6602520221\ldots$. The density of the corresponding bars is plotted along the vertical axis. The area of all bars at each stage of the partition is equal to unity. The process of construction of the Cantor set is described in Sec. 2). What happens to the Cantor set in the limit $N\to\infty$ (N is the number of the step of the partition) under the condition that the normalization of the total area (to unity, for definiteness) under the remaining bars is conserved? We wish to show that in the limit $N\to\infty$ the Cantor set converges to a fractional integral with the exponent ν of the integral being a measure of the fraction of remaining states and equal to the fractal dimension of the set.

In other words, the Cantor set for the fractional integral is the analog of a δ-like sequence, which, as is well known, converges in the limit to the δ function.

2. CONNECTION BETWEEN FRACTIONAL INTEGRAL AND CANTOR SET

To establish the connection between the fractional integral and the fractal Cantor set it is convenient to use the step function

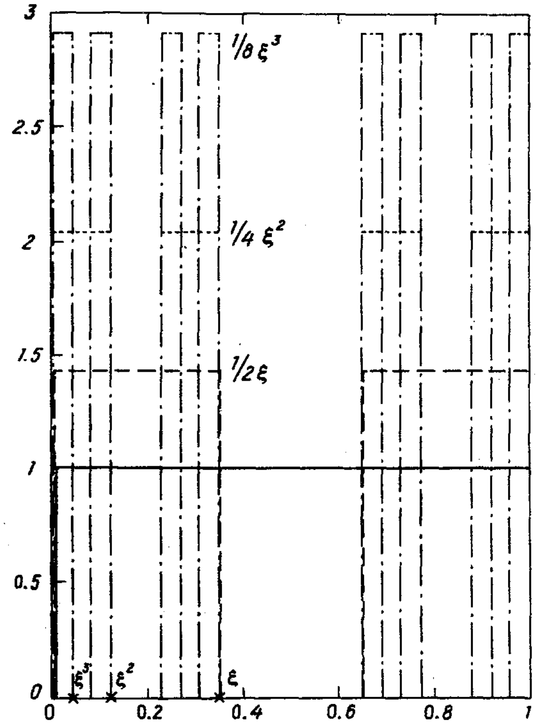

Fig 1

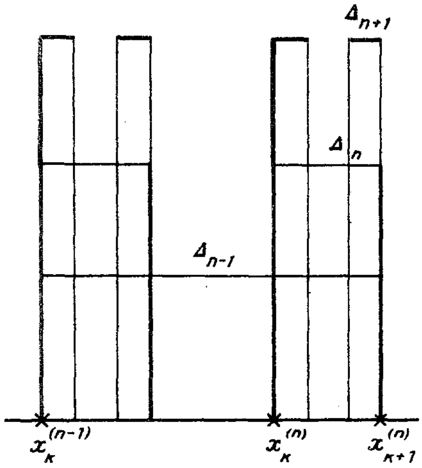

Fig 2

$$\eta(t_1<\tau<t_2)=\begin{cases}1, & \text{if} \quad \tau\in[t_1,t_2],\\ 0, & \text{if } \tau \text{ is outside } [t_1,t_2].\end{cases} \tag{3}$$

The Laplace transform of the expression (8) is

$$\eta(t_1<\tau<t_2)\doteqdot\frac{1}{p}e^{-pt_1}(1-\exp(-p(t_2-t_1))). \tag{9}$$

As is well known [1], the Cantor set is constructed in accordance with the following algorithm. The entire time interval of length (duration) t is chosen first. In the next stage, the central part of the interval is removed and at the end of the stage there remain two intervals of length ξt ($\xi < 1/2$). In the next stage, each remaining interval of length ξt is subjected to the same division process, etc. The process of constructing the Cantor set is shown in Fig. 1. The coordinates of the points after the first division stage are $[0, \xi t]$, $[t(1-\xi), t]$. The density of the remaining states after the first division stage is $(2\xi t)^{-1}$. In the second stage, the number of points is eight and the coordinates of the points, in ascending order, are $[0, \xi^2 t]$, $[(\xi-\xi^2)t, \xi t]$, $[t(1-\xi), t(1-\xi+\xi^2)]$, $[t(1-\xi^2), t]$. The density of states in the second stage of the division is determined by the expression $1/(2\xi)^2 t$. As in the first stage, the expression for the density is obtained from the condition of normalization of all the remaining states to unity. If in stage N we denote the coordinates of the points by $t_m^{(N)}$ ($m=1, 2, \ldots$), then in stage $N+1$ the coordinates of the points, in ascending order, are determined by the recursion relations

