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Flow in Porous Media I: A Theoretical Derivation of Darcy's Law

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Abstract. Stokes flow through a rigid porous medium is analyzed in terms of the method of volume averaging. The traditional averaging procedure leads to an equation of motion and a continuity equation expressed in terms of the volume–averaged pressure and velocity. The equation of motion contains integrals involving spatial deviations of the pressure and velocity, the Brinkman correction, and other lower-order terms. The analysis clearly indicates why the Brinkman correction should not be used to accommodate a *no slip* condition at an interface between a porous medium and a bounding solid surface.

The presence of spatial deviations of the pressure and velocity in the volume-averaged equations of motion gives rise to a *closure problem*, and representations for the spatial deviations are derived that lead to Darcy's law. The theoretical development is not restricted to either homogeneous or spatially periodic porous media; however, the problem of *abrupt changes* in the structure of a porous medium is not considered.

Key words. Volume averaging, Brinkman, correction, closure.

0. Nomenclature

Roman Letters

$\mathcal{A}_{\beta\sigma}$ interfacial area of the β-σ interface contained within the macroscopic system, m^2

$\mathcal{A}_{\beta e}$ area of entrances and exits for the β-phase contained within the macroscopic system, m^2

$A_{\beta\sigma}$ interfacial area of the β-σ interface contained within the averaging volume, m^2

$A^*_{\beta\sigma}$ interfacial area of the β-σ interface contained within a unit cell, m^2

$A_{\beta e}$ area of entrances and exits for the β-phase contained within a unit cell, m^2

$\mathbf{B}$ second order tensor used to represent the velocity deviation (see Equation (3.30))

$\mathbf{b}$ vector used to represent the pressure deviation (see Equation (3.31)), m^{-1}

d distance between two points at which the pressure is measured, m

$\mathbf{g}$ gravity vector, m/s^2

$\mathbf{K}$ Darcy's law permeability tensor, m^2

L characteristic length scale for volume averaged quantities, m
ℓ_β characteristic length scale for the β-phase (see Figure 2), m
ℓ_σ characteristic length scale for the σ-phase (see Figure 2), m
$\mathbf{n}_{\beta\sigma}$ unit normal vector pointing from the β-phase toward the σ-phase ($\mathbf{n}_{\beta\sigma} = -\mathbf{n}_{\sigma\beta}$)
$\mathbf{n}_{\beta e}$ unit normal vector for the entrances and exits of the β-phase contained within a unit cell
p_β pressure in the β-phase, N/m^2
$\langle p_\beta \rangle^\beta$ intrinsic phase average pressure for the β-phase, N/m^2
$\tilde{p}_\beta$ $p_\beta - \langle p_\beta \rangle^\beta$, spatial deviation of the pressure in the β-phase, N/m^2
r_0 radius of the averaging volume and radius of a capillary tube, m
$\mathbf{v}_\beta$ velocity vector for the β-phase, m/s
$\langle \mathbf{v}_\beta \rangle$ phase average velocity vector for the β-phase, m/s
$\langle \mathbf{v}_\beta \rangle^\beta$ intrinsic phase average velocity vector for the β-phase, m/s
$\tilde{\mathbf{v}}_\beta$ $\mathbf{v}_\beta - \langle \mathbf{v}_\beta \rangle^\beta$, spatial deviation of the velocity vector for the β-phase, m/s
$\mathcal{V}$ averaging volume, m^3
V_β volume of the β-phase contained within the averaging volume, m^3

Greek Letters

ϵ_β $V_\beta/\mathcal{V}$, volume fraction of the β-phase
ρ_β mass density of the β-phase, kg/m^3
μ_β viscosity of the β-phase, Nt/m^2
$\boldsymbol{\psi}$ arbitrary function used in the representation of the velocity deviation (see Equations (3.11) and (B1)), m/s
ξ arbitrary function used in the representation of the pressure deviation (see Equations (3.12) and (B2)), s^{-1}

1. Introduction

The process of flow through porous media is of interest to a wide range of engineers and scientists, in addition to politicians and economists who recognize the importance of groundwater flows and a variety of tertiary oil recovery processes. The one-dimensional empiricism discovered by Darcy in 1856 has served as a starting point for numerous practical applications and as a constant challenge for theoreticians. While the original conditions studied by Darcy are found in many practical situations, it is the extensions to more general cases that are especially deserving of theoretical analysis for they usually represent situations in which experiments are difficult to perform. The first such extension is to fully three-dimensional flows which abound in the practical world of groundwater flows and oil recovery processes. While this form of Darcy's law is used with great frequency, there appears to be no experimental verification of the obvious tensorial representation of Darcy's empiricism. On the other hand, there are many theoretical treatments that lead to this result (Gray and O'Neill, 1976) and summaries of the subject are available in the work of Bear (1972) or in the more recent treatise of Greenkorn (1984). Beyond the three-dimensional extension, there are two challenging areas of considerable practical importance: flow at moderate Reynolds numbers and multiphase fluid flow. In multidimensional form, these flows represent extremely difficult experimental problems and reliable theoretical results are of great value. Further extensions would include the case of deformable solids (Biot, 1962) and the flow of nonlinear, viscoelastic fluids (Slattery, 1967). All these extensions have received some type of theoretical treatment, but in every case, one or several constitutive assumptions were made enroute to the final result. The single exception would appear to be the work of Brenner (1968) in which Stokes flow in a spatially periodic porous medium was analyzed in order to produce Darcy's law for the case in which the volume-averaged velocity vector was a constant.

In this work we are not restricted to either spatially periodic or homogeneous porous media, and the analysis is devoid of any constitutive assumptions. Although the final result is a foregone conclusion, the analysis provides a framework for the study of important extensions such as those mentioned above. Two-phase flow is of particular interest since recent studies of drying granular porous media (Whitaker, 1984) indicate that the traditional ideas concerning relative permeabilities and irreducible saturations need a careful reexamination. The problem of two-phase flow in porous media is investigated in Part II of this paper and Part III treats the linear problem associated with deformable media.

2. Volume Averaging

The system under consideration is illustrated in Figure 1 in which the macroscopic length scale is identified by L and an averaging volume is indicated by $\mathcal{V}$. The details of the system are shown in Figure 2 in which ℓ_β has been used for the characteristic length of the liquid phase and ℓ_σ as the characteristic length of the solid phase. Traditionally one thinks of the method of volume averaging as being applicable for systems in which the length scales are constrained by (Whitaker, 1969)

$$\ell \ll r_0 \ll L \tag{2.1}$$

where r_0 is the radius of the averaging volume, $\mathcal{V}$. In reality, this inequality is nothing more than a convenient restriction that is satisfied by many systems of practical importance; however, it is *not* an inherent restriction in the method itself. The recent work of Ross (1983) concerning the flow in the region between a porous medium and a homogeneous fluid makes this point clear, and in the following theoretical development we will be careful to point out where length-

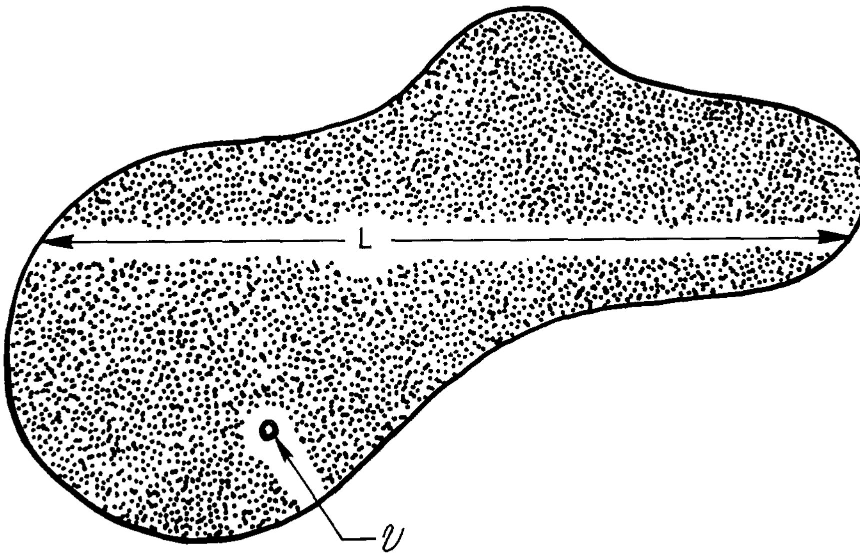

Fig. 1. Macroscopic system.

scale constraints are imposed and the consequences that result when the constraints fail.

