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A Continuum Approach to High Velocity Flow in a Porous Medium

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Abstract. A general macroscopic linear momentum balance equation is derived in the form of a constitutive relation for high velocity fluid flow in a porous medium. It shows the nonlinearity in Forchheimer's formula for nonDarcy flow arising primarily from microscopic inertial phenomena, and expresses the inertial force in terms of the macroscopic velocity in an anisotropic and nonlinear manner. The point of departure is Euler's first law of motion, valid at any point in the fluid phase which is assumed to completely occupy the void space. The geometry of the void space, i.e., of the solid matrix, is taken as arbitrary. By introducing an alternative description of the microscopic kinematic field, namely deviations of the local velocity magnitudes and directions from the macroscopic values of these quantities separately, a general macroscopic momentum equation for fluid flow in a porous medium is obtained after averaging over a REV. From the general equation, most of the established relations for nonDarcy flow can be recovered as special cases. Explicit analytic expressions are obtained for the involved inertial coefficients from which the origin and nature of nonlinear (inertial) effects for high velocity flow in a porous medium is clearly demonstrated. It is also shown that the coefficient associated with the quadratic term for nonDarcy flow is not material and is a function of the macroscopic flow. Finally, some previous results are discussed and an extension of the derived equation to include higher-order nonlinear effects, with regard to the resistivity force, is proposed.

Key words. Flow in porous media, high velocity flow, inertial effects, nonDarcy flow, constitutive equations.

1. Notation

a	coefficient in (1) and (67)
A^f	surface bounding the macroscopic fluid region V^f
A^{ff}	a part of A^f adjacent to the fluid
A^{fs}	a part of A^f adjacent to the solid
b	coefficient in Equation (1) and (67)
$\mathbf{C}$, C_{kl}	inertial coefficient tensor in Equation (57)
$\mathbf{d}$, d_{kl}	deformation rate tensor
$\mathrm{d}a$	microscopic infinitesimal surface area
$\mathrm{d}A^f$	boundary surface of $\mathrm{d}V^f$
$\mathrm{d}A^{fs}$	part of $\mathrm{d}A^f$ adjacent to the solid
$\mathrm{d}A^{ff}$	part of $\mathrm{d}A^f$ adjacent to the fluid
$\mathrm{d}A_x$	macroscopic surface element
$\mathrm{d}v$	microscopic infinitesimal volume element

$\mathrm{d}V$	a representative elementary volume (REV)
$\mathrm{d}V^f$	volume occupied by the fluid within a REV
$\mathrm{d}V^s$	volume occupied by the solid within a REV
$\mathrm{d}V_x$	macroscopic volume element
$\mathbf{e}$	unit vector in the direction of the microscopic velocity with components α_k
$\hat{\mathbf{e}} \equiv \hat{\alpha}_k \mathbf{i}^k$	where $\hat{\alpha}_k$ is defined in (22)
$\mathbf{f}$, f_k	body force vector
$\mathbf{i}^k$	unit vector in the direction of the coordinate axis 1, 2, 3
$\mathbf{I}$	unit tensor
$\tilde{\mathbf{I}}$	local inertia terms defined in (44)
$\mathbf{I}$	local inertia term defined in Equation (57)
$\mathbf{J}$, J_k	driving force of motion defined in (49)
j	integer with values $\geqslant 1$
$\lvert k \rvert$	curvature magnitude defined in (52)
$\mathbf{k}^*$, k^*_{kl}	coefficient tensor defined in (61)
$\mathbf{K}^*$, K^*_{kl}	coefficient tensor defined in (59)
m	real number
$\mathbf{n}$	microscopic surface normal unit vector for $\mathrm{d}a$ over $\mathrm{d}A^{fs}$, oriented from the fluid to the solid phase
$\mathbf{N}$	macroscopic surface normal unit vector for $\mathrm{d}A_x$
p	pressure
$\tilde{p}$	macroscopic fluid pressure as defined in Hassanizadeh and Gray (1980)
$\mathbf{P}$, P_{kl}	tensor defined in (44)
$\mathbf{q}$, q_k	fluid mass flux
$\mathbf{r}$	microscopic position vector
$\mathbf{r}^{fs}$	microscopic position vector on the surface $\mathrm{d}A^{fs}$
$\mathbf{R}$, R_{kl}	resistivity coefficient tensor defined in (33)
$\tilde{\mathbf{R}} \equiv -\mathbf{R}$	
$\mathbf{R}^*$, R^*_{kl}	coefficient tensor defined in (57)
$R^{(i)}$	tensors of order 3 and 4 defined in (81)
R', R, $\tilde{R}$	scalar coefficients defined in (85)
$\mathbf{S}$, S_{kl}	tensor defined in (44)
$\tilde{\mathbf{S}}$, $\tilde{S}_{kl}$	tensor defined in (79)
t	time
$\mathbf{t}$, t_k	stress vector
$\mathbf{t}$, t_{kl}	stress tensor
$\tilde{\mathbf{t}}$, $\tilde{t}_{kl}$	macroscopic fluid stress tensor as defined in Hassanizadeh and Gray (1980)
$\mathbf{t}^*$, t^*_{kl}	stress tensor defined in (76)
$\mathbf{T}$, T_k	resistivity force defined in (32)
$\tilde{\mathbf{T}}$, $\tilde{T}_k$	dissipative part of $\bar{\rho}\mathbf{T}$
$\mathbf{v}$, v_k	velocity vector

$\hat{\mathbf{v}}$	microscopic velocity deviation vector defined in (2)
$\hat{v}$	deviation of the microscopic velocity magnitude defined in (21)
V	a macroscopic portion of the porous region
V^f	part of V occupied by the fluid
V^s	part of V occupied by the solid
w_k	interphase surface velocity vector
$\mathbf{x}$, x_k	macroscopic position vector

Greek

α_k	components of $\mathbf{e}$
$\hat{\alpha}_k$	components of $\hat{\mathbf{e}}$, defined in (22)
γ	distribution function defined in (15)
δ_{kl}	Kronecker delta
$\delta(\mathbf{r}-\mathbf{r}^{fs})$	Dirac delta function
δ^k	angle between $\mathbf{i}^k$ and $\mathbf{e}$
∇	gradient operator, with respect to either $\mathbf{r}$ or $\mathbf{x}$
∇_r, ∇_x	gradient operator with respect to $\mathbf{r}$, i.e., $\mathbf{x}$
ϵ	porosity
θ	temperature
κ, κ'	scalar quantities defined in (39)
λ	viscosity coefficient
$\tilde{\lambda}$	macroscopic fluid viscosity coefficient as defined in Hassanizadeh and Gray (1980)
λ_j	coefficients defined in (86)
$\mathbf{\Lambda}$, Λ_{kl}	tensor defined in (34)
μ	viscosity coefficient
$\tilde{\mu}$	macroscopic fluid viscosity coefficient as defined in Hassanizadeh and Gray (1980)
$\mathbf{\Pi}$, Π_{kl}	coefficient tensor defined in (46)
$\tilde{\mathbf{\Pi}}$, $\tilde{\Pi}_{kl}$	coefficient tensor defined in (46)
ρ	fluid mass density
Σ	coefficient tensors of order 0, 2 and 3 defined in (89) and (91)
$\boldsymbol{\tau}$	stress vector defined in (II4)
$\boldsymbol{\tau}^*$	stress vector defined in (75)
ψ	an arbitrary quantity

Special Notation

$\sum$	summation
$\in$	belongs to
$\cup$	union

$\tilde{\psi}$	deviating quantity
$\bar{\psi}$	macroscopic quantity
$\psi_{,k}\psi_{k,l}$	spatial derivation with respect to either **r** or **x**
$\langle\ \rangle$	an averaging operator
$(\ )^T$	transposed
$(\ )^{-1}$	inverse
$\boldsymbol{\psi}$	vector
$\boldsymbol{\psi}$	tensor
$(\ )^{(i)}$	symbolic notation for quantities defined in (81)
k, l, m, n	take values 1 to 3 and denote Cartesian components of tensorial quantities

2. Introduction

The phenomena of fluid motion in a porous medium may be conceptualized as a two-phase flow system (solid-fluid), where the phases are mutually separated by interphase boundaries over which exchange of extensive quantities takes place (e.g., of mass, momentum, etc.).

Balance equations can be written for an arbitrary spatial point **r** which is at time t occupied by only one phase; this will be referred to as the microscopic scale. If one furthermore assumes constitutive equations for each phase, the problem of flow could, in principle, be solved on this scale if the boundary conditions, i.e., the geometry of the interphase boundaries and exchange conditions over them, could somehow be determined.

This, however, is a practically unattainable task, since the detailed geometry of the interphase boundaries cannot be described. In addition, for a vast majority of different porous media, measurements at the microscopic point **r** cannot be performed.

Fortunately, in most engineering problems one has no interest in changes within individual phases. The interest is primarily focused on an overall, gross or average effect around a mathematical point of the porous region, which can be measured in specific circumstances.

Therefore, when describing fluid motion in a porous medium, an alternative scale has to be introduced which is more consistent with our ability to observe the medium, viz. what we shall refer to as the macroscopic scale. The passage from the microscopic to the macroscopic scale is made through the application of an averaging procedure, and is further facilitated by the use of a representative elementary volume (REV), i.e., a volume over which the state variables and fluxes of the involved phases are averaged. Through averaging, single phase variables, discontinuous on the microscopic scale, become continuous on the macroscopic scale where, at each mathematical point, properties associated with both phases are assigned. Thus, through the change of scale, fields of variables differentiable both in space and time are obtained for the fluid and the solid phase. A fictitious flow system of overlapping continua is established within which an interchange of various quantities takes place.

At the macroscopic level fluid movement in a porous medium at low velocity magnitudes is described by Darcy's law. This law was originally obtained as an empirical relationship (e.g., Bear, 1972) and expresses the driving force, i.e., the gradient in the fluid potential, as being proportional to fluid velocity. A generalized form of Darcy's law has also been obtained from theoretical considerations, where a macroscopic linear momentum balance equation is combined with a constitutive relation for the resistivity force, which defines the exchange of linear momentum over the solid-fluid boundaries within a REV. It has been shown (Hassanizadeh and Gray, 1980) that for isothermal conditions the dissipative part of the resistivity force, within the linear theory development, is proportional to the average velocity of the fluid relative to the solid phase.

Darcy's law has been verified by numerous experiments and similarly, the limitations of its application are well known (Scheidegger, 1960; Bear, 1972). It is valid for flow situations where the magnitude of the fluid velocity is small, i.e., when inertial effects (forces) may be neglected. Thus, the Darcian relation implies that the driving force is entirely balanced by the resistivity force. The part of the driving force balanced by inertial forces is assumed to be negligible (e.g., Hubbert, 1940). As velocities increase in magnitude, however, the inertial force becomes more dominant.

Various relationships have been proposed and derived by authors to describe fluid motion in a porous medium at high velocities (see, e.g., Scheidegger, 1960; Bear, 1972; Hannoura and Barends, 1980). An early and widely used relationship was proposed as an empirical formula by Forchheimer, i.e.,

$$J \equiv |\mathbf{J}| = aq + bq^2, \tag{1}$$

where a and b are assumed to be material constants, $\mathbf{J}$ is the hydraulic gradient, and q is the magnitude of the fluid mass flux, i.e., $q \equiv |\mathbf{q}|$. Many of the relations proposed or derived by other authors after the work of Forchheimer, have a form analogous to Equation (1). Although various modifications have been suggested, the form of Equation (1) seems to be in sufficiently good agreement with experimental evidence (see, e.g., Scheidegger, 1960; Bear, 1972).

In the early analysis of nonDarcy flow, there was a tendency by authors to interpret the proportionality of the driving force J with q and q^2 by making analogies with turbulent flow in pipes. As is well known, at high velocities in pipes (i.e., at large Reynolds numbers), turbulence develops and J has a quadratic dependence in terms of the velocity. This analogy, however, is not adequate in explaining the inertial phenomena in porous media. Experiments confirm that when gradually increasing the macroscopic velocity, the nonlinear (quadratic) dependence of J on q appears much before the onset of real turbulence in porous media flow (see, e.g., Scheidegger, 1960; Bear, 1972). Thus, the nonlinearity in laminar flow had to be attributed to inertial effects, which primarily arise from the fact that fluid velocity streamlines on the microscopic level are not rectilinear, and a macroscopic force is generated proportional to q^2.

Apart from the proposed empirical and semi-empirical relations, an effort was directed toward a derivation of Equation (1) from theoretical considerations. These were based on different approaches, such as dimensional analysis, capillaric model, Kozeny theory, drag theory, and statistical theories, all of which are reviewed in Scheidegger (1960) and Bear (1972).

In addition to these approaches, there have been theoretical examinations of nonDarcy flow where considerations have been made on two scales and an averaging procedure is employed. For example, Irmay (1958) considers the Navier–Stokes equations on the microscopic scale, assuming an incompressible fluid and a specific geometry for the pore space. After averaging over a REV, this author obtained an equation analogous to Forchheimer's formula, where isotropic coefficients are derived and expressed in terms of geometric parameters of the assumed pore space.