$$t_{m+1}^{(N+1)}=t_m^{(N)}, \quad t_{m+2}^{(N+1)}=t_m^{(N)}+\xi^{N+1}t, \quad t_{m+3}^{(N+1)}=t_{m+1}^{(N)}-\xi^{N+1}t, \quad t_{m+4}^{(N+1)}=t_{m+1}^{(N)}, \tag{10}$$

with density $1/(2\xi)^{N+1}\cdot t$. The contribution to the integral from the 2^N bars in stage N of the division can be expressed as

$$J(t)=\frac{1}{(2\xi)^N\cdot t}\int_0^t d\tau\sum_{m=1}^{2^N}\eta(t_m^{(N)}<\tau<t_{m+1}^{(N)})f(\tau), \tag{11}$$

$$J(t)\doteqdot\Phi(p)=\frac{1-\exp(-pt\xi^N)}{(2\xi)^N pt}\sum_{m=1}^{2^N}e^{-pt_m^{(N)}}F(p). \tag{12}$$

The sum in the expression (12) can be transformed as follows. For this, we consider Fig. 2. The Laplace transform of the step function is expressed in stage n by

$$\eta(x_k^{(n)}<x<x_{k+1}^{(n)})\doteqdot\frac{1}{p}e^{-px_k^{(n)}}(1-\exp(-p\Delta_n)).$$

If this process is repeated for stage $n+1$, then we obtain

$$\eta(x_k^{(n)} < x < x_k^{(n)} + \Delta_{n+1}) + \eta(x_k^{(n)} + \Delta_n - \Delta_{n-1} < x < x_k^{(n)} + \Delta_n) \doteqdot$$

$$\frac{1}{p} e^{-px_k^{(n)}} (1 - e^{-p\Delta_{n+1}}) + \frac{1}{p} e^{-px_k^{(n)}} e^{-p(\Delta_n - \Delta_{n+1})} (1 -$$

$$\exp(-p\Delta_{n+1})) = \frac{1 - \exp(-p\Delta_{n+1})}{p_{(n+1)}} (1 + e^{-p(\Delta_n - \Delta_{n+1})}) e^{-px_k^{(n)}}.$$

Using the connection $x_k^{(n)} = x_{k+1}^{(n+1)}$ and repeating this procedure, we obtain the final result

$$\sum_{k=1}^{2^{n+1}} \eta(x_k^{(n+1)} < x < x_{k+1}^{(n+1)}) \doteqdot \frac{1}{p} (1 - \exp(-p\Delta_{n+1})) \prod_{k=1}^{n} (1 + \exp(-p(\Delta_{k-1} - \Delta_k))). \tag{13}$$

Using the relation (13) and remembering that $\Delta_n = \xi^n t$, we can transform the expression (12) to

$$\Phi(p) = \frac{1 - \exp(-pt\xi^N)}{pt\xi^N} Q_N(pt(1-\xi))F(p), \tag{14}$$

where

$$Q_N(z) = 2^{-N} \prod_{n=0}^{N-1} (1 + \exp(-z\xi^n)) \tag{15}$$

with $z = pt(1-\xi)$. For relatively large N ($N \ll 1$), $|pt\xi^N| \gg 1$ it follows from (14) that

$$\Phi(p) = Q_N(z)F(p). \tag{16}$$

It may be noted that $Q_N(z)$, determined by the expression (15), satisfies the equation

$$Q_N\left(\frac{z}{\xi}\right) = \frac{1 + \exp(-z/\xi)}{2} Q_{N-1}(z). \tag{17}$$