The boundary value problem under consideration can be expressed as

$$0 = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta , \tag{2.2}$$

$$\nabla \cdot \mathbf{v}_\beta = 0 \tag{2.3}$$

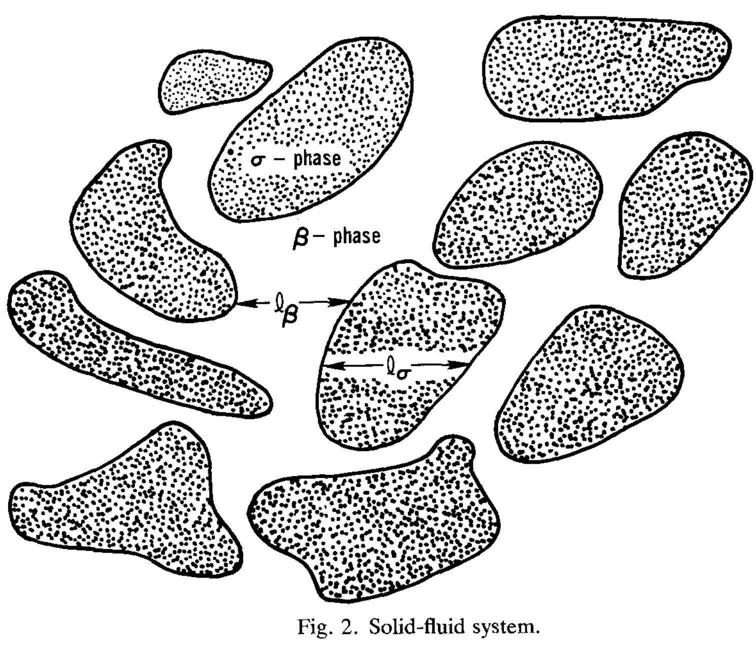

Fig. 2. Solid-fluid system.

$$\text{B.C.1} \quad \mathbf{v}_\beta = 0, \quad \text{on } \mathscr{A}_{\beta\sigma}, \tag{2.4}$$

$$\text{B.C.2} \quad \mathbf{v}_\beta = \mathbf{f}(\mathbf{r}, t), \quad \text{on } \mathscr{A}_{\beta e}. \tag{2.5}$$

Here we have carefully identified quantities associated with the β-phase with an appropriate subscript in preparation for the subsequent analysis of two-phase flow in porous media. The interfacial area between the fluid and solid phases has been identified by $\mathscr{A}_{\beta\sigma}$ while $\mathscr{A}_{\beta e}$ has been used to represent the area of entrances and exits for the macroscopic system illustrated in Figure 1. While the boundary condition at $\mathscr{A}_{\beta e}$ is generally unknown except in terms of averages, Equation (2.5) will serve as a reminder of what we do *not* know and will help to identify the assumptions made about the effect of this boundary on the flow field.

There are two types of volume averages that are commonly encountered in the study of multiphase transport phenomena, and both are used in the traditional formulation of Darcy's law. The first of these is the *phase average* which is defined by

$$\langle \psi_\beta \rangle = \frac{1}{\mathscr{V}} \int_{V_\beta} \psi_\beta \, \mathrm{d}V. \tag{2.6}$$

Here V_β represents the volume of the β-phase contained within the averaging volume $\mathscr{V}$. In general, $\langle \psi_\beta \rangle$ is not the preferred average since it is *not* equal to ψ_β when the latter is a constant. The *intrinsic phase average* is represented by

$$\langle \psi_\beta \rangle^\beta = \frac{1}{V_\beta} \int_{V_\beta} \psi_\beta \, \mathrm{d}V \tag{2.7}$$

and it is clearly more representative of the conditions in the β-phase. These two averages are related by

$$\langle \psi_\beta \rangle = \epsilon_\beta \langle \psi_\beta \rangle^\beta \tag{2.8}$$

in which ϵ_β is the volume fraction of the β-phase. This is given explicitly by

$$\epsilon_\beta = V_\beta / \mathscr{V}. \tag{2.9}$$

In this work, the averaging volume should be thought of as a sphere of constant radius, while V_β depends on the nature of the porous medium under consideration and will only be a function of the spatial coordinates for a rigid porous medium. The matter of time and space-dependent averaging volumes has been explored briefly by both Gray (1983) and Cushman (1983).

When the average of Equation (2.2) is formed, one obtains the average of a gradient while it is the gradient of an average that is desired. The latter can be obtained from the former by means of the spatial averaging theorem (Anderson and Jackson, 1967; Marle, 1967; Slattery, 1967; Whitaker, 1967). For some quantity ψ_β associated with the β-phase, this theorem takes the form

$$\langle \nabla \psi_\beta \rangle = \nabla \langle \psi_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \psi_\beta \, \mathrm{d}A. \tag{2.10}$$

Here $A_{\beta\sigma}$ represents the interfacial area contained within the averaging volume and $\mathbf{n}_{\beta\sigma}$ represents the unit outwardly directed normal vector for the β-phase. Veverka (1981) has raised questions about the validity of Equation (2.10), and in a recent study Howes and Whitaker (1985) have re-examined the derivation with great care. Their work confirms the correctness of Equation (2.10) and points out that under certain pathological conditions the derivative of the average may not be defined at a countable set of points. However, this does not prohibit the use of Equation (2.10) nor the integration of the partial differential equations that result from the method of volume averaging.

It is important to note that both Marle (1967) and Anderson and Jackson (1967) used weighting functions in order to define the averages used in their studies, and that Equations (2.6)–(2.10) represent special cases in which the weighting function is constant within the averaging volume and zero everywhere else. Baveye and Sposito (1984) and Cushman (1984) have pointed out that the approach of Marle and of Anderson and Jackson can be used to rigorously match theory and experiment since the weighting function can be chosen to correspond to the characteristics of the measuring device. This presents a situation in which *different* properties may be associated with *different* weighting functions and gives rise to additional complexities in the theoretical development. This aspect of spatial smoothing was considered earlier by Bear and Braester (1972) in terms of a property-dependent REV, and the problem of precise comparison between theory and experiment still remains as an important area of investigation.

In a theoretical and experimental study of heat conduction in porous media, Nozad *et al.* (1985) were able to show that experimental values of 'point temperatures' could be used to accurately determine changes in the spatial average temperature. A similar situation occurs during mass transfer in porous media (Ryan *et al.*, 1981; Carbonell and Whitaker, 1984), thus in these cases there is no need to precisely match the measuring instrument characteristics with the definition of the dependent variable in a spatially smoothed transport equation. The pressure field for *single-phase* flow in porous media represents another situation in which point values can be used to accurately deduce changes in averaged values and the arguments supporting this point of view are given in Appendix A.

Intuition suggests that the volume-averaged velocity for *single-phase* flow in porous media will be insensitive to the choice of $\mathscr{V}$ provided $\mathscr{V}$ is sufficiently large; however, two-phase flow processes are undoubtedly more complex and Baveye and Sposito cite the problem of measuring the moisture content as an example in which instrument characteristics need to be carefully considered. In general, one might think that local heterogeneities represent the most severe problem in effecting a closure between theory and experiment. One line of attack is the use of weighting functions to remove any uncertainty between the defined dependent variable and the measured quantity; however, the effects of these local heterogeneities must be captured theoretically if a precise correspondence between dependent variables and experiments is to be useful. An alternative may be the relaxation of the constraint suggested by Equation (2.1) and the inclusion of higher order terms in the spatially smoothed transport equations. The work of Ross (1983) represents an example of this approach.