In comparison, Bachmat (1965) takes a more general approach with regard to defining the pore space. Assuming a constitutive relation of a specific form at the microscopic level, namely the viscous force being proportional to the microscopic velocity, and considering an incompressible fluid, he derives an anisotropic form of Equation (1), where the coefficient a is a second-rank tensor and coefficient b is a scalar expressed analytically in terms of microscopic flow directions. As compared with previous studies, the results of Irmay (1958) and Bachmat (1965) have brought much light onto the nature and origin of nonDarcy flow in a porous medium, demonstrating that the nonlinearity in (1) arises mainly due to microscopic inertial phenomena. This is a view supported by various authors (see, e.g., the work of Ahmed and Sunada (1969), and the reviews given in Scheidegger (1960) and Bear (1972)).

In other investigations (e.g., Whitaker, 1969; Gray and O'Neill, 1976; Hassanizadeh and Gray, 1980; Shapiro, 1981), flow in a porous medium has been analysed based on general continuum considerations of the microscopic flow, where macroscopic balance equations for a mixture of phases are obtained through the application of an averaging procedure. Following the derivation of macroscopic balance equations, constitutive relations are determined in order for the theory to be applicable to particular flow systems, i.e., thermodynamic macroscopic quantities are related to thermokinetic macroscopic quantities in a materially dependent way, based on the axioms of constitutive theory. The procedure is similar to the one used in classical mixture theory (see, e.g., Eringen and Ingram, 1967).

Although this type of approach is fully general in treating fluid flow through a porous medium, the final form of the macroscopic linear momentum balance equation is not adequate for a theoretical interpretation of experimental results on nonDarcy flow.

When treating the linear momentum balance equation at the microscopic scale, the essential point in this approach is the introduction of a velocity deviation vector:

$$\hat{\mathbf{v}}(\mathbf{r}, t) \equiv \mathbf{v}(\mathbf{r}, t) - \bar{\mathbf{v}}(\mathbf{x}, t) \tag{2}$$

This definition is generally employed for various types of problems in fluid dynamics and multiphase flow systems, e.g., when analysing turbulence (Beran, 1968), where $\bar{\mathbf{v}}$ is an ensemble average, or in multiphase flow analysis (Ishii, 1975), where $\bar{\mathbf{v}}$ is a time average, or both space and time average (Drew, 1972), or only space average when treating flow in a porous medium (Whitaker, 1969; Gray and O'Neill, 1976; Hassanizadeh and Gray, 1980; Shapiro, 1981). With $\hat{\mathbf{v}}$ so defined, after averaging, one obtains a term $\langle \rho\hat{\mathbf{v}}\hat{\mathbf{v}} \rangle$, where $\langle \quad \rangle$ denotes an averaging operator. This is a tensorial quantity and has dimensions of stress (e.g., Reynolds stress in turbulence). It is usually incorporated with the ordinary stress tensor, thus forming a new quantity for which a constitutive relation is found (see, e.g., Hassanizadeh and Gray, 1980; Shapiro, 1981). Alternatively, it can be treated separately and a separate constitutive equation for $\langle \rho\hat{\mathbf{v}}\hat{\mathbf{v}} \rangle$ is assumed (see, e.g., Gray and O'Neill, 1976; Whitaker, 1969).

We can state that most of the information on convective microscopic inertial effects is present in this term. Moreover, the nonlinearity in Equation (1) probably arises mainly due to this term. However, with constitutive relations that have been obtained for $\langle \rho\hat{\mathbf{v}}\hat{\mathbf{v}} \rangle$, either directly, or through the definition of a generalized macroscopic stress tensor, one is not able to obtain the experimentally approved form of Equation (1). It seems, therefore, that this type of approach, i.e., employing $\hat{\mathbf{v}}$ as given in (2) and defining a constitutive relation for the term $\langle \rho\hat{\mathbf{v}}\hat{\mathbf{v}} \rangle$ based on mixture theory, is not adequate for treating high velocity flow in porous media. This point will be brought up in more detail later in the paper.

In addition to these investigations, flow in a porous medium has also been studied through direct application of the mixture theory (see, e.g., Mueller, 1971; Raats, 1971; Morland, 1972), where the medium is treated only macroscopically and microscopic considerations are not made. With regard to macroscopic balance of forces and inertial phenomena, this approach often leads to an incomplete description. This point will also be brought up later and briefly discussed.

On one side we thus have derivations based on assumptions not sufficiently general, but which finally result in experimentally confirmed equations. On the other, although a fully general approach is made, the analysis does not result in equations which can be reduced to the experimentally verified form for nonDarcy flow.

In an attempt to overcome a part of these difficulties, we present in this paper a derivation of a macroscopic linear momentum balance equation for fluid flow in a porous medium, based on continuum considerations. It will be shown that the nonlinear force in Forchheimer's formula arises primarily due to microscopic inertial phenomena and, moreover, that the inertial phenomena, as manifested on the macroscopic level, can be understood only through analysis of the microscopic flow. Instead of employing the conventional velocity deviation vector $\hat{\mathbf{v}}$ of (2), an alternative description of the microscopic kinematic field will be introduced, namely a deviation of local velocity magnitude and direction separately, finally resulting in an equation which can be considered as a generalization of Forchheimer's formula (1). Reducing this general equation to Equation (1) and other established and experimentally confirmed relations for nonDarcy flow as special cases, will be straightforward.

3. Microscopic Description of the Fluid Motion in a Porous Medium

Since our primary interest is to analyse nonDarcy flow, we shall only consider a porous medium saturated by a single fluid phase. In addition, we shall restrict our analysis by considering a nondeformable solid matrix whose void space is continuous. Furthermore, we shall assume that no mass exchange takes place between the solid and fluid phase, that interphase surface quantities (e.g., surface tension effects) are negligible, and that the solid and fluid phases are nonpolar.

At a microscopic spatial point in the porous medium, either the fluid or the solid phase will exist at time t (Figure 1). With $\mathbf{r}^{fs}$ we shall denote the position vector of a spatial point of the interphase surface, which we materially assume to belong to the solid phase.

The volume of the entire porous region, both fluid and solid, is denoted by V, which consists of the volumes occupied by the fluid V^f and by the solid V^s, where $V = V^f \cup V^s$. The boundary of V^f, i.e., A^f, consists of a material surface adjacent to the solid phase denoted by A^{fs}, and a geometrical surface, A^{ff}, which is a part of the external boundary of V, i.e., $A^f = A^{ff} \cup A^{fs}$ (see Figure 1). In general, A^{ff} is not a material surface with respect to mass or any of the other extensive quantities, such as momenta, energy, or entropy. In a similar manner, we can define the boundary of the solid phase A^s, such that $A^s = A^{ss} \cup A^{sf}$. It should be noted that $A^{sf} = A^{fs}$.

In the analysis which follows, we shall be primarily concerned with mechanical phenomena and, therefore, only consider the balances of mass and momenta, i.e., at each microscopic point $\mathbf{r}$ and time t, we shall assume the following balance equations to be valid (Eringen, 1965):

Mass balance

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{3}$$

Linear momentum balance

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \nabla \cdot \mathbf{t} + \rho \mathbf{f}. \tag{4}$$

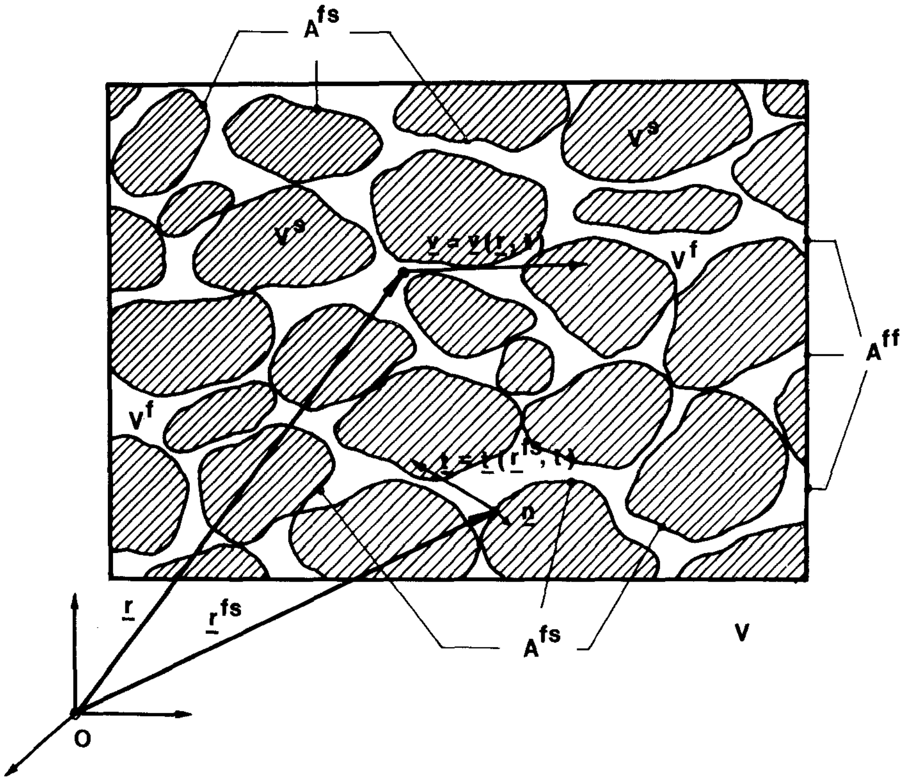

Fig. 1. A portion V of a porous region, composed of V^f and V^s; A^f bounds V^f and $A^f = A^{ff} \cup A^{fs}$.

Angular momentum balance

$$\mathbf{t} = \mathbf{t}^T. \tag{5}$$

Since we consider a nondeformable solid matrix, the above equations are only of interest at points in the fluid phase.

Initially, let us consider the fluid motion to be laminar. The assumption of laminar flow implies the existence of streamlines continuous within V^f. In addition, laminar flow provides a unique solution at time t for the fluid stress and fluid velocity fields within V^f, provided that the boundary conditions on A^f, i.e., on A^{ff} and A^{fs}, are known. The fluid stress and velocity fields can be obtained solving Equations (3) and (4), when a constitutive equation for the fluid stress is assumed. Later, however, we shall relax this restriction and discuss turbulent flow in a porous medium. On the surface A^{fs} we may take as the boundary condition $\mathbf{v}(\mathbf{r}^{fs}, t) = \mathbf{0}$ (no-slip condition), if the solid matrix is assumed to be fixed. Furthermore, the fluid stress $\mathbf{t}$ has to be known on A^{fs}, while the dissipative part of the microscopic fluid stress tensor will be related to the fluid velocity through some material coefficients, e.g., λ and μ, if the fluid is assumed Newtonian. It follows, therefore, that the actual solution of the microscopic flow, i.e., the velocity $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ and the corresponding field of streamlines in V^f is determined by the geometric configuration of V^f, and also by the material properties of the fluid and by the boundary conditions given over A^{ff}.

Consider a time t and a microscopic point $\mathbf{r}$ in the fluid of the velocity vector $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$. It can be expressed as

$$\mathbf{v} = \mathbf{v}(\mathbf{r}, t) = v_k(\mathbf{r}, t)\mathbf{i}^k = v(\mathbf{r}, t)\mathbf{e}(\mathbf{r}, t), \tag{6}$$

where $\mathbf{e}$ is a unit vector in the direction $\mathbf{v}$, and v is the magnitude of $\mathbf{v}$, i.e., $v = |\mathbf{v}|$ (see Figure 2). (Throughout the text the summation convention is valid for repeated indices.) After scalar multiplication of Equation (6) by $\mathbf{i}^l$, one has

$$v_k \delta_{kl} = v\mathbf{e} \cdot \mathbf{i}^l = v_l \tag{7}$$

or

$$v_k = v\alpha_k \tag{8}$$

where

$$\alpha_k \equiv \mathbf{e} \cdot \mathbf{i}^k. \tag{9}$$

From the definition it follows that α_k are cosines of angles between the unit vectors $\mathbf{e}$ and $\mathbf{i}^k$ (see Figure 2), i.e.,

$$\alpha_k \equiv \mathbf{e} \cdot \mathbf{i}^k = \cos(\mathbf{e}, \mathbf{i}^k) = \cos \delta^k. \tag{10}$$

By writing the three components of the velocity vector in the form $v_k = v\alpha_k$, we have expressed them in terms of four quantities which are functions of both space and time, i.e., $v = v(\mathbf{r}, t)$ and $\alpha_k = \alpha_k(\mathbf{r}, t)$, with an additional restriction on the components of the unit vector $\mathbf{e}$, i.e.,

$$\alpha_k \alpha_k = 1. \tag{11}$$

The α_k values contain the information on the geometry of the microscopic flow. If $\alpha_k = \text{const.}$ in V^f, the microscopic flow is parallel, corresponding to a porous medium composed of parallel stream tubes. If $\alpha_k = \alpha_k(t)$, the microscopic flow is also parallel but unsteady, i.e., the direction changes in time.