It follows from (17) that for the interval

$$\xi/(1-\xi) < |pt| < \xi^{-N} \tag{18a}$$

and for

$$0 \leqslant \tau/t \leqslant 1 \tag{18b}$$

the function $Q_N(z)$ satisfies the functional equation

$$Q_N(z/\xi) \simeq {}^1/_2 Q_{N-1}(z). \tag{19}$$

It is readily seen that in the limit $N \to \infty$ there exists the limit $Q_N(z)$. Using the inequality $0 < |\exp(-pt\xi^n(1-\xi))| < 1$, we can conclude that for any pt ($0 < |pt| < \infty$)

$$0 < \lim_{N \to \infty} Q_N(z) = \bar{Q}(z) < 1.$$

Therefore, in the limit $N \to \infty$ the functional equation (19) takes the form

$$\bar{Q}(z/\xi) = {}^1/_2 \bar{Q}(z). \tag{20}$$

The solution of Eq. (20) has the form

$$\bar{Q}(z) = A_\nu z^{-\nu}. \tag{21}$$

In the expression (21), $\nu = \ln 2/\ln(1/\xi)$ is the fractal dimension of the Cantor set.

Thus, we have shown that for the intermediate region $|pt|$ satisfying (18a) $\Phi(p)$ takes the form

$$\Phi(p) \simeq A_\nu (1-\xi)^{-\nu} (pt)^{-\nu} F(p). \tag{22}$$

To the expression (22) there corresponds a representation of $J(t)$ in the form of the fractional integral (11):

$$J(t)=A_\nu[t(1-\xi)]^{-\nu}[\Gamma(\nu)]^{-1}\int_0^t (t-\tau)^{\nu-1}f(\tau)d\tau=$$

$$\frac{A_\nu}{(1-\xi)^\nu\Gamma(\nu)}\int_0^1 (1-u)^{\nu-1}f(ut)du\equiv B_\nu t^{-\nu}D^{-\nu}f, \quad 0<\nu<1, \quad 0<\tau<t. \tag{23}$$

The constant A_ν cannot be obtained from Eq. (20), and it is therefore necessary to make $Q_N(z)$ into an integral in order to estimate A_ν approximately. Details of the estimate of the constant A_ν are given in the Appendix.

3. CORRESPONDENCE WITH TWO LIMITING CASES AND PHYSICAL INTERPRETATION OF THE RESULTS

The values of the parameter ξ are in the interval

$$0<\xi\leq 1/2. \tag{24}$$

For $\xi=1/2$, the fractal dimension is $\nu=1$. In this case, it follows from (23) that $J(t)$ is related to $f(t)$ through the complete integral and corresponds to the case of complete memory. If $\xi\to 0$ ($\nu\to 0$), then from the expression (15) it follows that

$$\bar{Q}(z)=1/2(1+\exp(-pt)). \tag{25}$$

In the t representation, the expression (25) corresponds to a linear combination of two delta functions of half intensity localized at the ends of the chosen interval [0, t]. This case corresponds to complete absence of memory, as followed from the preliminary discussion. Thus, it follows from the analysis of the limiting cases that the exponent ν of the fractional integral corresponds to the fraction of preserved states in the process of evolution of the considered physical system and encompasses the cases of completely closed ($\nu=1$) and Markov ($\nu=0$) systems when all states degenerate into one or two with infinitely high density. An interesting case for analysis is $\xi=1/4$. In this case $\nu=1/2$, which corresponds to classical diffusion in quasi-one-dimensional semi-infinite systems, in which the connection between the concentration and flux is always expressed through an integral or derivative of only half order [13—15]. The appearance of a fractional integral of half order in this case can be understood as follows. For one-dimensional diffusion, there exist, as is well known, two equivalent solutions [13]. One of them corresponds to equalization of the concentration in the forward direction ($x>0$); the second solution corresponds to the solution reflected from the boundary in the opposite direction ($x<0$). For semi-infinite space, the density of states for the forward process becomes predominant, and therefore half the states is lost. In other words, a fractional integral of half order indicates fraction of states preserved in a diffusion process for semi-infinite channels.