While unanswered questions remain concerning the comparison between theory and experiment, the approach taken here has proved to be useful in prior studies and has the desirable feature that a method of closure is available. This means that the *form* of the spatially smoothed transport equation is obtained without constitutive assumptions, and a method is available to predict the coefficients that appear in the smoothed equation.

2.1. CONTINUITY EQUATION

We begin the process of averaging with the continuity equation to obtain

$$\langle \nabla \cdot \mathbf{v}_\beta \rangle = \nabla \cdot \langle \mathbf{v}_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \mathbf{v}_\beta \, \mathrm{d}A = 0. \tag{2.11}$$

Use of the no-slip boundary condition given by Equation (2.4) leads to the traditional form of the averaged continuity equation for an incompressible flow

$$\nabla \cdot \langle \mathbf{v}_\beta \rangle = 0. \tag{2.12}$$

The solenoidal characteristic of the phase average velocity is certainly a motivating factor in the use of Equation (2.12) rather than the intrinsic phase average form which can be obtained by use of Equation (2.8) in Equation (2.12) to give

$$\nabla \cdot \langle \mathbf{v}_\beta \rangle^\beta = -\epsilon_\beta^{-1} \nabla \epsilon_\beta \cdot \langle \mathbf{v}_\beta \rangle^\beta. \tag{2.13}$$

However, we will need this latter form of the continuity equation in subsequent developments since prior experience (Gray, 1975) indicates that it is best to use the intrinsic phase average in defining spatial deviations. It is of some importance to note that *no length scale constraints* have been imposed in the derivation of either form of the continuity equation.

2.2. EQUATIONS OF MOTION

The phase average form of the Stokes equations is given by

$$0 = -\nabla \langle p_\beta \rangle - \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} p_\beta \, \mathrm{d}A + \epsilon_\beta \rho_\beta \mathbf{g} + \mu_\beta \langle \nabla \cdot \nabla \mathbf{v}_\beta \rangle. \tag{2.14}$$

While the phase average velocity is generally perferred in the analysis of flow in porous media, we require the *intrinsic* phase average of the pressure since this more closely corresponds to the measured value or the value imposed at a boundary. This requires the use of

$$\langle p_\beta \rangle = \epsilon_\beta \langle p_\beta \rangle^\beta \tag{2.15}$$

along with Gray's (1975) decomposition

$$p_\beta = \langle p_\beta \rangle^\beta + \tilde{p}_\beta . \tag{2.16}$$

Use of these relations in the first two terms of Equation (2.14) yields

$$-\nabla \langle p_\beta \rangle - \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} p_\beta \, \mathrm{d}A = -\epsilon_\beta \nabla \langle p_\beta \rangle^\beta - \langle p_\beta \rangle^\beta \nabla \epsilon_\beta - \\ - \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \langle p_\beta \rangle^\beta \, \mathrm{d}A - \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \tilde{\mathbf{p}}_\beta \, \mathrm{d}A. \tag{2.17}$$

It is intuitively appealing to think of $\langle p_\beta \rangle^\beta$ as a constant with respect to integration over $A_{\beta\sigma}$, and Carbonell and Whitaker (1984) have shown that the relation

$$\frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \langle p_\beta \rangle^\beta \, \mathrm{d}A = \frac{1}{\mathscr{V}} \left\{ \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \, \mathrm{d}A \right\} \langle p_\beta \rangle^\beta \tag{2.18}$$

is satisfactory when the following length scale constraint is valid

$$\left(\frac{r_0}{L} \right)^2 \ll 1. \tag{2.19}$$

Since the radius of the averaging volume is generally small compared to the macroscopic length scale illustrated in Figure 2, this constraint is usually satisfied. In addition, if the constraints given by Equation (2.1) are accepted at the outset, Equation (2.19) is automatically satisfied. The origin of the constraint given by Equation (2.19) is indicated in subsequent paragraphs by Equations (2.26) and (2.27).

One can use the averaging theorem (with $\psi_\beta = 1$) to prove that

$$\frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \, \mathrm{d}A = -\nabla \epsilon_\beta \tag{2.20}$$

and use of this relation with Equation (2.18) in (2.17) leads to

$$-\nabla \langle p_\beta \rangle - \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} p_\beta \, \mathrm{d}A = -\epsilon_\beta \nabla \langle p_\beta \rangle^\beta - \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \tilde{p}_\beta \, \mathrm{d}A. \tag{2.21}$$

This is essentially an application of Gray's (1975) modified averaging theorem, and it represents the first use of a length-scale constraint. A similar result is given by Raats and Klute (1968). It is important to note that the inequality given by Equation (2.19) results from an *order of magnitude analysis* and the precise nature of the constraint that allows one to use Equation (2.18) will require a detailed study of special cases.

Substitution of Equation (2.21) into (2.14) and use of the averaging theorem with the viscous term leads to

$$0 = -\epsilon_\beta \nabla \langle p_\beta \rangle^\beta - \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \tilde{p}_\beta \, dA + \epsilon_\beta \rho_\beta \mathbf{g} + \\ + \mu_\beta \left\{ \nabla \cdot \langle \nabla \mathbf{v}_\beta \rangle + \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{v}_\beta \, dA \right\}. \tag{2.22}$$

A second application of the averaging theorem provides

$$\langle \nabla \mathbf{v}_\beta \rangle = \nabla \langle \mathbf{v}_\beta \rangle + \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \mathbf{v}_\beta \, dA \tag{2.23}$$

with the *no slip* condition leading to the obvious simplification. Use of this relation in Equation (2.22) yields

$$0 = -\epsilon_\beta \nabla \langle p_\beta \rangle^\beta - \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \tilde{p}_\beta \, dA + \epsilon_\beta \rho_\beta \mathbf{g} + \\ + \mu_\beta \left\{ \nabla^2 \langle \mathbf{v}_\beta \rangle + \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{v}_\beta \, dA \right\}. \tag{2.24}$$

We can now repeat the development given by Equations (2.15)–(2.11) using the decomposition

$$\mathbf{v}_\beta = \langle \mathbf{v}_\beta \rangle^\beta + \tilde{\mathbf{v}}_\beta \tag{2.25}$$

in the last term in Equation (2.24). This leads to the following solution

$$\frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{v}_\beta \, dA = -\nabla \epsilon_\beta \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta + \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{v}}_\beta \, dA \tag{2.26}$$

provided the length-scale constraint indicated by Equation (2.19) is valid. It is of some interest to note that when Equation (2.19) is *not* valid, the operation associated with Equation (2.25) leads to higher order terms, i.e.,

$$\frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{v}_\beta \, dA = -\nabla \epsilon_\beta \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta + \\ + \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{v}}_\beta \, dA + \mathbf{H} : \nabla \nabla \langle \mathbf{v}_\beta \rangle^\beta + \cdots \tag{2.27}$$

The work of Carbonell and Whitaker (1984) indicates that $\mathbf{H}$ can be approximated by

$$\mathbf{H} \sim -\frac{r_0^2}{5} [\mathbf{I} \nabla^2 \epsilon_\beta + 2 \nabla \nabla \epsilon_\beta] \tag{2.28}$$

and this is the origin of the restriction imposed by Equation (2.19).

We now return to Equation (2.24), represent the phase average velocity in terms of the intrinsic phase average velocity, and make use of Equation (2.26) to obtain

$$0 = -\nabla\langle p\rangle^\beta + \rho_\beta \mathbf{g} + \mu_\beta\{\nabla^2\langle \mathbf{v}_\beta\rangle^\beta + \epsilon_\beta^{-1}\nabla\epsilon_\beta \cdot \nabla\langle v_\beta\rangle^\beta + \epsilon_\beta^{-1}\langle \mathbf{v}_\beta\rangle^\beta\nabla^2\epsilon_\beta\} - \\ -\frac{1}{V_\beta}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\tilde{p}_\beta \,\mathrm{d}A + \frac{\mu_\beta}{V_\beta}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{v}}_\beta\,\mathrm{d}A. \tag{2.29}$$

Here it becomes apparent that representations for both $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$ are required in order to obtain a closed form and, in general, constitutive assumptions have been used to obtain the closure. For example, one can evoke the linear transformation (Whitaker, 1969)

$$\mathbf{v}_\beta = \mathbf{M}\cdot\langle \mathbf{v}_\beta\rangle^\beta \tag{2.30}$$

in order to develop expressions for $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$. This approach can be fortified with the principle of material frame indifference (Slattery, 1980), and Gray and O'Neill (1976) have utilized this approach in order to include inertial effects in the analysis.