Employing (3) in Equation (4), and expressing v_k as given by Equation (8), one has in component form

$$\rho \frac{\partial}{\partial t}(v\alpha_k) + \rho v \alpha_l (v\alpha_k)_{,l} = t_{lk,l} + \rho f_k \tag{12}$$

which can be further expanded as

$$\rho \frac{\partial}{\partial t}(v\alpha_k) + \rho v v_{,l} \alpha_l \alpha_k + \rho v^2 \alpha_l \alpha_{k,l} = t_{lk,l} + \rho f_k. \tag{13}$$

The left-hand side of (13) contains the entire inertial force for the microscopic flow. Included in the inertial terms is a quantity which is quadratic in the velocity

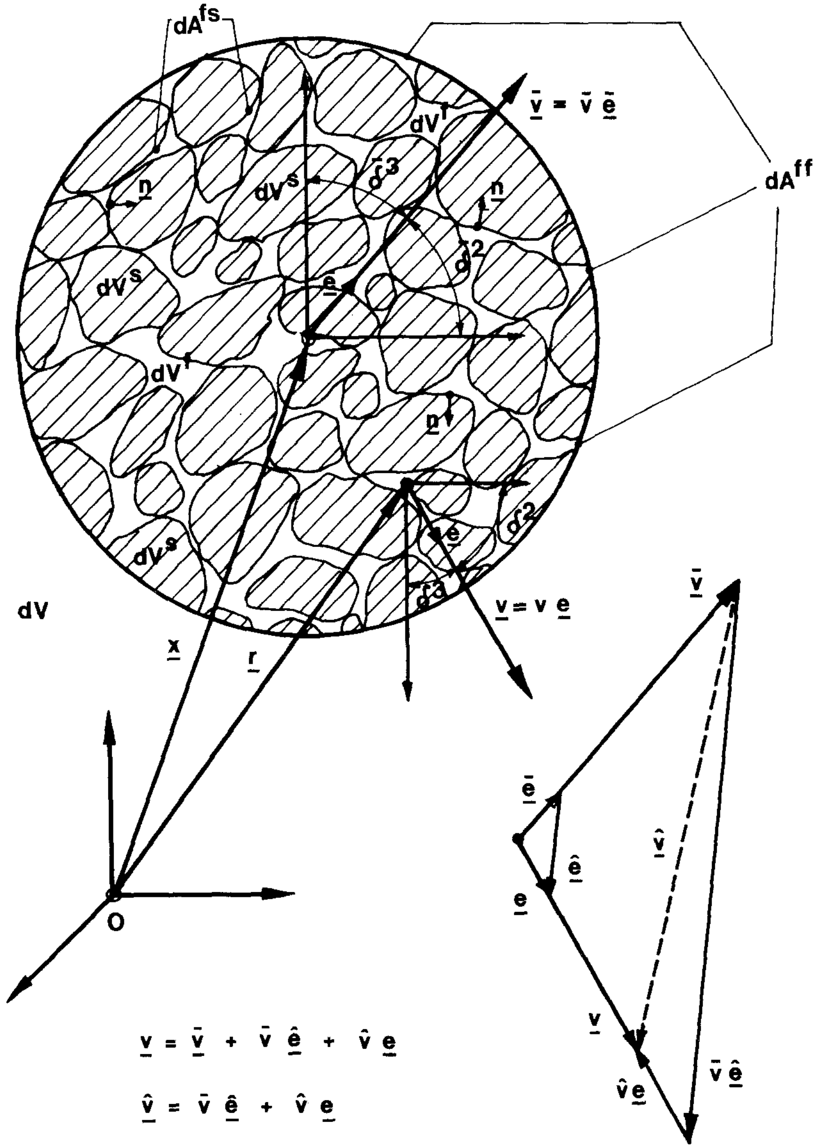

Fig. 2. A REV with the introduced deviating quantities $\hat{v}$ and $\hat{\mathbf{e}}$ for assumed vectors $\bar{\mathbf{v}}$ and $\mathbf{v}$; $\hat{\mathbf{v}}$ is the deviation vector defined in (2). (For simplicity, vectors $\mathbf{v}$ and $\bar{\mathbf{v}}$ have been chosen to lie in the $x_2 - x_3$ plane.)

magnitude v, i.e., $v^2\alpha_l\alpha_{k,l}$. The quantity $\alpha_l\alpha_{k,l}$ contains information on the curvature of the microscopic flow, as will be shown later. This quantity will vanish for $\alpha_k = \text{const.}$ or if $\alpha_k = \alpha_k(t)$.

4. Macroscopic Description of the Fluid Motion in a Porous Medium

The solution for the fluid movement in a porous medium on a microscopic level would require us to know the exact configuration of the void space, i.e., the geometry of the surface A^f (Figure 1). Since the precise knowledge of this configuration can never be attained, and since our interest in fluid flow through a porous medium is usually related to a scale much larger than the size of a typical pore opening, we consider flow on the macroscopic scale. That is, we are not concerned with an exact description of fluid motion in individual pore openings, but rather with a gross effect of the microscopic flow. Therefore, to provide a mathematical description which would involve quantities defined on the scale of our interest, an averaging procedure is introduced to proceed from the microscopic to macroscopic levels.

In place of a spatial (Eulerian) point $\mathbf{r}$, defined on the microscopic scale as to contain only one phase at time t, a spatial point $\mathbf{x}$ is defined for the macroscopic scale as the centroid of a REV (denoted by $\mathrm{d}V$ in Figure 2), which now contains both phases simultaneously. Consequently, macroscopic densities and fluxes at $\mathbf{x}$ are variables continuous in both space and time.

Referring to Figure 2, a macroscopic volume $\mathrm{d}V$ consists of $\mathrm{d}V^f$ and $\mathrm{d}V^s$. Similarly, the boundary of $\mathrm{d}V^f$ is $\mathrm{d}A^f$, with $\mathrm{d}A^f = \mathrm{d}A^{ff} \cup \mathrm{d}A^{fs}$.

The averaging procedure has been a topic of investigation by several authors (see, e.g., Bear, 1972; Gray and O'Neill, 1976; Hassanizadeh and Gray, 1979a; Shapiro, 1981; Slattery, 1969; Whitaker, 1969). In the following paragraphs, we present only those points which are necessary for the discussion. For a more detailed analysis, the reader should consider the above-mentioned references.

An average (macroscopic) mass density is defined in the form

$$\bar{\rho} = \bar{\rho}(\mathbf{x}, t) \equiv \frac{1}{\mathrm{d}V} \int_{\mathrm{d}V} \rho(\mathbf{r}, t) \gamma(\mathbf{r}, t) \, \mathrm{d}v \tag{14}$$

where $\gamma = \gamma(\mathbf{r}, t)$ is a distribution function defined as (e.g., Whitaker, 1969)

$$\gamma(\mathbf{r}, t) = \begin{cases} 1 & \text{for } \mathbf{r} \in \mathrm{d}V^f \\ 0 & \text{for } \mathbf{r} \in \mathrm{d}V^s \end{cases} \tag{15}$$

and where $\mathrm{d}v$ is the infinitesimal volume element associated with a spatial point $\mathbf{r}$ on the microscopic scale.

A macroscopic velocity vector may be defined in the form

$$\bar{\mathbf{v}}(\mathbf{x}, t) \equiv \frac{1}{\bar{\rho}\, \mathrm{d}V} \int_{\mathrm{d}V} \rho(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t) \gamma(\mathbf{r}, t) \, \mathrm{d}v. \tag{16}$$

In an analogous manner to that given at the microscopic level, we can express $\bar{\mathbf{v}}(\mathbf{x}, t)$ in terms of its magnitude $\bar{v}$, and a unit vector $\bar{\mathbf{e}}$ in the direction of $\bar{\mathbf{v}}$ (see Figure 2), i.e.,

$$\bar{\mathbf{v}} = \bar{v}_k \mathbf{i}^k = \bar{v} \bar{\mathbf{e}} \tag{17}$$

Both $\bar{v}$ and $\bar{\mathbf{e}}$ are macroscopic quantities. (Note that a macroscopic (average) quantity is not necessarily defined as a direct application of the integral (averaging) operation on the corresponding microscopic quantity, as it is the case for the density in (14).)

After scalar multiplication of (17) with $\mathbf{i}^l$, we obtain

$$\bar{v}_k\delta_{kl} = \bar{v}\bar{\mathbf{e}} \cdot \mathbf{i}^k = \bar{v}\bar{\alpha}_l = \bar{v}_l, \tag{18}$$

where we have defined $\bar{\alpha}_k$ as

$$\bar{\alpha}_k \equiv \bar{\mathbf{e}} \cdot \mathbf{i}^k = \cos \bar{\delta}^k. \tag{19}$$

Using (8) and (18), Equation (16) can be rewritten in the form

$$\bar{v}\bar{\alpha}_k = \frac{1}{\bar{\rho}\,\mathrm{d}V}\int_{\mathrm{d}V} \rho v \alpha_k \gamma \,\mathrm{d}v = \frac{1}{\bar{\rho}\,\mathrm{d}V}\int_{\mathrm{d}V^f} \rho v \alpha_k \,\mathrm{d}v. \tag{20}$$

At each microscopic point $\mathbf{r}$ in $\mathrm{d}V^f$, a deviation from the macroscopic velocity magnitude and its direction at point $\mathbf{x}$ with respect to the microscopic velocity magnitude and its direction, can be defined as

$$\hat{v}(\mathbf{r}, t) \equiv v(\mathbf{r}, t) - \bar{v}(\mathbf{x}, t) \tag{21}$$

and

$$\hat{\alpha}_k(\mathbf{r}, t) \equiv \alpha_k(\mathbf{r}, t) - \bar{\alpha}_k(\mathbf{x}, t). \tag{22}$$

We note that the values of $\hat{v}$ and $\hat{\alpha}_k$ are unique at $\mathbf{r}$ only for a specific macroscopic point $\mathbf{x}$ and a chosen size of $\mathrm{d}V$. Thus, we could have written $\hat{v} = \hat{v}(\mathbf{x}, \mathbf{r}, t)$ and $\hat{\mathbf{e}} = \hat{\mathbf{e}}(\mathbf{x}, \mathbf{r}, t)$ for a given $\mathrm{d}V$. In addition, although the introduced vector $\hat{\mathbf{e}} = \hat{\alpha}_k\mathbf{i}^k$ is not a unit vector, i.e., $\hat{\mathbf{e}} \cdot \hat{\mathbf{e}} \neq 1$, one can specify an upper limit for its magnitude as

$$|\hat{\mathbf{e}}| \leqslant 2; \tag{23}$$

however, values $|\hat{\mathbf{e}}| = 2$ and close to 2 are not very likely within a REV, since they physically imply a microscopic flow in the opposite direction to the macroscopic flow. In Figure 2, for assumed specific values of $\bar{\mathbf{v}}$ and $\mathbf{v}$, the microscopic velocity is expressed through the introduced deviating quantities $\hat{v}$ and $\hat{\mathbf{e}}$, in contrast to employing the conventional deviation vector $\hat{\mathbf{v}}$ defined in (2). It should be emphasized that apart from the points in $\mathrm{d}V^f$, where $\mathbf{v} = \mathbf{0}$ (at $\mathbf{r} = \mathbf{r}^{fs}$), $\hat{v}$ is not equal $|\hat{\mathbf{v}}|$.

Substitution of Equations (21) and (22) into (20), yields

$$\begin{aligned}\bar{\rho}\bar{v}\bar{\alpha}_k &= \frac{1}{\mathrm{d}V}\int_{\mathrm{d}V^f} \rho(\bar{v}\bar{\alpha}_k + \hat{v}\bar{\alpha}_k + \bar{v}\hat{\alpha}_k + \hat{v}\hat{\alpha}_k)\,\mathrm{d}v \\ &\equiv \bar{\rho}\bar{v}\bar{\alpha}_k + \frac{1}{\mathrm{d}V}\int_{\mathrm{d}V^f} \rho\hat{v}\bar{\alpha}_k\,\mathrm{d}v + \frac{1}{\mathrm{d}V}\int_{\mathrm{d}V^f} \hat{\alpha}_k\bar{v}\rho\,\mathrm{d}v + \\ &\quad + \frac{1}{\mathrm{d}V}\int_{\mathrm{d}V^f} \rho\hat{v}\hat{\alpha}_k\,\mathrm{d}v.\end{aligned} \tag{24}$$

For the above identity to be valid $\bar{v}$, $\hat{v}$, $\bar{\alpha}_k$ and $\hat{\alpha}_k$ must satisfy

$$\int_{\mathrm{d}V} \rho\gamma(\hat{v}\bar{\alpha}_k + \bar{v}\hat{\alpha}_k + \hat{v}\hat{\alpha}_k)\,\mathrm{d}v = 0. \tag{25}$$

Employing the definitions of $\hat{v}$ and $\hat{\alpha}_k$ given in (21) and (22) into Equation (13) gives

$$\rho \frac{\partial \bar{v}_k}{\partial t} + \rho\hat{\alpha}_k \frac{\partial \bar{v}}{\partial t} + \rho\bar{v}\frac{\partial \hat{\alpha}_k}{\partial t} + \rho\bar{\alpha}_k \frac{\partial \hat{v}}{\partial t} + \hat{v}\rho\frac{\partial \bar{\alpha}_k}{\partial t} + \rho\frac{\partial}{\partial t}(\hat{v}\hat{\alpha}_k) + \\ + \rho\bar{\alpha}_l\bar{v}^2\hat{\alpha}_{k,l} + \rho\hat{\alpha}_l\bar{v}^2\hat{\alpha}_{k,l} + \rho\bar{\alpha}_k\bar{\alpha}_l\hat{v}_{,l}\bar{v} + \rho\bar{\alpha}_k\hat{\alpha}_l\hat{v}_{,l}\bar{v} + \\ + \rho\hat{\alpha}_k\bar{\alpha}_l\hat{v}_{,l}\bar{v} + \rho\hat{\alpha}_k\hat{\alpha}_l\hat{v}_{,l}\bar{v} + 2\rho\bar{\alpha}_l\hat{\alpha}_{k,l}\hat{v}\bar{v} + 2\rho\hat{\alpha}_l\hat{\alpha}_{k,l}\hat{v}\bar{v} + \\ + \rho\alpha_k\alpha_l\hat{v}\hat{v}_{,l} + \rho\alpha_l\hat{v}^2\alpha_{k,l} = t_{lk,l} + \rho f_k \tag{26}$$

where we have eliminated the terms multiplied by $\bar{v}_{,l}$ since, by definition, $\bar{v}$ is not a function of $\mathbf{r}$ (only of $\mathbf{x}$) and since the gradients in (26) are defined with respect to $\mathbf{r}$.