From these arguments it can be understood that some physical systems that can be described by equations in fractional derivatives must contain channels belonging to some branching fractal structure. This was confirmed in [15], in which an "ultraslow" diffusion equation of the following type was obtained for the main channel:

$$\frac{\partial^\alpha c}{\partial t^\alpha}=\mathcal{D}_\alpha\frac{\partial^2 c}{\partial x^2}, \quad 0<\alpha<1. \tag{26}$$

The structure of the channels may differ and be generated by a definite fractal structure of the medium. In [10,16,17], such processes were classified as processes with "residual" memory. The reason for the use of the term is as follows. In statistical physics, one of the simple criteria of irreversibility of a process is change of the sign of the time under the substitution $t\to -t$. A specific feature of a process described by fractional derivatives is that under such a substitution

$$(-t)^\nu=t^\nu[\cos(\nu\pi)+i\sin(\nu\pi)] \tag{27}$$

part of the process is preserved, while the other part corresponds to irreversible losses. A process with "residual" memory corresponds to the energy principle formulated by Jonscher for dielectric relaxation [18] in the frequency domain.

From this point of view, transport processes in percolation clusters, fractal trees, and porous systems must be reanalyzed in order to obtain correct transport equations for such systems. In particular, the exponent of the fractional integral corresponds to the fraction of channels (branches) open for flow.

Another large class of physical systems in which one can expect the appearance of equations in fractional derivatives is represented by processes with loss due to collisions. We write Newton's equation in the form

$$\Delta\mathbf{v}_i=\frac{1}{m_i}\int_0^t \mathbf{F}_i(\mathbf{r},\mathbf{p},\tau)d\tau=\frac{t}{m_i}\int_0^1 \mathbf{F}_i(\mathbf{r},\mathbf{p},ut)du, \tag{28}$$

where m_i is the mass of particle i, and $\mathbf{F}_i$ is the force of the interaction of particle i with the medium. If the interaction with the medium is collisional in nature, then the force can be expressed in the form

$$\mathbf{F}_i(\mathbf{r}, \mathbf{p}, \tau) = \mathbf{F}_i(\mathbf{r}, \mathbf{p}, \tau) \sum_k \eta(t_k < \tau < t_{k+1}) \rho_k, \tag{29}$$

where ρ_k is the density of states, and $\eta\ (t_k < \tau < t_{k+1})$ is the step function defined by the expression (8). For a force acting for only a definite fraction of the time, we obtain, repeating the arguments of the previous section,

$$\Delta \mathbf{v}_i = \frac{t}{m_i} B_\nu D^{-\nu} \mathbf{F}_i, \tag{30}$$

where

$$B_\nu = A_\nu (1-\xi)^{-\nu}, \quad D^{-\nu} f = [\Gamma(\nu)]^{-1} \int_0^u (u-u_1)^{\nu-1} f(u_1 t)\, du_1$$

is a dimensionless fractional integral written down for the variable $u = \tau/t$. Using the commutative properties of the fractional derivative (11), we can rewrite Eq. (30) in the more elegant form

$$\frac{m_i}{t^2} \frac{d^{1+\nu}(\Delta \mathbf{r}_i)}{du^{1+\nu}} = B_\nu \mathbf{F}_i, \quad 0<\nu<1, \quad 0<u=\tau/t<1. \tag{31}$$

This equation can be used to describe Brownian motion and loss due to collisions. In particular, for the elastic force $\mathbf{F}_i = \varkappa \nabla^2(\Delta \mathbf{r}_i)$ we obtain a generalized transport equation of the form

$$\frac{m_i}{t^2} \frac{d^{1+\nu}(\Delta \mathbf{r}_i)}{du^{1+\nu}} = B_\nu \varkappa \nabla^2 (\Delta \mathbf{r}_i). \tag{32}$$

An equation of the form (32) was first obtained in [9], but from other considerations. From Eq. (32) there follows a new type of linear wave motion, intermediate between pure diffusion, $\nu = 0$, and classical wave motion, $\nu = 1$. It would be interesting to observe such waves experimentally.