In this work, closure is obtained directly in terms of the governing differential equations for $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$. This approach not only allows one to determine the correct form of the representations for $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$, but it also provides a means for calculating the coefficients that appear in these representations. Before proceeding with that development, we must make certain that Equation (2.29) is consistent with the length-scale constraint given by Equation (2.19). Since the length scale for $\tilde{\mathbf{v}}_\beta$ is ℓ_β as indicated in Figure 2, we have the estimate

$$\nabla\tilde{\mathbf{v}}_\beta = \mathbf{O}(\tilde{\mathbf{v}}_\beta/\ell_\beta) \tag{2.31}$$

From the decomposition given by Equation (2.25) and the no-slip condition imposed at the β-σ interface we know that

$$\tilde{\mathbf{v}}_\beta = \mathbf{O}(\langle \mathbf{v}_\beta\rangle^\beta) \tag{2.32}$$

and the last term in Equation (2.29) can be estimated as

$$\frac{\mu_\beta}{V_\beta}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{v}}_\beta\,\mathrm{d}A = \mathbf{O}(\mu_\beta\langle \mathbf{v}_\beta\rangle^\beta/\ell_\beta^2). \tag{2.33}$$

An examination of Equations (2.25)–(2.27) will indicate that the restriction given by Equation (2.19) is based on the idea that the length scale associated with averaged quantities, i.e., ϵ_β, $\langle p_\beta\rangle^\beta$, and $\langle \mathbf{v}_\beta\rangle^\beta$, is the macroscopic length scale L indicated in Figure 1. This leads to the following estimates

$$\mu_\beta\nabla^2\langle \mathbf{v}_\beta\rangle^\beta = \mathbf{O}(\mu_\beta\langle \mathbf{v}_\beta\rangle^\beta/L^2), \tag{2.34}$$

$$\mu_\beta\epsilon_\beta^{-1}\nabla\epsilon_\beta\cdot\nabla\langle \mathbf{v}_\beta\rangle^\beta = \mathbf{O}(\mu_\beta\langle \mathbf{v}_\beta\rangle^\beta/L^2), \tag{2.35}$$

$$\mu_\beta\epsilon_\beta^{-1}\langle \mathbf{v}_\beta\rangle^\beta\nabla^2\epsilon_\beta = \mathbf{O}(\mu_\beta\langle \mathbf{v}_\beta\rangle^\beta/L^2). \tag{2.36}$$

On the basis of Equation (2.19) and the natural requirement that $\ell_\beta < r_0$, we see that all the viscous terms estimated by Equations (2.34)–(2.36) are *small* compared to the last term in Equation (2.29). In addition, we can see that these three terms are *large* compared to the last term in Equation (2.27), thus it is *permissible* to retain these terms in Equation (2.29). However, in view of the length scale constraint given by Equation (2.19), there is absolutely no reason to retain these terms since their contribution to the determination of the velocity field will be negligible. In particular, it is *not permissible* to retain the term often referred to as the *Brinkman correction* (Brinkman, 1947) in order to develop a *no slip* condition for the average velocity. Under these circumstances the characteristic length scale for ϵ_β and $\langle \mathbf{v}_\beta \rangle^\beta$ is on the order of the radius of the averaging volume, r_0, and Equation (2.29) is no longer valid*. For this reason we express Equation (2.29) as

$$0 = -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \tilde{p}_\beta \, \mathrm{d}A + \frac{\mu_\beta}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{v}}_\beta \, \mathrm{d}A \tag{2.37}$$

and consider the analysis to be restricted to systems in which the length scale for ϵ_β, $\langle p_\beta \rangle^\beta$ and $\langle \mathbf{v}_\beta \rangle^\beta$ is large compared to r_0. This immediately raises a question about the relation of r_0 to ℓ_β since the latter is generally known *a priori*, and we would like to say something definitive about L and r_0 relative to ℓ_β. A reasonable requirement for r_0 would be that ϵ_β, $\langle p_\beta \rangle^\beta$ and $\langle \mathbf{v}_\beta \rangle^\beta$ are well behaved functions in the sense that they satisfactorily describe observed phenomena. Here we are confronted with the fact that $\langle p_\beta \rangle^\beta$ and $\langle \mathbf{v}_\beta \rangle^\beta$ are rarely, if ever, measured directly**; however, this is not the case with the void fraction and we have no other choice but to develop arguments on the basis of this variable. If we let the calculations of ϵ_β as a function of r_0 serve as a guide (Howes and Whitaker, 1985), we would require that $r_0 \geq 5\ell_\beta$. On this basis we think of Equation (2.37) as being restricted to situations in which averaged quantities undergo significant variations over distances that are at least fifty times larger than ℓ_β.

3. Closure

The general closure problem requires first that we obtain representations for $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$ in terms of the dependent variables $\langle p_\beta \rangle^\beta$ and $\langle \mathbf{v}_\beta \rangle^\beta$, and second that we be able to determine the coefficients that appear in these representations from a purely theoretical point of view. This sets the stage for a detailed comparison between theory and experiment in terms of parameters describing the geometry of the porous medium.

* This occurs because *higher order* derivatives, such as those indicated in Equation (2.27), must be included for both the velocity *and* the pressure if the restriction indicated by Equation (2.19) is not imposed. In addition, the *lower order* terms in Equation (2.29) must be retained if the Brinkman correction is to be used in the region near a solid boundary. Nield (1983), following a somewhat different line of analysis, has also concluded that the Brinkman correction is not justified.

** The matter of experimental measurement of $\langle p_\beta \rangle^\beta$ is discussed in Appendix A.

Our initial objective is to derive the governing differential equations for $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$, and to do so we make use of the decompositions given by Equations (2.16) and (2.25) in Equations (2.2)–(2.5) to obtain

$$-\nabla\tilde{p}_\beta+\mu_\beta\nabla^2\tilde{\mathbf{v}}_\beta=-[-\nabla\langle p_\beta\rangle^\beta+\rho_\beta\mathbf{g}+\mu_\beta\nabla^2\langle\mathbf{v}_\beta\rangle^\beta] \tag{3.1}$$

$$\nabla\cdot\tilde{\mathbf{v}}_\beta=-\nabla\cdot\langle\mathbf{v}_\beta\rangle^\beta \tag{3.2}$$

$$\text{B.C.1}\quad \tilde{\mathbf{v}}_\beta=-\langle\mathbf{v}_\beta\rangle^\beta,\quad \text{on } \mathscr{A}_{\beta\sigma} \tag{3.3}$$

$$\text{B.C.2}\quad \tilde{\mathbf{v}}_\beta=\mathbf{g}(\mathbf{r},t),\quad \text{on } \mathscr{A}_{\beta e}. \tag{3.4}$$

Here we have used $\mathbf{g}(\mathbf{r},t)=\mathbf{f}(\mathbf{r},t)-\langle\mathbf{v}_\beta\rangle^\beta$ to represent $\tilde{\mathbf{v}}_\beta$ at the entrances and exits of the macroscopic system. Even though Equation (2.37) is not applicable at the surface of the macroscopic system, the average velocity and pressure are defined on that surface and Equations (2.2) and (2.3) are valid on that surface. Thus Equations (3.1)–(3.4) represent a valid boundary value problem for $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$ even though $\mathbf{g}(\mathbf{r},t)$ may be difficult to specify from a practical point of view.