Equation (26) is the linear momentum balance equation at the microscopic level in which we have introduced macroscopic quantities $\bar{v}$ and $\bar{\mathbf{e}}$. The terms on the left-hand side of (26) are inertial forces expressed in terms of $\bar{v}$, $\hat{v}$, $\bar{\mathbf{e}}$, and $\hat{\mathbf{e}}$. From Equation (26), which is valid at each point $\mathbf{r}$, the next step is to obtain a macroscopic linear momentum balance equation valid at $\mathbf{x}$.

After employing an averaging procedure analogous to the one described by Hassanizadeh and Gray (1979a), a macroscopic linear momentum balance equation is derived in the form

$$\bar{\rho}\frac{\partial \bar{v}_k}{\partial t} + \langle \rho\hat{\alpha}_k\rangle \frac{\partial \bar{v}}{\partial t} + \bar{v}\left\langle \rho\frac{\partial \hat{\alpha}_k}{\partial t}\right\rangle + \left\langle \rho\frac{\partial \hat{v}}{\partial t}\right\rangle \bar{\alpha}_k + \\ + \frac{\partial \bar{\alpha}_k}{\partial t}\langle \rho\hat{v}\rangle + \left\langle \rho\frac{\partial}{\partial t}(\hat{v}\hat{\alpha}_k)\right\rangle + \bar{v}\langle \rho\hat{\alpha}_{k,l}\rangle \bar{v}_l + \\ + \bar{v}^2\langle \rho\hat{\alpha}_{k,l}\hat{\alpha}_l\rangle + \bar{\alpha}_k\langle \rho\hat{v}_{,l}\rangle \bar{v}_l + \langle \rho\hat{\alpha}_l\hat{v}_{,l}\rangle \bar{v}_k + \\ + \langle \rho\hat{\alpha}_k\hat{v}_{,l}\rangle \bar{v}_l + \langle \rho\hat{\alpha}_k\hat{\alpha}_l\hat{v}_{,l}\rangle \bar{v} + 2\langle \rho\hat{\alpha}_{k,l}\hat{v}\rangle \bar{v}_l + \\ + \langle 2\rho\hat{\alpha}_l\hat{\alpha}_{k,l}\hat{v}\rangle \bar{v} + \langle \rho\alpha_k\alpha_l\hat{v}\hat{v}_{,l}\rangle + \langle \rho\alpha_l\hat{\alpha}_{k,l}\hat{v}^2\rangle = \bar{t}_{lk,l} + \bar{\rho}T_k + \bar{\rho}\bar{f}_k \tag{27}$$

with $\langle\ \ \rangle$ denoting the averaging operation, i.e.,

$$\langle \psi\rangle \equiv \frac{1}{\mathrm{d}V}\int_{\mathrm{d}V} \psi\gamma\,\mathrm{d}v = \frac{1}{\mathrm{d}V}\int_{\mathrm{d}V^f} \psi\,\mathrm{d}v \tag{28}$$

The development of the first and second term on the right-hand side of (27) is given in Appendix II.

If the assumption that $\rho = \text{const.}$ is made, it is shown in Appendix I how (27) takes a simplified form, i.e.,

$$\rho\epsilon\frac{\partial \bar{v}_k}{\partial t} + \langle \rho\hat{\alpha}_{k,l}\hat{\alpha}_l\rangle \bar{v}^2 + \langle \rho\hat{\alpha}_l\hat{v}_{,l}\rangle \bar{v}_k + \langle \rho\hat{\alpha}_k\hat{\alpha}_l\hat{v}_{,l}\rangle \bar{v} + \\ + \langle \rho\hat{\alpha}_{k,l}\hat{v}\rangle \bar{v}_l + \langle 2\rho\hat{\alpha}_l\hat{\alpha}_{k,l}\hat{v}\rangle \bar{v} + \langle \rho\alpha_k\alpha_l\hat{v}\hat{v}_{,l}\rangle + \\ + \langle \rho\alpha_l\hat{\alpha}_{k,l}\hat{v}^2\rangle + \bar{v}_l\bar{v}_k\rho\epsilon_{,l} = \bar{t}_{lk,l} + \rho\epsilon T_k + \rho\epsilon\bar{f}_k \tag{29}$$

where ϵ is the porosity defined as

$$\epsilon \equiv \frac{\mathrm{d}V^f}{\mathrm{d}V} = \frac{1}{\mathrm{d}V}\int_{\mathrm{d}V} \gamma(\mathbf{r}, t)\,\mathrm{d}v = \epsilon(\mathbf{x}, t). \tag{30}$$

A constitutive equation for the macroscopic fluid stress tensor may be assumed in the form:

$$\bar{t}_{kl} = -\epsilon\bar{p}\,\delta_{kl} + \bar{\lambda}\,\bar{d}_{ll}\,\delta_{kl} + 2\bar{\mu}\,\bar{d}_{kl}; \tag{31}$$

$\bar{p}$ is the macroscopic fluid pressure, $\bar{\mathbf{d}}$ is the rate of macroscopic fluid deformation, and $\bar{\lambda}$ and $\bar{\mu}$ are macroscopic viscosity coefficients. (Note that although the constitutive equation for $\bar{\mathbf{t}}$ is assumed here to be analogous to the one given in the work of Hassanizadeh and Gray (1980), the definition of $\mathbf{t}$ is not the same. This point will be recalled later in the paper.)

The macroscopic body type of force $\bar{\rho}\mathbf{T}$ arises from the resistance which the fluid experiences due to the presence of the solid phase within a REV. Through thermodynamic considerations, in the work of Hassanizadeh and Gray (1980) $\bar{\rho}\mathbf{T}$ is shown to consist of a reversible and a dissipative part, i.e.,

$$\bar{\rho}T_k = \epsilon_{,k}\bar{p} + \tilde{T}_k \tag{32}$$

or, as a linear approximation

$$\bar{\rho}T_k = \epsilon_{,k}\bar{p} + R_{kl}\bar{v}_l \tag{33}$$

having assumed the solid matrix to be nondeformable and fixed.

The dissipative part of $\nabla \cdot \bar{\mathbf{t}}$ has its origin in fluid viscosity, i.e., it expresses the total microscopic fluid viscous forces within $\mathrm{d}V^f$ as manifested on the macroscopic level. It has been generally accepted that this force is much smaller as compared to $\tilde{\mathbf{T}}$. Moreover, considering high fluid velocity magnitudes, the inertial forces will tend to dominate even more viscous forces, as compared to the flow with lower velocities. In our following analysis, we shall therefore neglect the dissipative part of $\nabla \cdot \bar{\mathbf{t}}$. In addition, since the microscopic flow field is taken as laminar, it seems reasonable to assume (33) valid. Later, however, a higher-order expansion of $\tilde{\mathbf{T}}$ in terms of $\bar{\mathbf{v}}$ will be considered.

Alternative forms of both Equations (27) and (29) can be derived if one expresses the microscopic velocity field $\mathbf{v}(\mathbf{r}, t)$ through a mapping

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{\Lambda}(\mathbf{r}, t) \cdot \bar{\mathbf{v}}(\mathbf{x}, t). \tag{34}$$

Several authors have discussed the nature of tensor $\mathbf{\Lambda}$ (see, e.g., Nikolaevskii, 1959; Whitaker, 1969; Neuman, 1976; Stokes, 1983). It has been shown that tensor $\mathbf{\Lambda}$ may be considered as constitutive (i.e., depending only on the structure of the porous media) under assumptions which are not valid for the flow analysed here, and consequently the tensor field $\mathbf{\Lambda}$ cannot be considered as constitutive, i.e., independent of the microscopic and macroscopic velocities. However, as one can observe from (27) and (29), the inertial effects as manifested on the macroscopic level, include quantities in form of macroscopic continuum coefficients, which are necessarily functions of both the microscopic and macroscopic flows, i.e., functions of $\bar{v}$, $\bar{\mathbf{e}}$, $\hat{\mathbf{e}}$ and $\hat{v}$. In the following, we shall express the macroscopic inertial quantities of (27) and (29) by employing a scalar κ in place of the magnitude deviation $\hat{v}$, where κ is given as a function of $\mathbf{\Lambda}$, $\mathbf{e}$ and $\bar{\mathbf{e}}$.

Rewriting Equation (34), using definitions (18) and (8), and multiplying both sides with $\mathbf{e}$, we have, in component form,

$$v = \bar{v}\Lambda_{kl}\bar{\alpha}_l\alpha_k. \tag{35}$$

With (35), Equation (21) becomes

$$\hat{v} = (\Lambda_{kl}\bar{\alpha}_l\alpha_k - 1)\bar{v} \equiv \kappa\bar{v}. \tag{36}$$

Thus, the deviation of the microscopic velocity magnitude from the macroscopic velocity magnitude is expressed in terms of a scalar multiplicant κ and the macroscopic velocity magnitude $\bar{v}$, where for a specific $\mathbf{x}$ and $\mathrm{d}V$, $\kappa = \kappa(\mathbf{r}, t)$.

From the definition of $\bar{\mathbf{v}}$ given in (16), it follows that

$$\bar{\rho}\bar{v}_k = \langle \rho v_k \rangle = \langle \rho\Lambda_{kl} \rangle \bar{v}_l, \tag{37}$$

i.e., the condition

$$\langle \rho\Lambda_{kl} \rangle = \bar{\rho}\,\delta_{kl}. \tag{38}$$

Furthermore, the scalar quantity κ defined as

$$\kappa \equiv \Lambda_{kl}\bar{\alpha}_l\alpha_k - 1 \equiv \kappa' - 1 \tag{39}$$

has to satisfy the following relation

$$\langle \rho\kappa\alpha_k \rangle = -\langle \rho\hat{\alpha}_k \rangle \tag{40}$$

which emerges from the condition given by (25). Thus, we see from (40) that when $\hat{\mathbf{e}} = \mathbf{0}$, i.e., $\mathbf{e} = \bar{\mathbf{e}}$ (corresponds to a porous medium composed of straight parallel tubes), we have

$$\langle \rho\kappa \rangle = 0. \tag{41}$$

Apart from condition (38), one is not able to specify any additional information on the nine components of $\mathbf{\Lambda}$ and, thus, its introduction might seem disadvantageous. Considering, however, the scalar quantity κ', defined in terms of $\mathbf{\Lambda}$, and the microscopic and macroscopic flow directions in (39), the physical implications become more observable.

The scalar quantity κ' is related to the distribution of microscopic velocity magnitudes within $\mathrm{d}V^f$ at $\mathbf{x}$, expressed through $\bar{v}$ at $\mathbf{x}$. A restriction on κ' could be written in the form

$$0 \leqslant \kappa' < \kappa_0' \tag{42}$$

where κ_0' denotes an upper limit of the relation $v/\bar{v}$ at $\mathbf{x}$, and where we have included the points $\mathbf{r}^{fs}$ on the interphase surfaces, with the no-slip condition assumed (i.e., $\kappa'(\mathbf{r}^{fs}, t) = 0$). Although the exact value of κ_0' cannot be determined for the general case, an estimation, assuming a specific model of the void space (e.g., interconnected random tubes, Bear (1972)), could be made. In addition, considering κ' as a stochastic variable, one could assume for the same model a specific distribution function for κ' within $\mathrm{d}V^f$.