The arguments of the previous section lead to new types of relaxation, which follow from the equations for the harmonic oscillator and the classical equation with exponential relaxation law:

$$\frac{d^{1+\nu}(\Delta \mathbf{r})}{du^{1+\nu}} + B_\nu (\omega_\alpha t)^2 \Delta \mathbf{r} = 0, \tag{33a}$$

$$\frac{d^\nu F}{du^\nu} + B_\nu (\lambda t) F = 0, \quad 0<\nu<1. \tag{33b}$$

In Eq. (33a), ω_α $(\alpha = x, y, z)$ are the oscillator eigenfrequencies, and $\Delta \mathbf{r}$ is the displacement vector; in (33b), λ is the reciprocal relaxation time, and B_ν is a dimensionless constant of order unity. These equations were actually given in [10] but without proper derivation. Equation (33b) is of particular interest; it predicts "ultraslow" relaxation, in which some physical quantity F changes more slowly than the first derivative. Are such processes observed anywhere experimentally? At the least, one can draw attention to the existence of "ultraslow" relaxation of induced electric polarization in dielectrics [18—19]. As is well known [19], some experimental data are well described by the empirical Cole—Cole expression for the complex susceptibility:

$$\chi(\omega) = \frac{\chi_0}{1 + (j\omega/\omega_p)^{1-\alpha}} \tag{34}$$

Without making use of the hypothesis of a distribution of the relaxation time [19], about the validity of which there are strong doubts [18], we obtain the expression (34) from Eq. (33b). Equation (33b), written down for the polarization P in the presence of an alternating external electric field, can be expressed in the form

$$(\lambda t)^{-1} \frac{d^\nu P}{du^\nu} + B_\nu P = \chi_2 E. \tag{35}$$

By means of Fourier transformation of the fractional derivative (11), Eq. (35) can be solved by the standard methods and an expression of the form (34) obtained from it for the susceptibility with parameters $\nu = 1-\alpha$, $\omega_p = (B_\nu \lambda t)^{1/\nu} t^{-1}$, $\chi_0 = \chi_2 / B_\nu$. This result clearly indicates the existence of "ultraslow" relaxation processes and necessitates re-examination of a number of processes "hidden" beneath distributions of relaxation times.

The results of the previous section can also be applied to the Liouville equation. The ordinary Liouville equation describes the evolution of a closed system, in which the total number of states is conserved. A Liouville equation of the type

$$\frac{i\hbar}{t}\frac{\partial^{\nu}\rho}{\partial u^{\nu}}=[H,\rho] \tag{36}$$

for given time interval $[0, t]$ with loss of the part $(1-\nu)$ of the states takes into account irreversibility naturally and does not require the introduction of an infinitesimally small source [20] to construct the nonequilibrium operator. A detailed investigation of the thermodynamics of systems described by Liouville equations of the type (36) will be made very soon.

4. SOME GENERALIZATIONS OF THE RESULTS

In Sec. 2, we obtained a fractional integral for a function determined by the expression (15). The result can be generalized to the case when the Cantor set consists at each stage of k bars of width $k\xi<1$. The previous result was obtained for the case $k=2$. Using the same ideas that led to the expression (15) and Fig. 2, we can show that for k bars ($k=2, 3, \ldots$) $Q_N^{(k)}(z)$ takes the form

$$Q_N^{(k)}(z)=\prod_{n=0}^{N-1}\frac{1-\exp\left(-\frac{k}{k-1}z\xi^n\right)}{k\left[1-\exp\left(-\frac{1}{k-1}z\xi^n\right)\right]} \tag{37}$$

for $0<\xi\leq 1/k$, $z=pt(1-\xi)$. Instead of investigating the function $Q_N^{(k)}(z)$m we prove a more general result. We consider the product

$$G_N(z)=\prod_{n=0}^{N-1}f(z\xi^n), \tag{38}$$

where $f(x)$ is some arbitrary function. We show that if certain conditions are imposed on $f(x)$ the product (38) possesses "universal" behavior and leads to a dependence of the form (21) for a large class of divisions of the Cantor set. To find the condition on the function $f(x)$, we represent $G_N(z)$ in the form of the sum

$$G_N(z)=\exp\left[\sum_{n=0}^{N-1}\ln f(\xi^n z)\right]\simeq\exp\left[\int_0^{N-1}\ln f(\xi^u z)du\right]=\exp\left[\frac{1}{\ln(1/\xi)}\int_{\varepsilon z}^{z}\ln[f(y)]\frac{dy}{y}\right]. \tag{39}$$