One of the first simplifications we can make in the closure problem concerns the continuity equation for $\tilde{\mathbf{v}}_\beta$. In the general case, each of the three terms on the left-hand side of Equation (3.2) will be on the order of $\langle\mathbf{v}_\beta\rangle^\beta/\ell_\beta$ while each of the three terms on the right-hand side will be on the order of $\langle\mathbf{v}_\beta\rangle^\beta/L$. Under these circumstances the *source term*, $\nabla\cdot\langle\mathbf{v}_\beta\rangle^\beta$, in the continuity equation for $\tilde{\mathbf{v}}_\beta$ will have no effect on the $\tilde{\mathbf{v}}_\beta$-field and the latter can be treated as solenoidal. This allows us to write Equation (3.2) as

$$\nabla\cdot\tilde{\mathbf{v}}_\beta=0 \tag{3.5}$$

without requiring that the volume-averaged velocity field be uniform. This type of argument is peculiar to the continuity equation and cannot be used to simplify Equation (3.1) since $\nabla\langle p_\beta\rangle^\beta$ and $\rho_\beta\mathbf{g}$ are generally not negligible compared to the terms on the left-hand side. In this case we follow the method of Crapiste *et al.* (1986) and form the intrinsic phase average of Equation (3.1) to obtain

$$\frac{1}{V_\beta}\int_{V_\beta}[-\nabla\tilde{p}_\beta+\mu_\beta\nabla^2\tilde{\mathbf{v}}_\beta]\,\mathrm{d}V=-[-\nabla\langle p_\beta\rangle^\beta+\rho_\beta\mathbf{g}+\mu_\beta\nabla^2\langle\mathbf{v}_\beta\rangle^\beta]. \tag{3.6}$$

Here we have made use of ideas presented in the previous section and treated the right-hand side of Equation (3.1) as a constant with respect to integration over V_β. A comparison of Equations (3.6) and (3.1) allows us to express the boundary value problem for $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$ as

$$-\nabla\tilde{p}_\beta+\mu_\beta\nabla^2\tilde{\mathbf{v}}_\beta=\frac{1}{V_\beta}\int[-\nabla\tilde{p}_\beta+\mu_\beta\nabla^2\tilde{\mathbf{v}}_\beta]\,\mathrm{d}V, \tag{3.7}$$

$$\nabla\cdot\tilde{\mathbf{v}}_\beta=0, \tag{3.8}$$

$$\text{B.C.1}\quad \tilde{\mathbf{v}}_\beta=-\langle\mathbf{v}_\beta\rangle^\beta,\quad \text{on } \mathscr{A}_{\beta\sigma}, \tag{3.9}$$

$$\text{B.C.2}\quad \tilde{\mathbf{v}}_\beta=\mathbf{G}\cdot\langle\mathbf{v}_\beta\rangle^\beta,\quad \text{on } \mathscr{A}_{\beta e}. \tag{3.10}$$

Here we have used $\mathbf{g}(\mathbf{r}, t) = \mathbf{G} \cdot \langle \mathbf{v}_\beta \rangle^\beta$ as a matter of convenience in writing the second boundary condition. It is important to note that Equations (3.7) and (3.8) *are not restricted to single phase flows* and we will use both of these results, without change, in a subsequent study of two-phase flow.

At this point we seek a solution of the form

$$\tilde{\mathbf{v}}_\beta = \mathbf{B} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \boldsymbol{\psi}, \tag{3.11}$$

$$\tilde{p}_\beta = \mu_\beta \mathbf{b} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mu_\beta \xi \tag{3.12}$$

in which $\boldsymbol{\psi}$ and ξ are completely arbitrary functions. This allows us to specify $\mathbf{B}$ and $\mathbf{b}$ in any way we wish, and we choose to specify these two functions by means of the following boundary value problem

$$-\nabla \mathbf{b} + \nabla^2 \mathbf{B} = \frac{1}{V_\beta} \int_{V_\beta} [-\nabla \mathbf{b} + \nabla^2 \mathbf{B}] \, dV, \tag{3.13}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3.14}$$

$$\text{B.C.1} \quad \mathbf{B} = -\mathbf{I}, \quad \text{on } \mathscr{A}_{\beta\sigma}, \tag{3.15}$$

$$\text{B.C.2} \quad \mathbf{B} = \mathbf{G}, \quad \text{on } \mathscr{A}_{\beta e}, \tag{3.16}$$

$$\langle \mathbf{b} \rangle^\beta = \langle \mathbf{B} \rangle^\beta = 0. \tag{3.17}$$

It is worthwhile to note that we expect that the boundary condition given by Equation (3.16) will have little influence on $\mathbf{b}$ and $\mathbf{B}$, and in practice these fields will be determined in some *representative region* of a porous medium. Under these circumstances Equation (3.16) will be replaced with a spatially periodic condition (Brenner, 1980); however, at this point we wish to keep the analysis as general as possible.

The boundary value problem for $\boldsymbol{\psi}$ and ξ is obtained by the substitution of Equations (3.11) and (3.12) into Equations (3.7)–(3.10). When this is done, and Equations (3.13)–(3.17) are applied, we find that $\boldsymbol{\psi}$ and ξ are determined by

$$\begin{aligned} -\nabla \xi + \nabla^2 \boldsymbol{\psi} &= \frac{1}{V_\beta} \int_{V_\beta} [-\nabla \xi + \nabla^2 \boldsymbol{\psi}] \, dV + \nabla \langle \mathbf{v}_\beta \rangle^\beta \cdot \mathbf{b} \\ &\quad - 2[\nabla \langle \mathbf{v}_\beta \rangle^\beta]^T [\nabla (\mathbf{B}^T) - \langle \nabla (\mathbf{B}^T) \rangle^\beta] - \mathbf{B} \cdot \nabla^2 \langle \mathbf{v}_\beta \rangle^\beta, \end{aligned} \tag{3.18}$$

$$\nabla \cdot \boldsymbol{\psi} = -\nabla \langle \mathbf{v}_\beta \rangle^\beta \cdot \mathbf{B}^T, \tag{3.19}$$

$$\text{B.C.1} \quad \boldsymbol{\psi} = 0, \quad \text{on } \mathscr{A}_{\beta\sigma}, \tag{3.20}$$

$$\text{B.C.2} \quad \boldsymbol{\psi} = 0, \quad \text{on } \mathscr{A}_{\beta e}, \tag{3.21}$$

$$\langle \xi \rangle^\beta = \langle \boldsymbol{\psi} \rangle^\beta = 0. \tag{3.22}$$

It is appealing to *assume* that the solution to the homogeneous problem★ associated with Equations (3.18)–(3.22) is the null solution, $\xi = \boldsymbol{\psi} = 0$. While we

★ This is obtained by requiring that $\langle \mathbf{v}_\beta \rangle^\beta$ is constant so that all terms involving $\nabla \langle \mathbf{v}_\beta \rangle^\beta$ are zero.

have no general proof of this, it can be proved for spatially periodic porous media and this is done in Appendix B. Because of this, we need only estimate (Whitaker (1983), Sec. 2.9) the magnitude of the nonhomogeneous terms in order to develop estimates for ξ and $\boldsymbol{\psi}$. A little thought will indicate that Equations (3.18) and (3.19) can be expressed as

$$-\nabla \xi + \nabla^2 \boldsymbol{\psi} = \mathbf{O}\left\{\frac{\mathbf{B}\cdot\langle\mathbf{v}_\beta\rangle^\beta}{\ell_\beta L}, \frac{\mathbf{b}\cdot\langle\mathbf{v}_\beta\rangle^\beta}{L}\right\}, \tag{3.23}$$

$$\nabla\cdot\boldsymbol{\psi} = \mathbf{O}\left\{\frac{\mathbf{B}\cdot\langle\mathbf{v}_\beta\rangle^\beta}{L}\right\} \tag{3.24}$$

since we seek only an *estimate* for ξ and $\boldsymbol{\psi}$. Here we have made use of the idea that the volume integral on the right-hand side of Equation (3.18) cannot be larger than the two terms on the left-hand side, and we have retained only the largest of the nonhomogeneous terms.