Employing (36), Equation (27) can be written in the form

$$\tilde{I}_k + \bar{v}P_{kl}\bar{v}_l + S_{kl}\bar{v}_l = -\epsilon\bar{p}_{,k} + R_{kl}\bar{v}_l + \bar{\rho}\bar{f}_k \tag{43}$$

where

$$\tilde{I}_k \equiv (\bar{\rho} + \langle\rho\kappa\rangle)\frac{\partial\bar{v}_k}{\partial t} + (\langle\rho\hat{\alpha}_k\rangle + \langle\rho\kappa\hat{\alpha}_k\rangle)\frac{\partial\bar{v}}{\partial t},$$

$$\begin{aligned} P_{kl} \equiv{}& \bar{\alpha}_k\langle\rho\kappa_{,l}\rangle + \langle\rho\hat{\alpha}_{k,l}\rangle + \delta_{kl}\langle\rho\hat{\alpha}_m\kappa_{,m}\rangle + \\ &+ \langle\rho\hat{\alpha}_k\kappa_{,l}\rangle + \langle 2\rho\hat{\alpha}_{k,l}\kappa\rangle + \{\langle\rho\hat{\alpha}_{k,m}\hat{\alpha}_m\rangle + \\ &+ \langle\rho\hat{\alpha}_k\hat{\alpha}_m\kappa_{,m}\rangle + \langle 2\rho\hat{\alpha}_m\hat{\alpha}_{k,m}\kappa\rangle + \\ &+ \langle\rho\alpha_k\alpha_m\kappa\kappa_{,m}\rangle + \langle\rho\alpha_m\hat{\alpha}_{k,m}\kappa^2\rangle\}\bar{\alpha}_l, \end{aligned}$$

$$\begin{aligned} S_{kl} \equiv{}& \left(\left\langle\rho\frac{\partial\hat{\alpha}_k}{\partial t}\right\rangle + \left\langle\rho\frac{\partial\kappa}{\partial t}\right\rangle\bar{\alpha}_k + \right. \\ &\left. + \left\langle\rho\frac{\partial\kappa}{\partial t}\hat{\alpha}_k\right\rangle + \left\langle\rho\frac{\partial\hat{\alpha}_k}{\partial t}\kappa\right\rangle\right)\bar{\alpha}_l. \end{aligned} \tag{44}$$

If the microscopic condition $\rho = \text{const.}$ is valid, i.e., $\partial\rho/\partial t = \rho_{,l} = 0$, one may employ (36) in Equation (29) to obtain

$$\epsilon\rho\frac{\partial\bar{v}_k}{\partial t} + \bar{v}\tilde{\Pi}_{kl}\bar{v}_l = -\epsilon\bar{p}_{,k} + R_{kl}\bar{v}_l + \epsilon\rho\bar{f}_k, \tag{45}$$

where

$$\tilde{\Pi}_{kl} \equiv \Pi_{kl} + \bar{\alpha}_k\epsilon_{,l},$$

$$\begin{aligned} \Pi_{kl} \equiv{}& \{\delta_{kl}\langle\hat{\alpha}_m\kappa_{,m}\rangle + \langle\hat{\alpha}_{k,l}\kappa\rangle + \\ &+ (\langle\hat{\alpha}_{k,m}\hat{\alpha}_m\rangle + \langle\hat{\alpha}_k\hat{\alpha}_m\kappa_{,m}\rangle + \langle\hat{\alpha}_m\hat{\alpha}_{k,m}\kappa\rangle + \\ &+ \langle\alpha_k\alpha_m\kappa\kappa_{,m}\rangle + \langle\alpha_m\hat{\alpha}_{k,m}\kappa^2\rangle)\bar{\alpha}_l\}\rho. \end{aligned} \tag{46}$$

Summarizing the derivation of the macroscopic linear momentum equation for fluid motion in a porous medium, four different forms have been obtained. Two of these equations, (27) and (29), are expressed in terms of $\hat{v}$, one for $\rho = \rho(\mathbf{r}, t)$ and the other for $\rho = \text{const.}$ Alternatively, Equations (43) and (45) have been written employing the tensor $\mathbf{\Lambda}$, i.e., the scalar quantity κ. As has been shown, when describing inertial phenomena as manifested on the macroscopic level for fluid flow in a porous medium, one cannot avoid quantities such as $\hat{v}$, or $\hat{\mathbf{e}}$ and $\nabla\hat{\mathbf{e}}$, since the macroscopic inertial forces arise mainly due to these quantities. Hence, although Λ is not constitutive, we find it more appropriate to express the final form of the macroscopic linear momentum balance equation in terms of κ, rather than in terms of $\hat{v}$, within the present development.

Thus, Equations (43) and (45), expressed through the tensor Λ defined in (34), i.e., through the scalar quantity κ, one for $\rho = \rho(\mathbf{r}, t)$ and the other for $\rho = \text{const.}$, can be respectively rewritten as

$$\tilde{I}_k + \bar{v}P_{kl}\bar{v}_l + R'_{kl}\bar{v}_l = J_k \tag{47}$$

and

$$\epsilon\rho \frac{\partial \bar{v}_k}{\partial t} + \bar{v}\tilde{\Pi}_{kl}\bar{v}_l + \tilde{R}_{kl}\bar{v}_l = J_k \tag{48}$$

where

$$R'_{kl} \equiv S_{kl} + \tilde{R}_{kl}, \qquad J_k \equiv -\epsilon\bar{p}_{,k} + \bar{\rho}\bar{f}_k, \qquad \tilde{R}_{kl} \equiv -R_{kl} \tag{49}$$

with tensors $\mathbf{P}$, $\tilde{\mathbf{\Pi}}$, $\mathbf{S}$, and the vector $\tilde{\mathbf{I}}$ defined in (44) and (46).

5. Discussion on Derived Continuum Coefficients

When solving problems of fluid motion in a finite porous region, an important step is to constitutively relate measurable field quantities, i.e., the macroscopic velocity $\bar{\mathbf{v}}$ to the pressure gradient $\mathbf{J}$. Essentially, this means that for a specific porous region a relation between $\bar{\mathbf{v}}$ and $\mathbf{J}$ is established, independent of the macroscopic boundary conditions. Darcy's law is valid for low velocity magnitudes and the relationship between $\mathbf{J}$ and $\bar{\mathbf{v}}$ is maintained through a constitutive coefficient $\mathbf{R}$, which is assumed invertable and its inverse (usually referred to as hydraulic conductivity) is, for practical purposes, further expressed through the permeability tensor and the viscosity coefficient of the fluid involved (see, e.g., Bear, 1972). The permeability tensor is assumed as a constitutive quantity to depend only on the configuration of $\mathrm{d}V^f$ (i.e., on the geometry of the void space).

Inertial forces, however, which macroscopically become significant as velocity magnitudes increase, have a primarily kinematical origin, i.e., they directly depend on the exact velocity field $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ in $\mathrm{d}V^f$, but also on the scalar field $\rho = \rho(\mathbf{r}, t)$ for the general case. Exact velocity and density fields, in turn, depend not only on the microscopic boundary conditions (configuration of $\mathrm{d}A^{fs}$), but also on the macroscopic boundary conditions (given over $\mathrm{d}A^{ff}$) and the material properties of the fluid. From the preceding analysis, we are able to observe in Equation (47) (i.e., (48)), more precisely the origin of inertial forces as manifested on the macroscopic level. These forces mainly arise due to the fact that, first, the microscopic velocity changes locally; second, the microscopic streamlines are curved; and third, the magnitude of the microscopic velocity differs from the macroscopic velocity magnitude, i.e., a deviation $\hat{v} \neq 0$ (i.e., $\kappa \neq 0$) exists and, in addition, $\nabla \hat{v} \neq \mathbf{0}$ (i.e., $\nabla \kappa \neq \mathbf{0}$).

Let us now, for simplicity, assume the case $\rho = \text{const.}$ and consider Equation (48) which has been derived following microscopic and macroscopic analysis. This equation relates $\mathbf{J}$ and $\bar{\mathbf{v}}$ through quantities written in the form of continuum coefficients. One can observe from definition (46) that the structure of inertial continuum coefficients is complex, and that their exact values could be calculated only if the field $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ is known.

A practical approach, however, when treating high velocity flow in a porous medium, has been to assume the inertial coefficient of (48) to be isotropic and material, i.e., constitutive for a specific porous region, and then experimentally determine its value (see, e.g., Scheidegger, 1960; Bear, 1972). It follows from Equation (48) and definitions (46) that this approach essentially involves an approximation. In the following paragraphs, we will, therefore, discuss the nature of the derived inertial coefficients, finally emphasizing the caution one should have when assuming the coefficient, generally employed for nonDarcy flow, to be material.

The coefficient tensor $\tilde{\mathbf{\Pi}}$ in (48) is associated with the quadratic velocity and expresses macroscopic inertial forces due to microscopic flow phenomena within $\mathrm{d}V^f$ at a macroscopic point $\mathbf{x}$. It is composed of two parts: $\mathbf{\Pi}$, which is a function of the microscopic and macroscopic flows, and $\bar{\mathbf{e}}\nabla\epsilon$, where $\nabla\epsilon$ is constitutive and depends on the spatial variation of the void space.

If one assumes a simplified flow situation with $\hat{\alpha}_k = 0$ ($\bar{\alpha}_1 = 1$, $\bar{\alpha}_2 = \bar{\alpha}_3 = 0$), i.e., a medium composed of straight tubes in the direction x_1, of possibly varying cross-sections, the coefficient becomes

$$\tilde{\Pi}_{11} = \left\langle \rho\kappa \frac{\partial \kappa}{\partial x_1} \right\rangle + \frac{\mathrm{d}\epsilon}{\mathrm{d}x_1} \equiv \tilde{\Pi} \tag{50}$$

and the inertial effects are then due to changes of porosity and velocity magnitudes along x_1. If the cross-section areas of the tubes are constant, we have

$$\tilde{\Pi} = 0 \tag{51}$$

and inertial effects could only arise from local inertia, i.e., if

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t}\rho\epsilon \neq 0.$$

In the definition of $\tilde{\mathbf{\Pi}}$ (46), two terms do not contain the scalar quantity κ, namely $\langle \hat{\mathbf{e}} \cdot \nabla \hat{\mathbf{e}} \rangle \bar{\mathbf{e}}$ and $\bar{\mathbf{e}}\nabla\epsilon$. The former contains information only on the geometry of the microscopic flow. (Bachmat (1965) derived an equation analogous to (48), but with a scalar coefficient associated with the quadratic term. This coefficient is defined solely in terms of the void space geometry.) Furthermore, one can show that the curvature magnitude $|k|$ of a streamline curve, as defined in three-dimensional space, can be expressed in the form (Stoker, 1969)

$$|k| = \sqrt{\alpha_l \alpha_{k,l} \alpha_m \alpha_{k,m}}. \tag{52}$$

In addition, we can write

$$\alpha_l \alpha_{k,l} = \alpha_l \hat{\alpha}_{k,l} = \bar{\alpha}_l \hat{\alpha}_{k,l} + \hat{\alpha}_l \hat{\alpha}_{k,l} \tag{53}$$

employing the definition of $\hat{\mathbf{e}}$. Thus, we see that one part of $\tilde{\mathbf{\Pi}}$, associated with the squared velocity magnitude, is directly proportional only to the curvature of the microscopic streamline field integrated over $\mathrm{d}V^f$.

The tensorial character of $\tilde{\mathbf{\Pi}}$ stems from the tensor $\langle \nabla \hat{\mathbf{e}} \kappa \rangle$ and the fact that vectors of the remaining five terms in (46), as well as the gradient in porosity, will not, in general, give a resultant in the same direction as $\bar{\mathbf{e}}$. Clearly, with various possible irregular configurations of $\mathrm{d}A^{fs}$, there is no *a priori* reason why these vectors should be colinear with $\bar{\mathbf{e}}$.

Estimating the magnitude of individual terms of (44), or (46), is difficult and one is not able to say whether the terms containing κ are small as compared to $\langle \hat{\mathbf{e}} \cdot \nabla \hat{\mathbf{e}} \rangle$, which is a function of the flow geometry only. In general, for a random configuration of a porous medium, the microscopic velocity magnitudes change considerably within $\mathrm{d}V^f$ and the terms containing κ could make a significant contribution to the total value of $\tilde{\mathbf{\Pi}}$. Considering Equation (47) for the case when $\rho = \rho(\mathbf{r}, t)$, we see that besides the resistivity term assumed proportional to $\bar{\mathbf{v}}$, there appears an additional term proportional to $\bar{\mathbf{v}}$, namely $\mathbf{S} \cdot \bar{\mathbf{v}}$. The tensor $\mathbf{S}$ is defined in terms of local changes of κ and $\hat{\mathbf{e}}$. Although one can perhaps assume this term to be small, it seems that for the general case of microscopically varying fluid density, when one measures the resistance (i.e., the conductivity), one is really measuring $\mathbf{R}$ (i.e., $\{\mathbf{R}\}^{-1}$) of Equation (47). In other words, considering the definition of $\mathbf{R}$, one can say that the total resistance proportional to $\bar{\mathbf{v}}$ consists of two additive parts: one directly associated with physical properties of the fluid (e.g., μ) through friction over $\mathrm{d}A^{fs}$ and with the configuration of $\mathrm{d}A^{fs}$, while the other is directly associated with the kinematical field only, i.e., with local changes of the flow pattern within $\mathrm{d}V^f$.

From the definition of $\tilde{\mathbf{\Pi}}$ it follows that

$$\tilde{\Pi}_{kl} = \tilde{\Pi}_{kl}(\mathbf{x}, t) \tag{54}$$

or, for steady-state flow,

$$\tilde{\Pi}_{kl} = \tilde{\Pi}_{kl}(\mathbf{x}). \tag{55}$$

If one now considers the macroscopically uniform and unidirectional flow in a porous region with a homogeneous and isotropic configuration of the void space $\mathrm{d}V^f$ (where $\tilde{\mathbf{\Pi}} = \mathbf{\Pi}$ since $\nabla \epsilon = \mathbf{0}$), one could probably assume $\mathbf{\Pi}$ to be constant and isotropic. However, if one treats high velocity fluid flow which is not uniform and unidirectional, i.e., macroscopic streamlines are, for example, curved and possibly change in time, then even for a homogeneous and isotropic porous region

$$\Pi_{kl} = \Pi_{kl}(\mathbf{x}, t), \tag{56}$$

i.e., the continuum coefficient $\mathbf{\Pi}$ will also depend on the macroscopic boundary conditions.