Here $\varepsilon=\xi^{N-1}\gg 1$, $0<\xi<1$. Integrating the expression (39) by parts, we obtain

$$G_N(z)=\exp\left[\frac{1}{\ln(1/\xi)}\ln y\cdot\ln[f(y)]\,|_{\varepsilon z}^{z}-\frac{1}{\ln(1/\xi)}\int_{\varepsilon z}^{z}\frac{f'(y)}{f(y)}\ln y\,dy\right]. \tag{40}$$

We consider the limits $z\ll 1$, $\varepsilon z\gg 1$ and, assuming that

$$\lim_{\varepsilon z\to 0}f(\varepsilon z)=1,\quad \lim_{z\to\infty}f(z)=\bar{f}<1, \tag{41}$$

we obtain from (40)

$$G_N(z)\simeq z^{-\nu}\exp\left[-\frac{1}{\ln(1/\xi)}\int_0^{\infty}\frac{f'(y)}{f(y)}\ln y\,dy+\frac{1}{\ln(1/\xi)}\int_0^{\varepsilon z}\frac{f'(y)}{f(y)}\ln y\,dy+\frac{1}{\ln(1/\xi)}\int_z^{\infty}\frac{f'(y)}{f(y)}\ln y\,dy\right]. \tag{42}$$

Here $\nu=\ln(1/\bar{f})/\ln(1/\xi)$. To estimate the two integrals, we assume that

$$f(y)=\sum_{k=0}^{\infty}c_k y^k\quad\text{for}\quad y\ll 1, \tag{43a}$$

$$f(y)=\bar{f}+\sum_{k=1}^{\infty}C_k y^{-k}\quad\text{for}\quad y\gg 1. \tag{43b}$$

From the expansions (43), we readily conclude that

$$\frac{f'(y)}{f(y)}=\sum_{k=0}^{\infty}a_k y^k \quad \text{for} \quad y\ll 1, \tag{44a}$$

$$\frac{f'(y)}{f(y)}=\sum_{k=2}^{\infty}A_k y^{-k} \quad \text{for} \quad y\gg 1. \tag{44b}$$

The coefficients a_k and A_k are, respectively, functions of the coefficients c_k and C_k of the expansions (43). Using the expansions (44), we can find conditions of smallness of the last two integrals in the expression (42),

$$\frac{1}{\ln(1/\xi)}\int_0^{\varepsilon z}\frac{f'(y)}{f(y)}\ln y\,dy=\sum_{k=0}^{\infty}\frac{a_k}{\ln(1/\xi)}\int_0^{\varepsilon z}y^k\ln y\,dy=\frac{\varepsilon z}{\ln(1/\xi)}\sum_{k=0}^{\infty}\frac{a_k}{k+1}(\varepsilon z)^k\ln\left(\varepsilon z\exp\left(-\frac{1}{k+1}\right)\right).$$

This integral will be small if

$$\left|a_0(\varepsilon z)\ln\left(\frac{\varepsilon z}{e}\right)\right|\ll 1. \tag{45}$$

Here and in what follows, e is the base of natural logarithms. The second integral in (42) can be estimated similarly:

$$\frac{1}{\ln(1/\xi)}\int_z^{\infty}\sum_{k=2}^{\infty}A_k y^{-k}\ln y\,dy=\frac{1}{\ln(1/\xi)}\sum_{k=2}^{\infty}A_k\frac{\ln(z\exp(1/(k-1)))}{(k-1)z^{k-1}}.$$

It will be small if

$$\left|A_2\frac{\ln(ez)}{z}\right|\ll 1. \tag{46}$$

If the integral

$$\int_0^{\infty}(\ln[f(y)])'\cdot\ln y\,dy=\int_0^{\infty}\frac{f'(y)}{f(y)}\ln y\,dy=I \tag{47}$$

has finite value, then the product (38) leads to the result

$$G_N(z)\simeq A_\nu z^{-\nu} \tag{48}$$

for interval of the variable z satisfying the conditions (45) and (46) with constant $A_\nu=\exp[-I/\ln(1/\xi)]$. The fractional exponent ν is determined by

$$\nu=\ln(1/\bar{f})/\ln(1/\xi). \tag{49}$$

In particular, for the function $f(z\xi^n)$ in (37) the exponent $\nu=\ln k/\ln(1/\xi)$, and this generalizes the result (21).