We begin our estimation of the magnitude of $\boldsymbol{\psi}$ with Equation (3.24) and note that in general the left-hand side will consist of three terms of order $\boldsymbol{\psi}/\ell_\beta$. Under these circumstances the contribution of the *source term* in Equation (3.24) to the $\boldsymbol{\psi}$-field will be $\mathbf{B}\cdot\langle\mathbf{v}_\beta\rangle^\beta\ (\ell_\beta/L)$. Moving on to Equation (3.23) we use the estimates

$$\nabla\xi = \mathbf{O}(\xi/\ell_\beta), \qquad \nabla^2\boldsymbol{\psi} = \mathbf{O}(\boldsymbol{\psi}/\ell_\beta^2), \tag{3.25}$$

to obtain

$$\boldsymbol{\psi} = \mathbf{O}\left\{\mathbf{B}\cdot\langle\mathbf{v}_\beta\rangle^\beta\left(\frac{\ell_\beta}{L}\right), \mathbf{b}\cdot\langle\mathbf{v}_\beta\rangle^\beta\frac{\ell_\beta^2}{L}\right\}, \tag{3.26}$$

$$\xi = \mathbf{O}\left\{\mathbf{b}\cdot\langle\mathbf{v}_\beta\rangle^\beta\left(\frac{\ell_\beta}{L}\right), \mathbf{B}\cdot\langle\mathbf{v}_\beta\rangle^\beta/L\right\}. \tag{3.27}$$

These results must be compared with Equations (3.11) and (3.12), and when we do so we see that the *first* terms on the right-hand side of Equations (3.26) and (3.27) are smaller by a factor of ℓ_β/L than the comparable terms in Equations (3.11) and (3.12). In order to compare the *last* terms in these two estimates with the representations given by Equations (3.11) and (3.12), we return to Equations (3.7) and assume that the pressure and viscous forces are of comparable magnitude. This leads to the relation

$$\tilde{p}_\beta = \mathbf{O}\{\mu_\beta\tilde{\mathbf{v}}_\beta/\ell_\beta\}. \tag{3.28}$$

and use of the representation given by Equations (3.11) allows us to estimate the pressure deviation as

$$\tilde{p}_\beta = \mathbf{O}\left\{\frac{\mu_\beta\mathbf{B}\cdot\langle\mathbf{v}_\beta\rangle^\beta}{\ell_\beta}, \frac{\mu_\beta\boldsymbol{\psi}}{\ell_\beta}\right\}. \tag{3.29}$$

At this point we need only draw upon the inequality $\ell_\beta \ll L$ to see that the term

$\mathbf{B} \cdot \langle \mathbf{v}_\beta \rangle^\beta / L$ in Equation (3.27) makes a negligible contribution to $\tilde{p}_\beta$ through the term $\mu_\beta \xi$ in Equation (3.12). This line of thinking can be continued to conclude that $\boldsymbol{\psi}$ and ξ are always smaller by a factor of ℓ_β / L than the other terms in Equations (3.11) and (3.12) and this means that $\tilde{\mathbf{v}}_\beta$ and $\tilde{p}_\beta$ can be expressed as

$$\tilde{\mathbf{v}}_\beta = \mathbf{B} \cdot \langle \mathbf{v}_\beta \rangle^\beta, \tag{3.30}$$

$$\tilde{p}_\beta = \mu_\beta \mathbf{b} \cdot \langle \mathbf{v}_\beta \rangle^\beta. \tag{3.31}$$

While the *form* of these results offers nothing new in terms of prior studies, we now possess a boundary value problem for $\mathbf{b}$ and $\mathbf{B}$ and are in a position to determine the permeability tensor by direct theoretical means.

At this point we return to the volume-averaged momentum equation given by Equation (2.37) and make use of Equations (3.30) and (3.31) to obtain

$$0 = -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \left\{ \frac{\mu_\beta}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (\nabla \mathbf{B} - \mathbf{I}\mathbf{b}) \, \mathrm{d}A \right\} \cdot \langle \mathbf{v}_\beta \rangle^\beta. \tag{3.32}$$

Here we have used the approximations

$$\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} (\mathbf{b} \cdot \langle \mathbf{v}_\beta \rangle^\beta) \, \mathrm{d}A = \left\{ \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \mathbf{b} \, \mathrm{d}A \right\} \cdot \langle \mathbf{v}_\beta \rangle^\beta, \tag{3.33}$$

$$\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla (\mathbf{B} \cdot \langle \mathbf{v}_\beta \rangle^\beta) \, \mathrm{d}A = \left\{ \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{B} \, \mathrm{d}A \right\} \cdot \langle \mathbf{v}_\beta \rangle^\beta \tag{3.34}$$

both of which can be justified on the basis of Equations (2.19) and the length-scale restriction $\ell_\beta \ll L$. It is convenient to use the definition

$$\mathbf{C} = -\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (\nabla \mathbf{B} - \mathbf{I}\mathbf{b}) \, \mathrm{d}A \tag{3.35}$$

and express Equation (3.32) in the *intrinsic phase average* form of Darcy's law

$$\langle \mathbf{v}_\beta \rangle^\beta = -\frac{\mathbf{C}^{-1}}{\mu_\beta} \cdot [\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g}]. \tag{3.36}$$

Since the traditional form is given in terms of the *phase average* velocity, we define the permeability tensor by

$$\mathbf{K} = \epsilon_\beta \mathbf{C}^{-1} \tag{3.37}$$

and express our final result as

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathbf{K}}{\mu_\beta} \cdot [\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g}] \tag{3.38}$$

Here it is important to note that the permeability tensor is given directly in terms of the solution to the boundary value problem expressed by Eqs. (3.13)–(3.17) and the definitions given by Eqs. (3.35) and (3.37).

SOLUTION OF THE CLOSURE PROBLEM

In practice, one would *never* solve the boundary value problem in its present form in order to determine **K**. Instead, one would make use of the fact that the boundary conditions imposed at $\mathscr{A}_{\beta e}$ will have little influence on the fields under consideration, and one would determine these fields in some *representative region* of a porous medium such as the one illustrated in Figure 3. This region would quite naturally be treated as a unit cell in a spatially periodic porous medium (Brenner, 1980) and the boundary value problem under consideration would take the form

$$-\nabla \mathbf{b} + \nabla^2 \mathbf{B} = \frac{1}{V_\beta} \int_{V_\beta} [-\nabla \mathbf{b} + \nabla^2 \mathbf{B}] \, dV, \tag{3.39}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3.40}$$

$$\text{B.C.1} \quad \mathbf{B} = -\mathbf{I}, \quad \text{on } A^*_{\beta\sigma}, \tag{3.41}$$

$$\text{B.C.2} \quad \mathbf{B}(\mathbf{r} + \ell_i) = \mathbf{B}(\mathbf{r}), \qquad \mathbf{b}(\mathbf{r} + \ell_i) = \mathbf{b}(\mathbf{r}), \; i = 1, 2, 3, \tag{3.42}$$

$$\langle \mathbf{b} \rangle^\beta = \langle \mathbf{B} \rangle^\beta = 0. \tag{3.43}$$

Here we have used $A^*_{\beta\sigma}$ to represent the area of the β-σ interface contained

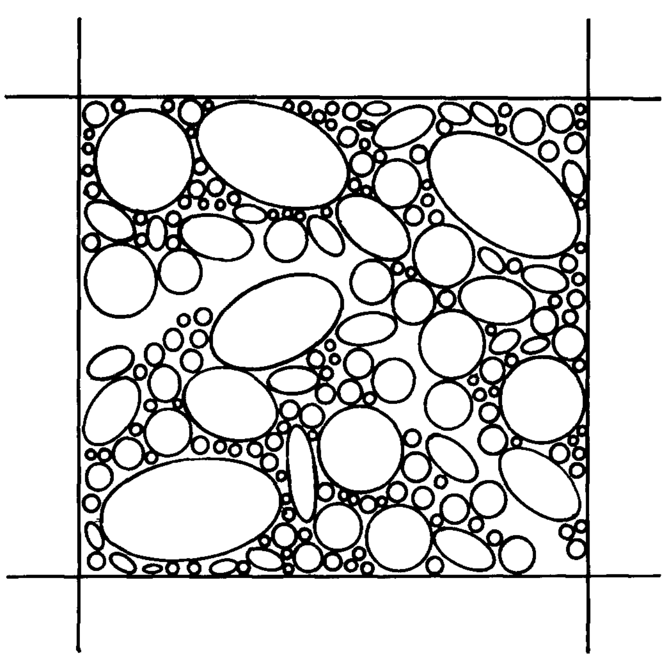

Fig. 3. Representative region of a porous medium.

within a unit cell and ℓ_i to represent the three nonunique lattice vectors that are needed to characterize a spatially periodic porous medium. For a system such as that illustrated in Figure 3, it should be clear that **b** and **B** will be dominated by the governing differential equations and the boundary condition imposed at $A^*_{\beta\sigma}$. The periodicity condition, used as a boundary condition at $A^*_{\beta e}$, will have little or no influence on the **b** and **B**-fields.