It follows from the definition that $\mathbf{\Pi}$ is not a constitutive coefficient in the same sense as $\mathbf{R}$, due to the dependence of both $\mathbf{e} = \mathbf{e}(\mathbf{r}, t)$ within a REV at $\mathbf{x}$ on the macroscopic flow curvature at $\mathbf{x}$ and, in general, of $\kappa = \kappa(\mathbf{r}, t)$ on $\bar{\mathbf{v}}$ (through $\mathbf{\Lambda}$). Hence, variations of $\mathbf{\Pi}$ in space will not only result from possible heterogeneities in the configuration of the pore space $\mathrm{d}V^f$ within the macroscopic flow region, but also due to possible changes in the macroscopic flow. Employing the averaging procedure, a microscopic differential field equation (4) (Euler's first law) has been transformed into a macroscopic linear momentum balance equation (47), i.e., (48). In this form, (47), i.e., (48), can be considered as a constitutive equation for high velocity fluid flow in a porous medium, relating the macroscopic dynamic field (pressure) to the macroscopic kinematic field (velocity) in an anisotropic and nonlinear manner, with two material type of coefficients, $\tilde{\mathbf{\Pi}}$ and $\mathbf{R}$. However, in contrast to the latter coefficient, which may be taken as independent of the macroscopic velocity field, the former should, in general, be treated as a function of the macroscopic velocity field, e.g., of the macroscopic streamline curvature. Although this essentially supports the major conclusion of Barak and Bear (1981), it is clearly in contrast with the conclusions drawn, for example, by Ahmed and Sunada (1969), and with the assumptions made by other authors (see the reviews in Scheidegger (1960) and Bear (1972)), namely, that the coefficient associated with the quadratic term for high velocity flow in a porous medium is material, i.e., constant for a specific porous region.

Based on definition (46), one is not able to directly evaluate components of $\tilde{\mathbf{\Pi}}$. However, one could possibly quantitatively estimate their values if a specific, simplified flow pattern, and a specific distribution of microscopic velocity magnitudes (i.e., of κ), within $\mathrm{d}V^f$, is assumed. Nevertheless, definitions given in (46), give a good qualitative insight into the nature and origin of coefficients generally employed for nonDarcy flow in a porous medium.

6. Comparison with Previously Developed Relations

Due to the analogous forms of Equations (47) and (48), we shall write them both as

$$I_k + \bar{v}C_{kl}\bar{v}_l + R^*_{kl}\bar{v}_l = J_k \tag{57}$$

where tensors $\mathbf{C}$, $\mathbf{R}^*$ and the vector $\mathbf{I}$ will take, respectively, values $\mathbf{P}$, $\mathbf{R}'$ and $\tilde{\mathbf{I}}$, i.e., values $\tilde{\mathbf{\Pi}}$, $\tilde{\mathbf{R}}$ and $\epsilon\rho(\partial\bar{\mathbf{v}}/\partial t)$, depending on whether we are assuming the microscopic flow to satisfy, respectively, $\rho = \rho(\mathbf{r}, t)$, i.e., $\rho = \text{const}$.

Based on Equation (57), we show, in the following steps, how various established relations for nonDarcy flow in a porous medium can be recovered as special cases. This will be mainly based on assuming isotropy conditions.

Neglecting for the moment the local inertia term **I**, we shall assume the tensor $\mathbf{R}^*$ to be invertible and multiply Equation (57) by its inverse to obtain

$$(\bar{v}Q^*_{kl} + \delta_{kl})\bar{v}_l = K^*_{kl}J_l \tag{58}$$

where

$$K^*_{kl} \equiv \{R^*_{kl}\}^{-1} \tag{59}$$

and

$$Q^*_{kl} \equiv K^*_{km}C_{ml}\,. \tag{60}$$

If one now assumes that $\mathbf{K}^*$ can be expressed as a multiplication of a scalar μ (microscopic fluid viscosity constant) and a tensorial quantity as

$$K^*_{kl} = \frac{k^*_{kl}}{\mu}, \tag{61}$$

we have

$$Q^*_{kl} = \frac{1}{\mu}\,k^*_{km}C_{ml} = \frac{1}{\mu}\,Q_{kl}\,. \tag{62}$$

Assuming a further isotropy of $\mathbf{Q}$, i.e.,

$$Q_{kl} = \eta\,\delta_{kl}\,, \tag{63}$$

we write Equation (58) in the form

$$\mu\left(\bar{v}\,\frac{\eta}{\mu} + 1\right)\bar{v}_k = k^*_{kl}J_l\,. \tag{64}$$

Employing the specific discharge vector defined as $\mathbf{q} = \epsilon\bar{\mathbf{v}}$ instead of $\bar{\mathbf{v}}$ in Equation (64), we get

$$\mu\left(q\,\frac{\eta}{\epsilon\mu} + 1\right)q_k = \epsilon k^*_{kl}J_l\,. \tag{65}$$

If the tensor $\mathbf{k}^*$ is also isotropic and

$$k^*_{kl} = k^*\,\delta_{kl}\,, \tag{66}$$

Equation (65) can be written for a specific flow direction as

$$J = aq + bq^2, \tag{67}$$

where

$$a \equiv \frac{\mu}{\epsilon k^*}, \qquad b \equiv \frac{\eta}{\epsilon^2 k^*}. \tag{68}$$

Equation (67) could have been obtained directly from (57) by assuming the isotropy of both **C** and **R***, i.e.,

$$C_{kl} = C\,\delta_{kl}, \qquad R^*_{kl} = R^*\,\delta_{kl}. \tag{69}$$

Furthermore, if one can write

$$R^* = \frac{\mu}{k^*} \tag{70}$$

we have from (57), in terms of **q**, and written in one-dimensional form:

$$J = \frac{\mu}{k^*\epsilon}\,q + \frac{C}{\epsilon^2}\,q^2. \tag{71}$$

Thus,

$$C = \frac{\eta}{k^*}. \tag{72}$$

Including the local inertia force for the case $\rho = \text{const.}$, we can write

$$J = aq + bq^2 + \rho\epsilon^2\frac{\partial q}{\partial t} \tag{73}$$

for a specific flow direction, since the solid matrix has been assumed nondeformable.

Most of the relations for nonDarcy flow in a porous medium proposed by different authors, based either on semi-empirical or theoretical grounds, have the form given by Equation (67) (see, e.g., Scheidegger, 1960, Bear, 1972, Hannoura and Barends, 1980). Equation (73) has been derived for unsteady flow when local inertia should be included which is analogous to most relations given for this case. The coefficient associated with local inertia term here is independent of the microscopic flow only for $\rho = \text{const.}$ Generally it is claimed that the effect of local inertia in a porous medium is small. In the review article by Hannoura and Barends (1980), a contrary observation is reported, i.e., in some types of porous media (coarse granular media), the term associated with local inertia can be significant.

Equation (65) has been derived for a more general case where the porous medium is anisotropic in terms of **K***. The coefficient tensor **k*** is referred to as the permeability tensor and is obtained from **K***, if **R*** is assumed to be invertible. Equation (65) has a form analogous with the result of Bachmat (1965). The definition of the scalar coefficient η in the first term on the left-hand side is, however, different and, as we have shown, was obtained with isotropy assumptions from its more general expression given in (57), which accounts for an inertial type of anisotropy. One of the important assumptions in the work of Bachmat (1965) is fluid incompressibility at the microscopic level. Even when employing the assumption $\rho = \text{const.}$, (i.e., $\partial\rho/\partial t = \rho_{,k} = 0$, which is a stronger restriction than microscopic incompressibility implying

$$\frac{\partial \rho}{\partial t} + v_l \rho_{,l} = 0), \tag{74}$$

the coefficient associated with the quadratic term in Equation (48) has a tensorial character. In other words, the inertial anisotropy associated with this term has a different nature than the anisotropy associated with $\mathbf{k}^*$, and although $\mathbf{k}^*$ may be for a specific medium, taken as isotropic, the coefficient $\mathbf{C}$ will, in general, maintain a tensorial character. This is partly in agreement with the result of Barak and Bear (1981). These authors propose a third-order tensor associated with the quadratic term which could account for the inertial type of anisotropy. However, we shall see further in the paper that a third- and fourth-order tensors can be obtained from a higher-order expansion of the dissipative part of the resistivity force $\tilde{\mathbf{T}}$, in terms of $\bar{\mathbf{v}}$.

Although we have shown that most of the established relations for nonDarcy flow can be recovered with isotropic assumptions from (57), an open question remains as to how justified are, in general, the approximations made by introducing isotropic conditions.

In the following paragraphs we shall recall an alternative and more standard approach for obtaining a macroscopic linear momentum balance equation for flow through a porous medium, which is also based on continuum considerations. The reason for this is to show that from such a procedure the extension to nonDarcy flow is not straightforward and not clear. This approach is presented systematically in the work of Hassanizadeh and Gray (1980), although similar equations to their final form of the macroscopic linear momentum balance equation may be found elsewhere (see, e.g., Raats and Klute, 1968; Raats, 1972; Whitaker, 1969; Slattery, 1967; Mueller, 1971; Bear, 1972; Gray and O'Neill, 1975).

Instead of separately defining deviations of v (microscopic velocity magnitude) and $\mathbf{e}$ (microscopic velocity direction), i.e., $\hat{v}$ and $\hat{\mathbf{e}}$, we could have defined a deviation velocity vector $\hat{\mathbf{v}}$ as (see Figure 2)

$$\hat{\mathbf{v}}(\mathbf{r}, t) \equiv \mathbf{v}(\mathbf{r}, t) - \bar{\mathbf{v}}(\mathbf{x}, t) \tag{2}$$

for a specific $\mathbf{x}$ and $\mathrm{d}V$. After applying an averaging procedure on the microscopic linear momentum balance equation, and employing (2), a term with dimensions of stress is obtained and incorporated with the stress tensor as

$$\boldsymbol{\tau}^* = \frac{1}{\mathrm{d}A} \int_{\mathrm{d}A} \gamma \mathbf{t}^* \cdot \mathbf{n}\, \mathrm{d}a, \tag{75}$$

where

$$\mathbf{t}^* \equiv \mathbf{t} - \rho \hat{\mathbf{v}} \hat{\mathbf{v}} \tag{76}$$

The macroscopic stress tensor $\tilde{\mathbf{t}}$ is then obtained from $\boldsymbol{\tau}^*$ in an analogous manner as $\bar{\mathbf{t}}$ is obtained from $\boldsymbol{\tau}$ (see Appendix II). Furthermore, assuming a macroscopically Newtonian fluid and a linear expansion of the resistivity force in terms of $\bar{\mathbf{v}}$, a macroscopic linear momentum balance equation is finally derived in the form

$$\bar{\rho}\frac{\partial \bar{v}_k}{\partial t} + \bar{\rho}\bar{v}_l\bar{v}_{k,l} = -\epsilon\tilde{p}_{,k} + (\bar{\lambda}\,\bar{\mathrm{d}}_{ll})_{,k} + 2(\bar{\mu}\,\bar{\mathrm{d}}_{kl})_{,l} + R_{kl}\bar{v}_l + \bar{\rho}\bar{f}_k\,. \tag{77}$$

In their paper, Hassanizadeh and Gray (1980) state that Equation (77) accounts for all inertial effects. Neglecting inertial terms (left-hand side) and macroscopic viscous forces (second and third term on the right-hand side), and assuming the invertibility of $\mathbf{R}$, they obtain Darcy's law. It seems, however, that Equation (77) accounts only for what may be considered as a macroscopic part of inertial effects. To demonstrate this point, one may consider a case of a macroscopically uniform, unidirectional and steady flow through a homogeneous porous medium. According to Equation (77), since the left-hand side will identically vanish, we have

$$-\epsilon\tilde{p}_{,k} + \bar{\rho}\bar{f}_k = -R_{kl}\bar{v}_l\,. \tag{78}$$

It would thus follow that for macroscopically uniform, unidirectional and steady flows, only Darcy's law is valid (assuming $\mathbf{R}$ invertible), irrespective of the macroscopic velocity magnitude. This clearly contradicts the experimental evidence for high macroscopic velocities. Therefore, we conclude that Equation (77) is incomplete and does not account for all inertial effects. It accounts for what may be considered as the macroscopic part of inertia, while significant remaining, or microscopic inertial effects, have been neglected in Equation (77) within the linear theory development.

A natural question arises as to whether a higher-order expansion of either $\tilde{\mathbf{t}}$ (which contains $\langle\rho\hat{\mathbf{v}}\hat{\mathbf{v}}\rangle$), or $\tilde{\mathbf{T}}$, in terms of $\bar{\mathbf{v}}$, could ensure the experimentally observed quadratic dependence of $\mathbf{J}$ on $\bar{\mathbf{v}}$ for high values of $\bar{v}$. One can show, however, that for an isotropic case, the form of Equation (1) cannot be obtained within the above-mentioned investigation.

Considering the term $\langle\rho\hat{\mathbf{v}}\hat{\mathbf{v}}\rangle$ which arises employing (2) in (4) and averaging, one could have alternatively treated it as a separate macroscopic quantity, instead of incorporating it in the macroscopic stress tensor, and developed a separate constitutive relation for $\langle\rho\hat{\mathbf{v}}\hat{\mathbf{v}}\rangle$. This has been, for example, presented in the work of Gray and O'Neill (1976). However, similar problems arise when one attempts to use their results to describe the simplest flow situation, namely a uniform, unidirectional and steady flow in a homogeneous medium, with isotropic conditions; one will not be able, for this case, to obtain the quadratic dependence of $\mathbf{J}$ in terms of the velocity $\bar{\mathbf{v}}$.