The results of Sec. 2 obtained for the regular Cantor set can be generalized to the case when the parameters are random and can be expressed in the form

$$\xi_i=\xi+\delta_i. \tag{50}$$

In (50), δ_i are small random deviations from the mean ξ,

$$\xi=n^{-1}\sum_i^n\xi_i,\ |\xi-\xi_i|=|\delta_i|\ll\xi<1.$$

In this case, it follows from the relation (13) that

$$\Delta_n=\xi_1\xi_2\ldots\xi_n\cdot t\simeq\xi^n\left(\exp\left[\sum_i^n\ln(1+\delta_i/\xi)\right]\right)\cdot t. \tag{51}$$

Taking into account the expansion

$$\ln(1+x)\simeq x-\frac{x^2}{2}+\ldots+(-)^{m+1}\frac{x^m}{m}$$

for $x=\delta_i/\xi\gg 1$ in (51), we obtain the result

$$\Delta_n=\xi^n\left[\exp\left(\frac{\langle\delta\rangle}{\xi}-\frac{\langle\delta^2\rangle}{2\xi^2}+\ldots\right)\right]^n\cdot t=\tilde{\xi}^n t. \tag{52}$$

Here,

$$\langle\delta^s\rangle=n^{-1}\sum_i\delta_i^s\quad(s=1,2,\ldots,m..)$$

are the mean values of the set $\{\delta_i\}^i$. It follows from the expression (52) that all the previous results remain valid if in the corresponding expressions we make the substitution $\xi\to\tilde{\xi}$.

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APPENDIX: ESTIMATE OF THE CONSTANT A_ν IN THE EXPRESSIONS (21)—(23)

For $k=2$, the function $f(y)$ is determined by the expression $f(y)=(1+e^{-y})/2$. It follows from (40) that

$$Q_N(z)=\exp\left[\frac{\ln y}{\ln(1/\xi)}\ln\left(\frac{1+e^{-y}}{2}\right)\Bigg|_{\varepsilon z}^{z}+\frac{1}{\ln(1/\xi)}\int_{\varepsilon z}^{z}\frac{\ln y\cdot e^{-y}}{1+e^{-y}}dy\right]=$$

$$z^{-\nu}\exp\left[\frac{1}{\ln(1/\xi)}\int_0^\infty\frac{e^{-y}}{1+e^{-y}}\ln y\cdot dy-\left(\int_0^{\varepsilon z}+\int_z^\infty\right)\frac{e^{-y}\ln y}{1+e^{-y}}dy\right],$$

$$\int_0^\infty\frac{e^{-y}\ln y}{1+e^{-y}}dy=\int_0^\infty\ln y\left[\sum_{n=1}^\infty(-)^{n+1}e^{-ny}\right]dy=\sum_{n=1}^\infty(-)^{n+1}\frac{\gamma+\ln n}{n}=-\gamma\ln 2+\sum_{k=2}^\infty(-)^k\frac{\ln k}{k}=-\frac{1}{2}\ln^2 2.$$

Thus, the value of the constant A_ν is determined by the expression

$$A_\nu=\exp\left(-\frac{1}{2}\frac{\ln^2 2}{\ln(1/\xi)}\right)=2^{-\nu/2}.$$

The last two integrals will be small if

$$\left|(z\varepsilon)\cdot\ln\left(\frac{z\varepsilon}{e}\right)\right|\ll 1,\quad |e^{-z}\ln z|\ll 1,$$

and this agrees approximately with the asymptotic behavior (18), which was established earlier by a different method.

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