The fact that a unit cell in a spatially periodic porous medium will be used to determine **K** in Equation (3.38), in no way limits that equation which is valid for any nonhomogeneous porous medium provided the length scale constraints are satisfied for ϵ_β, $\langle p_\beta\rangle^\beta$ and $\langle \mathbf{v}_\beta\rangle^\beta$. The determination of **K** by means of Equations (3.39)–(3.43) represents a formidable computational problem. However, a complex computational problem in one decade is often a routine problem in another, and at this time the problem of constructing unit cells that accurately characterize real systems represents a more challenging problem. For simple systems, such as a bundle of capillary tubes, Equations (3.39)–(3.43) are easily solved and this is done in Appendix C to obtain the Hagen–Poisseuille law.

4. Conclusions

The method of volume averaging has been used to derive Darcy's law from first principles without the use of any constitutive assumptions. The analysis has provided a means for the direct theoretical determination of the permeability tensor, and the necessary comparison between theory and experiment requires some complex numerical calculations and experiments that go beyond the original one-dimensional, macroscopic studies of Darcy.

Appendix A: Pressure Measurement

A detailed comparison between experiment and Darcy's law is difficult to achieve because of the problems associated with measuring volume-averaged quantities. Experimental determination of the volume-averaged velocity would appear to be extremely difficult; however, measurement of the pressure is another matter. As an example, let us assume that the *point* pressure is determined experimentally at two points a distance d from one another, and we would like to know when this measurement will provide a reasonable approximation for the change in the intrinsic phase average pressure. Use of Equation (2.16) allows us to express the pressure change as

$$\Delta p_\beta = \Delta\langle p_\beta\rangle^\beta + \Delta\tilde{p}_\beta \tag{A.1}$$

and we seek the constraint associated with $\Delta\tilde{p}_\beta \ll \Delta\langle p_\beta\rangle^\beta$. Since the *change* in $\tilde{p}_\beta$ is on the order of $\tilde{p}_\beta$ we can use Equations (3.28) and (3.3) to obtain the estimate

$$\Delta\tilde{p}_\beta = \mathbf{O}(\mu_\beta\langle \mathbf{v}_\beta\rangle^\beta/\ell_\beta). \tag{A.2}$$

From Darcy's law we have

$$\nabla \langle p_\beta \rangle^\beta = \mathbf{O}(\mu_\beta \langle \mathbf{v}_\beta \rangle^\beta / \ell_\beta^2) \tag{A.3}$$

and the first term of a Taylor series expansion yields

$$\Delta \langle p_\beta \rangle^\beta = \mathbf{O}\left[\frac{\mu_\beta \langle \mathbf{v}_\beta \rangle^\beta}{\ell_\beta} \left(\frac{d}{\ell_\beta}\right)\right]. \tag{A.4}$$

In order that $\Delta \tilde{p}_\beta$ be small compared to $\Delta \langle p_\beta \rangle^\beta$ we require that

$$\ell_\beta \ll d. \tag{A.5}$$

This constraint should be easy to satisfy in any experimental study of the pressure field for flow in homogeneous porous media; however, difficulties may arise when there are dramatic changes in the structure of the porous media. Under these circumstances $\Delta \tilde{p}_\beta$ could be significantly larger than the estimate given by Equation (A.2).

Appendix B: Uniqueness of the Closure Scheme

In the closure problem described in Section 3, the velocity and pressure deviations were represented as

$$\mathbf{v}_\beta = \mathbf{B} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \boldsymbol{\psi}, \tag{B.1}$$

$$\tilde{p}_\beta = \mu_\beta \mathbf{b} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mu_\beta \xi \tag{B.2}$$

and an important part of the closure scheme required a demonstration that $\boldsymbol{\psi}$ and ξ made negligible contributions to $\tilde{\mathbf{v}}_\beta$ and $\tilde{p}_\beta$. For the general case, order-of-magnitude analysis was used to demonstrate that $\boldsymbol{\psi}$ and ξ were negligible when certain length-scale constraints were satisfied. Here we wish to prove that $\boldsymbol{\psi}$ and ξ are zero for uniform flow in a spatially periodic porous medium.

If $\langle \mathbf{v}_\beta \rangle^\beta$ is taken to be constant, the nonhomogeneous terms in Equations (3.18) and (3.19) are zero and the boundary-value problem for a spatially periodic porous medium takes the form

$$-\nabla \xi + \nabla^2 \boldsymbol{\psi} = \frac{1}{V_\beta} \int_{V_\beta} [-\nabla \xi + \nabla^2 \boldsymbol{\psi}] \, \mathrm{d}V, \tag{B.3}$$

$$\nabla \cdot \boldsymbol{\psi} = 0, \tag{B.4}$$

$$\text{B.C.1} \quad \boldsymbol{\psi} = 0, \quad \text{on } A_{\beta\sigma}^*, \tag{B.5}$$

$$\text{B.C.2} \quad \boldsymbol{\psi}(\mathbf{r} + \ell_i) = \boldsymbol{\psi}(\mathbf{r}), \qquad \xi(\mathbf{r} + \ell_i) = \xi(\mathbf{r}), \quad i = 1, 2, 3, \tag{B.6}$$

$$\langle \xi \rangle^\beta = \langle \boldsymbol{\psi} \rangle^\beta = 0. \tag{B.7}$$

Here $A_{\beta\sigma}^*$ represents the area of the β-σ interface contained within a unit cell. Formation of the scalar product of Equation (B.3) with $\boldsymbol{\psi}$ and application of

Equation (B.4) leads to

$$-\nabla \cdot (\boldsymbol{\psi}\xi) + \tfrac{1}{2}\nabla^2(\boldsymbol{\psi}\cdot\boldsymbol{\psi}) - \nabla\boldsymbol{\psi} : \nabla\boldsymbol{\psi}^T = \boldsymbol{\psi} \cdot \left\{\frac{1}{V_\beta}\int_{V_\beta} [-\nabla\xi + \nabla^2\boldsymbol{\psi}]\,\mathrm{d}V\right\}. \quad \text{(B.8)}$$

One normally thinks of the averaging volume as a sphere; however, this is not necessary and in this analysis we take $\mathscr{V}$ to be constructed by the lattice vectors, ℓ_i. Note that this does not compromise the length-scale constraint, $\ell_\beta \ll r_0$, since one can always choose a unit cell having a characteristic length that is large relative to ℓ_β. The situation under consideration is illustrated in Figure 4, and a little thought will indicate that integration over V_β always takes place over the β-phase contained within an *entire* unit cell, although not necessarily the *same* unit cell. When $\mathscr{V}$ is chosen in this manner, the conditions given by Equations (B.6) require that the integral on the right-hand side of Equation (B.3) be constant. Under these circumstances, the integration of Equation (B.8) over V_β^* leads to

$$-\int_{V_\beta^*} \nabla \cdot (\boldsymbol{\psi}\xi)\,\mathrm{d}V + \tfrac{1}{2}\int_{V_\beta^*} \nabla^2(\boldsymbol{\psi}\cdot\boldsymbol{\psi})\,\mathrm{d}V = \int_{V_\beta^*} \nabla\boldsymbol{\psi} : \nabla\boldsymbol{\psi}^T\,\mathrm{d}V. \quad \text{(B.9)}$$

Here V_β^* represents the volume of the β-phase contained within a *single* unit cell, and in arriving at Equation (B.9) we have made use of the second of Equations (B.7). One should note that V_β^* and V_β coincide only for the special case in which the centroid of $\mathscr{V}$ coincides with the centroid of the unit cell.