Both Gray and O'Neill (1976) and Hassanizadeh and Gray (1980) employ an averaging procedure. As discussed earlier, the change of scale through averaging is the procedure of redefining a spatial (Eulerian) point, which finally results in a mixture of phases, thus transforming an inconvenient discrete system into a convenient system of overlapping continua. Following the derivation of the macroscopic balance equations, constitutive relations are defined in order for the theory to be applicable to particular flow systems. The procedure for obtaining constitutive equations, employed by the above authors, is essentially analogous to the one used in classical mixture theory (see, e.g., Eringen and Ingram, 1967), and is based on the axioms of constitutive theory (Eringen, 1965). Our concluding remark, however, is that the above-mentioned treatment is not adequate when dealing with inertial phenomena. More precisely, inertial effects for flow in a porous medium should not be treated in the same manner as, for example, the macroscopic resistivity force, or macroscopic stress, with regard to defining their macroscopic constitutive equations. Both the resistivity force and the macroscopic stress have their origin in microscopic forces, either over $\mathrm{d}A^{fs}$ or within $\mathrm{d}V^{f}$, while the macroscopic inertial forces, as it has been shown, are of different nature, i.e., they are of kinematical origin and are direct functions of the microscopic flow.

Apart from the continuum considerations which employ an averaging procedure, numerous works have appeared in the literature analysing fluid flow in a porous medium through application of the mixture theory (see, e.g., Mueller, 1971; Fulks *et al.*, 1971; Bedford and Ingram, 1971; Raats and Klute, 1968; Raats, 1971; Morland, 1972; Kenyon, 1976a, b). In terms of inertial phenomena, however, this approach will essentially lead to an incomplete description of the macroscopic balance of forces. To demonstrate this point more clearly, we shall briefly discuss some of the results on flow through porous media obtained by direct application of the classical mixture theory.

In these considerations the spatial point $\mathbf{x}$ is initially assumed to contain both phases simultaneously, i.e., the media is treated only macroscopically. Local balance equations are taken analogous to the ones valid in mixture theory (see, e.g., Truesdell, 1965), where exchange terms for different thermodynamic quantities are present. Based on axioms of constitutive theory, the constitutive relations are obtained. With regard to the linear momentum balance equation, however, a similar problem can be observed as the one mentioned earlier related to the work of Hassanizadeh and Gray (1980), namely that these equations are incomplete and do not include the microscopic inertial phenomena.

Mueller (1971), for example, derives a linear momentum equation resembling (77), where $\mathbf{R}$ is assumed isotropic and the macroscopic viscosity is neglected, reducing it to Darcy's law for an acceleration-free flow. Contrary to experience, however, for a macroscopic acceleration-free flow (namely, a macroscopically uniform, unidirectional and steady flow), one would have Darcy's law valid irrespective of the velocity magnitude.

Raats (1971) specifically discusses inertia in flow through a porous medium, based on a linear momentum equation essentially identical to the one considered by Mueller (1971). He then reduces it to Darcy's law for negligible inertial forces. As it has been shown, however, this form of the linear momentum balance equation is incomplete. It includes what one may refer to as macroscopic inertia, but does not account for significant microscopic inertial phenomena. Similar observations can be made in other works (see, e.g., Raats and Klute, 1968; Fulks *et al.*, 1971).

Thus, we may conclude that the macroscopic inertial forces for flow in a porous medium can be fully understood only if one considers the microscopic flow phenomena. In this respect, employing an averaging procedure provides an important advantage, since considerations are made on both the microscopic and macroscopic levels. This is in contrast to the direct application of the mixture theory, where one treats the porous medium only macroscopically, i.e., as a mixture of phases, without analysing the microscopic flow phenomena.

7. Some Possible Extensions of the Derived Equation

Equation (57) (which represents (47) for $\rho = \rho(\mathbf{r}, t)$, and (48) for $\rho = \text{const.}$) has been derived assuming the dissipative part of the resistivity force $\tilde{\mathbf{T}}$ as a linear function of $\bar{\mathbf{v}}$. One could suppose, however, this approximation to be insufficient for high macroscopic velocities. Thus, we shall propose in the following the macroscopic linear momentum balance equation in form of a constitutive relation for high velocity fluid flow in a porous medium, when a higher-order expansion of $\tilde{\mathbf{T}}$ in terms of $\bar{\mathbf{v}}$ is assumed.

We now write Equation (57) in the form

$$I_k + \bar{v}C_{kl}\bar{v}_l + \tilde{S}_{kl}\bar{v}_l = J_k + \tilde{T}_k \tag{79}$$

where $\tilde{\mathbf{S}} = \mathbf{S}$, i.e., $\tilde{\mathbf{S}} = \mathbf{0}$ if we assume, respectively, $\rho = \rho(\mathbf{r}, t)$, i.e., $\rho = \text{const.}$ Employing an expansion of $\tilde{\mathbf{T}}$ up to the third order in (79), yields

$$\begin{aligned} I_k &+ (\bar{v}C_{kl} + S_{kl})\bar{v}_l \\ &= J_k + R_{kl}\bar{v}_l + R^{(2)}_{klm}\bar{v}_l\bar{v}_m + R^{(3)}_{klmn}\bar{v}_l\bar{v}_m\bar{v}_n \end{aligned} \tag{80}$$

where

$$\begin{aligned} R^{(2)}_{klm} &\equiv \frac{\partial^2 \tilde{T}_k}{\partial \bar{v}_l \partial \bar{v}_m} \\ R^{(3)}_{klmn} &\equiv \frac{\partial^3 \tilde{T}_k}{\partial \bar{v}_l \partial \bar{v}_m \partial \bar{v}_n} \end{aligned} \tag{81}$$

evaluated for $\bar{\mathbf{v}} = \mathbf{0}$.

Neglecting the effect of $\mathbf{I}$ and $\tilde{\mathbf{S}}$, Equation (80) can be written as

$$J_k = \tilde{R}_{kl}\bar{v}_l + \bar{v}C_{kl}\bar{v}_l + R_{klm}\bar{v}_l\bar{v}_m + R_{klmn}\bar{v}_l\bar{v}_m\bar{v}_n \tag{82}$$

where

$$R_{klm} \equiv - R^{(2)}_{klm}, \qquad R_{klmn} \equiv - R^{(3)}_{klmn}. \tag{83}$$

Equation (82) is somewhat similar to a relation obtained in the work of Barak and Bear (1981). These authors maintain both the third- and fourth-order tensors, but do not have a third-order velocity term.

For an isotropic case, we have $R_{klm} = 0$ and Equation (82) can be written in the form

$$J_k = \tilde{R}\bar{v}_k + \bar{v}C\bar{v}_k + R'\bar{v}^2\bar{v}_k \tag{84}$$

where

$$C_{kl} = C\,\delta_{kl}, \qquad \tilde{R}_{kl} = \tilde{R}\,\delta_{kl} \tag{85}$$

and R' has been obtained from isotropy of R_{klmn}. Formally, one can write (84) as

$$J_k = \sum_{j=1}^{3} \bar{v}^{(j-1)}\lambda_j\bar{v}_k, \tag{86}$$

where λ_j are assumed constant and are to be determined experimentally for a specific medium. Equation (86), i.e., (84), partly gives support to the power series relation (Hannoura and Barends, 1980), but it is a question of whether one is justified in accepting

$$J_k = \sum_{j=1}^{N} \bar{v}^{(j-1)}\lambda_j\bar{v}_k, \tag{87}$$

with $N > 3$, as a generalized concept.

However, with local inertia negligible, we have a rational basis to consider (82) as a macroscopic linear momentum balance equation in form of a constitutive relation for high velocity flow through an anisotropic medium, that will include higher-order effects in terms of $\bar{\mathbf{v}}$ as compared to (57). For a fully isotropic case, Equation (84) should be employed.

In the work of Hassanizadeh and Gray (1980), it is shown that for nonisothermal conditions $\tilde{\mathbf{T}}$ is also a function of $\nabla\theta$. Thus, expanding $\tilde{\mathbf{T}}$ in terms of $\nabla\theta$ in a similar manner as it was done in terms of $\bar{\mathbf{v}}$, and maintaining terms up to second order, we can write (79) for the nonisothermal, anisotropic case as

$$\begin{aligned} &I_k + \bar{v}C_{kl}\bar{v}_l + R_{kl}\bar{v}_l \\ &\quad = J_k + \Sigma_{kl}\theta_{,l} + \Sigma_{klm}\theta_{,l}\theta_{,m} + R^{(2)}_{klm}\bar{v}_l\bar{v}_m + R^{(3)}_{klmn}\bar{v}_l\bar{v}_m\bar{v}_n \end{aligned} \tag{88}$$

where the coefficients

$$\Sigma_{kl} \equiv \frac{\partial \tilde{T}_k}{\partial \theta_{,l}}, \qquad \Sigma_{klm} \equiv \frac{\partial^2 \tilde{T}_k}{\partial \theta_{,l}\,\partial \theta_{,m}} \tag{89}$$

are both evaluated at equilibrium (i.e., for $\bar{\mathbf{v}} = \mathbf{0}$ and $\nabla\theta = \mathbf{0}$).

For a fully isotropic case, Equation (88) yields

$$I_k + \bar{v}C\bar{v}_k + R\bar{v}_k = J_k + \Sigma\,\theta_{,k} + R\bar{v}^2\bar{v}_k \tag{90}$$

where

$$\Sigma_{kl} = \Sigma\ \delta_{kl}\,. \tag{91}$$

An additional restriction for the derivation of Equation (57) was laminar flow within $\mathrm{d}V^f$. In other words, we have assumed the magnitude of microscopic fluid velocities to be within a range when turbulence has not developed in the flow region. With this assumption, the microscopic flow field can be considered as deterministic, i.e., for a specific configuration of the void space V^f and specific boundary conditions over A^{fs} (see Figure 1), a unique solution of the fields $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ and $\rho = \rho(\mathbf{r}, t)$ can be obtained from Equations (3) and (4), assuming the fluid Newtonian; from $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ a unique and continuous streamline field follows. However, if one considers microscopic velocities with magnitudes exceeding a certain limit for a specific region, turbulence will develop. In this case V^f and the boundary conditions over A^f do not uniquely determine the field $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$, and an essentially chaotic behaviour will take place (see, e.g., Rabinovich, 1978).

Although for a turbulent region we are not able to uniquely determine the velocity field, nevertheless, we know that developed turbulent flow is characterized by increased circulation. Instantaneous streamlines, in general discontinuous, will chaotically change their position, size and shape in the course of time, without external cause (i.e., without changing boundary conditions).

If we now consider a possible application of Equation (57) to fluid flow in a porous medium with developed turbulence, the main formal difficulty arises from the discontinuity of streamlines. Thus, quantities such as $\hat{\mathbf{e}}$, or $\nabla\hat{\mathbf{e}}$, associated with the streamline curvature, would have to be redefined through a limit as one approaches the discontinuity point along the streamline at time t.

Without attempting to introduce such formal analysis, we could consider Equation (57) essentially valid for a porous region with a developed turbulence. This is due to the very nature of the turbulence which implies an occurrence and increase in a similar type of inertial phenomena, namely the convective inertia is greater in a turbulent flow region as compared to a laminar one due to fluid particle circulation.

One could, therefore, expect that for turbulent flow in a porous medium, the coefficients $\mathbf{C}$ of Equation (57) would increase in magnitude, while the form of the equation would remain identical. This is what has been observed in experiments for the isotropic case (see, e.g., Scheidegger, 1960). With regard to the resistivity force, the higher-order expansion in terms of $\bar{\mathbf{v}}$, as given in (82), could be more appropriate as compared to the linear approximation given in (57), when one treats turbulence in a porous medium.

8. Summary

Based on microscopic continuum considerations, employing an averaging procedure, and introducing macroscopic quantities, Equation (57) has been derived, representing Equation (47) for the case when $\rho = \rho(\mathbf{r}, t)$, and Equation (48) for the case $\rho = \text{const}$.

If the dissipative part of the resistivity force $\tilde{\mathbf{T}}$ can be taken as a linear function of the macroscopic fluid velocity $\bar{\mathbf{v}}$, Equation (57) may be considered as a generalization of Forchheimers formula for nonDarcy flow that has a rational basis. It has been shown how one can recover from Equation (57) most of the established relations for nonDarcy flow as special cases, by employing isotropy assumptions. If the magnitude of the fluid velocity is small, the first and second terms on the left-hand side of (57) will be negligible, and a linear relationship between $\mathbf{J}$ and $\bar{\mathbf{v}}$ is obtained.

Although (57) is essentially the macroscopic linear momentum balance equation for the fluid phase, due to its form one can consider it as a constitutive type of relation for high fluid velocity magnitudes, when treating fluid flow in a porous medium as a system of overlapping continua. Equation (57) relates the dynamic quantity $\mathbf{J}$ to the kinematic quantity $\bar{\mathbf{v}}$ in a nonlinear and anisotropic manner, involving inertial continuum coefficients which should only be conditionally considered as material and, in general, depend on both the microscopic (configuration of the void space) and macroscopic boundary conditions (e.g., the curvature of the macroscopic streamline field).

Assuming a higher-order expansion of the resistivity force in terms of the macroscopic velocity, Equation (88) has been proposed for flow in a porous medium with possible temperature gradients.

Constitutive equations derived in the present work have been based on a specific conceptual model, that is, on a specific decomposition of the kinematic field, and are essentially qualitative in nature. The magnitude of obtained constitutive coefficients, as well as the significance of discussed effects (such as anisotropy due to macroscopic curvature), could further be quantified only through physical experiments.