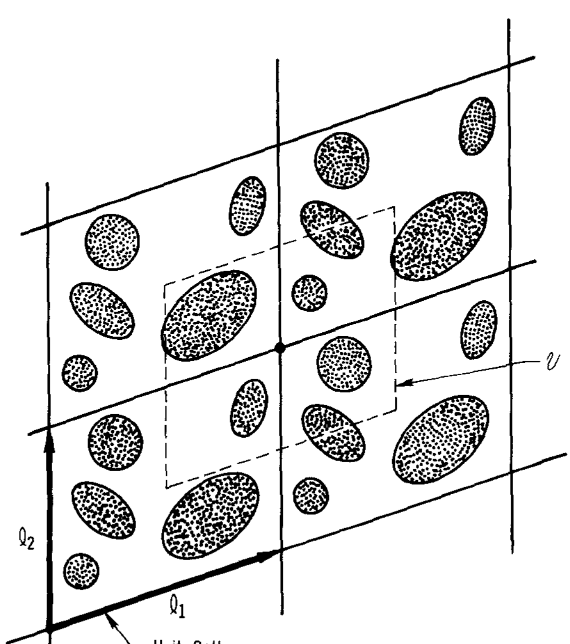

Fig. 4. Averaging-volume in a spatially periodic porous medium.

Use of the divergence theorem and the boundary condition given by Equation (B.5) allows us to express Equation (B.9) as

$$-\int_{A^*_{\beta e}} \mathbf{n}_{\beta e} \cdot \boldsymbol{\psi}\xi \, \mathrm{d}A + \int_{A^*_{\beta e}} (\mathbf{n}_{\beta e} \cdot \nabla \boldsymbol{\psi}) \cdot \boldsymbol{\psi} \, \mathrm{d}A = \int_{V^*_\beta} \nabla \boldsymbol{\psi} : \nabla \boldsymbol{\psi}^T \, \mathrm{d}V. \tag{B.10}$$

Here, $A^*_{\beta e}$ represents the area of entrances and exits of a unit cell and $\mathbf{n}_{\beta e}$ represents the outwardly-directed unit normal vector over $A^*_{\beta e}$. The spatial periodicity indicated by Equation (B.6) requires that

$$\int_{A^*_{\beta e}} \mathbf{n}_{\beta e} \cdot \boldsymbol{\psi}\xi \, \mathrm{d}A = \int_{A^*_{\beta e}} (\mathbf{n}_{\beta e} \cdot \nabla \boldsymbol{\psi}) \cdot \boldsymbol{\psi} \, \mathrm{d}A = 0 \tag{B.11}$$

and we are left with

$$0 = \int_{V^*_\beta} \nabla \boldsymbol{\psi} : \nabla \boldsymbol{\psi}^T \, \mathrm{d}V. \tag{B.12}$$

From this we deduce that $\boldsymbol{\psi}$ is a constant vector and from either Equation (B.5) or Equation (B.7) we conclude that

$$\boldsymbol{\psi} = 0. \tag{B.13}$$

Use of this result in Equation (B.3) yields

$$-\nabla \xi = \mathbf{c}_1 \tag{B.14}$$

in which $\mathbf{c}_1$ is a constant vector. The solution for ξ is given by

$$\xi = -\mathbf{r} \cdot \mathbf{c}_1 + c_2 \tag{B.15}$$

in which $\mathbf{r}$ is the position vector and c_2 is a constant of integration. On the basis of the second of Equations (B.6) we deduce that $\mathbf{c}_1$ is zero and then we make use of the first of Equations (B.7) to conclude that

$$\xi = 0. \tag{B.16}$$

Here we have proved that $\boldsymbol{\psi}$ and ξ are zero when the nonhomogeneous terms in Equations (3.18) and (3.19) are zero and the porous medium is spatially periodic. However, the proof has been achieved with the use of a *specific* averaging volume that led to the right-hand side of Equation (B.3) being a constant. Clearly one could choose other averaging volumes that would lead to the right-hand side of Equation (B.3) being a function of position, and the proof would fail. Escape from this dilemma rests with Equations (B.7) for, strictly speaking, they require that the averaging volume used in the closure problem be comprised of unit cells, and this allows us to pass from Equations (B.8) to (B.9) without hesitation.

Appendix C: Closure for Flow in a Capillary Tube

As an example, we wish to use Equations (3.39)–(3.43) to analyze uniform flow in a capillary tube with the thought that any theoretical analysis of Stokes flow in porous media should produce the correct result for a bundle of capillary tubes. A little thought will indicate that for this special case only b_z and B_{zz} are required and the boundary-value problem takes the form

$$-\frac{\partial b_z}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_{zz}}{\partial r}\right)=\frac{2}{r_0^2}\int_0^{r_0}\left[-\frac{\partial b_z}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_{zz}}{\partial r}\right)\right]r\,\mathrm{d}r, \tag{C.1}$$

$$\frac{\partial B_{zz}}{\partial z}=0, \tag{C.2}$$

$$\text{B.C.1} \quad B_{zz}=-1, \qquad r=r_0, \tag{C.3}$$

$$\text{B.C.2} \quad B_{zz}(z+\ell)=B_{zz}(z), \qquad b_z(z+\ell)=b_z(z), \tag{C.4}$$

$$\int_0^{r_0} b_z r\,\mathrm{d}r=\int_0^{r_0} B_{zz} r\,\mathrm{d}r=0. \tag{C.5}$$

Here the length ℓ is arbitrary and we have made the obvious assumption that

$$\tilde{\mathbf{v}}_\beta=\mathbf{k}\,\tilde{v}_{\beta z}\,. \tag{C.6}$$

On the basis of Equation (C.4) we see that neither b_z nor B_{zz} is function of z and Equation (C.1) simplifies to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_{zz}}{\partial r}\right)=\frac{2}{r_0}\left(\frac{\partial B_{zz}}{\partial r}\right)_{r=r_0}. \tag{C.7}$$

The solution for B_{zz} is given by

$$B_{zz}=\frac{r^2}{4}\left(\frac{2}{r_0}\right)\left(\frac{\partial B_{zz}}{\partial r}\right)_{r=r_0}+C_1\ln r+C_2 \tag{C.8}$$

and we need only impose Equation (C.3) and require that B_{zz} is finite to obtain

$$B_{zz}=-\frac{r_0}{2}\left(\frac{\partial B_{zz}}{\partial r}\right)_{r=r_0}\left[1-\left(\frac{r}{r_0}\right)^2\right]-1. \tag{C.9}$$

Use of the second of Equations (C.5) provides the final solution for B_{zz} given by

$$B_{zz}=1-2\left(\frac{r}{r_0}\right)^2. \tag{C.10}$$

We now return to Equation (3.32) and extract the z-component to obtain

$$0=-\frac{\partial\langle p_\beta\rangle^\beta}{\partial z}+\rho_\beta g_z+\frac{2\mu_\beta}{r_0}\langle v_{\beta z}\rangle^\beta\left(\frac{\partial B_{zz}}{\partial r}\right)_{r=r_0}. \tag{C.11}$$

Use of Equation (C.10) leads to

$$\langle v_{\beta z}\rangle^\beta=-\frac{r_0^2}{8\mu_\beta}\left[\frac{\partial\langle p_\beta\rangle^\beta}{\partial z}-\rho_\beta g_z\right] \tag{C.12}$$

which is the Hagen–Poiseuille law for flow in a capillary tube. For a bundle of capillary tubes of uniform diameter, this represents the intrinsic phase average form of Darcy's law.

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