Appendix I

Rewriting condition (25),

$$\int_{\mathrm{d}V} \rho\gamma(\hat{v}\bar{\alpha}_k + \bar{v}\hat{\alpha}_k + \hat{v}\hat{\alpha}_k)\,\mathrm{d}v = 0 \tag{I.1}$$

we first derive the integral locally with respect to time and get

$$-\frac{1}{\mathrm{d}V}\int_{\mathrm{d}V}\frac{\partial}{\partial t}(\rho\gamma)(\hat{v}\bar{\alpha}_k + \bar{v}\hat{\alpha}_k + \hat{v}\hat{\alpha}_k)\,\mathrm{d}v$$

$$= \bar{\alpha}_k \left\langle \rho \frac{\partial \hat{v}}{\partial t} \right\rangle + \frac{\partial \bar{\alpha}_k}{\partial t} \langle \rho \hat{v} \rangle + \frac{\partial \bar{v}}{\partial t} \langle \rho \hat{\alpha}_k \rangle + + \bar{v} \left\langle \rho \frac{\partial \hat{\alpha}_k}{\partial t} \right\rangle + \left\langle \rho \frac{\partial}{\partial t} (\hat{v} \hat{\alpha}_k) \right\rangle \tag{I.2}$$

Deriving Equation (I.1) spatially with respect to **r** and multiplying both sides with $\bar{v}_l$, we have

$$-\frac{\bar{v}_l}{\mathrm{d}V} \int_{\mathrm{d}V} (\rho\gamma)_{,l} (\hat{v}\bar{\alpha}_k + \bar{v}\hat{\alpha}_k + \hat{v}\hat{\alpha}_k)\,\mathrm{d}v = \{\bar{\alpha}_k \langle \rho \hat{v}_{,l} \rangle + \bar{v} \langle \rho \hat{\alpha}_{k,l} \rangle + + \langle \rho \hat{v} \hat{\alpha}_{k,l} \rangle + \langle \rho \hat{v}_{,l} \hat{\alpha}_k \rangle \} \bar{v}_l . \tag{I.3}$$

Adding (I.2) and (I.3) gives

$$-\frac{1}{\mathrm{d}V} \int_{\mathrm{d}V} \frac{D^f}{Dt} (\rho\gamma)(\hat{v}\bar{\alpha}_k + \bar{v}\hat{\alpha}_k + \hat{v}\hat{\alpha}_k)\,\mathrm{d}v = \bar{\alpha}_k \left\langle \rho \frac{\partial \hat{v}}{\partial t} \right\rangle + \frac{\partial \bar{\alpha}_k}{\partial t} \langle \rho \hat{v} \rangle + \left\langle \rho \frac{\partial}{\partial t} (\hat{v}\hat{\alpha}_k) \right\rangle + + \frac{\partial \bar{v}}{\partial t} \langle \rho \hat{\alpha}_k \rangle + \bar{v} \left\langle \rho \frac{\partial \hat{\alpha}_k}{\partial t} \right\rangle + \bar{\alpha}_k \langle \rho \hat{v}_{,l} \rangle \bar{v}_l + + \bar{v} \langle \rho \hat{\alpha}_{k,l} \rangle \bar{v}_l + \langle \rho \hat{\alpha}_k \hat{v}_{,l} \rangle \bar{v}_l + \langle \rho \hat{\alpha}_{k,l} \hat{v} \rangle \bar{v}_l \tag{I.4}$$

where

$$\frac{D^f}{Dt} (\rho\gamma) \equiv \frac{\partial}{\partial t} (\rho\gamma) + \bar{v}_l (\rho\gamma)_{,l} . \tag{I.5}$$

Identity (I.5) can be further developed as

$$\frac{D^f}{Dt} (\rho\gamma) = \frac{\partial \rho}{\partial t} \gamma - \rho w_l \gamma_{,l} + \bar{v}_l (\rho_{,l} \gamma + \rho \gamma_{,l}) = \left(\frac{\partial \rho}{\partial t} + \bar{v}_l \rho_{,l} \right) \gamma + \rho (w_l - \bar{v}_l) \delta(\mathbf{r} - \mathbf{r}^{fs}) n_l \tag{I.6}$$

where

$$\frac{\partial \gamma}{\partial t} = - w_l \gamma_{,l} \tag{I.7}$$

and

$$\gamma_{,l} = - n_l \delta(\mathbf{r} - \mathbf{r}^{fs}) \tag{I.8}$$

have been employed (see Gray and Lee, 1977).

From previous assumptions, it follows that the velocity of the interphase surface is zero, i.e.,

$$w_l = 0 \tag{I.9}$$

and we shall assume the fluid velocity at the interphase boundary as zero (no-slip condition), i.e.,

$$\mathbf{v}(\mathbf{r}^{fs}, t) = \mathbf{0} \tag{I.10}$$

which also implies

$$v(\mathbf{r}^{fs}, t) = \bar{v}(\mathbf{x}, t) + \hat{v}(\mathbf{r}^{fs}, t) = 0. \tag{I.11}$$

Thus, the left-hand side of (I.4) can be written as

$$\begin{aligned} -\frac{1}{\mathrm{d}V}\int_{\mathrm{d}V}\frac{D^f}{Dt}(\rho\gamma)(\hat{v}\bar{\alpha}_k + \bar{v}\hat{\alpha}_k + \hat{v}\hat{\alpha}_k)\,\mathrm{d}v \\ = -\frac{1}{\mathrm{d}V}\int_{\mathrm{d}V}\left(\frac{\partial\rho}{\partial t} + \bar{v}_l\rho_{,l}\right)\gamma(\hat{v}\bar{\alpha}_k + \bar{v}\hat{\alpha}_k + \hat{v}\hat{\alpha}_k)\,\mathrm{d}v - \\ -\frac{1}{\mathrm{d}V}\bar{v}_l\bar{v}_k\int_{\mathrm{d}A^{fs}}\rho n_l\,\mathrm{d}a \end{aligned} \tag{I.12}$$

where Equation (I.11) has been employed.

Applying a theorem which relates microscopic and macroscopic gradients (see, e.g., Gray and Lee, 1977), the integral of the last term on the right-hand side of (I.12) can be written in the form

$$\frac{1}{\mathrm{d}V}\int_{\mathrm{d}A^{fs}}\rho\mathbf{n}\,\mathrm{d}a = \frac{1}{\mathrm{d}V}\int_{\mathrm{d}V^f}\nabla_r\rho\,\mathrm{d}v - \nabla_x\bar{\rho} \tag{I.13}$$

where ∇_r and ∇_x denote the gradients with respect to $\mathbf{r}$ and $\mathbf{x}$, respectively. Furthermore, the average fluid density $\bar{\rho}$ can alternatively be expressed in terms of the intrinsic fluid density $\bar{\bar{\rho}}$ and porosity as

$$\bar{\rho} \equiv \frac{1}{\mathrm{d}V}\int_{\mathrm{d}V^f}\rho\,\mathrm{d}v = \epsilon\frac{1}{\mathrm{d}V^f}\int_{\mathrm{d}V^f}\rho\,\mathrm{d}v \equiv \epsilon\bar{\bar{\rho}}. \tag{I.14}$$

Thus, in component form, Equation (I.13) becomes

$$\frac{1}{\mathrm{d}V}\int_{\mathrm{d}A^{fs}}\rho n_l\,\mathrm{d}a = \frac{1}{\mathrm{d}V}\int_{\mathrm{d}V^f}\rho_{,l}\,\mathrm{d}v - (\bar{\bar{\rho}}\epsilon)_{,l}. \tag{I.15}$$

If one now assumes local and spatial variations of the microscopic fluid density to be negligible, i.e., $\partial\rho/\partial t = \rho_{,k} = 0$, Equation (I.12) can be written, employing (I.15) in the form

$$-\frac{1}{\mathrm{d}V}\int_{\mathrm{d}V}\frac{D^f}{Dt}(\rho\gamma)(\hat{v}\bar{\alpha}_k + \bar{v}\hat{\alpha}_k + \hat{v}\hat{\alpha}_k)\,\mathrm{d}v = \bar{v}_l\bar{v}_k\rho\epsilon_{,l} \tag{I.16}$$

where

$$\rho \equiv \bar{\bar{\rho}} = \text{const.} \tag{I.17}$$

Therefore, when (I.17) is valid, one can write (I.4) as

$$\begin{aligned}
&\bar{\alpha}_k \left\langle \rho \frac{\partial \hat{v}}{\partial t} \right\rangle + \frac{\partial \bar{\alpha}_k}{\partial t} \langle \rho \hat{v} \rangle + \left\langle \rho \frac{\partial}{\partial t} (\hat{v} \hat{\alpha}_k) \right\rangle + \\
&\quad + \frac{\partial \bar{v}}{\partial t} \langle \rho \hat{\alpha}_k \rangle + \bar{v} \left\langle \rho \frac{\partial \hat{\alpha}_k}{\partial t} \right\rangle + \bar{\alpha}_k \langle \rho \hat{v}_{,l} \rangle \bar{v}_l + \\
&\quad + \bar{v} \langle \rho \hat{\alpha}_{k,l} \rangle \bar{v}_l + \langle \rho \hat{\alpha}_k \hat{v}_{,l} \rangle \bar{v}_l + \langle \rho \hat{\alpha}_{k,l} \hat{v} \rangle \bar{v}_l \\
&= \bar{v}_l \bar{v}_k \rho \epsilon_{,l}
\end{aligned} \tag{I.18}$$

Appendix II

The derivation of the macroscopic stress tensor $\bar{\mathbf{t}}$ is analogous to the one presented in Hassanizadeh and Gray (1979a, b), the essential difference, however, being in the definition of the microscopic stress tensor. In the above reference it is defined as

$$\mathbf{t}^* = \mathbf{t} - \rho \hat{\mathbf{v}} \hat{\mathbf{v}}, \tag{76}$$

i.e., it contains a significant microscopic part of inertia, while in this work the microscopic stress tensor is $\mathbf{t}$.

Integrating the average $\langle \gamma \nabla \cdot \mathbf{t} \rangle$ over the entire volume of the medium V, with a volume element at $\mathbf{x}$ denoted by $\mathrm{d}V_x$, we have

$$\begin{aligned}
\int_V \langle \gamma \nabla_r \cdot \mathbf{t} \rangle \, \mathrm{d}V_x &= \int_V \langle \nabla_r \cdot (\mathbf{t}\gamma) \rangle \, \mathrm{d}V_x - \int_V \langle \mathbf{t} \cdot \nabla_r \gamma \rangle \, \mathrm{d}V_x \\
&= \int_V \langle \nabla_r \cdot (\mathbf{t}\gamma) \rangle \, \mathrm{d}V_x + \int_V \langle \mathbf{t} \cdot \mathbf{n} \, \delta(\mathbf{r} - \mathbf{r}^{fs}) \rangle \, \mathrm{d}V_x
\end{aligned} \tag{II.1}$$

where $\langle \quad \rangle$ is defined as

$$\langle (\quad) \rangle \equiv \frac{1}{\mathrm{d}V} \int_{\mathrm{d}V} (\quad) \, \mathrm{d}v \tag{II.2}$$

and (I.8) has been employed. Relation (II.1) can further be written using Theorem IV in Hassanizadeh and Gray (1979a), as

$$\begin{aligned}
&\int_V \left\{ \frac{1}{\mathrm{d}V} \int_{\mathrm{d}V} (\nabla_r \cdot \mathbf{t}) \gamma \, \mathrm{d}v \right\} \mathrm{d}V_x \\
&= \int_A \left\{ \frac{1}{\mathrm{d}A} \int_{\mathrm{d}A} \gamma \mathbf{t} \cdot \mathbf{n} \, \mathrm{d}a \right\} \mathrm{d}A_x + \int_V \left\{ \frac{1}{\mathrm{d}V} \int_{\mathrm{d}A^{fs}} \mathbf{n} \cdot \mathbf{t} \, \mathrm{d}a \right\} \mathrm{d}V_x .
\end{aligned} \tag{II.3}$$

It is now assumed that the stress vector $\boldsymbol{\tau}$, defined at the macroscopic boundary surface element $\mathrm{d}A_x$ and obtained from

$$\boldsymbol{\tau} = \frac{1}{\mathrm{d}A} \int_{\mathrm{d}A} \mathbf{n} \cdot \mathbf{t}\gamma \, \mathrm{d}a, \tag{II.4}$$

may be expressed through an existing stress tensor $\bar{\mathbf{t}}$ as

$$\boldsymbol{\tau} = \mathbf{N} \cdot \bar{\mathbf{t}} \tag{II.5}$$

where $\mathbf{N}$ is a unit normal on the surface element $\mathrm{d}A_x$, of A that bounds V.

Using the divergence theorem, the left-hand side of (II.1) is finally written in the form

$$\begin{aligned}\int_V \langle \gamma \nabla_r \cdot \mathbf{t} \rangle \, \mathrm{d}V_x &= \int_V \left\{ \frac{1}{\mathrm{d}V} \int_{\mathrm{d}A^{fs}} \mathbf{n} \cdot \mathbf{t} \, \mathrm{d}a \right\} \mathrm{d}V_x + \int_V \nabla_x \cdot \bar{\mathbf{t}} \, \mathrm{d}V_x \\ &= \int_V \bar{\rho} \mathbf{T} \, \mathrm{d}V_x + \int_V \nabla_x \cdot \bar{\mathbf{t}} \, \mathrm{d}V_x, \end{aligned} \tag{II.6}$$

where $\bar{\mathbf{t}}$ is the macroscopic fluid stress tensor, and $\mathbf{T}$ is defined as

$$\mathbf{T} \equiv \frac{1}{\bar{\rho} \, \mathrm{d}V} \int_{\mathrm{d}A^{fs}} \mathbf{n} \cdot \mathbf{t} \, \mathrm{d}a \tag{II.7}$$

with $\mathbf{n}$ being oriented from the fluid to the solid phase (see Figure 2).

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